Kochina and Hele-Shaw in Modern Mathematics, Science and Industry

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Abstract

This paper reviews the profound effect of P. Ya. Kochina's ideas about free boundary problems on modern science, industry and mathematics.

1 Introduction

This brief article reviews some aspects of P. Ya. Kochina's most influential research, namely her investigations into free boundary problems for harmonic functions. Her ideas have had implications for many areas of quantitative science, including materials science, the environment, medicine and finance. Even within mathematics, they have stimulated many new developments in the areas of complex analysis, asymptotic analysis, and partial differential equations with free boundaries.

1.1 Hele-Shaw cells

One year before Kochina's birth, Hele-Shaw [14] first described his "cell", which was an experimental device for studying fluid flow by pumping a viscous liquid into the gap between two closely-separated glass plates. Using dye-lines, he was able to observe the flow patterns generated when the flow was impeded by various kinds of obstacles, such as aerofoil sections, placed between the plates. Thus, he was able to verify, with great accuracy, Stokes' prediction (in an appendix to Hele-Shaw's paper) that, assuming the Reynolds number is not too large, the velocity $\mathbf{u}$ in the plane of this cell is irrotational. The pressure plays the role of a potential which satisfies homogeneous Neumann data on the obstacle[1] so

[1] It was only in 1968 [36] that Stokes' analysis was modified to account for the three-dimensional flow that is necessary to satisfy the no-slip condition on the obstacle.

that, with a suitable scaling,

$$\mathbf{u} = -\nabla p, \quad \Delta p = 0 \quad \text{in the fluid,} \tag{1}$$

and

$$\frac{\partial p}{\partial n} = 0 \quad \text{on the obstacle.} \tag{2}$$

The Hele-Shaw cell became famous as an analogue computer for Laplace's equation, and thus it was particularly useful for visualising two-dimensional flows in porous media, assuming they are slow enough to be governed by Darcy's law. However, for the next fifty years, this was thought to be the only scientific value of the Hele-Shaw cell, at least by many Western scientists.

1.2 Kochina's free boundary models in filtration

In the 1930's and 1940's, Kochina realised that many groundwater flows, especially dam problems, led to models in which the saturated region must be separated from the dry region by a free boundary Γ, which has to be determined as part of the problem [26, 27]. She, along with Muskat [22] and Galin [13], argued that at such an interface the pressure should be approximately constant and equal to that in the dry region, and that mass conservation requires that the liquid velocity normal to Γ be proportional to v_n, the normal velocity of Γ. Thus, without loss of generality, and with y in the vertical direction,

$$\mathbf{u} = -\nabla(p + \rho g y), \quad \Delta p = 0 \quad \text{in the fluid} \tag{3}$$

with

$$p = 0, \quad -\frac{\partial}{\partial n}(p + \rho g y) = v_n \quad \text{on } \Gamma, \tag{4}$$

where g is a measure of the strength of gravity.[2]

This realisation immediately cast the Hele-Shaw cell in a new role. It meant that, as well as in its traditional use as an analogue computer for linear potential problems, a Hele-Shaw cell with cavities enables the solutions of (3), (4) to be readily visualised. Indeed, Kochina [26, p. 243] was quickly able to obtain close agreement between some of her ingenious exact solutions to (3), (4) and the flows she observed in her cell.

Kochina's discovery that such nonlinear problems could be simulated so easily was a revelation in itself, but there were to be far more dramatic consequences.

1.3 The applicability of the Hele-Shaw model in modern science and technology

The list of scientific problems whose mathematical models can be reduced to (3), (4) is increasing year by year. Here we will only cite some of those that have

[2]The model (3), (4) is now often referred to as the "Hele-Shaw free boundary problem", despite the inappropriateness of this name.

appeared since the groundwater model was first suggested: the list is intended only to give a rough idea of the scope of the research that is encompassed.

First and foremost is the dramatic increase in new "Stefan" models that have been proposed. The original Stefan model concerned the albedo, but now it appears in many simulations in materials science, chemistry and biology. The prototype is for the melting or freezing of a material initially at its phase-change temperature, under the assumption that heat flows purely by conduction and that there is a prescribed latent heat. Then, as long as the specific heat is negligible, p in the zero-gravity limit of (3), (4) can be interpreted as the temperature (or concentration) in the new phase, with the melting temperature equal to zero and the dimensionless latent heat (or concentration jump) equal to unity. This model is the basis for the scientific study of processes ranging from steel making [12] to semiconductor fabrication [19], or from food freezing to laser welding [1]. On the other hand, the identification of p as the electric potential leads to models for electrochemical machining or forming [21], while tumour necrosis can be modelled by setting p to be the concentration of a biological agent, again with the incorporation of diffusion as in the Stefan problem [25].[3]

In a completely different vein, it is easy to verify that, if we define $\omega(x, y)$ to be the time at which Γ reaches the point (x, y) in the plane of a Hele-Shaw cell modelled by (3), (4) with $g = 0$, then the function

$$u(x, y, t) = \int_{\omega}^{t} p(x, y, \tau)\, d\tau \tag{5}$$

satisfies $\Delta u = 1$ and is thus the transverse displacement of a membrane under a uniform pressure. Moreover, the free boundary conditions (4) imply that $u = \partial u/\partial n = 0$ on Γ. Thus u has an interpretation in contact mechanics as the displacement of an inflated membrane pressed against a smooth rigid plane, and varying time leads to a one-parameter family of such static contact problems. (With diffusion reinstated, this problem emerges in the theory of optical stopping times for financial options, when p is related to the value of the option and the space variables are the values of the underlying stocks [38].) It is ironic that when the map $p \mapsto u$, which is commonly called a "Baiocchi transformation", was suggested in the 1960's in one of the first applications of the theory of variational inequalities (see [2]), it was used to prove existence and uniqueness results for the weak solution of the very same dam problem for which Kochina had obtained the explicit classical solution twenty years earlier.

This list could be extended almost indefinitely, but here we must go on to remark that there is also a long catalogue of intensively-studied mathematical models which are not free boundary problems but of which (3), (4) is a singular limit, as indicated below. Hence, it behoves the many p.d.e. researchers who study the following models to be aware of Kochina's ideas. Some of these models have more physical relevance than others, but they have all stimulated exciting new investigations in p.d.e. theory:

[3] It is noteworthy that in many of the "Stefan" generalisations of (3), (4), the dimensionless parameter that compares temporal rates of change in the bulk with the rate of change of Γ is small, so that (3), (4) is a relevant approximation.

- The Allen–Cahn equation

$$\tau \frac{\partial u}{\partial t} = \epsilon^2 \Delta u + u - u^3$$

as $\epsilon, \tau \to 0$;

- The Cahn–Hilliard equation

$$\tau \frac{\partial u}{\partial t} = -\Delta(\epsilon^2 \Delta u + u - u^3)$$

as $\epsilon, \tau \to 0$;

- The phase-field equations

$$\delta \frac{\partial u}{\partial t} = \epsilon^2 \Delta u + u - u^3 + \alpha T,$$
$$\tau \frac{\partial T}{\partial t} + \frac{\partial u}{\partial t} = \Delta T$$

as $\epsilon, \tau \to 0$;

- The porous medium equation[4]

$$\tau \frac{\partial u}{\partial t} = \nabla \cdot (u^m \nabla u)$$

as $m \to \infty$, $\tau \to 0$.

In each case, all the parameters are constants and limits have to be taken appropriately (see [3] for the first three and [9] for the last).

Again, this list could be extended considerably but we must now comment on what is the most fundamental theoretical attribute of (3), (4).

1.4 Ill-posedness and well-posedness

Kochina's famous explicit solutions of (3), (4), to be discussed further in §2.4, highlighted the "blow-up" properties of the solution in the unstable or suction cases, when the fluid enclosed by Γ is shrinking in area. Conversely, Kochina's solutions all gain smoothness when the fluid region expands. This is in accord with the linear stability analysis of Hill [15], and in 1958 Saffman & Taylor [33] encountered one of the most dramatic illustrations of this irreversibility. By extracting the fluid from one end of a cell in the form of a long, parallel-sided channel, they found that the "finger" of air that was eventually sucked towards the end of the channel occupied approximately one half of the channel width. However, their travelling wave analysis of (3), (4), which used a simplification of Kochina's conformal mapping procedure, led to a one-parameter family of

[4]This equation has even been proposed as a model for the spread of galactic civilisation [24]. However, it is often used in the theory of various kinds of groundwater flow, as was well-known to Kochina.

fingers and the pattern-selection problem thus posed has challenged engineers, mathematicians and physicists ever since. Again we encounter a situation where Kochina has stimulated an area of research now addressed in hundreds of papers and books. Indeed, the "Saffman–Taylor" instability, which is the phrase now commonly used to describe the ill-posedness of (3), (4) where the fluid region shrinks, has prompted many ingenious studies of regularised versions of (3), (4) whose objective is to understand phenomena such as dendritic growth and mushy regions. The morphologies that occur in such phenomena are subject to an unpredictability akin to that in turbulence, and (3), (4) with all its mathematical structure, is at the heart of the scientific basis of this unpredictability. Hence, we now make some more detailed remarks about this mathematical structure.

2 The Mathematics of Hele-Shaw free boundary problems

2.1 Explicit solutions via complex variable methods

We have already mentioned that Kochina found the explicit solution to the celebrated canonical problem of flow under gravity through a rectangular porous dam. This solution is probably the best known application of the method that she devised to deal with a commonly occurring class of free boundary problems for Laplace's equation, in which there are two independent linear relations between the independent variables (x, y) and the potential $-p$ and streamfunction ψ on each separate segment of the flow domain. The key point is that when the physical plane and the complex potential plane are mapped onto an auxiliary half-plane, the resulting Riemann–Hilbert problem can be solved in terms of Riemann P-functions. This link between free boundary problems and Fuchsian differential equations was a prominent feature of Kochina's work throughout her career [26], and it remains an active research area [6].

Far more influential, however, was the complex variable method that Kochina and Galin developed to deal with unsteady zero-gravity Hele-Shaw flows. Again, the key idea involves a conformal mapping of the physical $(x + iy)$ and complex potential $(-p + i\psi)$ planes onto an auxiliary domain, usually the unit disc $|\zeta| < 1$. Because the real part of the complex potential vanishes on the free boundary, one of these mappings is trivial, but the determination of the map $f(\zeta, t)$ from $|\zeta| < 1$ onto the fluid domain leads to the problem of finding a univalent conformal map satisfying the nonlinear boundary condition

$$\Re\left(\frac{1}{\zeta}\frac{\partial f}{\partial \zeta}\overline{\frac{\partial f}{\partial t}}\right) = \Re W(\zeta, t), \tag{6}$$

in which the right-hand side represents the transformed pressure which, being a solution of Laplace's equation in the known domain $|\zeta| < 1$, is fully determined by the driving mechanism imposed combined with the condition $p = 0$ on Γ.

It is a remarkable feature of this formulation of the problem that, for simple driving mechanisms such as point sources/sinks, exact solutions can be found using many simple maps (polynomials, rational and log-rational functions). The procedure is to assume a specific functional form for $f(\zeta, t)$ with time-dependent coefficients — Kochina's paper of 1945 [27] gave the "limaçon" example $f(\zeta, t) = a_1(t)\zeta + a_2(t)\zeta^2$ — and, on substitution into (6), observe a cancellation of extraneous terms which leaves precisely as many equations as there are unknown coefficients. There is a very large literature describing solutions of this type (in many cases rederivations of solutions published in the Russian literature in the 1950's); see the online bibliography at `www.maths.ox.ac.uk/~howison/Hele-Shaw/` .

2.2 Moments

Hele-Shaw flows have a deceptively simple geometric structure, in that the 'moments' of the fluid domain evolve in a predictable way [28]. To see this, in the simple case where the flow is driven by a single point sink at the origin of strength Q, we write the field equation as

$$\Delta p = -Q\delta(x)\delta(y) \qquad \text{in } \Omega(t).$$

For any function $L(z)$ analytic on $\Omega(t)$, use of Green's theorem shows that [29, 30]

$$\frac{d}{dt}\iint_{\Omega(t)} L(z)\, dxdy = \int_{\Gamma(t)} L(z) v_n\, ds = -\int_{\Gamma(t)} L(z)\frac{\partial p}{\partial n}\, ds = QL(0).$$

In particular, taking the integrand $L(z) = z^k$ for positive integers k we obtain the infinite set of moments $M_k(t)$, $k = 0, 1, \ldots$, which satisfy

$$\frac{d}{dt}M_k(t) \equiv \frac{d}{dt}\iint_{\Omega(t)} z^k\, dx\, dy = Q\delta_{0k}. \tag{7}$$

Thus, all the moments are constant except the area ($k = 0$), which changes at the rate Q. Indeed, the evolution of the moments gives the solution of the differential equations that arise when comparing coefficients in a 'brute force' analysis by direct substitution into (6).

The moment result for a point source/sink generalises easily to the case of a system of sources/sinks within Ω [29], or to multipole singularities [11]. It also leads immediately to a connection with problems of inverse gravitation in two space dimensions. If we define the *Cauchy transform* of Ω by

$$\Theta(z, \bar{z}, t) = \frac{1}{\pi}\iint_{\Omega} \frac{dx'dy'}{z - z'}$$

then it is easy to see that Θ is proportional to the z-derivative of the gravitational potential generated by a uniform density in Ω. Its Laurent expansion for large $|z|$ is

$$\sum_0^\infty \frac{M_k}{z^{k+1}},$$

and indeed this approach can easily be used to generate multiple solutions to the problem of recovering a domain from its moments [37].

2.3 The Baiocchi transform, the Schwarz function, and variational inequalities

In Section 1.3 we introduced the Baiocchi transform $u(x, y, t)$ of the pressure via (5), showing that it satisfies the free boundary problem

$$\nabla^2 u = 1 \quad \text{in regions crossed by } \Gamma \text{, and elsewhere by analytic continuation,}$$

with

$$u = \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma.$$

We can also introduce the Schwarz function [7] of the free boundary, writing Γ in the form $\bar{z} = g(z, t)$; this is always possible for (piecewise) analytic curves and $g(z, t)$ is analytic in a neighbourhood of any smooth point of Γ. Following [20], since $4\partial^2 u/\partial z \partial \bar{z} = 1$ in Ω and both $\partial u/\partial z = \partial u/\partial x - i\partial u/\partial y$ and $\bar{z} - g(z, t) = 0$ on Γ, we have by analytic continuation that

$$\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{1}{2}\left(\bar{z} - g(z, t)\right) \quad \text{in } \Omega.$$

Since also $\partial u/\partial t = p$, we see that the singularities of u, and hence of $g(z, t)$ inside Ω must either be constant in time or time-integrals of specified singularities of p. In a similar vein, following [28], the Cauchy transform can be recast in the form

$$\Theta(z, \bar{z}, t) = \begin{cases} \Theta_e(z, t) & \text{outside } \Omega, \\ \bar{z} + \Theta_i(z, t) & \text{inside } \Omega, \end{cases}$$

where Θ_e and Θ_i are analytic inside and outside Ω respectively. Indeed, from the definition of $g(z, t)$, we have $\Theta_e(z, t) - \Theta_i(z, t) = g(z, t)$, from which it is clear that the two approaches are essentially equivalent (but note that if the fluid domain is multiply connected, only the Cauchy transform is useful [31]).

If we additionally specify the constraint $u \geq 0$, then as mentioned above u satisfies a well-posed variational inequality [10]. Once a one-parameter family $u(x, y, t)$ of such solutions has been found, with specified singularities (or boundary values) varying with t, its time derivative is the pressure in a Hele-Shaw flow. Conversely, any Hele-Shaw suction flow whose solution exists until all the fluid has been removed or, in an infinite domain, for all t, is the time-derivative of such an obstacle problem [8]. The blow-up mentioned earlier is associated with negative regions of $u(x, y, t)$ reaching Γ, and (with some exceptions to be discussed later) it cannot occur when the constraint $u \geq 0$ is satisfied.

2.4 Blow-up

We have already mentioned Kochina's realisation that receding Hele-Shaw flows without surface tension can exhibit finite-time blow-up, a famous example being the limaçon solution [27]. In fact, as any zero-surface tension problem is time-reversible, blow-up solutions can, in principle, be generated by injecting with a non-smooth free boundary and then reversing the sequence of solutions so obtained. This procedure can reveal unexpected features, an example being the 'waiting time' that can occur when injection takes place into an initial domain with a corner [18]. The analysis of this situation again involves the Baiocchi transform of the pressure, and this device is also instrumental in the analysis of allowable cusps in *injection* problems [16]. It can be shown [34] that the obstacle problem (for the Baiocchi transform) can have singularities in its free boundary of $(4n+1)/2$-power type, and no others, and by virtue of the discussion above, it is also possible for a Hele-Shaw free boundary to develop such a cusp at one time while remaining smooth before and after this time; an example of a $5/2$-power cusp is given in [16]. Note that in these examples, the Baiocchi transform of the pressure does not breach the constraint $u \geq 0$ near the cusp.

2.5 Exponential asymptotics

The instabilities inherent in Kochina's solutions for Hele-Shaw flows in shrinking regions have given a new stimulus to the theory of asymptotic expansions, and more particularly to "asymptotics beyond all orders" or "exponential asymptotics" [23]. We have already remarked that the Hele-Shaw model is a simple paradigm for the delicate phenomenon of crystal growth, a shrinking (resp. expanding) fluid region being identified with a supercooled (resp. normal) liquid melt from which the crystal grows. It has long been a goal of materials science to understand the thermodynamic and mechanical balances that select the crystal shape, and a famous analogous "pattern selection" problem for Hele-Shaw flows arises out of the work of Saffman and Taylor [33] mentioned earlier. Their theory indicated that, with no surface tension effects at the interface, there is a one-parameter family of fingers penetrating the channel; but their experiments, and many others [32], suggest that when surface tension effects are small, the finger of asymptotic width approximately half that of the channel is selected.

These considerations prompt the question of what is the behaviour of the system (3), (4) when the condition $p = 0$ on Γ is replaced by a regularisation such as $p = -\epsilon v_n$ or $p = \epsilon\kappa$, where κ is the (appropriately signed) curvature of Γ and ϵ is a small positive parameter. (The Baiocchi transform with the constraint that $u \geq 0$, and the "smoothed" models listed in §1.3 can also be thought of as regularisations, but in a different spirit.)

Twenty years ago this would have been regarded as an open problem in singular perturbation theory. For example, in the Saffman–Taylor problem a straightforward expansion in powers of ϵ does not lead to a selection principle for the finger, and indeed it does not suggest that the effect of the regularisation is any other than a small perturbation of the unregularised solution. However, the new methodology (see [23] for an early collection of results and [5] for recent work on the Saffman–Taylor problem) has shown how dramatic the effect of the regularisation can be, at least for steady states or travelling waves. The ingenious procedure involves first reformulating the problem as a mixed boundary value problem in a half-space and then as a nonlinear integrodifferential equation for the slope of Γ. Next, the independent variable s is complexified and a WKB expansion is carried out in terms of ϵ in order to reveal the Stokes lines of the solution as a function of s; these lines are born at the singularities of the unregularised solution. (The structure of the solution near these singularities can, by itself, be used to find a solvability condition, as shown in [5].) Finally, the only admissible solutions are those whose Stokes lines behave in such a way that the relevant fixed boundary and symmetry conditions can be satisfied, and this turns out to be the case when the regularisation parameter ϵ takes on one of a discrete set of values. For the Saffman–Taylor problem, this shows that there is a countably infinite set of finger widths, whose limit as $\epsilon \to 0$ is $\frac{1}{2}$.

We are thus led to yet another aspect of the interplay between Kochina's work and complex variable theory. This is another story that is far from complete, because there is no generally accepted theory of exponential asymptotics for evolution problems, especially those that blow up at a finite time t^*. Controversy still rages over whether or not unregularised solutions can give useful information for times less than t^*; one theory [35] proposes that "daughter singularities" can emerge from the singularities of the analytic continuation of the unregularised problem, and propagate in such a way as to make the regularised solution differ from the unregularised solution by $O(1)$ for times that are $O(1)$ before t^*, and numerical evidence for this is given in [4].

2.6 The two-phase (Muskat) problem

An obvious generalisation of the Hele-Shaw problem described above is to introduce a second fluid of non-negligible viscosity into the cell, the free boundary Γ now being the interface between the two fluids. Thus the fluid domain Ω consists of two regions Ω_1 and Ω_2, separated by an interface Γ, and the fluid velocity in region i, $i = 1, 2$ is given by $\mathbf{u}_i = -k_i \nabla p$, where k_i are the mobilities, inversely proportional to the viscosities. The pressures p_i are again harmonic in Ω_i, but now the free boundary conditions are

$$p_1 = p_2, \qquad -k_1 \frac{\partial p_1}{\partial n} = -k_2 \frac{\partial p_2}{\partial n} = v_n, \tag{8}$$

expressing continuity of pressure and normal velocity respectively. The first of these conditions makes this problem very much harder than the one-phase problem, since we no longer have a constant pressure on Γ. In particular, the complex variable methods that work so well for the simpler problem become much less helpful (for a brief description of what can be done see [17]), and other theoretical approaches appear difficult. For example, the moment approach described above leads to an apparently unstudied generalisation of the classical moment problem, as we now show.

Suppose, for definiteness, that fluid 1 occupies a simply-connected domain Ω_1 containing a point source/sink of strength Q at the origin, and that fluid 2 occupies an annular region Ω_2, bounded inside by the interface Γ between the fluids and outside by a second interface Γ', the region exterior to Ω_2 being at constant (zero) pressure. (The point of this configuration is that it avoids difficulties associated with infinite regions and/or fixed boundaries.) Thus, in addition to (8) holding on Γ, we have

$$\Delta p_1 = -Q\delta(x)\delta(y) \quad \text{in } \Omega_1, \qquad \Delta p_2 = 0 \quad \text{in } \Omega_2, \tag{9}$$

and

$$p_2 = 0, \quad -k_2 \frac{\partial p_2}{\partial n'} = v_n' \qquad \text{on} \quad \Gamma'. \tag{10}$$

Now let $L(z)$ be analytic in Ω, and consider

$$\begin{aligned}
\frac{d}{dt}\left(\iint_{\Omega_1} \frac{L}{k_1}\,dxdy - \iint_{\Omega_2} \frac{L}{k_2}\,dxdy\right) &= \int_\Gamma \frac{Lv_n}{k_1}\,ds - \int_{\Gamma'} \frac{Lv_n}{k_2}\,ds' + \int_\Gamma \frac{Lv_n}{k_2}\,ds \\
&= \int_\Gamma -L\frac{\partial p_1}{\partial n}\,ds + \int_{\Gamma'} L\frac{\partial p_2}{\partial n}\,ds' - \int_\Gamma L\frac{\partial p_2}{\partial n}\,ds \\
&= QL(0) + \int_\Gamma -p_1\frac{\partial L}{\partial n}\,ds + \int_{\Gamma'} p_2\frac{\partial L}{\partial n}\,ds' + \int_\Gamma p_2\frac{\partial L}{\partial n}\,ds \\
&= QL(0);
\end{aligned}$$

we have used Green's theorem and the boundary conditions (9), (10), and we note the sign changes due to the fact that if $\mathbf{n}$ points out of Ω_1 it points into Ω_2.

It follows that the generalised moment

$$\iint_{\Omega_1} \frac{L}{k_1}\,dxdy - \iint_{\Omega_2} \frac{L}{k_2}\,dxdy$$

is constant if $L(0) = 0$ and changes linearly in t if $L(0) \neq 0$. As far as we know, this version of the moment problem has never been studied, but its solution could have important consequences. Although the linear stability analysis of a planar interface still predicts catastrophic instability when the mobility of the displacing fluid exceeds that of the displaced fluid, it is not known how the presence of the second fluid affects the blow-up that commonly occurs in contracting single fluid problems. It is possible that the effect of the second fluid on, say, blow-up via a 3/2–power cusp may be quite dramatic, since such a geometry can only occur in the two-fluid case if the displacing fluid can be squeezed out of the developing cusp sufficiently rapidly.

3 Conclusion

This article does scant justice to the multi-faceted nature of P. Ya. Kochina's work on free boundary problems. We have tried to highlight some of the principal mathematical innovations that have been stimulated by her research and to indicate the most glaring deficiencies in our current knowledge. The phenomenal growth of interest in free boundary problems of all kinds guarantees that Kochina's seminal ideas will affect the thinking of mathematicians and scientists for many decades to come.

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