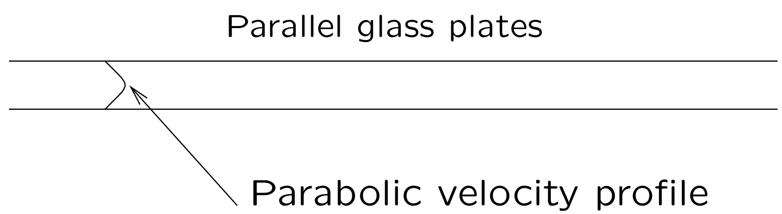
The Hele-Shaw Problem 1898–2004

History

HS Hele-Shaw, inventor of the Hele-Shaw cell
(and the variable-pitch propeller)

Physical set-up (1898)



- Locally plane Poiseuille flow of Newtonian viscous fluid. Take (x, y) coordinates in plane of cell, pressure approximately $p(x, y, t)$ (Stokes 1898).

- Gap-averaged equations

$$\mathbf{u} = -\frac{h^2}{12\mu}\nabla p, \qquad \nabla \cdot \mathbf{u} = 0 \quad \text{so} \quad \nabla^2 p = 0.$$

- Analogue for potential flow; no advance for half a century.

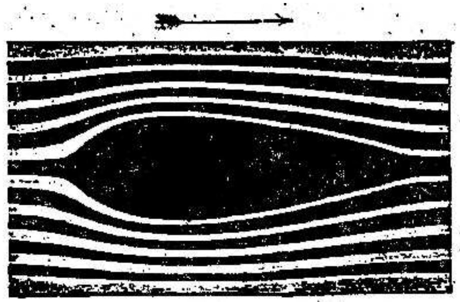


FIG. 9.—Section of screw shaft s rut (broad colour bands in thin sh

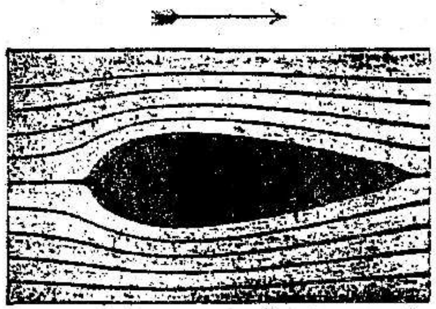


FIG. 10.—Section of screw shaft strut (narrow colour bands in t in

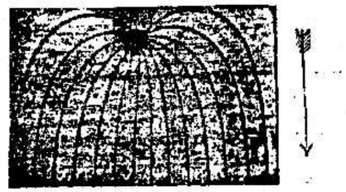


FIG. 7.—Uniform stream and "sink" in channel.

y comparing the theoretical lines of force w
s worked out for electrical and magi
Fig. 7 is a case of the flow of water throu
d in hydro-mechanics a "sink"), and w
is to the flow of electricity from an electr
one of the wires of a wire grating (see C
"Magnetism and Electricity," Fig. xiii.,

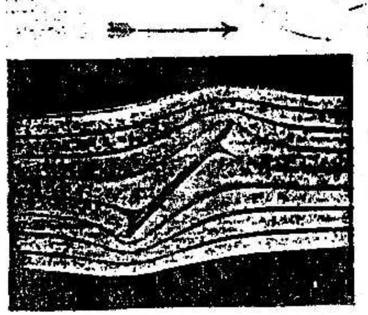


FIG. 8.—Inclined plate in thin sheet.

Flow past the propeller strut of one of Her Majesty's cruisers; Rankine body and flow past a flat plate.

Groundwater flow

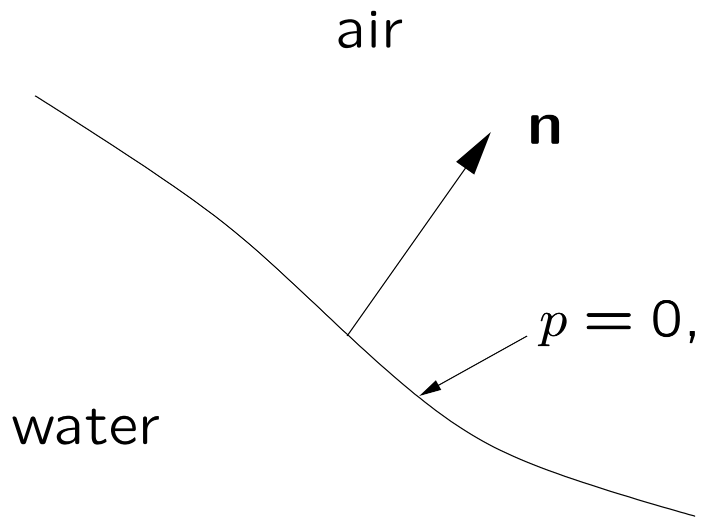
The 19th century French sewage engineer Darcy established the law

$$\mathbf{u} = -\frac{K}{\mu}\nabla(p + \rho g y), \quad y \text{ vertically upwards}$$

for flow of a liquid of viscosity μ through rock of permeability K. When $\nabla \cdot \mathbf{u} = 0$ we again have $\nabla^2 p = 0$ and vertical Hele-Shaw cells can be used to simulate groundwater flows, identifying K with $h^2/12\mu$.

From now on I use units where $K = 1$, $h^2/12\mu = 1$.

Later workers added free surfaces on which, in steady flow, we have the boundary conditions below. This opened the way to complex variable/hodograph methods.



$$p = 0,\ \frac{\partial}{\partial n}(p + \rho g y) = 0$$

Note that the condition $p = 0$ ignores any kind of surface tension effects at the free boundary.

Classical theory

The dam problem

A famous canonical problem was the flow from one lake to another through a rectangular dam. It was solved by P.Ya. Polubarinova–Kochina (also by Hamel). P–K developed a connection with the Riemann $\mathcal{P}$–function, Hilbert problems and Fuchsian differential equations which is still an active area of research.

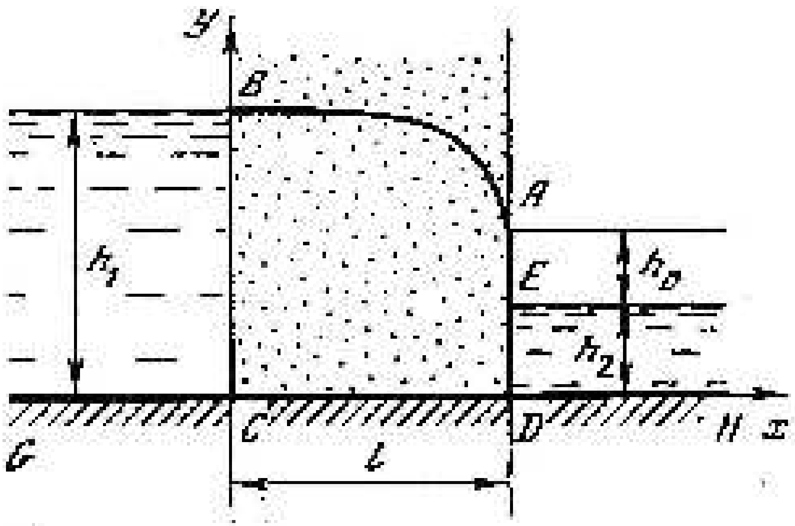


Рис. 3

лением p таким образо

$$\varphi = -k\left(\frac{p}{\rho g} + y\right),$$

где k — коэффициент ф
ции земляного слоя, p
ление, ρ — постоянная
ность, g — ускорение
тяжести, y — высота, от
ваемая от непроницаемо
нования.

Рассмотрим условия
типа

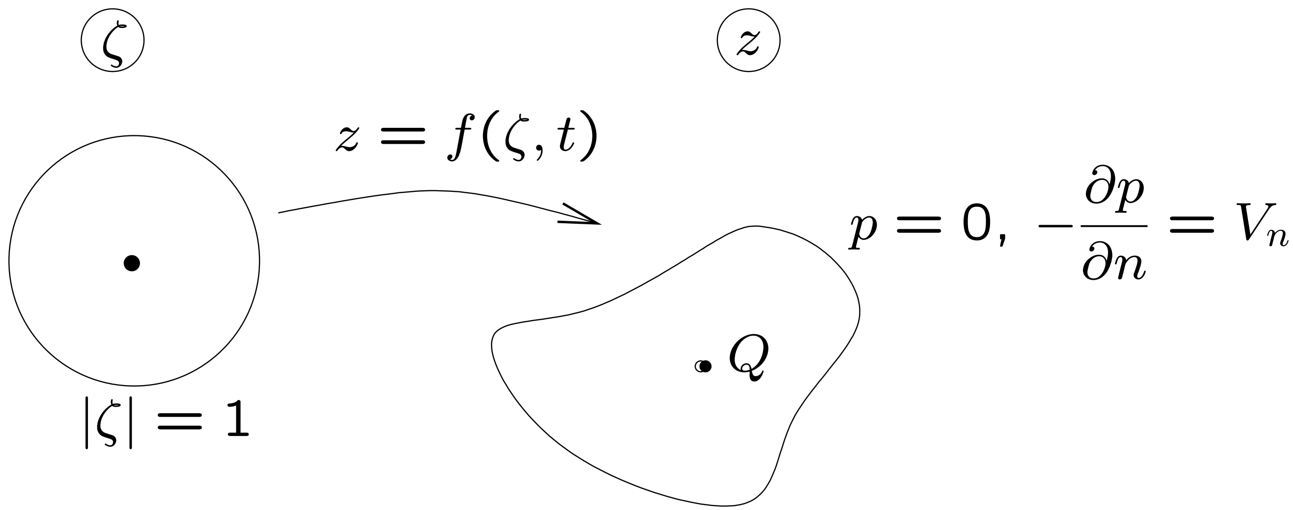
Around this time the *Muskat-Leibenzon* problem for two immiscible fluids with an interface was formulated. It is *much* harder than the one-fluid case because the pressure is not constant on the interface.

Pelageya Yakovlevna Polubarinova–Kochina (1899–1999).
Her published papers span the interval (1924, 1999).

Unsteady flows: the Polubarinova–Galin equation

1945: P–K and Galin independently wrote down a reformulation of *unsteady* one-fluid groundwater flows (with $g = 0$). They mapped the flow domain onto a canonical domain such as the unit circle (essentially the potential plane).

Consider flow driven by source/sink of strength Q at $z = 0$. Again take $p = 0$ on the free boundary.



Complex potential is $w(z,t) = \frac{Q}{2\pi}\log(\zeta(z,t))$. The free boundary condition $p = 0$ gives $Dp/Dt = 0$ so

$$\frac{\partial p}{\partial t} - |\nabla p|^2 = 0 = \Re\left(\frac{\partial w}{\partial t} - \left|\frac{\partial w}{\partial z}\right|^2\right)$$

on free boundary becomes

$$\Re\left(\frac{Q}{2\pi\zeta}\frac{\partial \zeta}{\partial t} - \frac{Q^2}{(2\pi)^2}\right) = 0$$

on $|\zeta| = 1$. Since $0 = \dfrac{\partial f}{\partial \zeta}\dfrac{\partial \zeta}{\partial t} + \dfrac{\partial f}{\partial t}$, we get

$$\Re\left(\zeta\frac{\partial f}{\partial \zeta}\overline{\frac{\partial f}{\partial t}}\right) = \frac{Q}{2\pi} \quad \text{on} \quad |\zeta| = 1,$$

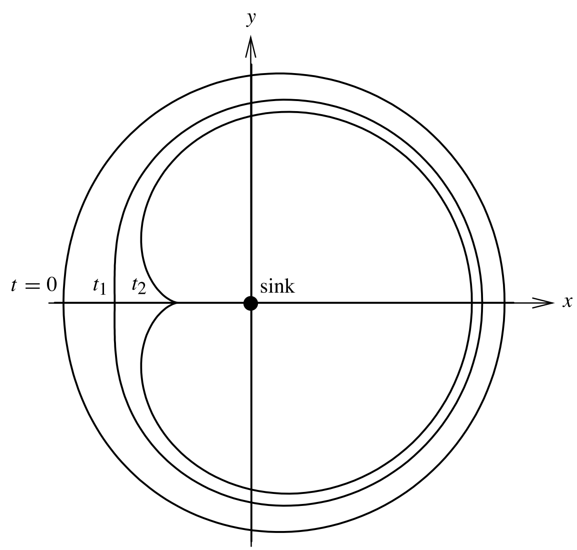
the *Polubarinova–Galin* or *Laplacian growth* equation. This reformulation is nonlinear and nonlocal but it is in a fixed domain.

Note immediately that the problem is completely time-reversible.

Using this method P–K and others found many explicit solutions, for example

$$z = f(\zeta, t) = \sum_{1}^{N} a_n(t)\zeta^n;$$

also rational functions work.* The case $N = 2$ and a sink at $z = 0$ starts with a limacon which becomes a cardioid with a cusp. There is no continuation for later times.



*Many of these solutions have been rediscovered, sometimes more than once.

Saffman and Taylor 1958

Saffman & Taylor 1958

They used a Hele-Shaw cell as an analogue for free surface flow in porous media.

less viscous fluid more viscous fluid

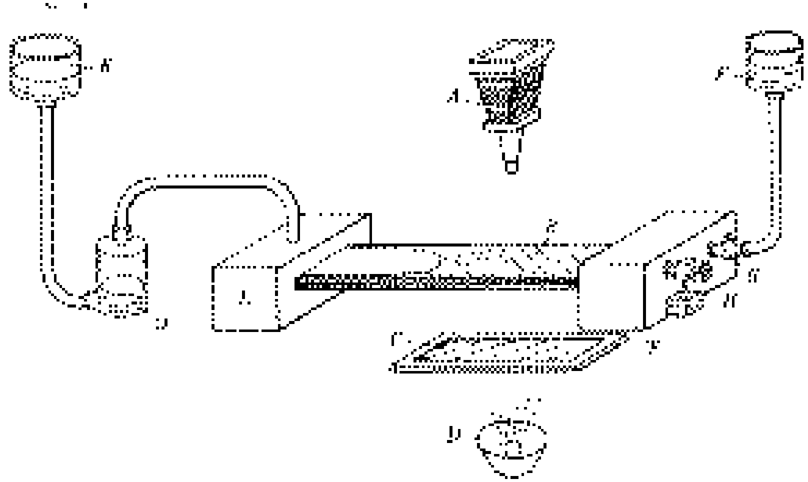
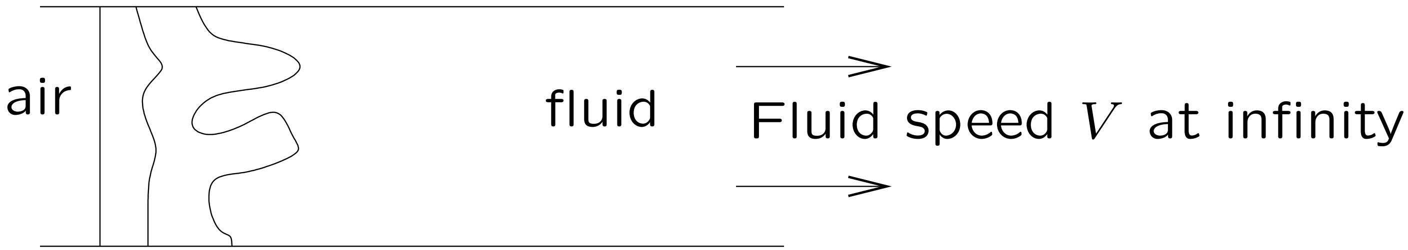


Figure 10. Arrangement of long Hele-Shaw cell for photographing fingers

Their experiments showed:

(1) The instability of an interface moving towards the *more viscous* fluid.



(2) Growth of a single long finger.

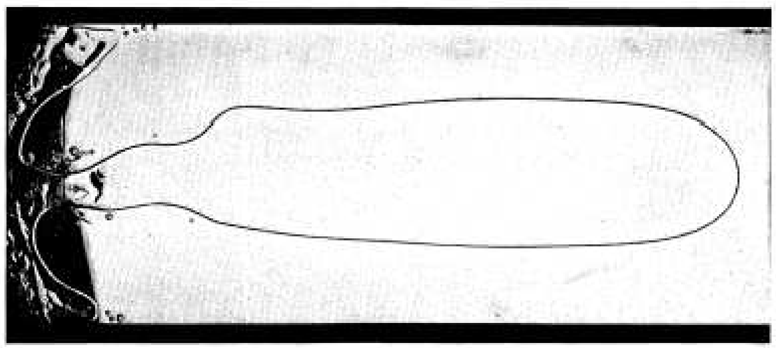


Figure 8

The Saffman–Taylor instability

S & T carried out a linear stability analysis of the moving interface $x = Ut + \epsilon e^{\alpha t} \sin ny$ to find

$$\text{growth rate} \quad \alpha = nU \quad (U > 0 \text{ is receding fluid})$$

This is catastrophically unstable (cf. Kelvin–Helmholtz for a vortex sheet). Blow-up is at least plausible since a smooth initial interface with exponentially decaying Fourier coefficients may lose smoothness because of the exponential growth.

S & T also introduced *surface tension* by saying

$$p = -\gamma\kappa \qquad \gamma \text{ is surface tension, } \kappa \text{ is curvature)}$$

Then the linear analysis gives

$$\alpha = nU - \gamma n^3.$$

This removes the short wavelength instability.

S & T finger solutions

S & T use complex variable (cf Helmholtz flows) to find a one-parameter family of travelling wave finger solutions, mapped from $|\zeta| < 1$ onto the fluid by

$$z = \frac{Vt}{\lambda} + \zeta + 2(1-\lambda)\log\tfrac{1}{2}(1+e^{-\zeta}), \qquad 0 < \lambda \le 1.$$

The parameter λ, the fraction of the finger occupied by air, is undetermined in the solution. In experiment, λ was repeatedly found to be close to $\frac{1}{2}$ except for very slow flow. Saffman later found unsteady solutions which start with a nearly flat interface and end up as a finger. Again, λ is arbitrary.

This is the selection problem: why is $\lambda = \frac{1}{2}$ selected?

The curve corresponding to $\lambda = \frac{1}{2}$ has many curious mathematical properties but none obviously gives selection.

The upper plot shows λ against capillary number $\mu U/\gamma$ where U is the speed of withdrawal. The lower plot compares theory and experiment, with λ fitted to data.

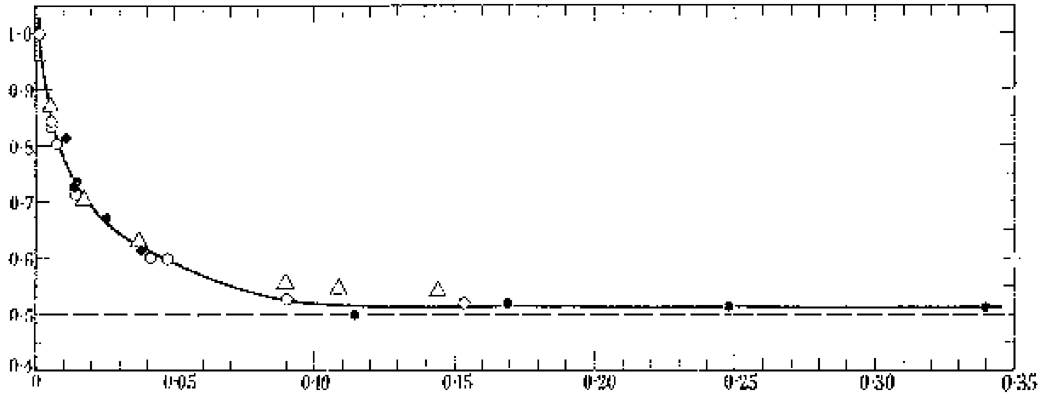


FIGURE 14. Measured values of λ for water penetrating into two oils: △, Diala (photographic measurements); ○, Diala (direct measurement); ●, Talpa (direct measurements).

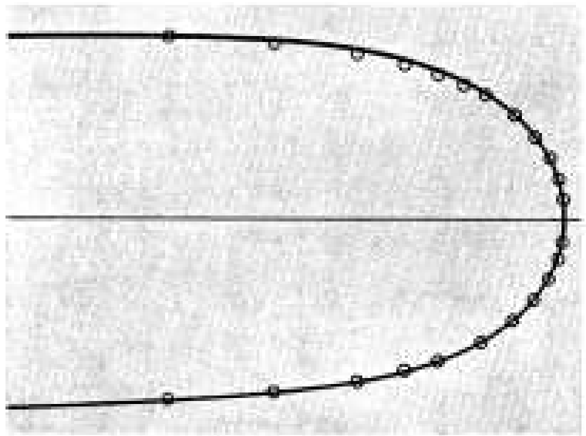


FIGURE 9

(Facing p. 318)

In summary, two main issues were raised:

- Instability and the nature of blow-up.

- Selection principles for finger solutions.

Contexts

Beyond Hele-Shaw: other models

The Hele-Shaw free boundary problem, or a variant of it, occurs in a huge variety of other areas. Among the most important:

• The one-phase Stefan problem for phase-change of a pure material (say ice) in water at the melting temperature is

$$\frac{\rho c}{k}\frac{\partial T}{\partial t} = \nabla^2 T \quad \text{in the ice}$$

with

$$T = 0, \quad \frac{\partial T}{\partial n} = -LV_n \quad \text{on the phase-change boundary}$$

where L is the latent heat. Setting $\rho c = 0$ recovers the Hele-Shaw problem; also the Hele-Shaw surface tension condition $p = \gamma\kappa$ is the Gibbs-Thomson surface energy condition.

• Squeeze films, in which the upper plate is moved normally, lead to the problem

$$\nabla^2 p = \frac{1}{h^2}\frac{dh}{dt}, \qquad p = 0, \quad -\frac{\partial p}{\partial n} = V_n \quad \text{on the boundary}$$

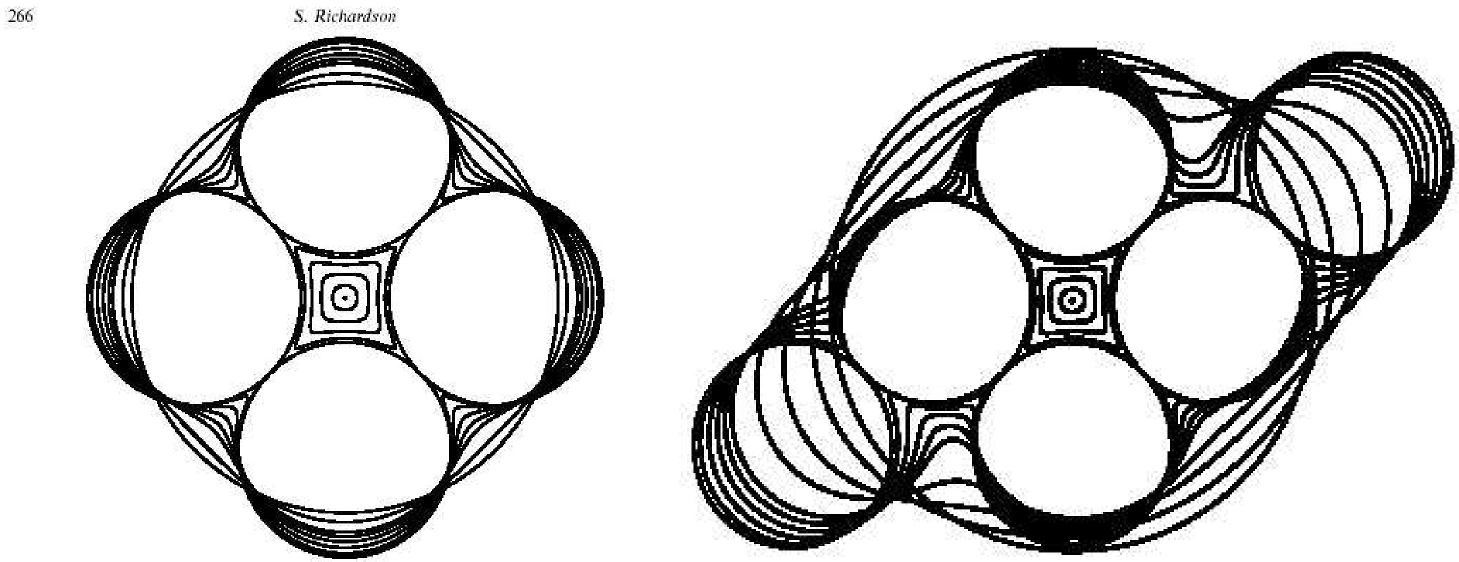
with many interesting properties (also the inviscid version of this problem).

• 2-D Stokes flow with free surfaces. Complex variables work for Stokes flow because the solutions of the biharmonic equation $\nabla^4 \psi = 0$ can be written

$$\psi = \Re(\overline{z}F(z) + G(z)), \qquad F \text{ and } G \text{ analytic.}$$

The free boundary conditions come out nicely *even with surface tension* and the theoretical structure is remarkably similar to Hele-Shaw. Explicit *unsteady solutions with surface tension* can be constructed using conformal maps.

Sintering of circles under surface tension (S Richardson)

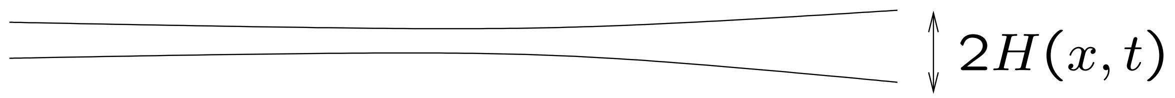


Mathematical contexts

• Hele-Shaw is an extreme example of nonlinear diffusion, when written as

$$\frac{\partial}{\partial t}\mathcal{H}(p) = \nabla^2 p, \qquad \mathcal{H}(\cdot) = \text{Heaviside function.}$$

Also a long thin thread in a Hele-Shaw cell



$$p \sim -\gamma H_{xx}, \quad u \sim -p_x \sim -\gamma H_{xxx}, \quad H_t + (uH)_x = 0,$$

gives

$$H_t + \gamma(HH_{xxx})_x = 0.$$

• Univalent function theory and other complex analysis areas.

• Weak and variational solutions (variational inequalities) for free boundary problems.

• Balayage and inverse potential problems (given the gravitational field, what is the body shape?).

• Scales of Banach spaces and abstract Cauchy-Kowalewskaya theory.

For zero-surface-tension Hele-Shaw, local-in-time existence of classical solutions is known, and global existence for weak solutions in the injection case. Only local results are known with positive surface tension.

Solution behaviour

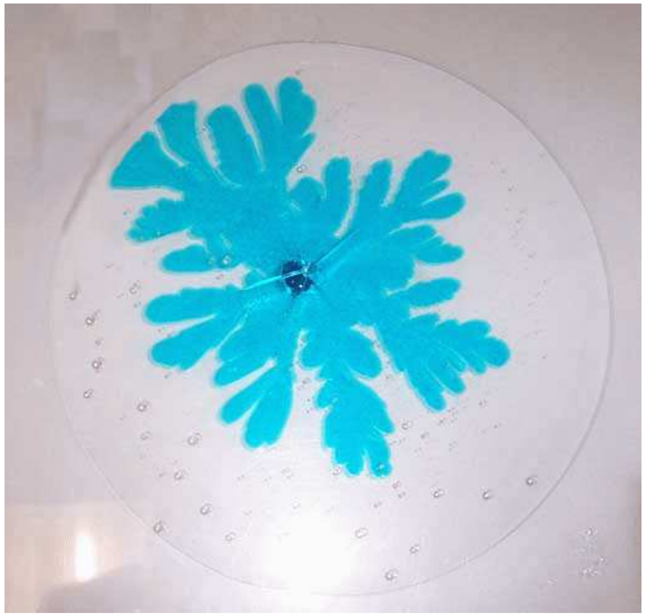
Blow-up and regularisation

Many explicit solutions show the general rule ‘suction is bad, injection is good’, corresponding to the Saffman-Taylor instability, and time-reversibility.

When $\gamma = 0$ (zero surface tension, ZST) and the fluid is finite, blow-up is guaranteed unless we start with a shape generated by injection.

For infinite regions there are solutions that leave some fluid behind, for example fingers, but there are also blow-up solutions.

Experiment shows that suction is very unstable. Here the less viscous liquid is injected in the middle of the cell.



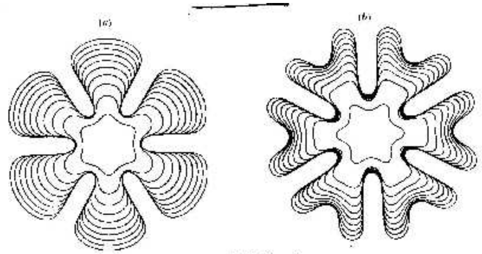


FIGURE 5. Radial fingering.

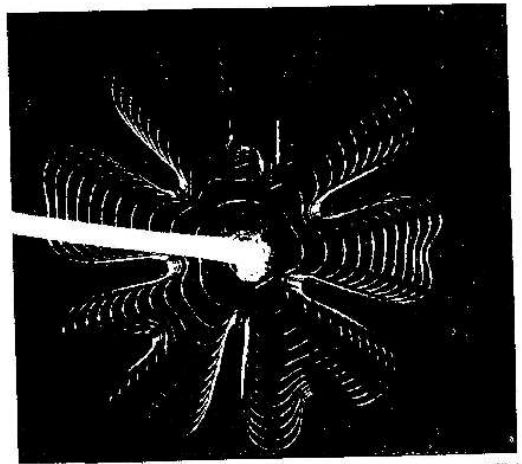


FIGURE 6. A multiple-exposure photograph of an air bubble growing in a Hele-Shaw cell filled with glycerine (kindly supplied by Dr L. Paterson).

Kinds of blow-up

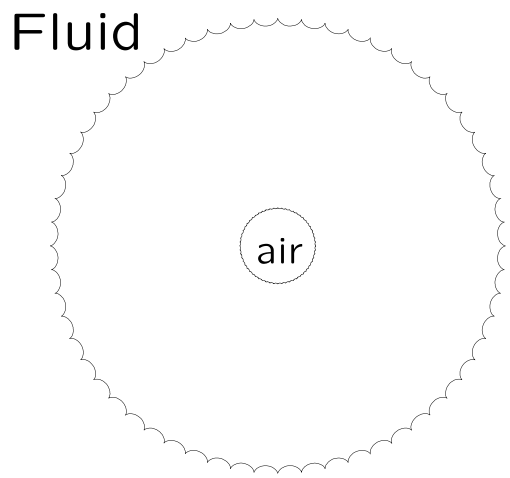
When $\gamma = 0$ (no surface tension) we know that blow-up is possible in suction problems.

Blow-up is by run-away: the pressure gradient ahead of a bulge in the boundary is large so the velocity is greater.

By time-reversal, almost anything *can* happen, but is always associated with the arrival of a singularity or derivative-zero of the mapping function $z = f(\zeta, t)$ at the unit circle.

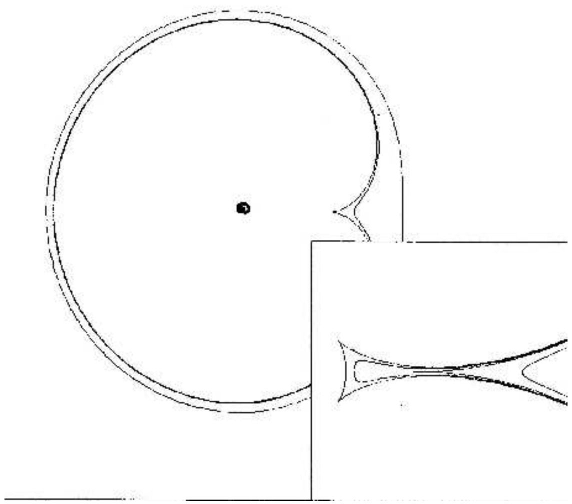
Some interesting cases:

• $\frac{3}{2}$–power cusps: 'generic' for breakdown by a zero derivative. The air bubble below breaks down with 60 simultaneous cusps.

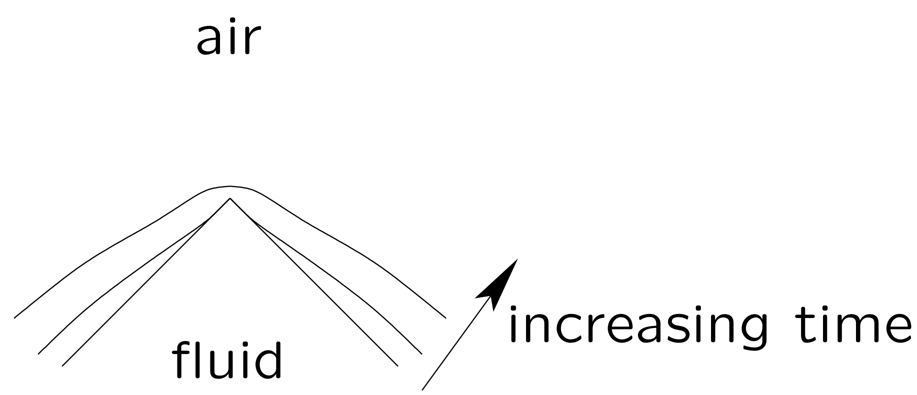


• $\frac{5}{2}$–power cusp. This is not generic, but with the right initial data it can happen. The solution *continues after the cusp forms*. In the example below the fluid region is the image of $|\zeta| < 1$ under a cubic map; it eventually blows up by two $\frac{3}{2}$–power cusps (inset).

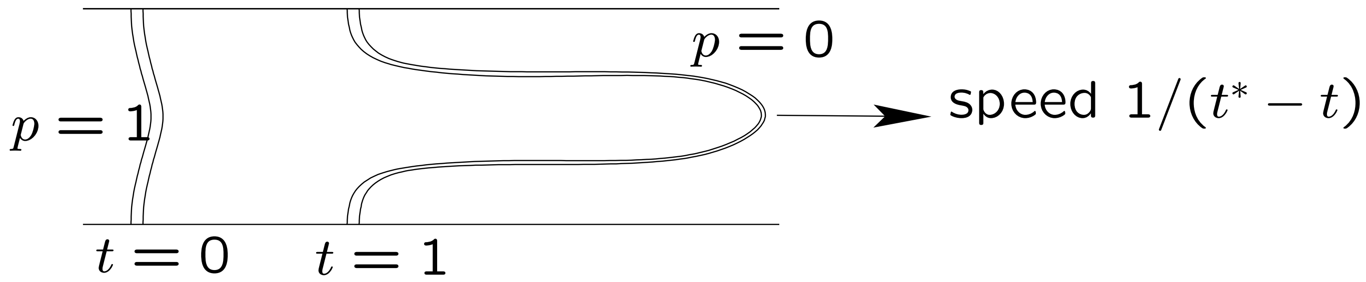
These (transient) $\frac{5}{2}$ power cusps can also happen in Stokes flow *even with surface tension*.



• 'Waiting times': if the initial boundary has a corner and we inject (forward, well-posed direction), the corner sits there for a finite time before smoothing off:



- A thin thread driven by a pressure difference can shoot off to infinity in finite time (solution by Feigenbaum 2003):



Indeed, motion of a thin thread can be reduced to the Cauchy–Riemann equations for $x(\xi, t) + iy(\xi, t)$ where ξ is a Lagrangian parameter (Farmer & SDH 2004).

Regularisation

What are the effects of small positive surface tension on blow-up? What is

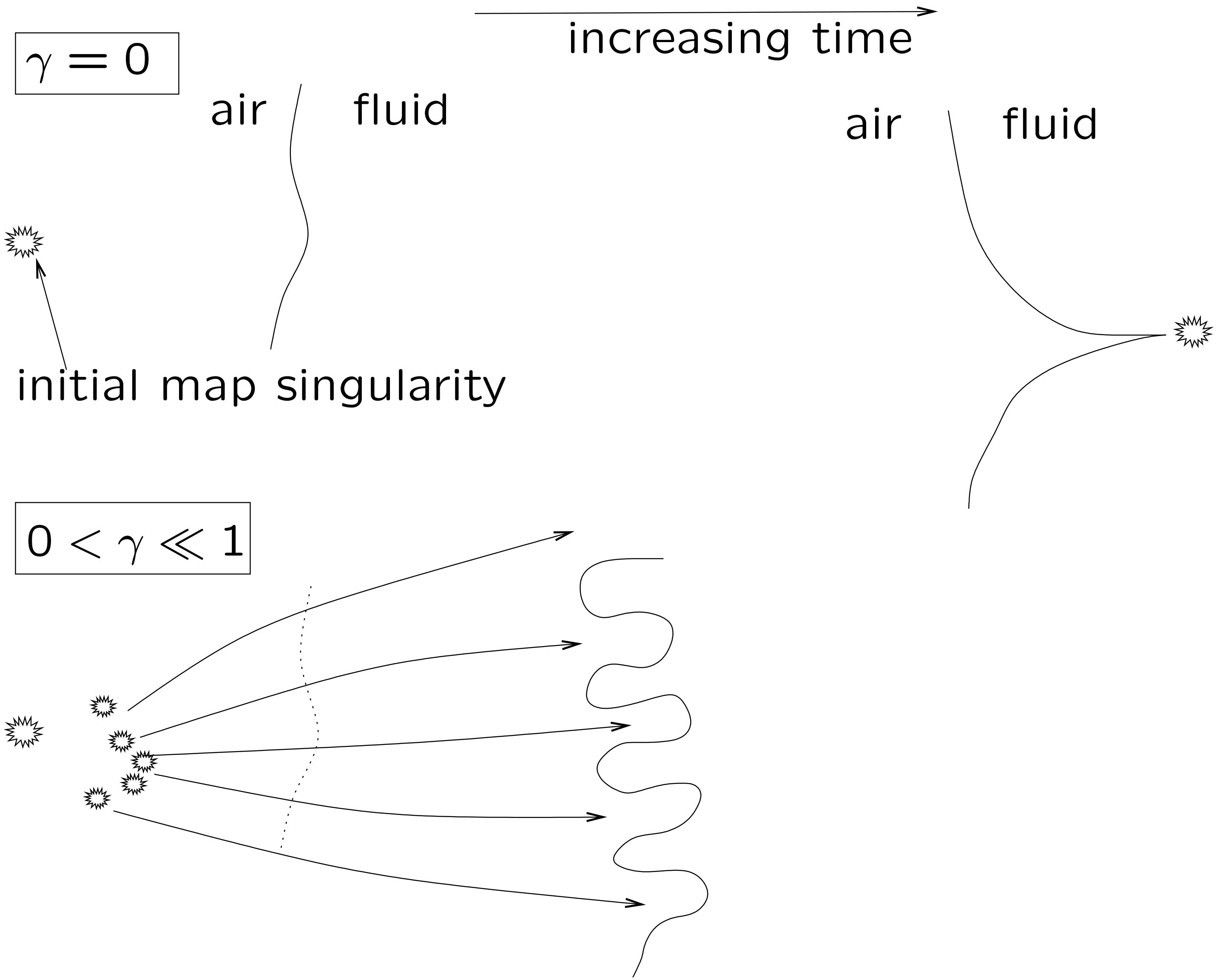
$$\lim_{\gamma \downarrow 0} (\gamma = 0 \text{ solution})?$$

For Hele-Shaw, surface tension probably stops all singularities in the free boundary* although this is not proved. But what do solutions look like?

*Except for instantaneous, ‘geometrically necessary’ cusps.

This is a very vexed issue. For Stokes flow, the small surface tension solution is close to the zero surface tension solution until the latter blows up, and then there is a *persistent cusp*.

For Hele-Shaw, the most plausible candidate so far is the idea of *daughter singularities* (Tanveer & coworkers). Singularities in the analytic continuation of the initial conformal map split into lots of 'daughters' and these cause the free boundary to change by $O(1)$ *at an* $O(1)$ *time before blow-up when* $\gamma = 0$.


$\gamma = 0$
increasing time
air
fluid
air
fluid
initial map singularity
$0 < \gamma \ll 1$

Selection problems

One might hope that putting small surface tension would resolve the $\lambda = \frac{1}{2}$ problem. Physicists in the 1980s, followed by mathematicians, developed the 'asymptotics-beyond-all-orders' selection principle.

Famous example: the Kruskal–Segur equation

$$\epsilon^2\theta''' + \theta' = \cos\theta, \quad ' = \frac{d}{ds}, \quad \theta(\pm\infty) = \pm\pi/2, \quad \theta(0) = 0.$$

When $\epsilon = 0$ the $\lambda = \frac{1}{2}$ curve is the solution (also a curvature flow). A regular expansion in powers of ϵ appears to work but in fact $\theta(0) = O(e^{-c/\epsilon})$ for all $\epsilon > 0$ and there is no solution unless $\epsilon = 0$.

This can all be analysed in terms of the Stokes line structure generated by the $\epsilon = 0$ solution. For Saffman–Taylor a similar approach gives a discrete family of solutions for $\gamma > 0$ (figure by SJ Chapman):

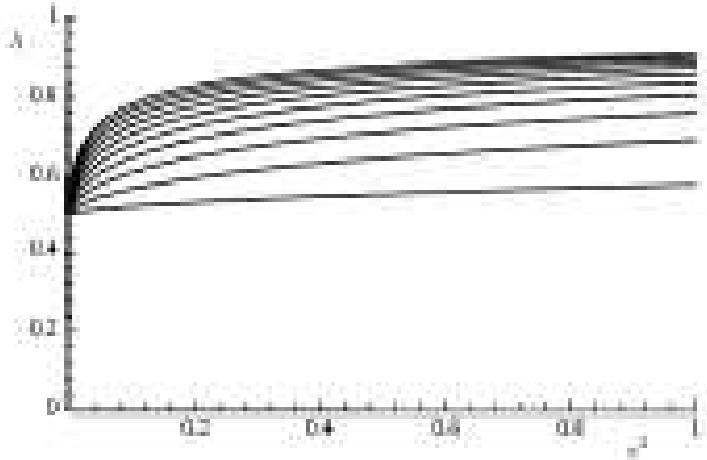
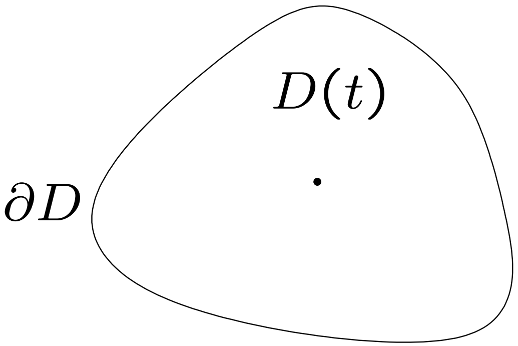


FIGURE 6. The relative finger width λ as a function of surface tension ϵ^2 for the first ten solution branches. The curves are valid asymptotically in the limit $\epsilon \to 0$, ϵN order one.

Mathematical structure

Conserved quantities: the moments

Flow with a source/sink at $z = 0$:



$$\nabla^2 p = Q\delta(\mathbf{x}) \quad \text{in } D$$

$$p = 0, \qquad -\frac{\partial p}{\partial n} = V_n \quad \text{on } \partial D$$

$$\begin{aligned}
\frac{d}{dt}\iint_{D(t)} z^k \, dxdy &= \int_{\partial D} z^k V_n \, ds \\
&= -\int_{\partial D} z^k \frac{\partial p}{\partial n} \, ds \\
&= \iint_D p\nabla^2 z^k - z^k \nabla^2 p \, dxdy - \int_{\partial D} p \frac{\partial z^k}{\partial n} \, ds \\
&= \begin{cases} Q & \text{if } k = 0 \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

So, defining the *moments* M_k as $M_k = \iint_D z^k \, dxdy$,

$$\frac{dM_0}{dt} = Q, \qquad \frac{dM_k}{dt} = 0, \quad k > 0.$$

Thus (Richardson 1972) we have an infinite set of conserved quantities. The moments are related to the *Cauchy Transform* of D,

$$C(z,t) = \frac{1}{\pi} \iint_D \frac{dx'dy'}{z'-z} \quad \left(= \frac{\partial}{\partial z} \frac{2}{\pi} \iint \log|z'-z| \, dx'dy' \right)$$

because as $z \to \infty$, C has the Laurent expansion

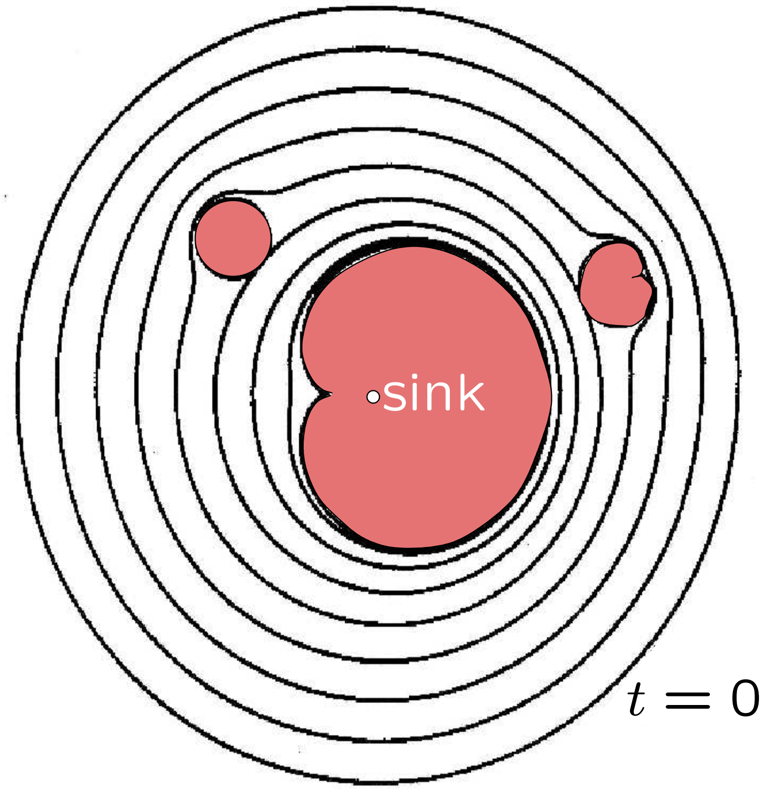
$$C(z,t) = -\sum_0^\infty \frac{M_k}{z^{k+1}}.$$

Hence if the flow is driven by one or more point sources/sinks,

$$\frac{\partial C}{\partial t} + \frac{Q}{z} = 0 \qquad \left(\text{or} \quad \frac{\partial C}{\partial t} + \sum \frac{Q_j}{z - z_j} = 0 \right).$$

So $C(z,t)$ evolves in a known way. It is known that if $C(z,0)$ is meromorphic in D then D is the image of the unit disk under a rational map. If the driving singularities of the flow coincide with those of C then we can calculate the flow explicitly.

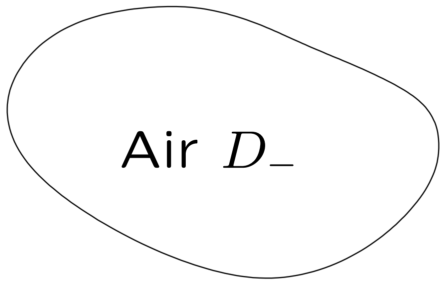
The Cauchy transform can also be used in Stokes flow and a variety of other problems including inviscid flows (Richardson, Crowdy, Tanveer).



(Richardson 2001)

Connection with integrable systems

Recent developments in mathematical physics (Mineev, Zabrodin,. . .), inspired by conserved quantities (moments). Consider bubble configuration:



Fluid D_+

Source/sink at infinity

Define

$$t_k = \frac{1}{|k|} \iint_{D_+} z^k \, d^2 z, \quad k < 0,$$
$$t_0 = \text{area},$$
$$t_k = \frac{1}{k} \iint_{D_-} z^k \, d^2 z, \quad k > 0,$$

with corresponding inner and outer moments M_k.

Now think of the t_k as varying to generate flows. This leads to an integrable system called the *dispersionless Toda lattice* and the Polubarinova–Galin equation

$$\Re\left(\zeta\frac{\partial f}{\partial \zeta}\overline{\frac{\partial f}{\partial t}}\right) = \frac{Q}{2\pi}, \quad \text{or} \quad \left\{\frac{\partial f}{\partial w}, \overline{\frac{\partial f}{\partial t_0}}\right\} = 1, \quad w = \frac{Q}{2\pi}\log\zeta,$$

is the *string equation* and we have the equations

$$\frac{\partial M_{-k}}{\partial t_j} = \frac{\partial M_{-j}}{\partial t_k}, \quad \frac{\partial M_{-k}}{\partial \overline{t_j}} = \frac{\partial \overline{M_{-j}}}{\partial t_k},$$

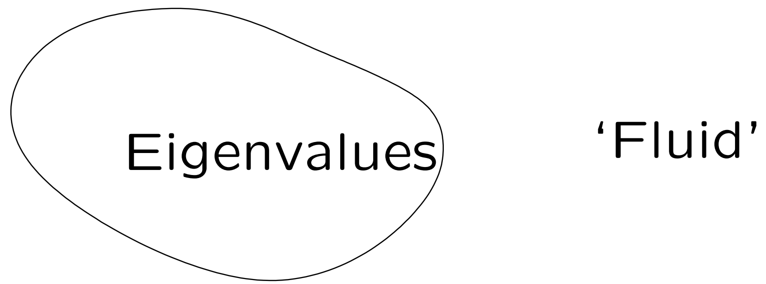
Solutions are equivalent to being able to determine a domain from its moments.

Random matrices

• If $\mathbf{M}$ is an $N \times N$ random Hermitian matrix, $\mathbf{M} = \mathbf{M}^*$, as $N \to \infty$ its (real) eigenvalues are distributed according to a semicircular density (Wigner) between $-\sqrt{N}$ and $\sqrt{N}$.

• *Normal* matrices satisfy $\mathbf{M}^*\mathbf{M} = \mathbf{M}\mathbf{M}^*$. They have complex eigenvalues: how are they distributed if $\mathbf{M}$ is random?

• Answer: in a domain D which, after scaling with $\sqrt{N}$, grows like a Hele-Shaw bubble (Wiegmann & Zabrodin 2002).



Introduce the partition function

$$Z_N = \int_{\text{normal matrices}} e^{N \mathrm{tr} V(\mathbf{M},\mathbf{M}^*)} d\mu(\mathbf{M}), \qquad d\mu = \text{measure}$$

where the ‘potential V tells us about the randomness.
Diagonalise and integrate out the ‘angular’ variables:

$$Z_N = \frac{1}{N!} \int |\Delta_N(z)|^2 \prod_1^N \exp(NV(z_j)) d^2 z_j, = \frac{1}{N!} \int \exp(\mathcal{E}(\mathbf{z}))\, d^2\mathbf{z}$$

where $\Delta_N =$ Vandermonde matrix. Now

$$\mathcal{E}(\mathbf{z}) = \sum_{i \neq j} \log |z_i - z_j| + \sum V(z_i)$$
$$= \text{energy of point charges... in the potential } V.$$

We minimise $\mathcal{E}$ (to get large N asymptotics by saddlepoint) and find the density (one-point correlation). (NB the energy at the saddle point is the tau-function of dToda.)

A variational argument (vary N with V fixed) then shows that the boundary moves according to the Hele-Shaw law.

There is also a connection with moments. The point charges analogy gives

$$\int \frac{\rho(z')\, d^2 z'}{z' - z} + \frac{\partial V}{\partial z} = 0$$

(note the Cauchy transform). The 'usual' form for V is

$$V(z) = -\tfrac{1}{2}|z|^2 \quad \text{(Gaussian)} \quad + \sum t_k z^k$$

where $t_k = M_k/k$ are identified with the *moments*. Thus

$$\frac{\partial V}{\partial z} = -t_o \overline{z} + \sum M_k z^k.$$

However,

$$\frac{\partial}{\partial \overline{z}} \int \frac{\rho(z')\, d^2 z'}{z' - z} + \frac{\partial^2 V}{\partial z \partial \overline{z}} = 0$$

and the second term is -1 so $\rho = 1$ and we recover the Cauchy transform in relation to the moments.

Open questions

• In what sense is blow-up for zero-surface-tension extraction problems generic? In a finite domain it is almost inevitable but in infinite domains one can exhibit families of solutions ($\sim$ polynomials) that are dense and do give blow-up, and other families ($\sim$ fingers) that are not. Which is generic?

• The 2-fluid Muskat problem is much harder and little is known. Global existence and uniqueness for the stable ZST problem with nearly planar initial data have recently been established (Siegel, Caflisch, SDH), and we can show blow-up via a curvature singularity in the unstable case: but can cusps form? Many other practical issues arise especially in inhomogeneous media.

• Is there an unsteady version of the small surface tension exponential asymptotics? (Daughter singularities.)

• Integrable systems and random matrices: a lot of open questions here. Eg, what happens if we are finding the domain of eigenvalues and the corresponding Hele-Shaw problem blows up?