BEYOND THE LASCAR GROUP

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ABSTRACT. We work in a first-order setting where structures are spread out over a metric space, with quantification allowed only over bounded subsets. Assuming a doubling property for the metric space, we define a canonical *core* $\mathfrak{J}$ associated to such a theory, a locally compact structure that embeds into the type space over any model. The automorphism group of $\mathfrak{J}$, modulo certain infinitesimal automorphisms, is a locally compact group $\mathfrak{G}$. The automorphism groups of models of the theory are related with $\mathfrak{G}$, not in general via a homomorphism, but by a *quasi-homomorphism*, respecting multiplication up to a certain canonical compact error set. This fundamental structure is applied to describe the nature of approximate subgroups. Specifically we obtain a full classification of (properly) approximate lattices of $SL_n(\mathbb{R})$ or $SL_n(\mathbb{Q}_p)$.

CONTENTS

1. INTRODUCTION

This paper consists of three successively more concrete parts, logically forming a progression. The first, purely model theoretic, constructs a canonical geometry associated with any first-order theory. This 'core space' was introduced in [27]; here we recast it in a 'local' setting that allows the core, and the attendant automorphism group, to

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be locally compact rather than compact. Beyond that, the relation between the core and the automorphism groups of a models of the theory is now brought out explicitly, in terms of a quasi-homomomorphism; the Lascar-Shelah neighbor relation plays the role of the error set. The second part uses this construction to give a structure theorem for approximate subgroups. The 'Lie model theorem' proved in [26] for near-subgroups, and used in [8], is extended in modified form to general approximate subgroups; a conjecture of Massicot-Wagner predicting that this should be possible is thus confirmed. The third part specializes to *approximate lattices* in locally compact groups. In the case of semisimple groups, a full classification can be achieved, solving in particular a problem of Björklund and Hartnick [5].

The rest of the introduction follows this outline in more detail. Type spaces - the Stone spaces dual to the Boolean algebra of formulas - occur for instance in the study of countable categoricity. These spaces contain all information about T, but only as a convenient reformulation of the syntax; they do not bring out well the geometry of definable sets; in particular, they are inert and admit no automorphisms. In Morley's proof of his theorem on uncountable categoricity, type spaces *over models* first played a leading role. Morley rank is immediately visible from their topology; and they carry a natural action of $Aut(M)$. This does not, however, give a canonical space or group associated with T, since a choice of M is required. For some purposes, this is alleviated by uniqueness theorems for saturated models, but then the group becomes large and unwieldy, falling outside any known structured family of groups.

Shelah [61] recognized the algebraic imaginaries as a critical geometry associated with a first-order theory T. The automorphism group is a profinite quotient of $Aut(M)$ (for M a sufficiently homogeneous model of T), and a perfect Galois duality holds between them. For T=ACF one recovers Galois's duality in full. This was for any theory; for stable T, Shelah showed how amalgamation of substructures is controlled by his group.

Kim and Pillay [35] defined the *bounded hyperimaginaries* as a generalization of the algebraic imaginaries; their automorphism group is a compact topological group. (See also [?].) Once again this is valid for any first-order theory; for simple theories, it controls amalgamation. This equally beautiful theory later became absorbed in continuous logic, and indeed in [4] is not even terminologically separated from Shelah's algebraic imaginaries.

It was Lascar, more than a decade earlier, who suggested that a richer Galois group is needed in unstable theories. He defined two candidates;

the compact one used by Kim-Pillay, and a a bigger common quotient of the groups $Aut(M)$, called the general Lascar group. The latter was much studied recently, but was never found to be the automorphism group of any meaningful geometry (see e.g. [32] for a related negative result.) Krupiński, Pillay, and Rzepecki [37], [38] , working group-theoretically, showed suggestively that the general Lascar group was in turn a quotient of a compact group; see the introduction to [27] for more detailed references.

In [27], a common subspace $\mathcal{J}$ of all type spaces over models was identified, generalizing the Shelah space and Kim-Pillay space for stable and simple theories, and having significant structure beyond the topology. While canonical and closely related to definability in T, the automorphism groups $Aut(M)$ of models of T do not naturally act on $\mathcal{J}$. We show here, however, that they do admit a useful *quasi-action*. A *quasi-homomorphism* $\phi : G \rightarrow H : K$ between two groups G, H, relative to an 'error set' $K \subset H$, is a map ϕ satisfying $\phi(1) = 1$ and $\phi(xy^{-1}) \in \phi(x)\phi(y^{-1})K$ for all $x, y \in G$. This is of interest when K is small in some sense. We find a quasi-homomorphism $\varphi : Aut(M) \rightarrow Aut(\mathcal{J}) : \triangle$. The error set $\triangle$ derives from a fundamental 'neighbor' relation going back to [40] (on $Aut(M)$ for highly saturated M) and [62] (for indiscernible sequences), and defined here on the core $\mathcal{J}$.

Given a quasi-homomorphism $\phi : G \rightarrow H : K$, one can of course factor out the group $\langle K \rangle$ generated by K so as to obtain an ordinary homomorphism $G \rightarrow H/\langle K \rangle$. In the case of the quasi-homomorphism $Aut(M) \rightarrow Aut(\mathcal{J}) : \triangle$ described above, this would recover the general Lascar group.

The stance taken here is that factoring out by $\langle \triangle \rangle$ is the wrong thing to do. An alternative way to forget unwanted information while discarding much less of value is offered by category theory: in place of changing the objects, add (iso)morphisms. This is set out in Appendix B. In the paper itself we will use the categorical approach only in spirit. We will however analyze mathematically the relevant morphisms.

Examples of quasi-homomorphisms include projective representations of a group G; here H is a unitary group, the group of automorphisms of a Hilbert space, and K is the one-dimensional center of H. Quasi-homorphisms with unitary targets were also studied notably in [66] (a 1938 paper first cited, according to MathSciNet, in 2011), and in [34], with K a set of elements of small operator norm in both cases. When G is finite (Turing) or amenable (Kazhdan), they were shown to be close to ordinary homomorphisms.

The quasi-homomorphisms of interest to us will have locally compact target groups, and normal, compact error sets K. The case $H = \mathbb{R}$ has

been very well studied; a quasi-homomorphism into $\mathbb{R}$, with compact error set, is called a *quasimorphism*; it is significant in several branches of geometry. (Readers unfamiliar with the notion may enjoy reading the very short introduction [41] at this point).

Returning to first-order theories, we can get a clear idea of the smallness of $\triangle$ if we allow a slight generalisation, to local logic. This is introduced in §2. In many-sorted logic, quantification is allowed only on a given sort, or finitely many sorts. If one thinks of a sort as compact (say via the type space), then a many-sorted model is only locally compact. Local logic is a slight variation, likewise restricting quantification, but refraining from destroying possible symmetries by naming specific bounded domains as sorts. In the simplest case, one has a metric m (a discrete metric is good enough for our purposes); and quantification is allowed over balls of bounded m-radius. We will assume a 'doubling' property - a 2-ball for the metric m is covered by finitely many 1-balls. The construction of $\mathcal{J}$ generalizes; $Aut(\mathcal{J})$ is now locally compact rather than compact; it has a natural locally compact Hausdorff quotient, which is what we will actually work with. The relation $\triangle$ on $\mathcal{J}$ remains local: $x \triangle y$ implies that their distance is at most 1. As a results, the set of automorphisms $\triangle^{Aut(\mathcal{J})}$ is a normal, compact subset of $Aut(\mathcal{J})$. In fact, compactness alone does not fully capture the tightness of $\triangle$. But along with conjugation invariance - arising from the canonical nature of $\triangle$ - it is sufficient for most of the group-theoretic applications.

At this point, the reader is encouraged to consult Theorem 3.10, summarizing the main properties of the locally compact core $\mathcal{J}$, and Theorem 3.16, including the basic properties of the quasi-homomorphism φ. The material is closely parallel to the non-local case of [27]. The exposition in sections 2 and 3 is a compromise, not repeating every lemma of [27], but attempting to be self-contained with respect to the main line and in particular the results that will be critical in the later sections.

In §4, we transpose to the case of definable groups. In the local setting, we look at groups that are direct limits of definable sets; in the model theory literature they are known as piecewise-definable, locally definable, or strict ind-definable. Thus $G = \cup X_n$, where each X_n is a definable set, and multiplication restricts to a definable map $X_n \times X_n \to X_{n+1}$. The doubling condition amounts here to the assumption that X_{n+1} is covered by finitely many translates of X_n This makes each X_n into an *approximate subgroup* of G, as defined in [64]. At least at first approach, approximate subgroups are best understood up to

commensurability, i.e. without distinguishing X and X', if each can be covered by finitely many translates of the other.

We will call an approximate subgroup *laminar* if it contains a sequence (X_n) as above indexed by *negative* integers. A commensurability class ω is *laminar* if it contains a laminar approximate subgroup. Model-theoretically, X is laminar iff it contains a subgroup cut out by an intersection of generic definable subsets. Quotienting such a a $\bigwedge$-definable subgroup out from a saturated version of G leads to a homomorphism $\varphi : G \to \mathsf{H}$ into a locally compact group H; and ω can be recovered from φ and H, since the elements of ω are intertwined with pullbacks of compact open neighborhoods of 1 in H. We obtain something like a Bourgain system ([22]) down to an arbitrarily fine scale. In the non-laminar case, to extend the metaphor, eddies of a certain size prevent further descent.

We call X *amenable* if there exists a translation invariant measure on X^3; more generally it is a *near subgroup* if an appropriate invariant ideal on definable subsets of X^3 exists. For amenable approximate subgroups and and near subgroups it was proved in [26], [59], [47], in various settings, that the commensurability class of X is laminar; in fact the sequence $X^n, n \geq 4$ can be continued downwards through the negative integers.

Here we prove, with no amenability assumptions, that *a map* $\varphi : G \to \mathsf{H} : K$ *into a locally compact group still exists, with the same property of intertwining* compact *with* definable. Only now φ is a quasi-homomorphism, a compact, normal error set K is allowed; and open neighborhoods of K (rather than of 1) are considered on the locally compact side. See Theorem 4.2.

Model-theoretically, the characteristic definability property of φ in the laminar case was *definable separation*: if C, C' are disjoint compact subsets of H, then $\varphi^{-1}(C), \varphi^{-1}(C')$ are separated by a definable set. If we give H a metric, C, C' are separated by some distance $\epsilon > 0$. In the general setting, we still have the separation property but now assuming $CK, C'K$ are disjoint; they must be seen to be distinct even if our resolution is limited to the radius of K. In the laminar case $\epsilon > 0$ is arbitrary, but in general a certain minimum separation is required.

Massicot and Wagner wrote, regarding approximate subgroups: "We conjecture that even without the definable amenability assumption a suitable Lie model exists" [47]. The term 'Lie model' was coined earlier in [8], where it was the starting point for the classification of pseudofinite approximate groups. The meaning of 'suitable' was not further clarified, but I view Theorem 5.15 as a confirmation of their intuition. It asserts that *any commensurability class of approximate subgroups arises*

*by pullback of compact neighborhoods under an quasi-homomorphism
into a Lie group L; moreover, the error set is contained a bounded
subset of a closed, normal abelian subgroup $A \trianglelefteq L$, $A \cong \mathbb{R}^N$, such that
L-conjugation respects a Euclidean structure on A.* (We refer to such
actions as *rigid*.)

It was not immediately apparent whether Theorem 5.15 has real con-
tent or merely replaces one kind of approximateness (for subgroups)
by another (for homomorphisms); the problems attacked in part 3, de-
scribed below, were intended as a testing ground. The latter turned
out to be a much tighter condition, thanks to the combined com-
pactness and normality of the error set and the finite dimensional-
ity of the target. To bring this out directly we give two additional
formulations within §5, a group-theoretic and a model-theoretic one.
The group-theoretic version, Theorem 5.24, points out that the quasi-
homomorphism $\varphi : G \to \mathsf{H} : K$ of Theorem 5.15 can be viewed as
an (ordinary) homomorphism from an 'almost' central extension $\tilde{G}$ of
G by $\mathbb{R}^n$, into the Lie group H. Here 'almost' means that the im-
age in $End(\mathbb{R}^n)$ of conjugation by elements of $\tilde{G}$, while not trivial, is
contained in a compact subgroup. This resembles the situation with
projective representations of Lie groups, seen as representations of a
central extension. The cocycle coding the group extension will be a
bounded one, forming one connection with the far-reaching results of
bounded cohomology (e.g. [23], [10], [11],[50].)

The second formulation is Theorem 5.20. The model-theoretic avatar
of an approximate subgroup in the laminar case is an $\bigwedge$-definable group
'of bounded index.' Here we show that in general, an approximate sub-
group is associated with an $\bigwedge$-definable set, obtained as the 'approx-
imate kernel' of finitely many (homogeneous) quasimorphisms on an
$\bigwedge$-definable group Γ of bounded index, i.e. the pullback of a compact
neighborhood of the error set in $\mathbb{R}^n$. This connects again, not in quite
the same way, to bounded cohomology.

For instance, since groups of bounded exponent admit neither dense
homomorphisms into connected Lie groups nor unbounded quasimor-
phism, any approximate subgroup of a bounded exponent group is com-
mensurable to an actual subgroup; the conclusion is the same as in [26],
where this was used as an illustration of the theory, but the hypothesis
includes no amenability assumption.

The $\bigwedge$-definable group Γ is the kernel of a continuous homorphism
into a locally compact group with respect to the logic topology; the
quasimorphism $\Gamma \to \mathbb{R}^n$ is also continuous; so that the bumpiness
alluded to above is felt only at the scale of the error set.

The (symmetrized) approximate kernel of a nontrivial quasimorphism is always an approximate subgroup (Proposition 5.11), but never a laminar one (Proposition 5.28)

The third part of the paper, consisting of §6, §7 and Appendix A, is concerned with the geometry of numbers; but this will not be obvious at the outset. Our working environment in §6 is an easily defined class of locally compact groups, that we call *abstractly semisimple.* It has two defining requirements: no abelian normal subgroups, no discrete conjugacy classes other than the trivial ones. These serve as ambient groups; within them, we consider discrete approximate subgroups. (Perhaps it is worth recalling here, though I am aware of no direct connection, the fundamental role of discreteness in the theory of pseudo-finite approximate subgroups of [8].) We call a commensurability class of discrete approximate subgroups *strictly dense* if the group generated by any element of the class is dense; this is at an opposite extreme from discrete subgroups; under certain assumptions, a decomposition exists into the two kinds. Building on the results of §5, we begin by showing that, for abstractly semisimple groups, *a strictly dense approximate subgroup is commensurably contained in a laminar one.* See Theorem 6.9.

Recall that a *lattice* is a discrete subgroup of finite co-volume; it is called *uniform* if it is co-compact. In the case of the Euclidean space $\mathbb{R}^3$, a lattice is the accepted mathematical model of a crystal, and their histories are inseparable. Yves Meyer in [48] defined, in the setting of $\mathbb{R}^n$, *approximate lattices*; this later became the mathematical home for quasicrystals. Meyer proved that they are laminar as approximate subgroups, with Lie model $\mathbb{R}^n$; he did so in the equivalent, geometrically suggestive language of *model sets.* On the other hand, lattices in non-commutative locally compact groups such as $SL_n(\mathbb{R})$ play a major role in many parts of geometry, ergodic theory and number theory; especially relevant to us are the arithmeticity theorems of [46].

Björklund and Hartnick [5] were first to investigate approximate versions. They defined uniform approximate lattices, and offered two "tentative" definitions in the non-uniform case. We will suggest another definition, defining an *approximate lattice in the sense of finite covolume* to be a discrete approximate subgroup Λ of G, such that $\Lambda C = G$ for some Borel set C of finite measure. As in [5], Λ is said to be *uniform* if C can be taken compact. It seems to work; the basic properties of approximate lattices, with this elementary definition, are developed in Appendix A. After writing this, I ran into Problem 6.4 of [12], asking for the "right" definition of an approximate lattice; Appendix §A, then, is our possibly naive proposal. In this paper, approximate lattice will

always mean, by way of abbreviation, 'approximate lattice in the sense of finite covolume'.

Among the approximate lattices, Björklund and Hartnick consider the *Meyer sets*. The definition (that we naturally extend from the uniform case) inolves not only G but also an additional locally compact H, that we will call the complementary group, and a lattice Γ in $G \times \mathsf{H}$. A Meyer set M is defined by the positive primitive formula of *local* logic: $(\exists h \in C)((x, h) \in \Gamma)$, where C is a compact neighborhood of 1 in H. This description is equivalent to laminarity of M, with H corresponding to the target of the homomorphism into a locally compact group.

I am grateful to Emmanuel Breuillard for introducing me to this beautiful theory, and explaining this equivalence between the language of laminarity, or Lie models, and of Meyer model sets.

Björklund and Hartnick posed the problem of proving laminarity when possible ([5], Problem 1, formulated there for uniform approximate lattices.) Simon Machado, in [42], [43], did so in the nilpotent and solvable cases, and for amenable groups. In §6, complementing this work, we solve Problem 1 for semisimple groups G. To do so, we will use a local logic where 'local' implies 'discrete'. This contrasts with the natural treatment of say $SL_n(\mathbb{R})$ in local logic, where "local" means "compact" (see e.g. [29], section 7.) Nevertheless the local compactness of $Aut(\mathcal{J})$ (as mediated by Theorems 4.11,5.10) allows us to recover compactness, effectively finding the complementary group and presenting the approximate lattice as a Meyer set. See Corollary 6.11. The duality between compactness and discreteness seen here is a key feature of this world.

At this point, Margulis arithmeticity becomes available. See arith-modelset for the definition of an *arithmetic approximate lattice*. It is just like the definition of an arithmetic lattice in [46], except that (as if in line with the recommendation of Hensel and Weil) all places are treated as equals; no special provision is made for archimedean primes. If we switch attention for a moment to rings, $\mathbb{Z}$ is the simplest lattice (in $\mathbb{R}$), while the approximate ring $\{a/p^n \in \mathbb{Z}[1/p] : |a| \le p^n\}$ is the simplest arithmetic approximate lattice (in $\mathbb{Q}_p$). In each case, the (approximate) lattice R is the set of rationals whose norm in every completion, other than the ambient locally compact group, is at most 1. And in both cases, appropriately understood, $G(R)$ gives rise to arithmetic lattices in $G(\mathbb{R})$ or $G(\mathbb{Q}_p)$ for semisimple groups G. Theorem 7.4 shows that *any approximate lattice in G is commensurable with a product of lattices and arithmetic approximate lattices of direct factors of G.* Note that the arithmeticity of the strictly approximate lattices is valid even for $n = 2$; higher rank assumptions are not required.

To illustrate the concrete character of this answer, let us describe, up to commensurability, the approximate lattices Λ in $SL_n(\mathbb{R})$. They depend on a choice of a number field $K \leq \mathbb{R}$ and an algebraic group $G \leq GL_N$ defined over K, and isomorphic to SL_n over $\mathbb{R}$ (the case $G = SL_n$, $n = N$ is interesting enough). Let Λ_K be the set of matrices in $G(K) \leq GL_N(K)$ of the form $I + M$, where I is the identity matrix and M is a matrix whose nonzero entries are Pisot numbers α with $\mathbb{Q}(\alpha) = K$. Then Λ_K is an approximate lattice in $SL_n(\mathbb{R})$. It follows from Theorem 7.4 that *any approximate lattice in $SL_n(\mathbb{R})$ is commensurable to a conjugate of one of these, or to a lattice.*

In positive characteristic, strictly approximate lattice seem unnecessary, and in fact, do not exist (Corollary 7.5).

The class of abstractly semisimple groups was introduced in §6 as simply a convenient home for groups like $SL_n(K)$, with K a local field, and their more demanding cousins; but it eventually becomes natural to take interest in the class itself. Theorem 7.7 shows that all strictly approximate lattices in abstractly semisimple groups are of arithmetic origin. In particular, by Remark 7.8, *an arbitrary abstractly semisimple group containing a strictly approximate lattice is virtually isogenous an adelic product of semisimple groups over local fields*, where 'virtually isogenous' allows homomorphisms with co-compact image and compact kernel. The proof of this recognition theorem uses Margulis arithmeticity again, and a method of pivoting over the complementary locally compact.

Finally, we take a peek beyond soluble or semisimple groups. We construct a simple example of a non-laminar approximate lattice. Returning to bounded cohmology, we also give a criterion for an approximate lattice $\Lambda \leq G$, where G is semisimple, ensuring laminarity of any approximate lattice Λ' lying above Λ in an algebraic group G' with semisimple part G.

Leitfaden: The appendices are, within the present paper, self-contained.Sections (2,3,4) form a logical sequence. The main results of the later sections require, from this development, only parts (1,2) of Theorem 4.2. The reader interested mostly in approximate lattices can read Appendix 5, the first few pages of §5, including Theorem 5.15, and turn to §6.

Questions of uneven scope and difficulty are scattered through the text, mostly at the end of the sections.

Thanks to Arturo Rodriguez-Fanlo and Anand Pillay for their reading and comments, and to Yves de Cornulier for explaining the literature around Proposition C.3.

Immediately after writing this paper I became aware that Simon Machado had been working on the structure of approximate lattices in semisimple groups at the same time. He obtained similar results including arithmeticity and rigidity results for strong approximate lattices in higher rank Lie groups, via what appears to be a beautiful and completely different approach via Zimmer cocycle rigidity; see [45]. See also the introduction to §A. The basic definitions differ slightly and still need to be compared in detail; see in particular the comments added to Questions 7.11 and A.22.

Machado also drew my attention, while I was lecturing on these results, to the paper [3] of Bader, Furman and Sauer; it is very close in both statement and proof to Theorem 7.7 and Remark 7.8. I left the treatment as it was to illustration of the naturality of the approximate lattice approach, and with future generalizations beyond the laminar framework in mind.

While the results that approximate lattices in semisimple groups are close to lattices (in other groups) are very satisfactory, they also mean that the additional flexibility of approximate lattices must find use outside the semisimple setting, or beyond the lattice framework. In particular Kazhdan suggested investigating discrete approximate subgroups in GL_n over local fields, with respect to growth rates of the number of points in increasing balls. Can growth rates be achieved for approximate subgroups that are not possible for subgroups?

Moving to the approximate lattice setting allows investigating an arithmetic lattice in $GL_n(\mathbb{R})$ within any $GL_n(\mathbb{Q}_p)$; it suffices to extend to an adelic lattice, cut with neighborhoods of the identity in the other places and project to $GL_n(\mathbb{Q}_p)$; little information is lost, and in particular the commensurability class of the original lattice can be reconstructed. Taking ultraproducts, and noting that the degree k of approximateness does not depend on p, one can even obtain such information within linear groups over equicharacteristic Henselian fields K. Can some of the theory of these lattices be seen more clearly when working in $GL_n(\mathbb{Q}_p)$ or $GL_n(K)$?

2. Local logic

We begin by reviewing the theory of existentially closed and saturated models. This theory belongs to the model theory of the 1960's, and was already recalled in detail in [27]. But we need a generalization here, permitting a localization of the quantifiers without disturbing a possible symmetry. The extension is slight in that upon adding a constant symbol c in $\mathcal{D}$, we return to the previous positive-primitive

setting, with balls of various radii around c are viewed as sorts. As usual more care is needed to subtract a constant than to add one.

Throughout this paper, unless explicitly specified otherwise, the word 'definable' will mean: by a formula of the given language, without parameters.

We assume given a relational language (with equality) $\mathcal{L}$, with a distinguished sort $\mathcal{D}$, and a distinguished family $\{\mu\}$ of binary relations on $\mathcal{D}$, called *locality relations*. For simplicity, we will think of $\mathcal{D}$ as the only sort, and assume we have only one locality relation μ_1. For any $\mathcal{L}$-structure A, we define a metric μ on A according to the Cayley graph of μ_1:

$$\mu(x,y) \le 1 \iff x = y \vee \mu_1(x,y) \vee \mu_1(y,x)$$

$$\mu(x,y) \le n+1 \iff (\exists z)(\mu(x,z) \le 1 \wedge \mu(z,y) \le n)$$

We write $\mu_n(x,y)$ for $\mu(x,y) \le n$ (replacing μ_1 by its symmetrization), and also $\mu_n(a) := \{y : \mu_n(a,y)\}$.

By a pp formula, we mean one built from atomic formulas using conjunction and *local existential quantification*: if $\phi(x,y,\cdots)$ is a pp formula with a free variable x and another variable or constant y (both of sort $\mathcal{D}$), then $(\exists x)(\mu_1(x,y) \wedge \phi)$ is also pp. Thus iterated quantification ranges over a ball of bounded radius. There are no pp sentences, though they do appear as soon a a constant is named.

A *primitive-universal theory* $\mathcal{T}$ includes negations of sentences of the form $(\exists x)(\bigwedge_{i=1}^m \psi_i)$, where x is a tuple of variables, and ψ_i a tuple of atomic formulas, or equivalently a pp formula; thus a typical sentence can be written $(\forall x)\neg\psi(x)$. A *primitive-existential theory* includes sentences of the form $(\exists x)(\bigwedge_{i=1}^m \psi_i)$. Given a primitive-universal theory $\mathcal{T}$, let $\mathcal{T}_\exists$ consist of all existential sentences $(\exists x)\psi(x)$ where $\neg(\exists x)\psi(x) \notin \mathcal{T}$.

A *weak model* of $\mathcal{T}$ is a structure A where the sentences of $\mathcal{T}$ hold. It is a *model* of $\mathcal{T}$, in the sense of local logic, if it is a weak model where any two elements of $\mathcal{D}(M)$ are at finite μ-distance. We will sometimes use the term *local model* for emphasis and clarity.

We say LJEP holds for $\mathcal{T}$ if for any two local models A, B of $\mathcal{T}$, there exists a local model C of $\mathcal{T}$ and homomorphisms $A \to C$ and $B \to C$. In this case, $\mathcal{T}^\pm := \mathcal{T} \cup \mathcal{T}_\exists$ is consistent, (and indeed any existentially closed model of $\mathcal{T}$ is a model of $\mathcal{T}^\pm$).

In our application to the core, LJEP will hold in a strong form, (LJEP$_k$): If $(\exists x)\psi(x), (\exists y)\psi'(y)$ are in $\mathcal{T}_\exists$, where ψ, ψ' are pp and x, y are variables (of sort $\mathcal{D}$), then $(\exists x)(\exists y)(\mu_k(x,y) \wedge \psi(x) \wedge \psi'(y)) \in \mathcal{T}_\exists$.

In the main application, in fact, we will have LJEP$_2$; in case of transitivity, when $\mathcal{D}$ has a single 1-type, LJEP$_0$ holds.

LJEP_k implies LJEP: If $A, B \models \mathcal{T}$, pick $a \in \mathcal{D}(A), b \in \mathcal{D}(B)$; then the union of the diagrams of A and of B, along with the sentence $\mu_k(a, b)$, is consistent. Any weak model of this union, restricted to the universe spanned by A, B, will be a model of $\mathcal{T}$.

Remarks on LJEP One can define a distance on maximal pp types, namely $d(p, q) \leq n$ if $\mu_k(x, y) \wedge p(x) \wedge q(y)$ is consistent with $\mathcal{T}$. Define an equivalence relation on types, $d(p, q) < \infty$. Then LJEP holds iff any two types are equivalent. Without the joint embedding property, the results of this section still hold, except for the uniqueness of the universal model in Proposition 2.1; instead, there is one universal model realizing each equivalence class.

Local logic ultrapowers of a model M of $\mathcal{T}$ are formed by first taking the ordinary ultrapower, then restricting to the elements at finite distance from some element of M. We can similarly define ultraproducts of models M of $\mathcal{T}_c$, i.e. models M of $\mathcal{T}$ with a distinguished element $a = c^M \in \mathcal{D}(M)$.

A model A of $\mathcal{T}$ is *existentially closed* (abbreviated e.c.) if $\mathcal{D}(A) \neq \emptyset$, and for every homomorphism $f : A \to B$, where $B \models \mathcal{T}$, and any $\mathcal{L}_A$-pp sentence ϕ, if $B_A \models \phi$ then $A_A \models \phi$. Here $\mathcal{L}_A$ is $\mathcal{L}$ expanded by constants for the elements $a \in A$; they are interpreted as a in A_A and as $f(a)$ in B_A.

In particular, if A is e.c. then A is *simple*, i.e. every homomorphism $f : A \to B$ is an $\mathcal{L}$- embedding (an isomorphism onto the image.) Thus A is e.c. iff it is simple, and existentially closed with respect to embeddings.

It is clear from Zorn's lemma that any model A of $\mathcal{T}$ admits a homomorphism into a simple one. The usual direct limit construction shows that any simple model A of $\mathcal{T}$ admits an embedding into an existentially closed model B of $\mathcal{T}$, with $|B| \leq |\mathcal{L}_A| + \aleph_0$. Write $M \overset{e.c.}{\models} \mathcal{T}$ if M is an e.c. model of $\mathcal{T}$. Also, for any local sentence ψ, write $\mathcal{T} \overset{e.c.}{\models} \psi$ if $M \models \psi$ whenever $M \overset{e.c.}{\models} \mathcal{T}$.

The class of e.c. models admits amalgamation: if $f_i : A \to B_i$, we may embed each B_i in an ultrapower A^* of A, then compose with a homomorphism to an e.c. model.

An e.c.-model $\mathcal{U} \models \mathcal{T}$ is λ-*saturated* if whenever A is an e.c. substructure of $\mathcal{U}$ and $h : A \to B$ an embedding with $B \overset{e.c.}{\models} \mathcal{T}$ with $|B| < \lambda$, there exists an embedding $g : B \to \mathcal{U}$ with $g \circ h = Id_A$.

For pp formulas $\phi(x), \psi(x)$ we write $\phi \perp \psi$ if $\mathcal{T} \models (\forall x)\neg(\phi \wedge \psi)$. If $M \overset{e.c.}{\models} \mathcal{T}$, for any pp formula $\phi(x)$ and any a from M, we have $M \models \phi(a)$

or $M \models \psi(a)$ for some pp formula $\psi \perp \phi$. This is proved just as in [27] 2.3(1), taking into account that pp formulas now use local quantifiers only.

Write $\phi \underset{e.c.}{\subset} \psi$ if $\theta \perp \phi$ whenever $\theta \perp \psi$; in e.c. models we have $\phi \underset{e.c.}{\subset} \psi$ iff $\phi(M) \subseteq \psi(M)$.

We will be interested in a generalized finiteness condition on $\mathcal{T}$:

(ecB): For some infinite cardinal κ, any e.c. model of $\mathcal{T}$ has cardinality $\leq \kappa$.

Here is a more concrete form of this condition; we will name it positive primitive compactness. Note the similarity, in complete metric setting, to the 'totally bounded' definition of local compactness; here one thinks of $\neg\phi$ as an open neighborhood of the diagonal, with ϕ ensuring a distance bounded away from 0, in any given bounded subset.

(ppC): Let $\phi(x, y)$ be a pp formula, implying $\mu_n(x, y)$ for some n, and with $\phi \perp =$; i.e. $\mathcal{T} \models \neg(\exists x)\phi(x, x)$. Then for some $k = k(\phi) \in \mathbb{N}$, $\mathcal{T} \models \neg(\exists x_0, \cdots, x_k) \bigwedge_{0 \leq i < j \leq k} \phi(x_i, x_j)$.

Walter Taylor isolated this condition (in the non-local setting) using the Erdoš-Rado theorem, in connection with the Hanf number for e.c. models; i.e. he proved the equivalence of (ecB) with (ppC). If (ppC) fails, it is clear from compactness that there exist arbitrarily large e.c. models of $\mathcal{T}$; indeed they contain arbitrarily long sequences (a_i) with $\phi(a_i, a_j)$ for all $i < j$, precluding any possible equality $a_i = a_j$. Conversely if (ppC) holds, (ecB) follows with $\kappa = 2^{\mathcal{L}}$; This is due (in the non-local setting) to Walter Taylor. we prove this below by constructing the universal model without any mention of cardinals bigger than $2^{\mathcal{L}}$.

Proposition 2.1. *Let $\mathcal{T}$ be a local primitive-universal theory enjoying (ppC). Then:*

(1) *Any $M \models \mathcal{T}$ admits a homomorphism to a unique $|\mathcal{L}|^+$-saturated e.c. model $\mathcal{U}$. If $M \underset{e.c.}{\models} \mathcal{T}$ then $\mathcal{U}$ is unique up to isomorphism over M.*

(2) *$\mathcal{U}$ is maximal: if $\mathcal{U} \leq B \underset{e.c.}{\models} \mathcal{T}$ then $U = B$. If $f : \mathcal{U} \to B \models \mathcal{T}$ is a homomorphism, there exists a homomorphism $r : B \to \mathcal{U}$ with $r \circ f = Id_{\mathcal{U}}$.*

(3) *Any homomorphism on $\mathcal{U}$ is an embedding, and any endomorphism $f : \mathcal{U} \to \mathcal{U}$ is an isomorphism.*

(4) *$\mathcal{U}$ is homogeneous for pp types.*

(5) *Any e.c. model A of $\mathcal{T}$ satisfies $|A| \leq 2^{|\mathcal{L}|}$.*

(6) if $\mathcal{T}$ has LJEP, then $\mathcal{U}$ is universal, i.e. , i.e. any $M \models \mathcal{T}$ admits a homomorphism $M \to \mathcal{U}$; and $\mathcal{T}$ is the unique universal e.c. model of $\mathcal{T}$.

Proof. We may assume $M \overset{e.c.}{\models} \mathcal{T}$, by finding a homomorphism from M to an e.c. model. Existence of an $|\mathcal{L}|^+$-saturated e.c. model $\mathcal{U}$, embedding M, with $|\mathcal{U}| \leq 2^{|\mathcal{L}|+|M|}$, now follows from the usual construction of saturated models, using amalgamation repeatedly and taking direct limits.

Claim . Let $\mathcal{U}$ be an $|\mathcal{L}|^+$-saturated e.c. model. Then $\mathcal{U}$ is maximal.

Proof. Suppose, for the sake of contradiction, that $B \overset{e.c.}{\models} \mathcal{T}$, $\mathcal{U} \leq B$, and $b \in B \setminus U$. By $|\mathcal{L}|^+$-saturation, one can find a sequence $(a_i : i < |\mathcal{L}|^+)$ with $a_i \in \mathcal{U}$ such that a_i has the same pp type as b over $\{a_j : j < i\}$. Since $a_i \in \mathcal{U}$ we have $a_i \neq b$, and so since B is e.c. there exists a pp formula $R_i \perp =$ with $R_i(a_i, b)$ and such that R_i implies μ_n for some $n = n_i$. By refining the sequence we may assume all R_i are equal to some R, and all $n_i = n$. Then $R(a_j, a_i) \wedge \mu_{2n}(a_i, a_j)$ holds for all $j < i$, contradicting (ppC). $\qquad\square$

Thus the first part of (2) holds. Now uniqueness in (1) follows too: if M admits a homomorphism into two such e.c. models $\mathcal{U}$ and $\mathcal{U}'$, by amalgamation over M they embed over M into a third model $\mathcal{U}''$; but then by the claim the embeddings $\mathcal{U} \to \mathcal{U}''$ and $\mathcal{U}' \to \mathcal{U}''$ are isomorphisms (over M.)

(3) Any homomorphism from any $A \overset{e.c.}{\models} \mathcal{T}$ into a model of $\mathcal{T}$ is an embedding, by definition of e.c..

Let $f : \mathcal{U} \to \mathcal{U}$ be an endomorphism, with image $U' \leq U$. Since $\mathcal{U}$ is e.c., f is an isomorphism $\mathcal{U} \to U'$; hence U' is also maximal. Since $U' \leq \mathcal{U}$, we have $\mathcal{U} = U'$. Thus f is an automorphism of $\mathcal{U}$.

As $\mathcal{U}$ is universal, if $\mathcal{U} \leq B \models \mathcal{T}$ there exists a homomorphism $f : B \to \mathcal{U}$; on $\mathcal{U}$ it induces an isomorphism g; so $r = g^{-1} \circ f : B \to \mathcal{U}$ is as required in the second sentence of (2).

(4) Homogeneity for pp types: let $C, C' \subset \mathcal{U}$ and let $g : C \to C'$ be a bijection preserving pp formulas. We may assume $C \neq \emptyset$, say $c \in C$. Using locality of $\mathcal{U}$, any existential sentence θ true in $\mathcal{U}_C$ is implied by a stronger pp sentence, specifying also a distance of the witnesses from c. Hence θ, transposed via g, is also true in $\mathcal{U}_{C'}$. Thus g extends to an embedding G of $\mathcal{U}$ in some elementary extension $\mathcal{U}^*$ of $\mathcal{U}$ (a weak model initially). Restricting to elements at finite distance from $g(c)$ will give a local model V including all of $g(\mathcal{U})$. We also have the usual

inclusion of $\mathcal{U}$ in $\mathcal{U}^*$, and again it is contained in V. Let $r : V \to \mathcal{U}$ be a retraction. Then $r \circ G : \mathcal{U} \to \mathcal{U}$ is an automorphism extending g.

For (5), we may assume that A is $|\mathcal{L}|^+$-saturated. By Löwenheim-Skolem, A contains an $|\mathcal{L}|^+$-saturated e.c. substructure A' of cardinality at most $2^{|\mathcal{L}|}$. By the Claim we have $A = A'$.

(6) Now assume LJEP. Then any two $|L|^+$-saturated e.c. models $\mathcal{U}, \mathcal{U}' \models T$ are isomorphic. Indeed by LJEP $\mathcal{U}, \mathcal{U}'$ both embed into some $\mathcal{U}''$, and then $\mathcal{U} \cong \mathcal{U}'' \cong \mathcal{U}'$ as in the proof of uniqueness in (1).

Any universal $U \overset{e.c.}{\models} \mathcal{T}$ is isomorphic to $\mathcal{U}$: since U is universal, there exists a homomorphism $f : U \to \mathcal{U}$, which must be an embedding; so we may assume $U \leq \mathcal{U}$. Then there exists a retraction $r : \mathcal{U} \to U$. But endomorphisms of $\mathcal{U}$ are isomorphisms, so $\mathcal{U} \cong U$.

$\square$

In the absence of LJEP, we can define a distance on maximal pp types, namely $d(p, q) \leq n$ if $\mu_k(x, y) \wedge p(x) \wedge q(y)$ is consistent with $\mathcal{T}$. Define an equivalence relation on types, $d(p, q) < \infty$. Then one can generalize (6) to (6'): there is one universal model realizing each equivalence class of types.

2.2. **The pp topology on** $\mathcal{U}$**.** Let us topologize the sorts of $\mathcal{U}$ (specifically, $\mathcal{D}^m$ for $m \in \mathbb{N}$). We give each sort of $\mathcal{U}$ the coarsest topology such that every pp-definable set, with parameters, is closed. Thus as a pre-basis for $\mathcal{D}^m$ we take the complements of sets of the form $\{x : R(x, c)\}$, with $x = (x_1, \ldots, x_m)$, $c = (c_1, \ldots, c_k) \in \mathcal{U}$, and R a pp relation. While the existential quantifiers in R must be bounded in line with the local definition of pp, there is no requirement that R imply a finite distance between its variables. In case every pp relation is equivalent to a conjunction of atomic formulas (as will hold for the pattern language, Theorem 3.10), we can of course restrict attention to atomic R. As we included equality in the language, singletons in $\mathcal{D}^n$ are pp-definable with parameters and hence closed, so each sort of $\mathcal{U}$ is T1.

We will now make a further assumption; essentially, it asserts that ∞ can be separated from any given point of $\mathcal{U}$ by disjoint open sets.

(*Loc*) There exists a pp formula $\mu_{>n}(x, y)$ such that $\mu_{>n} \perp \mu_n$, and for some $N > n$, $\mathcal{D} \overset{e.c.}{\subseteq} \mu_{>n} \cup \mu_N$.

Lemma 2.3. *Let* $a \in \mathcal{D}(\mathcal{U})$, $n \in \mathbb{N}$, *and let* $B = \mu_n(a) \subset \mathcal{D}$ *be a* μ-*ball. Then* $B^m \subset \mathcal{D}^m$ *is compact. Assuming (Loc) holds,* $\mathcal{D}$ *is locally*

compact;[1] *and every compact subset of* $\mathcal{D}(\mathcal{U})$ *is contained in some ball* $\mu_n(a)$.

Proof. Consider a family F_i of basic closed subsets of B^m with the finite intersection property. F_i is defined by $R_i(x, c_i)$ with R_i a finite disjunction of pp formulas. In an elementary extension U' of $\mathcal{U}$, one can find $d = (d_1, \ldots, d_m)$ with $R_i(d, c_i)$ holding for all i, as well as $\mu_n(a, d_i)$ for each $i \leq m$. We can initially allow U' to be a non-local extension, then discard all elements at infinite distance from a; this will not effect the truth value of local formulas for the remaining elements, and the d_i remain in the domain. By Proposition 2.1 there exists $r : U' \to \mathcal{U}$, $r|\mathcal{U} = Id_{\mathcal{U}}$. Then $R_i(r(d), c_i)$ holds for each i, so $r(d) \in \cap_i F_i$ and $\cap_i F_i \neq \emptyset$.

If (Loc) holds, then a lies in the interior of $\mu_N(a)$, since it is contained in the complement of $\mu_{>n}(a)$ which in turn is contained in $\mu_N(a)$. Hence a has a compact neighborhood, and so $\mathcal{D}$ is locally compact. If Y is a compact subset of $\mathcal{D}(\mathcal{U})$, as it is covered by neighborhoods $\mu_N(a')$, it must be covered by finitely many such neighborhoods, $\mu_N(a_1), \cdots, \mu_N(a_m)$; and as the 'centers' a_i are at finite distances from each other, it follows that Y is contained in a large enough ball. $\qquad\square$

Lemma 2.4. *Let* $A \models \mathcal{T}$, $a \in A$. *Assume:*

(1) *Each ball* $\mu_n(a)$ *is compact in the pp-topology.*
(2) *For some* n, *if* $\phi(x)$ *is a pp-formula in one variable then either* $\mathcal{T} \models \neg\exists x \phi$, *or* ϕ *is realized in* $\mu_k(a)$.

Then A is universal, i.e. every model of $\mathcal{T}$ admits a homomorphism to A; and $\mathcal{T}$ is pp-compact (ppC).

Proof. To prove universality it suffices to show that any $B \overset{e.c.}{\models} \mathcal{T}$ admits a homomorphism into A. Let $b_0 \in B$. Then any finite number of pp formulas true of b_0 are realized in $\mu_k(a)$; the sets of realizations are closed; hence by (1), they are realized simultaneously, i.e. some $a_0 \in \mu_k(a)$ realizes the pp-type of b_0. Let f be a map from C into A preserving pp-formulas, with maximal possible domain C containing b_0. To prove universality we have to show that $C = B$. Otherwise, let $b \in B \smallsetminus C$. Let p be the pp-type of b over C. Then p ensures $\mu_n(x, b_0)$ for some n. By compactness of $\mu_{n+k}(a)$, as in the first step, $f(p)$ is realized in A. This contradicts the maximality of C and proves that $C = B$, and so A is universal.

---

[1]We use the term to mean that every point has a compact neighborhood; it does not presuppose that $\mathcal{D}$ is Hausdorff.

Let us now prove ppC. Let $\phi(x, y)$ be a local pp formula contradicting equality, as in the definition of ppC; $\phi(x, y)$ implies $\mu_m(x, y)$. We may take ϕ symmetric (replacing it by $\phi(x, y) \vee \phi(y, x)$ if necessary.) Suppose there is no finite bound on the size ϕ-cliques. Then by compactness there exists a weak model B of $\mathcal{T}$ and an infinite ϕ-clique in B; it must be contained in a μ_m-ball, and so we can take B to be a local model. Then B embeds into A, so in fact A contains an infinite ϕ-clique within some ball $\mu_n(a)$. Extend it to a maximal ϕ-clique $I \subset \mu_n(a)$. Then $\mu_n(c)$ is covered by the open sets $\neg\phi(c, x)$, as c ranges over I. Hence $\mu_n(a)$ is covered by a finite number of sets $\neg\phi(c, x)$, where now $c \in I_0$ for some finite $I_0 \subset I$. If $d \in I \setminus I_0$ we have however $\phi(c, d)$ for each $c \in I_0$. This contradiction proves that a bound exists. $\qquad\square$

It follows from universality that $\mathcal{T}$ has LJEP; of course, condition (2) really implies LJEP_k.

2.5. The pp topology on $\mathrm{Aut}(\mathcal{U})$.

Let $G = \mathrm{Aut}(\mathcal{U})$. Give G the topology whose pre-basic open set have the form

$$\{g : \neg R(ga_1, \ldots, ga_n, b_1, \ldots, b_m)\}$$

where $a_1, \ldots, a_n, b_1, \ldots, b_m \in \mathcal{U}$ and R is pp. The action $G \times \mathcal{U} \to \mathcal{U}$ is then continuous in the first variable; for fixed g it is also continuous in the second variable, since g is an automorphism. It is also easy to see that multiplication $G \times G \to G$ is continuous in each variable, and that inversion $g \mapsto g^{-1}$ is continuous.

As $\mathcal{U}$ is T1, it follows from left-continuity of the action that G is T1.

Let $G(a, n) = \{g \in G : \mu_n(g(a), a)\}$.

Lemma 2.6. $G(a, n)$ is compact. Assuming (Loc), G is a locally compact space.

Proof. Fix a and n. Let u be an ultrafilter on a set I, and let $g_i \in G(a, n)$, $i \in I$; we need to find a limit point g of $(g_i)_i$ along u. Let $\mathcal{U}^*$ be the ultrapower of $\mathcal{U}$ along u, and let $g_* : \mathcal{U} \to \mathcal{U}^*$ be the ultraproduct of the maps $g_i : \mathcal{U} \to \mathcal{U}^*$. Since $\mu_n(g_i(a), a)$, we have $\mu_n(g_*(a), a)$ and hence $\mu_{n+k}(g_*(c), a)$ whenever $\mu_k(c, a)$. Thus g is defined on $\mathcal{U}$ into the local ultrapower. Let j be the diagonal embedding $\mathcal{U} \to \mathcal{U}^*$, ultrapower of $Id : \mathcal{U} \to \mathcal{U}$. As $\mathcal{U}^* \models \mathcal{T}^{\pm}$, Proposition 2.1 provides a homomorphism $r : \mathcal{U}^* \to \mathcal{U}$ with $r \circ j = Id_{\mathcal{U}}$. Let $g = r \circ g_*$. Then $g \in End(\mathcal{U}) = \mathrm{Aut}(\mathcal{U})$. If $R(g_i a_1, \ldots, g_i a_n, b_1, \ldots, b_m)$ holds for u-almost all $i \in I$, then $\mathcal{U}^* \models R(g_* a_1, \ldots, g_* a_n, j b_1, \ldots, j b_m)$ so $\mathcal{U} \models R(ga_1, \ldots, ga_n, b_1, \ldots, b_m)$. Hence g is indeed a limit point of $(g_i)_i$ along u.

Assuming (Loc), G is locally compact: since translations are continuous, it suffices to show that the identity element $1 \in G$ has a compact neighborhood. Indeed as in Lemma 2.3, the compact set $G(a, N)$ contains an open set including 1, namely the complement of $\{g : \mu_{>n}(a, g(a))\}$. $\qquad\square$

Let X be a G-invariant subset of $\mathcal{D}$. Define the infinitesimal elements of G, in its action on X, to be

$$\mathfrak{g}_X = \{g \in G : \text{for every nonempty open } U \subset X, \quad gU \cap U \neq \emptyset\}$$

Fix X to be either $\mathcal{D}(\mathcal{U})$ or a complete 0-definable pp type $P \subset \mathcal{D}(\mathcal{U})$; and let $\mathfrak{g} = \mathfrak{g}_X, \mathcal{G} = G/\mathfrak{g}, \pi : G \to \mathcal{G}$ the quotient map.

By Lemma C.1 of [27], $\mathfrak{g}$ is a closed normal subgroup of G, and $\mathcal{G} = G/\mathfrak{g}$ is Hausdorff. Note that $G/\mathfrak{g}$ remains locally compact, with continuous inversion and right and left multiplication. It follows from Ellis' joint continuity theorem [18] that $G/\mathfrak{g}$ is a locally compact Hausdorff topological group, acting continuously on $\mathcal{U}/\mathfrak{g}$. (A direct proof in our setting is given in [27] 3.27.)

We have shown the main part of:

Proposition 2.7. *Assume (ecB) and (Loc). Let $\mathcal{U}$ be the universal e.c. model, $G = Aut(\mathcal{U}), \mathfrak{g} = \mathfrak{g}_\mathcal{U}, \mathcal{G} = G/\mathfrak{g}$. Then $\mathcal{G}$ is a locally compact topological group. With $\pi : G \to \mathcal{G}$ the quotient map, we have:*

- *If $Y \subset \mathcal{G}$ is precompact, then $\pi^{-1}(Y)$ is bounded.*
- *If $W \subset G$ is bounded, then πW is precompact.*

Here $W \subset G$ is said to be bounded if for some/any $a \in \mathcal{D}(\mathcal{U})$, for some $N \in \mathbb{N}$, $Wa \subset \mu_N(a)$; see Definition C.1 12.

Proof. We saw that $\mathcal{G}$ is a locally compact topological group, and that $\mathfrak{g}$ is a closed normal subgroup of G. Fix $a \in \mathcal{D}(\mathcal{U})$. By Lemma 2.3, a has a compact neighborhood $W = \mu_n(a)$. Then by definition of $\mathfrak{g}$, if $g \in \mathfrak{g}$ then $gW \cap W \neq \emptyset$; so $\mu_n(a, gb)$ holds for some b with $\mu_n(a, b)$; hence $\mu_n(ga, gb)$, and $\mu_{2n}(a, ga)$. This shows that $\mathfrak{g} \subset G(a, 2n)$, which is compact by Lemma 2.6. Thus $\mathfrak{g}$ is compact. It follows easily[2] that if $Y \subset \mathcal{G}$ is compact, then so is the pullback to G. Thus for the first item it suffices to show that if $Y \subset G$ is compact, then $Y \subset G(a, N)$ for some N. Indeed as Y is compact, Ya is compact. By Lemma 2.3, $Ya \subset \mu_N(a)$ for some N; hence by definition $Y \subset G(a, N)$.

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[2][Let $\pi : G \to \mathcal{G}$ be the quotient map. Let $Y \subset G$ be closed with $\pi(Y)$ compact. Then Y is compact: Let $y_i \in Y, i \in I$, and u an ultrafilter on I. The $\lim_{i \to u} \pi(y_i) = \pi(b)$ for some b. Since $b\mathfrak{g}$ is compact, it has a compact neighborhood $U = U\mathfrak{g}$; and we have $y_i \in U$ for almost all i; so y_i approaches a limit along u.]

If $W \subset G$ is bounded, $W \subset G(a, N)$ for some N; we saw that $G(a, N)$ is compact; hence so is $\pi(G(a, N))$. So $\pi(W) \subset \pi(G(a, N))$ is precompact.

$\square$

Proposition 2.8. *Assume (ecB) and (Loc); also that $\mathcal{L}$ is countable. Let P be a complete pp type in $\mathcal{D}(\mathcal{U})$, $\mathcal{G}(P) = G/\mathfrak{g}_P$. Then $\mathcal{G}(P)$ is a second countable, locally compact topological group. If P' is another complete pp type in $\mathcal{D}(\mathcal{U})$, then $\mathcal{G}(P'), \mathcal{G}(P)$ differ only by a compact isogeny.*

Proof. We have the natural homomorphism $G \to \mathcal{G}(P)$, by restriction and then going to the quotient. It is clearly continuous, when $\mathcal{G}(P)$ is endowed with the topology described above, with respect to the action of G on P. The kernel is compact, indeed with n as in the proof of Proposition 2.7, for any $a \in P$, the kernel is contained in $G(a, n)$. It follows that $\mathcal{G}(P)$ is locally compact, and (regardless of P) differs by a compact isogeny from $\mathcal{G}$. When $\mathcal{L}$ is countable, the proof of [27] 3.24 shows that $\mathcal{G}(P)$ is second countable. (using σ-compactness via $G = \cup_{n \in \mathbb{N}} G(a, n)$ in place of compactness; note that Lemma C.4 remains valid when $\mathfrak{t}_1$ is σ-compact, and $X \times Y$ is $\mathfrak{t}_2$-compact for $\mathfrak{t}_1$-compact X, Y, as it the case here.) $\square$

3. The core space

Let T be a complete first-order theory, in a language L with a distinguished sort D and locality relation on D, that we take to be generated by a single reflexive, symmetric binary relation m_1 on D. Let $d_m(x, y)$ be the Cayley-graph distance formed from m_1, as in the discussion of local logic. A model of T will be said to be *m-local* if any two elements of D are at finite d_m- distance. We treat T in local logic. This means, first of all, that we restrict attention to m-local models M of T, and in particular to the part of the type space consisting of types realized in elementary extensions that are themesleves m-local. If a weak model N' is an elementary extension of M in the usual sense, and N is obtained by simply removing any points of $D(N')$ at infinite distance from points of $D(M)$, then by fiat, N is a (local) elementary extension of T in the local logic sense.

We will use only local formulas, i.e. formulas built up using local quantifers only; with the one exception that we assume T is complete with respect to sentences $(\forall x)\phi$ and $(\exists x)\phi$, where ϕ is local. [3]. Note

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[3]We can assume or arrange by Morleyzation that T admits local quantifier-elimination. i.e. every local formula is equivalent to a quantifier-free one. In this

that the 'local' restriction on types $p(x) \in S_D(M)$ amounts simply to saying that some formula $m_n(x, b)$ be represented in p. Following convention in enriched logics, we denote the usual quantifier-free type space by $S^{weak}(M)$; and the subspace of local types by $S(M)$.

For simplicity we will concentrate on the space of local 1-types in D, S_D. $S_D(M)$ is an open subset of $S_D^{weak}(M)$ with the usual topology; hence it is locally compact.

3.1. **The pattern language.** Let us recall the construction in [27] of the pattern language $\mathcal{L}$, and the $\mathcal{L}$-structure on the type spaces $S_x(M)$, $M \models T$. Here we concentrate on a single sort D of T, and will be interested only in one sort of $\mathcal{L}$, to be denoted $\mathcal{D}$, corresponding to local 1-types on D. Let x, x_i be variables ranging over D. Let y be an additional tuple of variables (the *parameter variables*.) Let $t = (\phi_1, \ldots, \phi_n; \alpha)$ be an n-tuple of formulas $\phi_i(x_i, y)$, and let $\alpha(y)$ be a formula. To each such t we associate a relation symbol Def_t of $\mathcal{L}$, taking variables $(\xi_1, \ldots, \xi_n)$ ranging over $\mathcal{D}$. For any $M \models T$, we define an $\mathcal{L}$-structure with universe $S = S_D(M)$. The interpretation of Def_t is:

$$\mathsf{Def}_t^S = \{(p_1, \ldots, p_n) \in S : (\forall a \in \alpha(M)) \bigvee_{i \leq n} (\phi_i(x, a) \in p_i)\}$$

Thus when $n = 1$, $\mathsf{Def}_{\phi;\alpha}$ indicates a partial definition scheme for ϕ, thus the notation Def. (In [27], $Om_{\neg\phi;\alpha}$ was used instead, where Om stands for *omitting* $\neg\phi$. These are equivalent in the presence of negation in L, but even then Def is more natural.) For typographical reasons, we will sometimes write $\mathsf{Def}[t]$ for Def_t.

3.2. **Locality structure on $\mathcal{D}$.**

Definition 3.3. A model M of T is *orbit-bounded* (in the sort D) if for some $k \in \mathbb{N}$, for all $a, b \in D(M)$, there exists $\alpha \in Aut(M)$ such that $a, \alpha(b)$ are at distance $\leq k$.

We will make two assumptions about T as a local theory.
(OB) T has an orbit-bounded model M.
(DP) For each $r \in \mathbb{N}$, for some $m = m(r) \in \mathbb{N}$, every $r + 1$-ball is a union of at most m r-balls.

The second assumption, (DP), is the *doubling* property for the metric d_m. Assuming any two points are connected by a finite chain of points at distance 1 (i.e. the locality is generated by a single relation), (DP)

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case T is determined by its universal part; the models of T are e.c. models of $T_\forall$. In [27] it was useful to allow more general universal theories, restricting attention to e.c. models; this could be done here too.

follows from the same statement for $r = 1$. If it holds for one ball in an orbit-bounded model, then the uniform version will be valid too.

(DP) is needed in order to define the locality structure for the pattern language, by a pp formula. It follows from doubling that every n-ball is a union of finitely many 1-balls; hence a type p of D over a model M is local iff it represents the formula $m(x, y)$, i.e. $m(x, c) \in p$ for some $c \in D(M)$. (In other words we will not have $\mathsf{Def}[\neg m(x, y); D(y)]$.)

The terminology orbit-bounded is short for 'the space of orbits of $Aut(M)$ on M is bounded, under the induced metric'. In our applications, (OB) will hold in every local model of T. In general, if T has one orbit-bounded model, then every $\aleph_0$-homogneous model will be orbit-bounded.

Definition 3.4. We define the locality generating relation $\mu_1(\xi_1, \xi_2)$ for $\mathcal{L}$ to be

$$\mathsf{Def}[\neg m_1(x_1, y_1), \neg m_1(x_2, y_2); \neg m_5(y_1, y_2)]$$

Towards (Loc), we also define $\mu_{>1}(\xi_1, \xi_2)$ by the formula:

$$\mathsf{Def}[\neg m_1(x_1, y_1), \neg m_1(x_2, y_2); m_5(y_1, y_2)]$$

(We used 5 above, where 3 would do, in order to ensure below that close Lascar neighbors have μ-distance at most 1; it is not an essential point.)

For the space of types over a model M, if $p_1, p_2 \in S(M)$ are local types, and $S(M) \models \mu_1(p_1, p_2)$, by (DP) there exist $b_1, b_2 \in D(M)$ with $m_1(x_i, b_i) \in p_i$; now $\mu_1(p_1, p_2)$ guarantees that $m_5(b_1, b_2)$ holds, so that any two realizations of p_1, p_2 will be at m-distance at most 7.

Conversely if realizations b_i of p_i exist with $m_1(b_1, b_2)$, then $\mu_1(p_1, p_2)$ holds (even with 3 in place of 5): we have to show that if $M \models \neg m_5(c_1, c_2)$ then $\neg m_1(x_1, c_1) \in p_1$ or $\neg m_2(x_1, c_2) \in p_2$. Otherwise, $m_1(x_1, c_i) \in p_i$ so $m_1(b_i, c_i)$. Then (initially in an elementary extension of M) the chain c_1, b_2, b_2, c_2 shows that $m_3(c_1, c_2)$ holds, a contradiction.

Lemma 3.5. *Assume M is orbit-bounded; then $S_D(M)$ is orbit-bounded too, even just with respect to the action of $Aut(M)$.*

Proof. Let $p, q \in S_D(M)$. Since $p(x)$ is local, for some $n \in \mathbb{N}$, $p(x)$ contains a formula defining an m-ball of radius n. By the doubling property (DP) it follows that $p(x)$ contains a formula defining an m-ball of radius 1; i.e. $m_1(x, c) \in p$ for some $c \in M$. Similarly $m_1(x, d) \in q$ for some $d \in M$. Let $\sigma \in Aut(M)$ be such that $m(d, \sigma(c)) \leq k$. Then σ induces an automorphism of $S(M)$, and $m_1(x, \sigma(c)) \in \sigma(p)$. Let $d = d_1, \cdots, d_k = \sigma(c)$ be a chain of elements of $D(M)$, with $m_1(d_i, d_{i+1})$.

(Say k is even.) Also write d_i for the algebraic type $x = d_i$. We have $\mu_1(q, d_2), \mu_1(d_2, d_4), \cdots, \mu_1(d_{k-2}, d_k), \mu_1(d_k, \sigma(p))$. Thus $\mu_{k/2+1}(\sigma(p), q)$ holds in $S(M)$. $\qquad\square$

3.6. **The pattern theory of** T. We define $\mathcal{T}$ be the set of local, primitive universal $\mathcal{L}$-sentences true in $S(M)$ for some $M \models T$. Defined this way, it is not clear that $\mathcal{T}$ is consistent; but this, along with LJEP, follows from Lemma 3.7.

Lemma 3.7. *Let M be an orbit-bounded model of T. Then $S(M) \models \mathcal{T}$.*

Proof. Let $N \models T$. If $S(M) \models (\exists x_1) \cdots (\exists x_n)\Theta$, where Θ is a finite conjunction of basic relations of $\mathcal{L}$, we have to show that $S(N) \models (\exists x_1) \cdots (\exists x_n)\Theta$. For simplicity, let us assume Θ is a basic relation $\mathsf{Def}[\phi_1, \ldots, \phi_n; \alpha]$.

Let $p_1, \ldots, p_n \in S(M)$ be such that $S(M) \models \Theta(p_1, \ldots, p_n)$. Since M and $S(M)$ are local, each p_i is at some finite m-distance d from some $a \in M$, i.e. $m_d(a, x_i) \in p_i$. Thus the following partial type over M is consistent:

$$\{\bigvee_{i=1}^{n} \phi_i(x_i, c) : c \in \alpha(M)\} \cup \{m_d(a, x_i) : i = 1, \ldots, n\}$$

Hence, for any $\sigma \in Aut(M)$, the same partial type is consistent if we replace a by $\sigma(a)$. By the orbit boundedness of M, it follows that for *any* $a' \in D(M)$, the type below is consistent too:

$$\{\bigvee_{i=1}^{n} \phi_i(x_i, c) : c \in \alpha(M) \cup \{m_{d+k}(a', x_i) : i = 1, \ldots, n\}$$

Thus for any m, we have in M:

$$(\forall u \in D)(\forall c_1, \ldots, c_m)(\bigwedge_j \alpha(c_j) \to (\exists x_1, \ldots, x_n) \bigwedge_i m_{d+k}(u, x_i) \wedge \bigwedge_j (\bigvee_{i=1}^{n} \phi_i(x_i, c_j)))$$

This applies equally to N (using the completeness of T), and so retracing out steps, we find that for any $b' \in N$ there exist types $q_1, \ldots, q_n$ with $\mu_{d+k}(b', x_i) \in q_i$ and $S(N) \models \mathsf{Def}_t(q_1, \ldots, q_n)$. The q_i are local types, and show that $S(N) \models (\exists x_1) \cdots (\exists x_n)\Theta$, as promised. $\qquad\square$

Regarding (*Loc*), clearly $\mu_1, \mu_{>1}$ are incompatible in any local type space. On the other hand, if $p_1, p_2 \in S(M)$, then either $\mu_{>1}$ holds or $m_i(x_i, b_i) \in p_i$ for some $b_1, b_2 \in \mathcal{D}(M)$ with $m_5(b_1, b_2)$. Then as in the proof of Lemma 3.5 we have $\mu_4(p_1, p_2)$. Thus $S(M) \models (\forall \xi)(\mu_{>1}(\xi) \vee \mu_4(\xi))$, so (say by Lemma 3.8) $\mathcal{T} \overset{e.c.}{\models} (\forall \xi)(\mu_{>1}(\xi) \vee \mu_4(\xi))$.

Lemma 3.8. *Let $M \models T$ be orbit-bounded. $A \models \mathfrak{T}$. Then there exists a homomorphism $f : A \to S(M)$. In particular any e.c. model of $\mathfrak{T}$ is isomorphic to a substructure of $S(M)$. $\mathfrak{T}$ has the properties (ecB) and (Loc).*

Proof. We have to check that the usual (locally compact) type space topology of $S_D(M)$ satisfies the hyptheses of Lemma 2.4. (2) there was already proved in Lemma 3.5.

The interpretation of $R = \mathsf{Def}[\phi_1, \ldots, \phi_n; \alpha]$ is closed, since if $p = (p_1, \ldots, p_n) \notin R$, then $\neg\phi_i(x_i, c) \in p_i$ for some $c \in \alpha(M)$; the set of all tuples $q = (q_1, \ldots, q_n)$ with $\neg\phi_i(x_i, c) \in q_i$ is an open neighborhood of p in the product topology, disjoint from R.

As for compactness of $\mu_n(p)$, where $p \in S_D(M)$: if $q \in \mu_n(p)$, there exists a chain $p = p_0, \cdots, p_n = q$ with $\mu_1(p_i, p_{i+1})$. Pick $a_i \in D(M)$ with $m_1(x, a_i) \in p_i$. Then $m_5(a_i, a_{i+1})$, so $m_{1+5n}(x, a_0) \in q$. The types q with $m_{1+5n}(x, a_0) \in q$ form a compact set W in the usual topology on the weak type space, which is entirely contained in the space of local types; hence W is compact. Now $\mu_n(p)$ is a subset of W defined by a pp relation, hence is closed in W and thus it is compact too.

$\square$

Proposition 2.1 and Proposition 2.7 are thus valid for $\mathfrak{T}$. We call $\mathcal{U}$ the core space of T. To summarize:

Let T be a local theory in the sort D, complete (with respect to local formulas), and satisfying (DP) and (OB). We have defined the *pattern language* $\mathcal{L}$, a local primitive universal theory in the sort $\mathcal{D}$, valid in $S_D(M)$ when $M \models T$.

Definition 3.9. A *core* for (type spaces of models of) T is an $\mathcal{L}$-structure J such that:

- For any orbit-bounded $M \models T$, there exists an $\mathcal{L}$- embedding $j : J \to S(M)$.
- For any such j, there exists a homomorphism $r : S(M) \to J$ such that $r \circ j = Id_J$.

Theorem 3.10. *Let T be a complete local theory in the sort D. Assume T has an orbit-bounded model, where every 2-ball is contained in finitely many 1-balls. Then:*

(1) A core exists and is unique up to isomorphism. We denote it $\mathfrak{J} = \mathrm{Core}\,(T)$.

(2) Reflection: let Θ be a universal sentence $(\forall p_1, \ldots, p_k)R$, where R is an arbitrary Boolean combination of local pp formulas in the pattern language $\mathcal{L}$. Let M be an orbit-bounded model of T. If $S(M) \models \Theta$ then $\mathfrak{J} \models \Theta$.

(3) *Quantifier elimination:* $\mathcal{L}$-*atomic types in* $\mathcal{J}$ *are orbits of* $Aut(\mathcal{J})$. *Every pp formula is equivalent (in any* $S(M)$ *as well as in* $\mathcal{J}$*) to a certain conjunction of atomic formulas.*
(4) $Aut(\mathcal{J})$ *has a natural locally compact topology:*

$$W(R; a, b) := \{g : \mathsf{Def}_t(ga_1, \ldots, ga_n, b_1, \ldots, b_m)\}$$

is a basic closed set. The canonical Hausdorff quotient $\mathcal{G}(T) := Aut(\mathcal{J})/\mathfrak{g}$ *is a locally compact topological group.*

We have just proved (1) and (4). By definition of a core, there exists an $\mathcal{L}$- embedding $j : \mathcal{J} \to S(M)$, as well as $r : S(M) \to \mathcal{J}$ with $r \circ j = Id_{\mathcal{J}}$. It follows that the truth value of any pp ψ formula is preserved by j; i.e. $\mathcal{J} \models \psi(a)$ if and only if $S(M) \models \psi(j(a))$. Hence for an arbitrary Boolean combinations Ψ of pp formulas, if $S(M) \models (\forall x)\Psi$, then $S(M) \models \Psi(j(a))$ so $\mathcal{J} \models \Psi(a)$, this for any a from $\mathcal{J}$. This proves (2). For (3) see [27], 3.12 (1).

3.11. **The local Lascar-Shelah relation.** We continue to assume T has an orbit-bounded model M, with bound k, and that doubling (DP) holds; and we concentrate on a single sort D.

A *definable family of finite local partitions* is a pair of formulas (B, E), with B a non-empty 0-definable subset of D^n for some $n \geq 1$, and $E(x, y; z_1, \ldots, z_n) \subset D^2 \times B$, such that for some $f \in \mathbb{N}$, with f strictly above the radius k of $M/Aut(M)$ we have:

(1) E implies $m_f(x, z_1)$ and $m_f(y, z_1)$.
(2) B implies $m_f(z_1, z_i)$ for each $i \leq n$.
(3) For any $b = (b_1, \ldots, b_n) \in B$, $E_b(x, y) := E(x, y; b)$ defines an equivalence relation on $m_f(b_1)$, with at most f classes.

Two elements $a, a' \in D$ are *Lascar-Shelah neighbors*, denoted $a \triangle a'$, if for each $n \in \mathbb{N}$, every non-empty 0-definable set $B \subset D^n$, and every definable family $E \subset D^2 \times B$ of finite local partitions,

$$M \models (\exists z_1, \cdots, z_n) E(a, a', z_1, \ldots, z_n).$$

If this holds for a single E, we will say that they are Lascar-Shelah neighbors with respect to E. The notation is intended to suggest an isosceles triangle, with a and b at the distinct endpoints of the equal sides. We will nevertheless sometimes write $\triangle (x, y)$ for $x \triangle y$.

We call two (local) types p_1, p_2 over a model N *Lascar neighbors* if there exist $a_i \models p_i$ such that a_1, a_2 are Lascar neighbors, i.e. if $p_1(x_1) \cup p_2(x_2) \cup \triangle(x_1, x_2)$ is consistent.

Call p_1, p_2 *close neighbors* if for every definable family of finite local partitions $(E_b)_{b \in B}$, for some $b \in B(N)$ and $d' \in D(N)$, $x_i E_b d' \in p_i$.

This implies that $p_1 \cup p_2 \models \triangle(x_1, x_2)$, and in particular p_1, p_2 are neighbors.

If p_i, p_{i+1} are Lascar neighbors for $i = 1, \ldots, n-1$, we will say that p_1, p_n are at Lascar distance at most n, and write $\triangle^n (p_1, p_n)$.

Lemma 3.12. *Let M be an orbit-bounded model of T, $a, a' \in D(M)$. Then $a \triangle a'$ iff for some elementary extension M' of M, and elementary submodel $N \prec M'$, $N \cong M$, we have $tp(a/N) = tp(a'/N)$. (Here by convention M' is local, and tp refers to the quantifier-free type.)*
  Also, $a\triangle a'$ implies $m(x, y) \leq 2$.

Proof. Assume $tp(a/N) = tp(a'/N)$. We have $D(N) \neq \emptyset$; pick $d \in D(N)$. Since M' is local, $M' \models m_l(a, d), m_l(a', d)$ for some l. By (DP), $m_l(d)$ is a finite union of balls $m_1(c_i)$, $c_1, \ldots, c_r \in D(N)$. So $m_1(a, c)$ holds for some $c = c_j \in D(N)$. Since $tp(a/N) = tp(a'/N)$, we have $m_1(a', c)$ too. This proves that $m(x, y) \leq 2$.
  Now let (B, E) be a definable family of finite local partitions.
  As $B \neq \emptyset$, and using (OB), there exists $b = (b_1, \ldots, b_n) \in B(N)$ with $m(c, b_1) \leq k$. Thus $m_{k+1}(a, b_1)$ and $m_{k+1}(a', b_1)$ hold. Since E_b defines a finite partition of $m_f(b_1)$ and $f \geq k+1$, and using that N is a model and $tp(a/N) = tp(a'/N)$, we have $E(a, a', b')$. This proves that $a \triangle a'$.
  Along the way we saw that $m(a, a') \leq 2$.
  Conversely, assume $a \triangle a'$. Let $y_c : c \in M$ be variables, and let Δ be the diagram of M, i.e. the set of L-local formulas in the variables y_c that become true in M under the assignment $y_c \mapsto c$. Fix one of the variables y_c, and call it y_1. Let Δ' be the set of all local formulas $\phi(x, y) \iff \phi(x', y)$, where the y are some of the y_c, including y_1, and ϕ is a formula of L. Since $a \triangle a'$, it is easy to see that $T \cup \Delta \cup \Delta' \cup \{m_{k+1}(a, y_1), m_{k+1}(a', y_1)\}$ is consistent. Hence this set of formulas is realized in some weak model, and restricting to elements at finite distance from a and a', it is realized in some model. This leads to $N \cong M$ with $N, M \prec M'$ for some $M' \succ M$. $\qquad\square$

3.13. **The neighbor relation in the pattern language.**

Lemma 3.14. *(1) ($\mathcal{L}$-$\bigwedge$-definability of $\triangle$.) There exist atomic formulas ψ_m of $\mathcal{L}$ such that for any $N \models T$, and any $p, q \in S(N)$, $p\triangle q$ iff for each $m \in \mathbb{N}$, $S(N) \models \psi_m(p, q)$.*
(2) The close neighbor relation on $S(M)$ is an intersection of $\leq |L|$ open sets in the pp topology.
(3) Let $\rho : S(N) \to S(N)$ be a retraction, i.e. a homomorphism with $\rho^2 = \rho$. Let $p \in S(N)$. Then $p, \rho(p)$ are close Lascar neighbors.

(4) Assume T is orbit-bounded. Let J be a copy of $\mathfrak{J}$ in $S(M)$, and assume $p, q \in J$ cannot be separated by disjoint open subsets of J. Then p, q are at Lascar distance at most 2.

Proof. (1) Let $p, q \in S(N)$. If $p \triangle q$ then for every definable family (B, E) of finite local partitions such that

$$T \models \neg(\exists x, x', u, y)(\phi(x, u) \wedge \phi'(x', u) \wedge \alpha(u) \wedge B(y) \wedge E(x, x', y))$$

we have

$$S(N) \models \mathsf{Def}[\neg\phi, \neg\phi'; \alpha](p, q)$$

Conversely, if $(p, q) \notin \triangle$, then $p(x) \cup q(x')$ (a set of formulas of $L(M)$, including the diagram of M) must be inconsistent with the following set of formulas:

$$\{(\exists y)(B(y) \wedge E(x, x', y)) : (B, E)\}$$

where (B, E) ranges over all definable families of finite local partitions, $y = (y_1, \ldots, y_k)$. Note that these formulas imply a bounded distance between x, x' and each y_i, and between them and the elements of M (the latter because p, q are local types.) By compactness, for some tuple c from N, $\phi(x, c) \in p$, $\phi(x', c) \in p'$, $\alpha \in tp_N(c)$, and for some finite set (B_j, E_j) of definable families of finite local partitions,

$$T \models \neg(\exists x, x', u, y_1, \ldots, y_r)(\phi(x, u) \wedge \phi(x', u) \wedge \alpha(u) \wedge \bigwedge_j B_j(y_j) \wedge E_j(x, x', y_j))$$

Let $y = (y_1, \ldots, y_r)$, $B(y) = \bigwedge_j B_j(y_j)$, $E(x, x', y) = \bigwedge_j E_j(x, x', y_j)$. Then

$$T \models \neg(\exists x, x', u, y)(\phi(x, u) \wedge \phi(x', u) \wedge \alpha(u) \wedge \bigwedge B(y) \wedge E(x, x', y))$$

yet c shows that $S(N)$ does not omit ϕ, ϕ', α. Thus $\mathsf{Def}[\neg\phi, \neg\phi'; \alpha](p, q)$ holds for each ϕ, ϕ', α as above iff $p \triangle q$.

(2) Let $q = \rho(p)$. Let $\bar{E} = (E_b)_{b \in B}$ be a definable family of local finite partitions. Consider the atomic formula $\psi = \psi_{\bar{E}}$,

$$\psi(\xi_1, \xi_2) = \mathsf{Def}[\neg x_1 E_y u, \neg x_2 E_y u; B(y) \wedge D(u)]$$

It is clear that $p, q \in S(M)$ are close Lascar neighbors with respect to $\bar{E}$ iff $S(M) \models \neg\psi(p, q)$. And $p, q \in S(M)$ close Lascar neighbors iff they are close Lascar neighbors with respect to each $\bar{E}$.

(3) Since a realization of q must be E_b-equivalent to some $d' \in N$ (as there are only finitely many classes, all represented in N), $S(N) \models \neg\psi(q, q)$. Now if we had $S(N) \models \psi(p, q)$, applying the homomorphism ρ would give $\psi(q, q)$, which we have just ruled out; so $S(N) \models \neg\psi(p, q)$. This means that for some $b \in B(N)$ and $d' \in D(N)$, $x_1 E_b d' \in p$ and

$x_2 E_b d' \in q$. As this holds for all definable families of finite partitions, $p, \rho(p)$ are close neighbors.

(4) By Lemma 3.8,$\mathcal{T}$ has Loc. Let N be a bounded open set containing both p and q. By compactness, it suffices to show that for each definable family $\bar{E}$ of local finite partitions, there exists $r \in cl(N)$ that is a Lascar neighbor to p and to q with respect to $\bar{E}$. In fact we will show that p, q have a common close Lascar neighbor in N, with respect to $\bar{E}$. By (2), the close neighbor relation with respect to $\bar{E}$ is pp-open. Let N' the $\bar{E}$-close neighbors of p within N; then N' is open and $p \in N'$. Similarly let N'' be the set of $\bar{E}$-close neighbors of p within N, an open neighborhood of q. Since p, q cannot be separated we have $N' \cap N'' \neq \emptyset$. Any $r \in N' \cap N''$ is a common close Lascar neighbor of p, q in N.

$\square$

Note that the ψ_m define the same relation on $S(N)$ and on any retract, notably any copy J of $\mathcal{J}$. We will refer to the relation they define on any model of $\mathcal{T}$ as the Lascar neighbor relation, and denote it by $\triangle$. Later we will use the same symbol on $\mathcal{J}$ and on $Aut(\mathcal{J})$ and $Aut(\mathcal{J})/\mathfrak{g}$; when the need arises to clarify, we will use a supersript, such as $\triangle^{\mathcal{J}}$.

3.15. Automorphisms of models and pattern automorphisms.

In this section, M, N will denote orbit-bounded (local) models of T. We say that a subset W of $Aut(M)$ is *bounded* if for some $a \in D(M)$, $Wa := \{w(a) : w \in W\}$ is contained in a d_m-ball of finite radius. Equivalently, for *all* $a \in D(M)$, Wa has finite radius. (Definition C.1 12.)

By Lemma 3.14 (1), $\triangle$ is defined by a conjunction Ψ of atomic sentences in any $S(N)$, N an orbit-bounded model of T. We define $\triangle$ by the same formula on $\mathcal{J}$. Since $\mathcal{J}$ embeds into $S(N)$, this does not depend on the choice of Ψ.

On $G = Aut(\mathcal{J})$, we let

$$\triangle = \triangle^{Aut(\mathcal{J})} = \{g \in Aut(\mathcal{J}) : (\forall a \in \mathcal{J})(a, g(a)) \in \triangle^{\mathcal{J}}\}$$

Let $\triangle^{\mathcal{G}}$ be the image of $\triangle^{Aut(\mathcal{J})}$ in $\mathcal{G}$.

$\triangle = \triangle^{\mathcal{G}}$ is a normal[4], symmetric, compact subset of $\mathcal{G}$. This follows from the same assertion in $Aut(\mathcal{J})$. In $Aut(\mathcal{J})$ we saw that the 2-ball is compact and contains $\triangle$ as a closed subset.

Theorem 3.16. *Assume (DP). Let M be an orbit-bounded model of T. $\iota : \mathcal{J} \to S(M)$ an embedding, and $\rho : S(M) \to \mathcal{J}$ a retraction. Then*

---

[4]I.e. conjugation-invariant. See Definition C.1 for group-theoretic terminology.

there exists a canonical quasi-homomorphism $\varphi : Aut(M) \to (\mathcal{G}(T) : \triangle)$. *We have:*

(1) When $C \subset \mathcal{G}$ *is precompact,* $\varphi^{-1}(C)$ *is bounded in* $Aut(M)$.
(2) When $W \subset Aut(M)$ *is bounded,* $\varphi(W)$ *is precompact in* $\mathcal{G}$.

Proof. Let $J = \iota(\mathcal{J})$, and identify J with $\mathcal{J}$; so that ι is the inclusion map $J \to S(M)$, and $\rho|J = Id_J$. The definitions of $\triangle$ on J and on $S(M)$ are compatible, via ι and ρ.

For $g \in Aut(M)$, define

$$\phi(g) = \rho \circ g \circ \iota \in Aut(\mathcal{J})$$

And let $\varphi = \pi \circ \phi$, where $\pi : Aut(\mathcal{J}) \to \mathcal{G}$ is the quotient map. Since π is a continuous and proper homomorphism, it suffices to prove the quasi-homomorphism property and (1,2) for ϕ.

We have $\phi(Id_M) = Id_J$ since $\rho \circ \iota = Id_J$. We have to show, for $g, h \in Aut(M)$, that $\phi(gh) \in \phi(g)\phi(h)\triangle$. For any $q \in S(M)$, we have $\rho(q)\triangle q$ in $S(M)$ by Lemma 3.14(3).

Thus

$$\rho h(q)\triangle h(q)$$

As $\rho g : S(M) \to \mathcal{J}$ is an $\mathcal{L}$-homomorphism, $\rho g \rho h(q) \triangle \rho g h(q)$; in particular, for $p \in \mathcal{J}$, $\rho g \rho h \iota(p) \triangle \rho g h \iota(p)$; so

$$\rho g \rho h \iota \triangle^{Aut(\mathcal{J})} \rho g h \iota$$

i.e. (using $\iota\rho = \rho$) $\phi(g)\phi(h)\triangle\phi(gh)$. Since $End(J) = Aut(J)$, we can write $\phi(gh) = \phi(g)\phi(h)k$ for some $k \in Aut(J)$, and we have $\phi(g)\phi(h)k\triangle\phi(g)\phi(h)$. Applying the $\mathcal{L}$-endomorphism $(\phi(g)\phi(h))^{-1}$ we see that $k \in \triangle^{Aut(J)}$.

This shows that $\phi : Aut(M) \to Aut(\mathcal{J}) : \triangle$ is a quasi-homomorphism. Composing with the Hausdorffization map $\pi : Aut(\mathcal{J}) \to \mathcal{G}(T)$, we see that $\phi : \mathcal{G}(T) : \triangle$ is also a quasi-homomorphism.

(1) In view of Proposition 2.7, it suffices to prove now that if $C \subset G = Aut(\mathcal{J})$ is bounded with respect to the action on $(\mathcal{J}, \mu)$, then $W := \phi^{-1}(C)$ is bounded with respect to the action on $D(M)$. Identifying $\mathcal{J}$ with a subset J of $S(M)$, the retraction ρ shows every point of $S(M)$ is at distance at most 1 from a point of J; moreover since ρ changes distances by at most 1, it is clear that W is bounded with respect to the action on $S(M)$. On the other hand, letting $i(a)$ denote the algebraic type $x = a$, we have an $Aut(M)$-invariant embedding $i : M \to S(M)$, and again we saw using (DP) that any element of $S(M)$ is at bounded distance from some element of $i(M)$. So if $a \in M$, then $Wi(a)$ has finite radius with respect to μ. But immediately following Definition 3.4, we saw that for $c, d \in i(M)$, the m-distance $m(c, d)$ is

at most $7\mu(i(c), i(d))$. Thus Wa has finite radius with respect to m, and so W is bounded with respect to the action on $D(M)$.

(2) Similarly, if $W \subset Aut(M)$ is bounded with respect to the action on (M, m), then it is bounded with respect to the induced action on types (to check boundedness we may choose a "center" a lying in M.) It follows as in (1) that the set $\rho \circ W$ has bounded action on J; so the image in $\mathcal{G}$ is precompact by Proposition 2.7. $\square$

Remark 3.17. If we introduce new sorts $\mathcal{D}_n$ corresponding to D^n, and define $\triangle_!^{Aut(\mathcal{J})}$ to be the set of automorphisms γ of $\mathcal{J}$ (on all these sorts) with $\gamma(a) \triangle a$ for any $a \in \mathcal{D}_n$, the proof still shows that $\phi : Aut(M) \to (Aut(\mathcal{J}) : \triangle_!)$. As we will have no immediate use for this apparent improvement, we will stick with $\triangle$.

Remark 3.18. In the various constructions of Theorem 3.16, we used the *fact* that the notion of locality is generated by a definable relation m, and we used the notion of finite distance in defining the notion of model and of type space. But m itself was not further used. Thus replacing m by another relation m', where $m'_1(x, y)$ implies $m_k(x, y)$ for some k, and vice versa, will change nothing: the class of models, the definition of the local type space, of the langauge $\mathcal{L}$, of $\mathcal{J}$, the notion of an embedding and a retraction between $\mathcal{J}$ and $S(M)$, and the maps ϕ and φ, all remain the same.

3.19. **Choices.** The homomorphism of Theorem 3.16 depends not only on M, but also on ι and ρ. Letting $\Sigma = S(M)$, the relevant spaces are $Hom(\mathcal{J}, \Sigma)$ (for choosing ι), and for $\iota \in Hom(\mathcal{J}, \Sigma)$, $J = \iota(\mathcal{J})$, the space of retractions $Hom_J(\Sigma, J)$. Their study is of considerable interest; in part, I think they are relevant to Gromov's questions about symmetry breaking in Ramsey theory. Here we will only mention a degree of canonicity that is nevertheless present, despite the choices. This will not be relevant to the main applications to approximate subgroups and lattices.

As discussed in [27], the choice of ι amounts to expanding M to a model of a certain universal theory T_{def}. T_{def} is no longer a local theory; it has a relation for each element of $\mathcal{J}$, and the action of $Aut(\mathcal{J})$ passes from the models to the langauge itself.

Remark 3.20. $\mathcal{J}$ is itself independent of the choices, and the quasi-homomorphism $Aut(M) \to \mathcal{J}$ (for a saturated M) can be made as unique as the saturated model itself: in an appropriate model of set theory, we fix a cardinal $\kappa \geq L_{def}$ with $2^\kappa = \kappa^+$, and let M_{def} be an existentially closed model of T_{def}, realizing every qf type over an existentially closed substructure of cardinality $\leq \kappa$. It was shown in

[27] (see 3.17) that T_{def} is an irreducible universal theory, i.e. the joint embedding property holds; this continues to apply sortwise; so M_{def} is unique up to isomorphism; and the reduct M to L is a κ-saturated model of T.

As for ρ, the arbitrariness is mitigated by Proposition 3.21 (1); in the spirit of Appendix B, it implies that ϕ gives a well-defined homomorphism $Aut(M) \to Aut_{\mathsf{ApprSp}}(\mathfrak{J}_h, \triangle_2)$.

Let us formulate a saturation hypothesis on a model M of T_{def}:
(SH)

(1) M embeds all models of T_{def} of size $\leq |\mathfrak{J}|$;
(2) M is homogeneous for L-elementary embeddings $M_1 \to M$ with $M_1 \models T$ with $|M_1| \leq |\mathfrak{J}|$. (I.e. for fixed M_1, all such embeddings are $Aut(M)$-conjugate.) (In particular, M is orbit-bounded.)

Note (SH) holds if either M is saturated as a model of T_{def}, or else if it is ec and saturated for qf types of L_{def}.

Proposition 3.21. *In Theorem 3.16, we further have:*

(1) If $f, f' : Aut(M) \to Aut(\mathfrak{J})$ are obtained from two choices of retractions ρ, ρ', then $f \triangle_2 f'$.

(2) Take M to satisfy (SH). Then ρ can be chosen so that the induced quasi-homomorphism $\phi : Aut(M) \to Aut(\mathfrak{J})$ (and $\varphi : Aut(M) \to \mathcal{G}(T)$) are surjective.

Proof. (1) Let $\alpha \in Aut(M)$. We have $\rho \circ \alpha \triangle \alpha$ by Lemma **??** (3). and $\rho' \circ \alpha \triangle \alpha$, so $\rho \circ \alpha \ \triangle_2 \ \rho' \circ \alpha$ and $\rho \circ \alpha \circ \iota \ \triangle_2 \ \rho' \circ \alpha \circ \iota$.

(3) The L_{def}-structure on M can be viewed as a homomorphism $\iota : \mathfrak{J} \to S(M)$. Let M_0 be a small elementary submodel of M as an L_{def}-structure, and let $r_0 : S(M) \to S(M_0)$ be the restriction map. Then $\iota_0 := r_0 \circ \iota : \mathfrak{J} \to S(M_0)$ admits, by the retraction property of $\mathfrak{J}$, a one-sided inverse $\rho_0 : S(M_0) \to \mathfrak{J}$, with $\rho_0 \iota_0 = Id_{\mathfrak{J}}$. Let $\rho : S(M) \to \mathfrak{J}$ be the composition $\rho_0 r_0$. Let $f(\alpha) = \rho \circ \alpha \circ \iota$.

Claim . f is surjective.

To see this, identify $\mathfrak{J}$ with $J = \iota(\mathfrak{J})$, so that ι becomes the inclusion map of J in $S(M)$, and $\iota_0 = r_0|J$. Let $\beta \in Aut(J)$. We can now view β also as a homomorphism $J \to S(M)$. According to [27] A.11(3), there exists $\sigma \in Aut(M)$ with $r_0 \circ \sigma|J = r_0 \circ \beta$. So $f(\sigma) = \rho\sigma\iota = \rho_0 r_0 \sigma\iota = \rho_0 r_0 \beta\iota = \rho_0 \iota_0 \beta\iota = \beta$. Since $\beta \in Aut(J)$ was arbitrary, f is onto. $\square$

3.22. **The pullback of** $\triangle$**.** Define the binary relation $\triangle_k$ to be the k-fold composition of $\triangle$ with itself; it is defined by a pp formula (the existential quantifiers are local since $\triangle \ (x, y)$ implies $m_2(x, y)$,

Lemma 3.12.) It is easy to check that the same expression defines the k-fold iteration of $\triangle$ on $\mathcal{J}$ (or see [27] 3.12.)

For any $\mathcal{A}, \mathcal{B} \overset{e.c.}{\models} \mathcal{T}$, we define a relation $\triangle_k$ on $Hom(\mathcal{A}, \mathcal{B})$ by

$$h_1 \triangle_k h_2 \iff (\forall a \in \mathcal{A})(h_1(a), h_2(a)) \in \triangle_k^{\mathcal{B}}$$

For an element of $Aut(\mathcal{A})$, we write $\triangle_k(h)$ for $h \triangle_k 1$, where $1 = Id_{\mathcal{A}}$. For $h_1, h_2 \in Aut(\mathcal{A})$, we have $h_1 \triangle_k h_2$ iff $(h_1(a), h_2(a)) \in \triangle_k^{\mathcal{A}}$ for all a; by definability of $\triangle_k$, this is iff $(a, h_1^{-1}h_2(a)) \in \triangle_k^{\mathcal{A}}$ for all a iff $\triangle_k(h_1^{-1}h_2)$. It follows that $\triangle_k$ is a normal, symmetric subset of $Aut(\mathcal{A})$.

The above notation applies to $Aut(\mathcal{J})$ and to $Aut(S(M)) = Aut(M)$ (via its action on $S(M)$.)

Finally, we define $\triangle_k^{\mathcal{G}(T)}$ to be the image of $\triangle_k^{Aut(J)}$ under the quotient homomorphism $Aut(J) \to \mathcal{G}(T)$.

Note that in the groups $Aut(J)$ or $\mathcal{G}(T)$ we have $\triangle_k \triangle_{k'} \subset \triangle_{k+k'}$, but (presumably) equality need not hold in general.

Proposition 3.23. *In Theorem 3.16, one can add:*

(1) $\varphi(\triangle_n^{Aut(M)}) \subset \triangle_n^{\mathcal{G}}$
(2) *If* $\phi(g_1)\phi(g_2)^{-1} \in \triangle_n^{Aut(\mathcal{J})}$*, then* $g_1 g_2^{-1} \in \triangle_{n+4}$*.*
(3) $\phi^{-1} \triangle_n^{Aut(\mathcal{J})} \subset \triangle_{n+3}^{Aut(M)}$
(4) $\varphi^{-1} \triangle_n^{\mathcal{G}(T)} \subset \triangle_{n+5}^{Aut(M)}$*.*
(5) *If* $\varphi(g_1)\varphi(g_2)^{-1} \in \triangle_n^{\mathcal{G}}$*, then* $g_1 g_2^{-1} \in \triangle_{n+6}$*.*

Proof. Identify $\mathcal{J}$ with $J \leq S(M)$, and let $\rho : S(M) \to J$ be a retraction.

(1) If $g \in \triangle^{S(M)}$, so that $g(p) \triangle p$ for all $p \in S(M)$, then for $p \in J$ we have $\phi(g)(p) = \rho g(p) L_n \rho(p) = p$. Thus $\phi(\triangle) \subset \triangle$. Similarly, $\phi(\triangle_n) \subset \triangle_n$.

(2) Let $g_1, g_2 \in Aut(M)$ and assume $\phi(g_1) \triangle_n \phi(g_2)$ in $Aut(\mathcal{J})$. Let $p \in S(M)$. By Lemma 3.14(3), $p \triangle \rho(p)$. Thus (using the fact that g_1, g_2, ρ are homomorphisms),

$$g_1(p) \triangle \rho g_1(p) \triangle \rho g_1(\rho(p)) \triangle_n \rho g_2(\rho(p)) \triangle \rho g_2(p) \triangle g_2(p)$$

So $g_1(p) \triangle_{n+4} g_2(p)$, and since p was arbitrary, $g_1 g_2^{-1} \in \triangle_{n+4}$.

(3) The penultimate step in the displayed formula in (2) can be skipped when $g_2 = Id$, since $\rho\rho p = \rho p$.

(4) Now suppose $\varphi(g) \in \triangle_n^{\mathcal{G}}$; i.e. $\phi(g) = \pi(h)$ for some $h \in \triangle_n^{Aut(J)}$. If $g \in \ker \pi$ then $g(p), p$ are inseparable by open sets. By Lemma **??** (4), we have $(p, g(p)) \in \triangle_2$ so $\ker \pi \subset \triangle_2$. So $\phi(g)h^{-1} \in \ker \pi \subset \triangle_2$. Hence $\phi(g) \in \triangle_{n+2}$, and by (3), $g \in \triangle_{(n+2)+3}$.

(5) Similarly, if $\varphi(g_1)\varphi(g_2)^{-1} \in \triangle_n^{\mathcal{G}}$, then by (2) we have $g_1 g_2^{-1} \in \triangle_{(n+2)+4}$.

$\square$

The example below is adapted from the basic component of the example of [14] of a non-compact Lascar group; local logic permits a simpler presentation. It is also in essence the same as Example 3 of a quasimorphism in [41], where one can find an illuminating discussion of its provenance in geometry.

Example 3.24. Let $T = Th_{loc;\mu_1}(\mathbb{R}, <, S)$. Here $S(x) = x + 1$,

$$\mu_1(x,y) \equiv x \le y \le S(x) \vee y \le x \le S(y),$$

and the subscript loc designates that we take the theory in local logic, i.e. using quantifiers with bounded range only.

The space Σ of (local) 1-types over $M = (\mathbb{R}, <, S)$ can be identified with $\{-, 0, +\} \times \mathbb{R}$; here $(0, \alpha)$ denotes the algebraic type $x = \alpha$; $(-, \alpha)$ denotes the type just below α, and dually $(+, \alpha)$. The two non-local 0-definable types, at ∞ and $-\infty$, are of course not part of the space Σ. Let $J = \{-, 0, +\} \times \mathbb{Z}$, a $\mathcal{L}$-substructure of Σ. Pick a cut $\gamma \in (0, 1]$ and define a $\mathcal{L}$-retraction $r : \Sigma \to J$ as follows: for any $n \in \mathbb{Z}$, $\delta \in [0, 1)$ and $* \in \{-, 0, +\}$, let $r(*, n+\delta) = (*, n)$ if $\delta \le \gamma$ and $r(*, n+\delta) = (*, n+1)$ if if $\gamma < \delta$. It is easy to see that no further collapse is possible, and J is the universal e.c. model of $\mathcal{T}$. We have $Aut(J) = \mathbb{Z}$.

$Aut(M)$ is the much bigger group of order preserving maps $\mathbb{R} \to \mathbb{R}$ commuting with S. The quasimorphism on $Aut(M)$, associated with the above retraction, is the translation number[5].

Restricted to the group of translations $\mathbb{R}$ as a subgroup of $Aut(M)$, we obtain a choice of 'nearest integer' map, which is of course a quasi-homomorphism $\mathbb{R} \to (\mathbb{Z}; \{-1, 0, 1\})$.

In connection with the discussion in §3.19, we note also that in this example, we have $Hom(\mathcal{J}, \Sigma) = \mathbb{R}$ and $Hom_J(\Sigma, J)$ is the space of cuts in $(0, 1)$.

Question 3.25. Is it possible to develop an analogous theory uniform in the doubling constant? Assume every 2-ball is uniformly a union of a Y-parameterized family of 1-balls, where the definable set Y is in some sense 'small' compared to D; at a minimum, there should exist types over M that properly increase M without increasing Y, so that finite distance still implies distance 1 from some point. According

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[5]I take this term from [?]. Emmanuel Breuillard pointed out to me that these ideas go back to Poincaré's *rotation number*. A homeomorphism ρ of the oriented circle $\mathbb{R}/\mathbb{Z} = M/S^{\mathbb{Z}}$ lifts uniquely to an element r of $Aut(M)$ fixing a given point m_0; the rotation number of ρ is the translation number of r.

to a theorem of Shelah [60], such types exist in pseudo-finite theories provided $|D|$ is not polynomially bounded by $|Y|$. Thus in theory, this could give a possible route to various questions of polynomial bounds. Developing a theory on this basis will not be easy, but even much stronger assumptions of relative smallness would be of interest. The theory of locally compact groups would need to be generalized, however, in any case, with the compact/ noncompact dichotomy replaced by a model-theoretic analogue.

4. DEFINABLE GROUPS

Recall Tao's definition of approximate subgroups.

Definition 4.1 ([64])**.** Let G be a group. A symmetric subset $\Lambda \subset G$ is an *approximate subgroup* if $\Lambda \sim \Lambda^2$.

Theorem 4.2. *Let G be a group, generated by an approximate subgroup Λ. Then there exists a second countable locally compact topological group H, a compact normal subset $\triangle \subset \mathsf{H}$ and a quasi-homomorphism*

$$f : G \to \mathsf{H} : \triangle$$

such that:

(1) For $C \subset \mathsf{H}$ compact, $f^{-1}(C)$ is contained in some Λ^i.
(2) For each i there exists a compact $C \subset \mathsf{H}$ with $\Lambda^i \subset f^{-1}(C)$.
(3) Specifically, $f^{-1}(\triangle) \subset \Lambda^{12}$.
(4) Let X, X' be compact subsets of H, with $\triangle^2 X \cap \triangle^2 X' = \emptyset$. Then there exist disjoint definable subsets D, D' of Λ^k for some k (with parameters in Λ) such that $f^{-1}(X) \subset D$, and $f^{-1}(X') \subset D'$.

The purely group-theoretic properties (1,2) will be the main engine of the group-theoretic applications, including Theorem 5.15 and Theorem 6.9; the second-countability will not be needed.

Note that H is compact iff Λ generates G in a finite number s of steps. Even in this case, when s is large, we obtain a nontrivial statement due to (3); in particular, $f : G \to \mathsf{H} : f(\Lambda^{12})$. See Corollary 4.13.

(4) should be compared to the familiar form of continuity or continuous-logic definability of a map into a compact space, asserting that the preimages of two disjoint compact sets can be separated by a definable set. Here this is asserted for two compacts at some discrete distance from each other.

We will begin with a proof of (1,2); the proofs of (3,4) are postponed till we discuss definability.

Proof. We define a local structure M. The universe $D(M)$ of M, as a set, is a copy of G. The relations on D include the binary relation

$yx^{-1} \in \Lambda$, denoted m_1: and the quaternary relation $yx^{-1} = zw^{-1} \in \Lambda$. The locality generator is m_1. There are no further relations in the structure M; thus the action of G on D from the right, $(g, d) \mapsto dg^{-1}$, is an automorphism of M. This gives a homomorphism $\delta : G \to Aut(M)$. Since G is transitive on D, it is clear that M is orbit-bounded. The balls can be identified as $m_1(a) = \Lambda a$, and $m_2(a) = \Lambda^2 a$. As Λ is an approximate subgroup, $\Lambda^2 \subset \cup_{i=1}^{k} \Lambda c_i$, so $m_2(a) = \cup_{i=1}^{k} m_1(c_i a)$; this gives the doubling property (DP). Thus Theorem 3.10 applies: $Th(M)$ admits a core $\mathcal{J}$, and $Aut(\mathcal{J})$ has a natural locally compact topology; the canonical Hausdorff quotient $\mathcal{G}(T) := Aut(\mathcal{J})/\mathfrak{g}_D$ is a locally compact topological group; we also defined a canonical compact, normal, symmetric subset $\triangle \subset \mathcal{G}(T)$. There exists an embedding $\iota : \mathcal{J} \to S(M)$, and a retraction $\rho : S(M) \to \mathcal{J}$. According to Theorem 3.16, there exists a quasi-homomorphism $\varphi : Aut(M) \to (\mathcal{G}(T) : \triangle)$ such that compact subsets of $\mathcal{G}$ pull back to bounded subsets of $Aut(M)$, and bounded subsets of $Aut(M)$ push forward to precompact subsets of $\mathcal{G}$.

Let $f = \varphi \circ \delta$. If $C \subset \mathcal{G}$ is compact, then $\delta(f^{-1}(C))$ is bounded in $Aut(M)$, being a subset of $\varphi^{-1}(C)$. Picking $d_0 \in D$ to be the identity element of G, we have $\delta(f^{-1}(C))(d_0) = d_0 f^{-1}(C) = f^{-1}(C)$; so $f^{-1}(C)$ is bounded in $D(M)$. This means that $f^{-1}(C) \subset \Lambda^i$ for some i. Conversely, $\delta(\Lambda^i)$ is by definition bounded, so $f(\Lambda^i) = \varphi(\delta(\Lambda^i))$ is precompact, hence $C = cl(f(\Lambda^i))$ is compact and $\Lambda^i \subset f^{-1}(C)$.

To obtain second countability, we modify the end of the construction slightly. Recall φ was the composition $\pi \circ \phi$, with $\pi : Aut(\mathcal{J}) \to \mathcal{G}$ the quotient map, and $\phi : Aut(M) \to (\mathcal{G}(T) : \triangle)$ as constructed in 3.16. Let P be the complete atomic type of $\rho(d_0)$. Let $\mathfrak{g} = \mathfrak{g}_P$ be the infinitesimal subgroup with respect to the action on P. By Proposition 2.8, $\mathfrak{g}_P$ is a closed, compact subgroup of $Aut(\mathcal{J})$, and $\mathcal{G}(P)$ is a second countable, locally compact topological group. Let $\pi_P : Aut(\mathcal{J}) \to \mathfrak{g}_P$ be the projection. We let $\varphi_P = \pi_P \circ \phi$, and again $f = \varphi_P \circ \delta$. We still have the above properties, and now according to Proposition 2.8, $\mathcal{G}(P)$ is also second countable.

$\square$

We will also mention a slightly more general version. The proof is identical once we have a generalization of the local logic of §2, where we do not assume that the locality relation is generated by a single relation; we will not repeat details but will describe the changes needed in the construction.

Proposition 4.3. *Let G be a group, ω a family of commensurable approximate subgroups of G. Assume: if $\Lambda, \Lambda' \in \omega$, then $\Lambda \cup \Lambda' \in \omega$, and*

$\Lambda^2 \in \omega$; and $\cup\omega = G$. Then there exists a locally compact topological group H, a compact normal subset $\triangle \subset \mathsf{H}$ and a quasi-homomorphism

$$f : G \to \mathsf{H} : \triangle$$

such that:

(1) For $C \subset \mathsf{H}$ compact, and $\Lambda \in \omega$, $f^{-1}(C)$ is covered by a finite union of right translates of Λ.
(2) For $\Lambda \in \omega$ there exists a compact $C \subset \mathsf{H}$ with $\Lambda \subset f^{-1}(C)$.
(3) If $\Lambda_0 \in \omega$ and $X \subset \Lambda \in \omega$, we have $f^{-1}f(X) \subset X\Lambda_0^{12}$. If $f(g_1) \in f(g_2) \triangle_k$ then $\delta(g_1 g_2^{-1}) \in \triangle_{12+2k}$.
(4) Let X, Y be compact subsets of H, with $\triangle^2 X \cap \triangle^2 Y = \emptyset$. Then there exists a definable subset D of some $\Lambda \in \omega$ (with parameters in Λ) such that $f^{-1}(X) \subset D$, and $f^{-1}(Y) \subset \Lambda \setminus D$.
(5) If ω is countable, we may take H to be second countable.

Proof. Define a structure M as in Theorem 4.2, with universe $D = G$; for *each* $\Lambda \in \omega$, we posit a binary relation $m_\Lambda : yx^{-1} \in \Lambda$; and a quaternary relation $yx^{-1} = zw^{-1} \in \Lambda$. A subset of D is viewed as local if it is contained in a ball $m_\Lambda(a) = \{b : m_\Lambda(a, b)\}$; or equivalently, in the union of finitely many such balls. As before, only existential quantifiers of the form $(\exists x)m_\Lambda(x, y) \wedge \cdots$ are allowed in the formation of pp formulas, but now all $\Lambda \in \omega$ are allowed. If we pick one $\Lambda_0 \in \omega$, it remains true that any ball $m_\Lambda(a)$ is covered by finitely many balls $m_{\Lambda_0}(b_i)$. A weak model is local if for any $a, b \in D(M)$, $m_\Lambda(a, b)$ holds for some $\Lambda \in \omega$. A type $p(x)$ is local if $m_\Lambda(x, y)$ is represented in p for some - or equivalently, by DP, for all - $\Lambda \in \omega$. The construction of $\mathcal{J}, Aut(\mathcal{J}), \mathfrak{g}, f$ and proof of their properties is identical.
□

Remark 4.4. Let ω' be any family of commensurable approximate subgroups of G, such that if $\Lambda \in \omega$ and $g \in G$ then $g^{-1}\Lambda g$ is commensurable with Λ. Then ω' may be completed to a class ω satisfying the hypotheses of Proposition 4.3, whose union is G.

Proof. If $\Lambda \in \omega$, then any two-sided translate $a\Lambda b$ is contained in a finite union of right translates of Λ: we have $a\Lambda a^{-1} \subset \cup_i \Lambda c_i$ for some $c_1, \ldots, c_l$; and $a\Lambda b = (a\Lambda a^{-1})ab \subset \cup_i \Lambda c_i ab$. In particular, each $c_i ab\Lambda$ is contained in a finite union $\cup\Lambda d_{ij}$, so $a\Lambda b\Lambda c \subset \cup_{i,j} \Lambda^2 d_{ij}$, and it follows that $a\Lambda b\Lambda c$ is contained in a finite union of right translates of Λ. From here it is clear that closing ω under the operations $\Lambda, \Lambda' \mapsto \Lambda \cup \Lambda'$, $\Lambda \mapsto \Lambda^2$, and $\Lambda \mapsto (a\Lambda \cup \{1\} \cup \Lambda a)$ will not damage the hypotheses.   □

4.5. **Definability.** We will now place our approximate subgroups in a definable setting more explicitly. A complete first-order theory T is

given with predicates Λ_i ($i \in I$, where I is a partially ordered index set.) If $i \leq j$, a definable injective map $\Lambda_i \to \Lambda_j$ is given, forming a direct limit system; the direct limit will be denoted G. We assume that for each $i, j \in I$ for some $k \in I$, a definable map $\Lambda_i \times \Lambda_j \to \Lambda_k$ is given; inducing a group multiplication on G. In this situation G is called a piecewise definable group of the theory T; or a strict Ind-object in Grothendieck's category of Ind-definable sets. We identify Λ_i with a subset of G, and assume further that any two Λ_i are commensurable in G; in particular each Λ_i is an approximate subgroup.

We say that a subset of G is *locally $\bigwedge$-definable* if the intersection with each Λ_i is $\bigwedge$-definable.

Given a theory T with a definable commensurable family of approximate subgroups, as above, we introduce a new local sort D, a torsor for G. For $M \models T$, $D(M)$ is a copy of $G(M)$. The new relations introduced are binary relations $m_i(x, y) : \equiv (yx^{-1} \in \Lambda_i)$ on D, and the relation $yx^{-1} = u \in \Lambda_i$ on $D^2 \times \Lambda_i$. We define a locality structure on D, using the m_i. Restricting attention to the induced (local) structure on D, we form $\mathcal{J}$ (in the corresponding sort $\mathcal{D}$.) Denote by L^+ the langauge obtained from L by adding the local sort D and these relations; M^+ the result of adding the torsor D to the structure as above; and $T^+ := Th(M^+)$.

Next recall the universal theory T_{def}^+ of [27] (see 3.17). A model N_{def}^+ is nothing more than a model N^+ of T^+, along with a homomorphism $\iota : \mathcal{J} \to S_D(N^+)$. The language L_{def}^+ is obtained from L^+ by adding a new relation $(d_p x)\phi$ for each $p \in \mathcal{J}$ and each $\phi(x; y_1, \ldots, y_k) \in L^+$, to be interpreted as a subset of D^k, namely as $\{(b_1, \ldots, b_k) \in D^k : \phi(x, b_1, \ldots, b_k) \in \iota(p)\}$. Fix some $p_0 \in \mathcal{J}$; the choice will not matter, up to bi-interpretation. We have in L_{def}^+ a unary relation $m_k^{p_0}$ in L^+ such that $m_k^{p_0}(b)$ holds iff $m_k(x, b) \in p_0$. We will only consider models of T_{def}^+ that omit the partial type: $\neg m_k^{p_0}(x, b), k = 1, 2, \cdots$. This amounts to treating the sort D of T_{def}^+ (a single local sort in T^+) as the union of infinitely many 'sorts' defined by $m_k^{p_0}$, with inclusion maps among them. Any model of T_{def}^+ (in this sense), restricted to an L-structure, is a (local) model of T^+.

Since some model of $(T_{def}^+)^{\pm}$ is a model of T, if $T_{def}^+ \models \neg(\exists x, y, z)(\alpha(x, y) \wedge \gamma(y, z))$ with $\alpha \in L^+$, and $T^+ \models (\forall y)(\exists x)(\alpha(x, y))$, then $T_{def}^+ \models \neg(\exists y, z)(\gamma(y, z))$.) Thus $T_{def}^+ \overset{e.c.}{\models} T^+$.

In Theorem 4.2 and Proposition 4.3, we used the action of G on D by right translation. This action depends on a choice of an element d_0 of D, used to identify G with D via $g \mapsto gd_0$. The homomorphism $\delta : G \to Aut(M^+)$ is thus also dependent on this choice, and may be

better denoted δ_{d_0}. However when X is a normal subset of $Aut(M^+)$, $\delta^{-1}(X)$ is well-defined and does not depend on the choice of d_0. In particular, $\delta^{-1}(\triangle_k)$ is well-defined. (Where $\triangle_k \subset Aut(M^+)$ is defined above Lemma 3.23.) Let us compare $\delta^{-1}(\triangle_k)$ to $\triangle_k$, as computed directly in T (i.e. in G and not in D.)

Consider

$$P := \{ab^{-1} : a, b \in G(M), a \triangle b\}$$

Here $\triangle$ is computed in $\aleph_0$-saturated models of T.

While there is no multiplication on D, we can define $ab^{-1} \in G$ for $a, b \in D$, so that $(ab^{-1})b = a$. Then $P = \{ab^{-1} : a, b \in D(M), a \triangle b\}$. (If $a, b \in D$ and $tp(a/N) = tp(b/N)$, then picking $d \in D(N)$ we have $ab^{-1} = (ad^{-1})(bd^{-1})^{-1}$ and $ad^{-1} \triangle^M bd^{-1}$.) From this it follows easily that $\delta^{-1}(\triangle_k^{Aut(M^+)}) \subset P^k$.

On the other hand, by locality and the doubling property, if $a, b \in D$ and $a \triangle^D b$ then $ab^{-1} \in \Lambda^2$. (Since $tp(a/N) = tp(b/N)$ implies that a, b lie in a common 1-ball.)

By Proposition 3.23 (2), we obtain:

Proposition 4.6. *In 4.2 and 4.3, one can add: If $f(g_1)f(g_2)^{-1} \in \triangle_k^{\mathcal{G}}$ then $\delta(g_1 g_2^{-1}) \in \triangle_{k+6}$ so $g_1 g_2^{-1} \in P^{k+6} \subset \Lambda^{2(k+6)}$. Likewise, if $f(g) \in \triangle_k^{\mathcal{G}}$ then $\delta(g) \in \Lambda^{2(k+5)}$.*

In particular, this proves Theorem 4.2 (3).

Remark 4.7. All conjugates of P are contained in P^3. Hence $P \subset Q \subset P^3$ for some normal $\bigwedge$-definable set Q.

Proof. Let $c \in P$, and $g \in G$. We have $c = c_1^{-1}c_2$ with $tp(c_1/N) = tp(c_2/N)$. Take N to be $\aleph_1$-saturated, and N_0 a countable elementary submodel; then $tp(g/N_0) = tp(m/N_0)$ for some $m \in N$; so $m^{-1}g \in P$. Now $tp(m^{-1}c_1m/M) = tp(m^{-1}c_2m/M)$, so $m^{-1}cm \in P$. Thus $g^{-1}cg = l^{-1}(m^{-1}cm)l \in P^3$. Now let Q be the intersection of all conjugates of P^3. $\square$

Proposition 4.8. *Let G be a piecewise definable group in a theory T, limit of definable approximate subgroups. Let $M_{def}^+ \models T_{def}^+$, let M^+ be the reduct to L^+, and $\iota : \mathcal{J} \to S(M^+)$ the embedding corresponding to M_{def}^+. Further let $\rho : S(M^+) \to \mathcal{J}$ be a retraction, and let d_0 be an element of $D(M^+)$, Let $\delta : G \to Aut(S(M)^+)$ the map induced by right multiplication if (D, d_0) is identified with $(G, 1)$: $\delta(g)(hd_0) = hg^{-1}d_0$. Finally let $\phi : Aut(S(M)^+) \to Aut(\mathcal{J})$ be defined by $\gamma \mapsto \rho \circ \gamma \circ \iota$, φ be the composition of ϕ with $Aut(\mathcal{J}) \to \mathcal{G}$, and $f = \varphi \circ \delta : G(M) \to \mathcal{G}$. Then:*

(1) *Let W be a compact subset of $\mathcal{G}$. Then there exists a $\bigwedge$-definable $W^* \subset G$ in $L_{def}^+(d_0)$ with $f^{-1}(W) \subset W^* \subset f^{-1}(W \,\triangle)$. W^* is defined by universal formulas of $L_{def}^+(d_0)$.*

(2) *Let W_1, W_2 be compact subsets of $\mathcal{G}$ with $\triangle W_1 \cap \triangle W_2 = \emptyset$. Then there exist definable sets $V_1, V_2 \subset G$ in $L_{def}^+(d_0)$ with $f^{-1}(W_i) \subset V_i$ and $V_1 \cap V_2 = \emptyset$.*

Proof. During the present proof, the word definable *and its attendants will refer to* $L_{def}^+(d_0)$.

We show first that we may assume some saturation of M_{def}^+; this will be needed for (2). Let $((M_{def}^+)^*, d)$ be an elementary extension of (M_{def}^+, d). Define $\rho^* = \rho \circ r$, where $r : S(M^+) \to S(M)$ is the restriction map. We obtain $\iota^*, \delta^*, \phi^*, \varphi^*$; and it is easy to verify that the various maps commute, and we have: $\phi \circ \delta = \phi^* \circ \delta^* | M$, and $f = f^* | M$. Thus by passing to an elementary extension we may assume M_{def}^+ is $\aleph_0$- saturated, and M^+ is $\aleph_0$- homogeneous.

We begin with the proof of (1).

Identify $\mathcal{J}$ with $\iota(\mathcal{J})$. Let f denote the composition

$$G(M) \to_\delta Aut(M^+) \to Aut(S_D(M^+)) \to_\phi Aut(\mathcal{J})$$

so that $f = \pi \circ \mathsf{f}$, with $\pi : Aut(\mathcal{J}) \to \mathcal{G}$. We will write $\hat{g}$ for $\delta(g)$ or for the image of g in $Aut(S_D(M^+))$ or $Aut(\mathcal{J})$. The homomorphism $Aut(\mathcal{J}) \to \mathcal{G}$ is continuous and proper; so it suffices to show that for each closed bounded subset V of $Aut(\mathcal{J})$, there exists a $\bigwedge$-definable set V^* with

$$\mathsf{f}^{-1}(V) \subset V^* \subset \mathsf{f}^{-1}(V \,\triangle)$$

Assuming the displayed statement holds for some family of sets V_i, closed under finite intersections, it will also be true for $\cap_i V_i$, with $(\cap_i V_i)^* = \cap_i V_i^*$; this, since by compactness $\cap_i V_i \,\triangle = (\cap_i V_i) \,\triangle$. Hence, recalling that basic closed sets are closed under finite intersections, it sufices to take V to be a basic closed set in $Aut(\mathcal{J})$, i.e.

$$V = \{\gamma \in Aut(\mathcal{J}) : R(a, \gamma(a))\}$$

Here (using Theorem 3.10 (3)) R can be taken to be a basic relation of $\mathcal{L}$ ensuring that γ lies in a bounded subset of G, when a is fixed; a is a tuple of elements of $\mathcal{J}$, and $\gamma((a_j : j \in J)) = (\gamma(a_j) : j \in J)$. While R will make use of only finitely many of the a_j and $\gamma(a_j)$, it will be convenient to allow a to be an infinite tuple, indeed a tuple enumerating all of $\mathcal{J}$. Let p be the atomic type of a in $\mathcal{J}$. For two tuples $(a_j : j \in J), (b_j : j \in J)$, write $\triangle^1 (a, b)$ if $\triangle (a_j, b_j)$ holds for each j. Define a subset V^* of G thus:

$$V^* = \{g \in G : S(M^+) \models (\exists \xi)(p(\xi) \wedge (\xi \,\triangle\, \hat{g}(a)) \wedge R(a, \xi))\}$$

Let $g \in G(M)$, $\gamma = \mathsf{f}(g)$ the image of g in $Aut(\mathfrak{J})$. We have $\gamma = (\rho \circ \hat{g})|\mathfrak{J}$.

If $\gamma \in V$, then letting $\xi = \gamma(a)$, we see (using the definition of V and Lemma 3.14(3)) that $g \in V^*$. Thus $\mathsf{f}^{-1}(V) \subset V^*$.

In the opposite directionm assume $g \in V^*$. Then

$$\mathfrak{J} \models (\exists \xi)(p(\xi) \wedge (\xi \bigtriangleup \rho\hat{g}(a)) \wedge R(a, \xi)).$$

Let b be a witness for ξ. Since $p(b)$ holds, $a \mapsto b$ defines an $\mathcal{L}$-homomorphism $\mathfrak{J} \to \mathfrak{J}$; by Proposition 2.1, it is an automorphism of $\mathfrak{J}$, call it γ'. Since $\gamma'(a) = b$ and $b \bigtriangleup \gamma(a) = \gamma(\gamma')^{-1}(b)$, we have $\gamma(\gamma')^{-1}(b) \bigtriangleup b$ and so $b \bigtriangleup \gamma'\gamma^{-1}(b)$. As b enumerates $\mathfrak{J}$, $\gamma'\gamma^{-1} \in \bigtriangleup^{Aut(\mathfrak{J})}$. But $R(a, b)$ shows that $\gamma' \in V$. So $\gamma \in V \bigtriangleup$.

Note that V^* is bounded. It remains only to see that V^* is (locally) $\bigwedge$-definable. Using local compactness of $S(M)$ we see that V^* is the intersection of all approximations to it, obtained by replacing $\bigtriangleup$ by a definable approximation inside the matrix of definition of V^*; these approximations can be chosen so as to still ensure that the set of g they define is bounded.

Hence V^* is a conjunction of pp relations concerning $(a, \hat{g}(a))$. By Theorem 3.10 (3), V^* can also be written as an (infinite) conjunction of atomic formulas Def_t; so it suffices to prove that each such formula is locally definable. To simplify notation, consider the atomic formula $\mathsf{Def}_t(p, \hat{g}(q))$, $t = (\phi_1(x_1, u), \phi_2(x_2, u))$, with ϕ_1, ϕ_2 formulas of T^+. Here $p, q \in \mathfrak{J}$ are fixed and g is the variable. We have $\mathsf{Def}_t(p, \hat{g}(q))$ iff there is no c with $\neg\phi_1(x, c) \in p$ and $\neg\phi_2(x, c) \in \hat{g}(q)$. Now $\phi_2(x, c) \in \hat{g}(q)$ iff $\phi_2(x, g^{\hat{}-1}(c)) \in q$. Thus

$$\mathsf{Def}_t(p, \delta(g)(q)) \iff \neg(\exists c)(d_p x)\neg\phi_1(x, c) \wedge (d_q x)\neg\phi_2(x, g^{\hat{}-1}(c))$$

The right hand side is definable in $L_{def}^+(d)$.

(2) Let W_1^*, W_2^* be as in (1). Then W_i^* is $\bigwedge$-definable (in particular contained in a definable approximation to G), and $W_1^* \cap W_2^* \subset f^{-1}(W_1 \bigtriangleup) \cap f^{-1}(W_2 \bigtriangleup) = \emptyset$. If W_1^*, W_2^* were not separated by definable sets $V_i \subset G$ (in $L_{def}^+(d_0)$), by compactness and since every type over $\emptyset$ is realized , we would have $W_1^* \cap W_2^* \neq \emptyset$, a contradiction. $\square$

Remark 4.9. The Beth-Craig-Robinson definability theory is valid for $\bigwedge$-definable relations. Here is the version we need: Let M be an $\aleph_0$-homogeneous L-structure. Let M' be an expansion to an L'-structure, in which every type is realized. Let X, Y be $\bigwedge$-definable subsets in M'. Assume $X \cap \sigma Y = \emptyset$ for every automorphism σ of M. Then X, Y are separated by a 0-definable set of M. [Proof: Let $SX = \{tp(a; M') :$

$a \in X\}$, and similarly SY. Then SX, SY are compact sets in the space S' of L'-types (over $\emptyset$.) Let $\pi : S' \to S$ be the projection to the space of L-types. Then $\pi SX, \pi SY$ must also be compact sets, and they are disjoint since if $tp_L(a) = tp_L(b)$ then there exists an L-automorphism σ with $\sigma(a) = b$. Hence there exists a 0-definable set D of L separating $\pi SX, \pi SY$ and thus also X, Y.]

It follows in particular that if X is $Aut(M)$-invariant, then X is $\bigwedge$-definable in L.

Definability in the following corollary (and beyond) refers again to L, not to L_{def}.

Corollary 4.10. *In Proposition 4.8, and hence also in 4.2 and 4.3, one can add that $f^{-1}(cl(\langle \triangle \rangle))$ is locally $\bigwedge$-definable in L. In fact it is locally an intersection of 0-definable sets commensurable with Λ, and the smallest subgroup with this property.*

Proof. As in Proposition 4.8, we may assume M is $\beth_2^+$-saturated in L_{def}, and $\aleph_0$-homogeneous in L. We will show that $f^{-1}(cl(\langle \triangle \rangle))$ is $Aut(M)$-invariant.

We can identify $Aut(M)$ with a subgroup of $Aut(M^+)$, namely the subgroup fixing d_0. Under this identification, we have, for $\sigma \in Aut(M)$ and $g \in G(M)$,

$$\delta(\sigma(g)) = \sigma^{-1}\delta(g)\sigma$$

We have $f = \varphi \circ \delta : G(M) \to \mathcal{G} : \triangle$; where $\varphi : Aut(M) \to \mathcal{G} : \triangle$ and $\delta : G \to Aut(M)$.

Let $\sigma \in Aut(M)$, $g \in G(M)$, and let Y be a normal subset of $\mathcal{G}$. Note first that $\sigma^{-1}(\varphi^{-1}(Y))\sigma \subset \varphi^{-1}(Y \triangle^3)$. Composing with δ, it follows that $\sigma(f^{-1}(Y)) \subset f^{-1}(Y \triangle^3)$. In case $Y = Y \triangle$, this shows that $f^{-1}(Y)$ is $Aut(M)$-invariant.

If in addition Y is closed, by Proposition 4.8, $f^{-1}(Y) = Y^*$ is also locally $\bigwedge$-definable in L_{def}; i.e. $\Lambda \cap f^{-1}(Y)$ is $\bigwedge$ - definable in L_{def}, for any $\Lambda \in \omega$. By Remark 4.9, (using the $Aut(M)$-invariance), $f^{-1}(Y)$ is then locally $\bigwedge$-definable in L.

In particular, this applies to $f^{-1}(cl(\langle \triangle \rangle))$. Since $\Lambda/f^{-1}(cl(\langle \triangle \rangle))$ has cardinality $\leq 2^{\aleph_0}$, it is clear that any definable subset of Λ containing $f^{-1}(cl(\langle \triangle \rangle))$ is commensurable with Λ.

In the other direction, let H be a locally $\bigwedge$–definable subgroup of G of bounded index. Then H induces a locally $\bigwedge$–definable equivalence relation E on the G-torsor D, namely $yx^{-1} \in H$; and thus an equivalence relation (also denoted E) on the type space $S_D(M)$. It is easy to see that E is an intersection of $\mathcal{L}$-definable relations, so that E is closed in the pp topology too; and we have $\rho(p)E\rho(q)$ iff pEq. The elements of

$Aut(J)$ preserving each E-class thus form a closed subgroup of $Aut(J)$. Identifying $\mathcal{J}$ with the image J of ρ, we see that the image of H in $Aut(\mathcal{J})$ contains a closed subgroup that in turn contains $\triangle^{Aut(\mathcal{J})}$; so the image contains $cl(\langle \triangle \rangle)$. This clearly remains true modulo $\mathfrak{g}$ too. $\square$

See [19] for a systematic treatment of piecewise hyper-definable groups. G_0^{00} is defined there to be the smallest locally - $\bigwedge$-definable subgroup of G of bounded index. In this notation, we have $f^{-1}(cl(\langle \triangle \rangle)) = G_0^{00}$.

4.11. **Proof of Theorem 4.2 (4).** Unlike Proposition 4.10, the conclusion here allows for parameters. Let M_0 be an elementary submodel of the given structure, with $|M_0| \leq |\Omega|$. Let M be a highly saturated model of T_{def}^+, with $\aleph_0$-homogeneous reduct to $L(M_0)$. We have the map $\varphi : Aut(M) \to \mathcal{G}$. By Proposition 3.23 (1), $\varphi(\triangle^{Aut(M)}) \subset \triangle^{\mathcal{G}}$. Now $Aut(M/M_0) \subset \triangle^{Aut(M)}$, by definition of $\triangle$. Thus if $\sigma \in Aut(M/M_0)$ and $a \in \varphi^{-1}(X)$ then $\sigma^{-1}a\sigma \in \varphi^{-1}(X \triangle^2)$. Let X^*, Y^* be as in Proposition 4.8. Then $X^* \subset \varphi^{-1}(X \triangle)$ so $\sigma^{-1}(X^*)\sigma \subset \varphi^{-1}(X \triangle^3)$. Since $\triangle Y \cap \triangle^3 X = \emptyset$, we have $\sigma^{-1}(X^*)\sigma \cap Y^* = \emptyset$. Thus by Beth, as formulated above, X^*, Y^* are separated by $L(M_0)$-definable sets.

It follows that the pullback of closed, $\langle \triangle \rangle$-invariant sets is $\bigwedge$-definable with parameters. Thus:

Corollary 4.12. *The induced homomorphism $G/f^{-1}(cl(\langle \triangle \rangle)) \to \mathsf{H}/cl(\langle \triangle \rangle)$ is continuous, i.e. the inverse image of any compact subset is $\bigwedge$-definable with parameters.*

Proposition 4.3 applies, in particular, to the top commensurability class of a group G, namely the generic definable subsets. Here the results of [27] apply directly, there is no need for locality, and H is compact; nevertheless, the distance between $\triangle$ and H reflects the level of approximateness. For instance we have, from Theorem 4.2 (3):

Corollary 4.13. *Let G be a group, let X be a symmetric definable subset commensurable to G. Let T^+ be the associated theory of a G-torsor D, $\mathcal{J}$ the core space of T^+ on the torsor sort D, and let $C = Aut(\mathcal{J})/\mathfrak{g}$ be the canonical compact Hausdorff group associated to $(G, \cdot, X)$. If $C = 1$ then $X^{12} = G$.*

It suffices of course that the image of G in C be trivial. The corollary gives, in principle, a compact model for slowly converging approximate subgroups Λ, i.e. where $\Lambda^m = G$ but the smallest such m is large. Whether it can be improved to a Lie model depends, in part, on an effectivization of the theory of normal conjugacy classes in locally compact groups.

Question 4.14. How far can the 12 in Theorem 4.2 be improved?

Question 4.15. Describe the image of f in Theorem 4.2. Is it equal to $Aut((\mathcal{J}_D)_h/(\mathcal{J}_G)_h)$?

5. Approximate subgroups

Let G be a group. For $\Lambda \subset G$, define $\Lambda^{-1} = \{a^{-1} : a \in \Lambda\}$, $\Lambda^2 = \{ab : a, b \in \Lambda\}$. $\widetilde{\Lambda} = \cup \Lambda^n$ will denote the group generated by Λ. Λ is *symmetric* if $1 \in \Lambda = \Lambda^{-1}$.

If $Y \subset G$ is covered by finitely many right translates of X, $Y \subset \cup_{j=1}^n Xb_j$, we will say that X *commensurably covers* Y. Two subsets $X, Y \subset G$ are *commensurable*, $X \sim Y$, if each is covered by finitely many translates of the other: $X \subset \cup_{i=1}^m Ya_i$, $Y \subset \cup_{j=1}^n Xb_j$.

Let Λ be an approximate subgroup of a group G (Definition 4.1.) The *commensurator* Comm (Λ) of an approximate subgroup $\Lambda \subset G$ is the group of elements $g \in G$ with $g\Lambda g^{-1}$ commensurable to Λ. The *commensurability class* ω *of* Λ is the set of approximate subgroups commensurable to Λ. The commensurator of Λ depends only on the commensurability class, and can be denoted Comm (ω) or $\langle\omega\rangle_{max}$. it is the pullback of the subgroup of $Aut(G)$ fixing ω, under the natural map $G \to Inn(G) \le Aut(G)$.

Lemma 5.1. *Let Λ be an approximate subgroup of G, ω the commensurability class. Then $\langle\omega\rangle_{max} = \cup\omega$; Comm (Λ) is the union of all approximate subgroups commensurable to Λ.*

Proof. Note first that $\Lambda \subset$ Comm (Λ): if $g \in \Lambda$ then $g^{-1}\Lambda g \subset (\Lambda)^3$, so $g^{-1}\Lambda g$ is commensurably covered by Λ, and conjugating back by g we obtain the converse. If $\Lambda' \sim \Lambda$ then Comm $(\Lambda') =$ Comm (Λ), so $\Lambda' \subset$ Comm (Λ). Conversely, if $g \in$ Comm (Λ), then $g\Lambda \cup \Lambda \cup \Lambda g^{-1}$ is an approximate subgroup in ω; see the proof of Remark 4.4. $\square$

Let H be a locally compact topological group. Let U be a compact neighborhood of 1. Then U is an approximate subgroup.

The compact, symmetric neighborhoods of 1 form a unique commensurability class of approximate subgroups; we call it the *fundamental commensurability class* of H. If $f : G \to \mathsf{H}$ is a homomorphism, the commensurability class in G of the pullback of any compact open neighborhood of the identity in H is denoted $\chi(f)$, and again called the fundamental commensurability class of f.

If H is a subgroup of G, any commensurability class ω' of approximate subgroups of H extends to a commensurability class ω of approximate subgroups of G. We say in this case that ω *belongs to* H. (This

terminology does not preclude belonging to still smaller subgroups.; in fact if ω belongs to H, then it belongs to any finite index subgroup of H. In many cases of interest, there does exist a unique smallest commensurability class of subgroups to which ω belongs; see Proposition 5.8.)

Lemma 5.2. *Let $H \leq G$. If $X, Y \subset H$ are commensurable as subsets of G, they are commensurable in H.*

Proof. Say $X \subset \cup Y a_i$; then either $Y a_i$ can be deleted, or $a_i \in Y^{-1} X \subset H$. thus X is also contained in a union of H-translates of Y; and dually. $\qquad \square$

5.3. **Laminar commensurability classes.** A commensurability class $\omega = \chi(f)$ arising from a homomorphism f on a subgroup of G into a locally compact group H will be said to be *laminar*. It is shown in [26] that in this case, H can be taken to be a connected Lie group with no compact normal subgroups; this Lie group is then uniquely determined by the class ω. This uses Yamabe's theorem, that guarantees that any locally compact group has an open subgroup compactly isogenous to a Lie group. This subgroup however is neither canonical nor normal, and making use of it exacts a technical toll later on, before one regains a measure of normality.

In the opposite direction, it is sometimes desirable to extend the domain of f as much as possible. It follows from Lemma 5.1 that the domain of any f with $\chi(f) = \omega$ is contained in the commensurator group ω; and indeed it is possible to find f with this domain.

Lemma 5.4. *If ω is a laminar commensurability class of approximate subgroups, there exists a homomorphism F with domain $\mathrm{Comm}\,(\omega)$ into a locally compact group, with $\chi(F) = \omega$.*

Proof. This can be seen via Proposition 4.3. More directly, view $\mathrm{Comm}\,(\omega)$ as an ind-definable group, where the inductive system is the family of all approximate subgroups in ω; by Lemma 5.1, their union of $\mathrm{Comm}\,(\omega)$. Let $G*$ be a saturated elementary extension, $\lambda^* = \lambda(G^*)$, and $\omega^* = \cup_{\lambda \in \omega} \lambda^*$. Let I be the set of $\lambda_1 \in \omega$ such that there exist $\lambda_2 \cdots \in \omega$ with $\lambda_{k+1}^2 \subset \lambda_k$, $\lambda_k^{-1} = \lambda_k$. If $\lambda, \lambda' \in \omega$, then we saw that $\lambda^2 \cap (\lambda')^2 \in \omega$. Hence if $\lambda_1, \lambda_1' \in I$ and λ_k, λ_k' are as in the definition of I, since $\lambda_{k+1}^2 \cap (\lambda_{k+1}')^2 \in \omega$, we have $\lambda_k \cap \lambda_k' \in \omega$ for each k, and so $\lambda_1 \cap \lambda_1' \in I$. It is clear that I is conjugation-invariant. Let N be the $\bigwedge$-definable subgroup of ω^*, formed by the intersection of all $\lambda \in I$. Then N is a normal subgroup of ω^*, $N \subset \lambda_1$ (for any choice of $\lambda_1 \in I$), and ω^*/N is locally compact in the logic topology. Let F be the natural map $\omega^* \to \omega^*/N$, restricted to ω. $\qquad \square$

For example, the approximate subgroup $\{a/2^n : n \in \mathbb{N}, a \in \mathbb{Z}, -2^n \le a \le 2^n\}$ of $\mathbb{Q}$ has commensurability class $\chi(f)$, where $f : \mathbb{Z}[1/2] \to \mathbb{R}$; and also $\chi(g)$, with $g : \mathbb{Q} \to \mathbb{R} \times \Pi_{p \ne 2} \mathbb{Q}_p$ the natural inclusion.

We note immediately a uniqueness statement for the Lie group explanation of a given commensurability class of approximate subgroups, with a given subgroup as domain.

If N is a compact normal subgroup of H, $\pi : \mathsf{H} \to \mathsf{H}/N$ the quotient map, then the composed map $\pi \circ f : G \to \mathsf{H}/H$ determines the same commensurability class of approximate subgroups as f. This is however the only ambiguity, i.e. $f : G \to \mathsf{H}$ is unique up to compact isogeny:

Proposition 5.5. *Let $h_i : G \to H_i$ be homomorphisms with dense image into locally compact groups, and assume $\chi(h_1) = \chi(h_2)$. Then there exist compact normal subgroups N_i of H_i and an isomorphism $\phi : H_1/N_1 \to H_2/N_2$, with $p_2 h_2 = \phi \circ p_1 h_1$ (where $p_i : H_i \to H_i/N_i$ is the natural projection.)*

Proof. Let $F_0 \le H_1 \times H_2$ be the image of G under the map $(h_1, h_2) : \Lambda \to H_1 \times H_2$. Let F be the closure of F_0. Then F is a closed subgroup of $H_1 \times H_2$. Let $\pi_i : F \to H_i$ be the projection, let $N_1 = \{x \in H_1 : (x, 1) \in F\}$, $N_2 = \{y \in H_2 : (1, y) \in F\}$.

Claim 1. N_i is compact; and indeed π_i is proper.

Proof. Let U_i be a precompact open neighborhood of 1 in H_i. Then $h_2^{-1}(U_2)$ is commensurable with Λ, and hence with $h_1^{-1}(U_1)$; so $h_2^{-1}(U_2)$ is covered by finitely many translates of $h_1^{-1}(U_1)$; hence $h_1 h_2^{-1}(U_2)$ is covered by finitely many translates of U_1, so it is precompact.

So $N_1 \subset \pi_1(F \cap (H_1 \times U_2)) \subset \pi_1 cl(F_0 \cap (H_1 \times U_2)) \subset cl(h_1 h_2^{-1}(U_2))$ and so N_1 is precompact. Being closed it is compact, and likewise N_2. Along the way we showed properness: $\pi_2^{-1}(U_2) \subset cl(h_1 h_2^{-1}(U_2)) \times U_2$ so it is pre-compact.

$\square$

Claim 2. F projects onto H_1 and onto H_2.

Proof. Let U be an open subset of H_2, with compact closure $\mathbf{U}$. By Claim 1, π_2 is proper, so $\pi_2^{-1}(\mathbf{U})$ is compact. Thus $\pi_2(\pi_2^{-1}(\mathbf{U}))$ is compact. By the density assumption, $\pi_2(\pi_2^{-1}(U))$ is dense in U. Thus $\pi_2(\pi_2^{-1}(\mathbf{U}))$ is closed, and contains a dense subset of U, hence it contains U. It follows that F projects onto H_2, and similarly H_1. $\square$

Now $N_i \trianglelefteq H_i$, and F induces a group isomorphism $\phi : H_1/N_1 \to H_2/N_2$. By the same argument as in Claim 1, $\phi(X)$ is compact iff X is compact. Let V_i be a compact neighborhood of 1 in H_i, and

$W_1 = V_1 \cup \phi^{-1}(V_2)$. Then W_1 is compact, as is $W_2 = \phi(W_1) = \phi(V_1) \cup V_2$. Thus ϕ restricted to W_1 is a homeomorphism near 1, which suffices.

$\square$

There is a one-sided version of Proposition 5.5:

Remark 5.6. If $h : G \to H$ and $h' : G \to H'$ are homomorphisms with dense image into locally compact groups, and $\chi(h)$ lies below $\chi(h')$ in the sense that there exist $\Lambda \in \chi(h), \Lambda' \in \chi(h')$ with $\Lambda \leq \Lambda'$, then there exists a compact normal subgroup N' of H' and a homomorphism $\phi : H \to H'/N'$ such that $\phi \circ h = p' \circ h'$.

This could be proved in the same way, but also follows from the statement of Proposition 5.5: define $h_2 : G \to H \times H'$ by $h_2(x) = (h(x), h'(x))$, and let H_2 be the closure of the image of h_2. Then $\chi(h_2) = \chi(h)$ and Proposition 5.5 applies, and proves the remark.

Given a commensurability class θ of approximate subgroups, if it arises from a homomorphism $f : G \to \mathsf{H}$, there is always a *universal* one with class θ, mapping with compact kernel onto any other. Often, there is also a minimal one, namely $f : G \to \mathsf{H}$ such that H has no compact normal subgroups. Both are unique up to a unique isomorphism under G.

5.7. **Intersections.** (See [44]). Let ω be a commensurability class of approximate subgroups of a group G, and let H be a subgroup of G. If $X, Y \in \omega$, then $Y \cap H$ is commensurably covered by $XX^{-1} \cap H$. To see this, say $Y \subset \cup_{i=1}^m Xc_i$. Then $Y \cap H \subset \cup_{i=1}^m (Xc_i \cap H)$. Now if $Xc_i \cap H \neq \emptyset$, let $h_i \in Xc_i \cap H$. Then $c_i \in X^{-1}h_i$ so $Xc_i \subset XX^{-1}h_i$. Hence the H-commensurability class of $XX^{-1} \cap H$ does not depend on the choice of $X \in \omega$. In particular, as $(XX^{-1} \cap H)^2 \subset (XX^{-1})^2 \cap H$ and $(XX^{-1})^2 \in \omega$, $XX^{-1} \cap H$ is an approximate subgroup of H. We denote this class by $\omega \cap H$. If H is normal, we also have the class ω/H, represented by X/H for any $X \in \omega$. Then ω belongs to H iff $\omega \cap H \sim \omega$. When H is normal, this is equivalent to: $\omega/H \sim \{1\}/H$.

One can similarly define an intersection of commensurability classes: if X, Y are approximate subgroups of G, then so is $XX^{-1} \cap YY^{-1}$, and the commensurability class of $XX^{-1} \cap YY^{-1}$ depends only on the classes of X and of Y. In case X, Y are commensurable, $XX^{-1} \cap YY^{-1}$ is commensurable to both. This can be seen directly using a Rusza-style argument, starting from a maximal subset $\{a_1, \ldots, a_n\}$ of X such that the sets a_iY are disjoint. A more conceptual derivation will soon be available: we will see below (Theorem 4.2) that any commensurability class of approximate subgroups can be written as $\chi(f)$, for some

approximate homomorphism into a locally compact group; then it is easy to check that $\chi(f \times g) = \chi(f) \cap \chi(g)$, and $\chi(f \times f) = \chi(f)$.

Lemma 5.8 was obtained only for laminar approximate lattices in an earlier version of this paper; Simon Machado saw that it holds for all approximate subgroups. Here we include, for completeness of the argument, the intermediate ase of laminar approximate subgroups.

Lemma 5.8. *For any commensurability class ω of laminar approximate subgroups of G, there exists a unique commensurability class $\langle \omega \rangle_{min}$ of subgroups of G such that ω belongs to a subgroup S iff S contains some element of $\langle \omega \rangle_{min}$. We have $S \in \langle \omega \rangle_{min}$ iff $\omega = \chi(f)$ for some homomorphism $f : S \to L$ with L a* connected *locally compact group.*

Proof. Since ω is laminar, there exists a subgroup H of G and a homomorphism $f : H \to L$, L a connected locally compact group, with dense image, and with $\omega = \chi(f)$. We have to show that if χ belongs to a subgroup S of G, then S contains a finite index subgroup of H. Since χ belongs to H and to S it belongs to $H \cap S$, so we may assume $S \leq H$. Let U be a compact neighborhood of 1 in L. Then $f^{-1}(U) \in \omega$, so $f^{-1}(U)$ is covered by finitely many cosets of S; hence a dense subset of U is covered by finitely many cosets of $f(S)$ and in particular of $L' := cl(f(S))$. A finite union of nowhere dense sets is nowhere dense; hence not all of these cosets of L' can be nowhere dense; so L' itself must have nonempty interior. Being a subgroup, L' is open. But L is connected, so $L' = L$.

We have $f^{-1}(U) \subseteq SF$ for some finite F. We claim that $H \subseteq SF$. To see this let $g \in H$. Then by density of $f(S)$ in L we have $f(s) \in U f(g)^{-1}$ for some $s \in S$, so $f(sg) \in U$. Thus $sg \in f^{-1}(U) \subset SF$ so $g \in SF$. Thus $H = SF$ so S has finite index in H.

$\square$

5.9. **The fundamental commensurability class.** More generally, if $f : G \to \mathsf{H} : K$ is a quasi-homomorphism, with H locally compact and $K = K^{-1}$ compact, we consider pullbacks $Y = f^{-1}(W)$ of compact neighborhoods W of K in H. We will see below that YY^{-1} is an approximate subgroup, whose commensurability class is uniquely determined. We continue to use the notation $\chi(f)$ for this class, and to refer to it as the fundamental class of f. We will write $X \sim \chi(f)$ to mean that the commensurability class of X equals $\chi(f)$. Uniqueness again holds but now in the category of compact correspondences.

Proposition 5.10. *Let H be a locally compact topological group, K a symmetric compact subset of H, and assume*

$$f : \Lambda \to H : K$$

is a quasi-homomorphism, with

$$f(xy^{-1}) \in f(x)f(y)^{-1}K$$

Then all sets $X = f^{-1}(UK)$, with U a compact neighborhood of the identity in H, are right- commensurable in Λ. Thus any symmetric set sandwiched between two such is an approximate subgroup. In particular XX^{-1} is one.

Proof. We begin by comparing translation in Λ and in H.

Claim 1. For any $V \subset H$ and $a \in H$, $f^{-1}(Va) \subset f^{-1}(VV^{-1}K)c$ for some c.

If $f^{-1}(Va) = \emptyset$, there is nothing to prove. Otherwise, let c be such that $f(c) \in Va$. If $f(x) \in Va$, then $f(xc^{-1}) \in (Va)(Va)^{-1}K = VV^{-1}K$; so $xc^{-1} \in f^{-1}(VV^{-1}K)$ and $x \in f^{-1}(VV^{-1}K)c$.

Claim 2. Let $Z \subset H$ be compact and let U be a neighborhood of 1 in H. Then finitely many right cosets of $f^{-1}(UK)$ cover $f^{-1}(Z)$.

Proof. Let V be a neighborhood of 1, with $VV^{-1} \subset U$. As Z is compact, it is covered by finitely many cosets Va; so it suffices to show that $f^{-1}(Va)$ is covered by a right coset of $f^{-1}(VV^{-1}K)$; this follows from (2) above. $\square$

Let U be a compact neighborhood of 1 in H, $X = f^{-1}(UK)$ and $Y = XX^{-1}$. Note Y is symmetric ($1 \in Y = Y^{-1}$). We have $Y \subset f^{-1}(UK(UK)^{-1}K)$; taking $Z = UK(UK)^{-1}K$ in Claim 2, we see that Y is covered by finitely many right translates of X. Claim 2 also implies the right-commensurability of any two such X, and hence of any set sandwiched W between two such. Thus any such W is an approximate subgroup; in particular this applies to Y. $\square$

Remark 5.11. (1) Proposition 5.11 did not assume normality of K.
(2) If we do assume that K is normal and that $f(xy) \in Kf(x)f(y)$ (as is the usual assumption for quasimorphisms), we obtain (with x, y replaced by $x, y^{-1}x$) that $f(x) \in f(y)f(y^{-1}x)K$, and so changing sides and using $K = K^{-1}$, we have $f(y^{-1}x) \in f(y)^{-1}f(x)K$, and similarly $f(xy^{-1}) \in f(x)f(y)^{-1}K$.
(3) It is worth noting that although we began with an approximate homomorphism into a locally compact group in the general sense of a *compact* error set K, the associated commensurability class in

Λ consists of approximate subgroups in Tao's original sense of a *finite* set of translates.

(4) Responding to a question of Kobi Peterzil during a talk, we consider the case of quasimorphisms, i.e. take $\mathsf{H} = \mathbb{R}$. we may take $K = [-1,1]$. Modifying f to be homogeneous, it becomes symmetric, $f(x^{-1}) = -f(x)$ and $f(1) = 0$. Given $e > 0$, let $k(e)$ be the smallest natural number such that $X = f^{-1}(-1-e, 1+e)$ is a k-approximate subgroup (Proposition 5.11). What kind of function is $k(e)$?

Following the above proof, we have $XX \subset f^{-1}(-3-2e, 3+2e)$; this interval is covered by m translates of $(-e/2, e/2)$, if $m > 4(1+ 6/e)$; the pullback of each such translate is contained in a translate of X. Hence X is a 5-approximate subgroup if $e > 24$.

Question 5.12. Is there an absolute bound on the level of approximateness, for small $e > 0$?

In Example 3.24, with Λ the central extension of the homeomorphism group of the circle, the translation number f commutes with translation by $\mathbb{Z}$; for any $e > 0$, if $X = f^{-1}(-1-e, 1+e)$, then $XX \subset f^{-1}(-3-2e, 3+2e) \subset X \pm \{0,1,2,3\}$. What about Brooks' examples, [41], Example 1?

Question 5.13. What is the minimal k such that the approximate kernel of any quasimorphism is commensurable to a k-approximate subgroup?

By the above discussion, the answer is one of the numbers 2,3,4,5.

We have already proved the principal structure theorem for approximate subgroups, Theorem 4.2; we will now refine it. See Definition C.1 (6) for the definition of rigidity, and §5.10 for χ.

Theorem 5.14. *Let ω be a commensurability class of approximate subgroups of a group G. Then there exists a subgroup $\widetilde{\Lambda} \leq G$, a connected Lie group L, a rigid abelian subgroup $A \trianglelefteq \mathsf{L}$, a normal compact subset K of A, and a quasi-homomorphism $f : \widetilde{\Lambda} \to \mathsf{L} : K$ with $\omega = \chi(f)$.*

Some supplements:

(i) One can demand that $A \cong \mathbb{R}^N$ for some $N \in \mathbb{N}$, and that $\langle f(\widetilde{\Lambda}) \rangle A$ be dense in L.
(ii) One can add that L has no compact normal subgroups.
(iii) In place of connectedness of L, and as an alternative to (i), one can ask that $\langle f(\widetilde{\Lambda}) \rangle$ is dense in L, and $A \cong \mathbb{R}^n \times \mathbb{Z}^k$ for some n, k.

(iv) The induced homomorphism $\widetilde{\Lambda} \to \mathsf{L}/A$ is definable in the sense of continuous logic.

Following the proof, we will give two ways to decompose the quasi-homomorphism f into better studied kinds. In Theorem 5.20, we will describe the situation in terms of a $\bigwedge$-definable group H and n definable quasimorphisms on H, i.e. a definable quasi-homomorphism on H into $\mathbb{R}^n$. In Proposition 5.23, we will instead decompose f in terms of a rigid group extension $\widehat{\Lambda}$ of $\widetilde{\Lambda}$, and a group homomorphism $\widehat{f} : \widehat{\Lambda} \to \mathsf{L}$.

5.15. **Refining the target.** The proof of Theorem 5.15 will start with a quasi-homomorpism $f : \Gamma \to \mathsf{H} : K$ as in Theorem 4.2, and gradually seek to simplify it, without changing the commensurability class $\chi(f)$ associated with f. Let us review in advance a few of the allowed modifications, along with an explanation of why the pullback class $\chi(f)$ does not change. Some of the steps parallel [20], where Fujiwara and Kapovich proved that any quasi-homomorphism with discrete target can be perturbed to one with central error. Here the best we can aim for is a rigid error set.

(1) Replace H by an open subgroup H_1 containing K; let $\Gamma_1 = f^{-1}(\mathsf{H}_0)$, $f_1 = f|\Gamma_1$; $K_1 = K \cap \mathsf{H}_1$. Then Γ_1 is a subgroup (if $x, y \in \Gamma_1$ then $f(xy) \in f(x)f(y)K \subset \Gamma_1$.) Since a compact open neighborhood of the identity in H_1 is also one in H, we have $\chi(f|\Gamma_1) = \chi(f)$.
(2) Factor out a compact normal N subgroup of H: let $f' : \Gamma \to \mathsf{H}/N : K/N$ be the composition of f with the quotient map $\pi : \mathsf{H} \to \mathsf{H}/N$. We have $(f')^{-1}(YN/N) = f^{-1}(YN)$ for any compact neighborhood Y of K in H, so $\chi(f) = \chi(f')$.
(3) Let H_2 be a closed subgroup of H, with $\mathsf{H}_2 K^m = \mathsf{H}$. Let $K_2 = \mathsf{H}_2 \cap K^{3m+1}$. For any $x \in \Gamma$ pick $c(x) \in K^m$ with $f(x)c(x) \in \mathsf{H}_2$, $c(1) = 1$, and define $g : \Gamma \to \mathsf{H}_2$ by $g(x) = f(x)c(x)$. For any $x, y \in \Gamma$ we have, for some $k = k(x,y) \in K$:

$$g(xy) = f(xy)c(xy) = f(x)f(y)kc(xy) = g(x)c(x)^{-1}g(y)c(y)^{-1}kc(xy) =$$

$$= g(x)g(y)c(x)^{-g(y)}c(y)^{-1}kc(xy) \in g(x)g(y)K^{3m+1}$$

so $g(xy) \in g(x)g(y)K_2$.

For a compact neighborhood U of 1 in H, $UK^m \cap \mathsf{H}_2$ is a compact neighborhood of 1 in H_2, and $f^{-1}(U) \subset g^{-1}(UK^m \cap \mathsf{H}_2) \subset f^{-1}(UK^{2m})$. Thus $\chi(f) = \chi(g)$.
(4) Let H_0 be an open subgroup of H of finite index. Let $\mathsf{H}_3 = H_0\langle K \rangle$. Then $\mathsf{H}_3 = H_0 K^m$ for some m. Let $\Gamma_1 = f^{-1}(\mathsf{H}_3)$. Then Γ_1 is a finite index subgroup of Γ, and by (1) $\chi(f) = \chi(f|\Gamma_1)$. By the previous step (3), we can find $g : \Gamma_1 \to H_0$ with $\chi(f) = \chi(g)$.

Remark 5.16. To a model theorist, the apparent arbitrariness in the choice of f_0 in step (3) is disturbing. It will only be used via (4), so in a finite index setting, where only finitely many parameters need be chosen. Note also in general that if $\langle K \rangle$ is centerless, the correction function c_0, and hence f_0, are uniquely defined in the construction. The general case could be made definable using a relational presentation of the notion of a quasi-homomorphism, namely the category CoGr of Appendix B, in effect allowing $f(x)$ to be a compact subset rather than an element; and taking $k(x) = f(x)^{-1} H_2 \cap K^m$.

Proof of Theorem 5.15. Let Λ_0 be an approximate subgroup in ω, and $\widetilde{\Lambda} = \langle \Lambda_0 \rangle$. By Theorem 4.2, there exists $f : \widetilde{\Lambda} \to \mathsf{H} : K$ with H locally compact, K normal, symmetric, compact, with $\Lambda_0 \sim \chi(f)$.

By Proposition C.3, there exist closed normal subgroups $N \leq A \leq H_1 \leq \mathsf{H}$ with H/H_1 finite, N compact, with A/N basic abelian and rigid in H_1/N, such that $\langle K \rangle \cap H_1 \leq A$. By 5.16 (4), we may pass from H to the finite index subgroup H_1 (replacing K by $K^m \cap H_1$ for appropriate m); so we may assume $K \subset A$, $N \leq A \leq \mathsf{H}$. Since N is compact, by 5.16 (2) we may factor it out. Now A is basic abelian and rigid in H, and $K \subset A$.

We have $A = DV$ with $D \cong \mathbb{Z}^l$ central in H, and V a rigid vector group. Let $D_{\mathbb{R}} = D \otimes \mathbb{R}$, so that $(A, D) \cong (\mathbb{Z}^l, \mathbb{R}^l)$. Let $\bar{\mathsf{H}} = \mathsf{H} \times_D D_{\mathbb{R}} := (\mathsf{H} \times D_{\mathbb{R}})/D$, with D embedded into $\mathsf{H} \times D_{\mathbb{R}}$ by $d \mapsto (d, -d)$. This diagonal image of D is a closed, discrete subgroup of $\mathsf{H} \times D_{\mathbb{R}}$ and so the quotient is still locally compact. Now H embeds into $\bar{\mathsf{H}}$ as a closed subgroup. Let $\bar{A} = A \times_D D_{\mathbb{R}}$. If U is a compact neighborhood of 1 in $\bar{\mathsf{H}}$, then $U \cap \mathsf{H}$ is a compact neighborhood of 1 in H. Hence we may replace H, A, K by $\bar{\mathsf{H}}, \bar{A}, K$; we have the same situation, and in addition $\bar{A} \cong \mathbb{R}^N$ for some N. We are of course not claiming any density at this point.

By Yamabe's theorem [68], H has an open subgroup H' and a compact subgroup N', normal in H', with H'/N' a Lie group. As V is connected and H' is open, $V \subset H'$; i.e. by the above paragraph, $A \subset H'$. By 5.16 (1), we may pass from H to H'; so we may assume $\mathsf{H} = H'$. Factoring out the compact normal subgroup N' using 5.16 (2) again, we have that L is a Lie group.

Replacing H by the closed subgroup $\mathsf{L} = cl(f(\widetilde{\Lambda})A)$, we may assume $f(\widetilde{\Lambda})A$ is dense in L. Now L^0 is an open subgroup of L; $\mathsf{L}^0 \cap f(\widetilde{\Lambda})A$ is thus still dense in L^0: since A is connected, $A \subset \mathsf{L}^0$; by 5.16 (1), we may replace L by L^0, so that we may assume L is connected. Finally, by 5.16(2), we may, without changing $\chi(f)$, factor out the maximal

normal compact subgroup of L, to obtain that L has no nontrivial normal compact subgroups.

Now the continuity (iv) comes from Corollary 4.12; the various modifications made to the quasi-homomorphism of Theorem 4.2, to wit restricting the domain to a subgroup, factoring out a compact normal subgroup of the image, and modifying by a map into A, will not effect this.

$\square$

Remark 5.17. If desired, say in order to connect with cohomological results phrased for unitary modules, we can make the error set contained in a complex (unitary) rather than real rigid space. Take first the semi-direct product of L with the complexification $A_{\mathbb{C}}$ (where L acts on A by conjugation, and using the derived action on $A_{\mathbb{C}}$), and then factoring out the image of the homomorphism $A \to \mathsf{L} \times A_{\mathbb{C}}$, $x \mapsto (-x, x)$, which becomes a normal subgroup. The three variants $\mathsf{L}, \mathsf{L}', \mathsf{L}''$ do not detract from the canonicity of the construction, but give three canonical forms that may be convenient for different purposes.

5.18. **Model-theoretic formulation.** Let us recall some definitions. As usual, we interpret $\bigwedge$ and $\bigvee$-definable sets in a sufficiently saturated model of the theory.

(1) Let $\widetilde{\Lambda} = \cup H_n$ be a $\bigvee$-definable group. A set $X \subset \widetilde{\Lambda}$ is *locally $\bigwedge$-definable* if for each N, $H_n \cap X$ is $\bigwedge$-definable.
(2) A subgroup Γ of $\widetilde{\Lambda}$ is *locally $\bigwedge$-definable* if $H_n \cap \Gamma$ is $\bigwedge$-definable, for all n.
(3) A $\bigwedge$-definable approximate subgroup W is said to *determine a commensurability class ω* if $W = \cap_n W_n$, with $W_1 \supset W_2 \supset \cdots$ all in ω.
(4) Let X be a $\bigwedge$-definable set. A function $\phi : X \to \mathbb{R}$ is said to be *continuous* (or: *definable in the sense of continuous logic*) if it is bounded, and for any two disjoint compact sets $C, C' \subset \mathbb{R}$ there exists a definable D with $\phi^{-1}(C) \subset D$ and $D \cap \phi^{-1}C' = \emptyset$.
(5) Let X be a locally $\bigwedge$-definable subset of $\widetilde{\Lambda}$. Then $\phi : X \to \mathbb{R}$ is *continuous* if $\phi|(X \cap H_n)$ is continuous for each n.

Theorem 5.19. *Let G be a group, ω a commensurability class of approximate subgroups of a group G. Then there exists a $\bigvee$-definable group $\widetilde{\Lambda}$, a locally $\bigwedge$-definable subgroup $\Gamma \leq \widetilde{\Lambda}$ of bounded index, and finitely many continuous quasimorphisms $\phi_1, \ldots, \phi_n : \Gamma \to \mathbb{R}$, such that the approximate kernel*

$$\Gamma_\phi := \cap_{i=1}^n \phi_i^{-1}[-1, 1]$$

is a $\bigwedge$-definable approximate subgroup, and ω is the commensurability class determined by Γ_ϕ.

Proof. By Theorem 5.15, there exists a quasi-homomorphism $f : \widetilde{\Lambda} \to \mathsf{L} : K$, where K is a compact subset of a rigid abelian subgroup $A \trianglelefteq \mathsf{L}$, $A \cong \mathbb{R}^n$, $\omega = \chi(f)$. Let $\bar{\mathsf{L}} = \mathsf{L}/A$, let $\bar{f} : \widetilde{\Lambda} \to \bar{\mathsf{L}}$ be the composed homomorphism $\widetilde{\Lambda} \to \mathsf{L} \to \bar{\mathsf{L}}$; and let Γ be the kernel. By Corollary 4.12, since A contains $cl(\langle \triangle \rangle)$, the induced homomorphism $\widetilde{\Lambda}/\Gamma \to \mathsf{L}/A$ is continuous, and Γ is a locally $\bigwedge$-definable group.

Let $\psi = f|\Gamma$. Identifying A with $\mathbb{R}^n$ and projecting to coordinates, we can view ψ as an n-tuple of quasimorphisms $\psi_1, \ldots, \psi_n$. Let ϕ_i be the homogeneous quasimorphism at bounded distance from ψ_i; and let $\phi = (\phi_1, \ldots, \phi_n)$. We have:

(1) $\phi^{-1}(-m, m)^n$ is an approximate subgroup, for large enough m (by Proposition 5.11).

(2) If $C \subset A$ is compact, then $\psi^{-1}(C) \subset f^{-1}(C)$ so this set is contained in a definable subset of $\widetilde{\Lambda}$. (Theorem 4.2 (4)).

(3) for some m, the pullbacks under ψ of two compact subsets of $\mathbb{R}^n$ whose Euclidean distance is at most m are separated by a definable set

(4) $\psi^{-1}(-m, m)^n$ has bounded index in $\widetilde{\Lambda}$ (even if we work in a large saturated M, if $(a_i : i < (2^{\aleph_0})^+)$ are elements of $\widetilde{\Lambda}$, then some a_i, a_j with $i < j$ must have the same image in L, hence $a_i^{-1} a_j \in A$, and $\psi(a_i) \in \psi(a_j) \triangle \subset (-m, m)^n$.))

Properties (2-4) of ψ above are all also inherited by ϕ.

Replacing ϕ by $\frac{1}{m}\phi$ we may assume $\phi^{-1}((-1, 1)^n)$ is an approximate subgroup, and the inverse images of two compacts that are 1-separated (i.e. $[-1, 1]^n C \cap C' = \emptyset$) are separated by a definable set. Along with the homogeneity of ϕ, this implies that ϕ is continuous: if C, C' are disjoint compact subsets of $\mathbb{R}^n$, they are ϵ-separated for some $\epsilon > 0$; let N be an integer with $N\epsilon > 1$. Note $\phi^{-1}(NC), \phi^{-1}(NC')$ are contained in a definable subset of $\widetilde{\Lambda}$. As NC, NC' are 1-separated, there exists a definable set D with $\phi^{-1}(NC) \subset D, D \cap \phi^{-1}(NC') = \emptyset$. Let $D^* = \{g \in \Gamma : g^n \in D\}$. Then $\phi^{-1}(C) \subset D^*$, $D^* \cap \phi^{-1}(C') = \emptyset$. Thus the definable sets separate pullbacks of an arbitrary pair of disjoint compacts, proving the continuity of ϕ. Since $\phi^{-1}[-1, 1]^n$ has bounded index in $\widetilde{\Lambda}$, any definable set containing $\phi^{-1}[-1, 1]^n$ must be commensurable with the definable approximations to $\widetilde{\Lambda}$.

$\square$

Remarks 5.20. (1) In case $n = 0$, Γ is $\bigwedge \omega$-definable. Existence of such a Γ is assured in this case by Theorem 3.5 of [26].

(2) We may choose $\widetilde{\Lambda}, \Gamma$ so that $\widetilde{\Lambda}/\Gamma$ is a connected Lie group with no nontrivial compact normal subgroup.

Corollary 5.21. *Let G be a group of bounded exponent. Then any approximate subgroup of G is commensurable to a subgroup.*

Proof. Let notation be as in Theorem 5.20; by Remark 5.21, we may take $\widetilde{\Lambda}/\Gamma$ to be a connected Lie group. Since it has bounded exponent, it must be trivial. Thus by the Theorem, ω is the commensurability class associated with a quasimorphism $\widetilde{\Lambda} \to \mathbb{R}^n$; we can take the quasimorphism to be homogeneous. In this case, the bounded exponent again implies that the image is 0. So ω is the commensurability class of the group $\widetilde{\Lambda}$. $\qquad\qquad\square$

5.22. **Group-theoretic formulation.** We restate Theorem 5.15 in more standard group-theoretic terms, somewhat in the spirit of the Bargmann-Wigner theorem on projective representations of Lie groups.

Let G be a group, let E_n be Euclidean space, O_n the group of isometries of E_n fixing 0. Assume a homomorphism $\rho : G \to O_n$ is given; ρ makes E_n into a G-module V; we call a *rigid G-module* to indicate that the image of ρ preserves the Euclidean norm.

Let $C^n(G, \rho) = C^n(G, V)$ be the $\mathbb{R}$-space of functions from $G^n \to V$ view this as a complex with the usual differential, $(d^{n+1}\varphi)(g_1, \ldots, g_{n+1}) = g_1\varphi(g_2, \ldots, g_{n+1}) + \sum_{i=1}^n (-1)^i \varphi(g_1, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1}\varphi(g_1, \ldots, g_n)$.

The kernel of d^{n+1} is denoted Z^n, the group of n-cocycles. In particular any $\alpha \in Z^2(G, V)$ determines canonically a group $G \underset{\alpha}{\times} V$ with universe $G \times V$, and multiplication: $(g, e)(h, d) = (gh, \rho(h)e + d + \alpha(g, h))$. It fits into an exact sequence

$$0 \to E_n \to G \underset{\alpha}{\times} V \to G \to 1$$

In case ρ is trivial, this is a central extension of G by $\mathbb{R}^n$; in general we may call it a rigid extension of G.

When G is discrete, we give $G \underset{\alpha}{\times} V$ the product topology of $G \times \mathbb{R}^n$. Let C_{bdd} be the subcomplex of the bounded chains, and H^*_{bdd} or H^*_b the cohomology.

Theorem 5.23. *Let ω be a commensurability class of approximate subgroups of a group G. Then there exists a subgroup $\widetilde{\Lambda} \le G$, a connected Lie group L without compact normal subgroups, a rigid $\widetilde{\Lambda}$-module V,*

a cocycle bounded 2-cocycle $\alpha : \widetilde{\Lambda}^2 \to V$, and a continuous homomorphism $\phi : \widetilde{\Lambda} \underset{\alpha}{\times} V \to \mathsf{L}$, with ω commensurable to $\{g \in \widetilde{\Lambda} : \phi(g, 0) \in U\}$, for U a compact open neighborhood of 1 in L.

Above, continuity refers to the product topology with $\widetilde{\Lambda}$ discrete; it amounts to continuity on $1 \times V$.

Proof. By Theorem 5.15 there exists a subgroup $\widetilde{\Lambda} \leq G$, a connected Lie group L, a normal compact subset K of a rigid abelian subgroup $A \leq \mathsf{L}$, $A \cong \mathbb{R}^N$, and a quasi-homomorphism

$$f : \widetilde{\Lambda} \to \mathsf{L} : K$$

with $\omega = \chi(f)$. We will identify A with $\mathbb{R}^N$, and write multiplication additively within A. Rigidity of A means that there exists a homomorphism $\rho' : \mathsf{L} \to O_N$ with $x^{-1}ax = \rho'(x)a$, for all $x \in \mathsf{L}, a \in A$. For $g \in \widetilde{\Lambda}$, let $\rho(g) = \rho'(f(g))$. As $K \subset A$, $\phi(K) = 1$ and hence $\rho : \widetilde{\Lambda} \to O_N$ is a homomorphism. Let V be A with this $\widetilde{\Lambda}$-module structure. Define $\alpha : \widetilde{\Lambda}^2 \to A = \mathbb{R}^N$ by $\alpha(x, y) = f(xy) - f(x) - f(y)$. It is easy to check that $d^3\alpha = 0$ in $C^3(\widetilde{\Lambda}, \rho)$. Since $\alpha(x, y) \in K$, it is bounded, i.e. $\alpha \in Z^2_{bdd}(\widetilde{\Lambda}, \rho)$.

Define $\phi : \widetilde{\Lambda} \underset{\alpha}{\times} V \to \mathsf{H}$ by $\phi(g, a) = f(g)a$. It is easy to check that ϕ is a homomorphism, and that $\chi(f) \sim \{g \in \mathsf{L} : \phi(g, 0) \in U\}$, for U a compact open neighborhood of 1 in L.

$\square$

Let β be the bounded cohomology class represented by α. If $\beta = 0$, then ω is laminar: we have, above, $\alpha = d^2(b)$ for some bounded $b : \widetilde{\Lambda} \to A$. Define $f'(x) = f(x)b(x)^{-1}$. Again it is easy to check that $f' : \widetilde{\Lambda} \to \mathsf{L}$ is a homomorphism. As $b(\widetilde{\Lambda})$ is a bounded subset of A, and hence a precompact subset of L, we have $\chi(f) = \chi(f')$. This shows that ω is laminar.

Similarly, if β is represented by α and also by α', then the corresponding approximate subgroups are commensurable.

An alternative description can be given in terms of cohomological data on G and on the quotient group $\bar{L} = \mathsf{L}/A$. In this case, the data consists of a homomorphism $\bar{\phi} : \widetilde{\Lambda} \to \bar{L}$, a finite-dimensional Euclidean space V, a homomorphism $\rho : \bar{L} \to O_n = Aut(V)$, a class $\beta \in H^2_b(G, V)$, agreeing in $H^2(G, V)$ with the $\bar{\phi}$-pullback of the class describing the extension L of $\bar{L}$. More precisely:

Corollary 5.24. *Let G be a group, ω a commensurability class of approximate subgroups of G. Then $\omega = \chi(\bar{f}, \bar{L}, L, \beta)$ is determined by:*

- *A connected Lie group L with no compact normal subgroup;*
- *A rigid abelian normal subgroup A of L; let $\bar{L} = L/A$, and let M be the $\bar{L}$-module A. Let $\alpha \in H^2(\bar{L}, A)$ be the cohomology class corresponding to the extension $1 \to A \to L \to \bar{L} \to 1$, viewed as discrete groups.*
- *A subgroup H of G, to which ω belongs.*
- *A homomorphism $\bar{f} : H \to \bar{L}$.*
- *A class $\beta \in H_b^2(H, M)$, such that the image of α in $H^2(H, M)$ coincides with the image $(\bar{f}|H)^*\beta$ of β in $H^2(H, M)$.*

ω is laminar if $\beta = 0$.

In particular if $H_b^2(H, M) = 0$ for all H such that ω belongs to H, then ω is laminar.

Except perhaps for the choice of H, that is determined only up to finite index, the data is canonically determined by ω: we begin with the locally compact pattern automorphism group; L is the unique connected Lie group with no compact normal subgroups; A is the closed subgroup arising from the Lascar neighbour relation $\triangle$; β is defined as above.

Conversely, given data $L, A, H, \beta, \bar{f}$, we can construct an approximate subgroup in H. Let L^δ be the discrete group underlying L, and represent it as $\bar{L} \times_\alpha A$. Then α, β determine classes in $H^2(H, A)$, and by assumption we have $\alpha - \beta = 0 \in H^2(H, A)$. Let $\gamma : H \to A$ be a map with $d\gamma = \alpha - \beta$. Define $\phi(x) = (\bar{g}(x), \gamma(x))$. The error set is still given by that of β, and thus is contained in A and bounded.

Burger and Monod have proved vanishing theorems for lattices in semisimple groups L. This will not immediately help us in understanding approximate lattices Λ of L, since the criterion of Corollary 5.25 requires vanishing of the bounded cohomology not of L itself, but of the subgroups G generated by approximate subgroups. Prima facie, these groups G seem perfectly likely to be free and hence have very large bounded cohomology. We will analyze Λ by other methods. But after the fact, we will know that G is in fact isomorphic to a lattice in a semisimple group (not in L itself but in a product of L with a 'complementary' Lie group); making the results of [11] and [49] relevant. It will be too late for G itself, but along with Gromov's mapping theorem, we will be able to apply the criterion to soluble extensions of G. See Proposition 7.10.

With Proposition 5.28, we will see a partial converse to the statement that the vanishing of β implies laminarity. Let Λ be an approximate subgroup of G, let $L, A, \bar{L}$ be as above with $A \cong \mathbb{R}$ central in L, and let $\beta \in \ker(H_b^2(G, \mathbb{R}) \to H^2(G, \mathbb{R}))$ be the associated class. If Λ is laminar,

then Λ belongs to a subgroup H of G such that the image of β in $H_b^2(H, \mathbb{R})$ vanishes. Indeed we have $\beta = d\alpha$ for some quasimorphism $\alpha : G \to \mathbb{R}$. Let α' be a homogeneous quasimorphism with $\alpha - \alpha'$ bounded. By Proposition 5.28, $d\alpha'$ is a homomorphism on some subgroup H representing the commensurability class of Λ. Passing to this subgroup H, we have $d\alpha' = 0$ so $d\alpha = d(\alpha - \alpha')$ and thus β vanishes as a class in $H_b^2(H)$.

Remark 5.25. It follows that any nonzero class in $H_b^2(G, \mathbb{R})$ indicates a non-laminar approximate subgroup, if not in G itself then in a central extension: the image of β in $H^2(G, \mathbb{R})$ determines a central extension $\widetilde{G}$ of G by $\mathbb{R}$; the image $\widetilde{\beta}$ of β in $H_b^2(\widetilde{G}, \mathbb{R})$ remains nontrivial by the Gromov mapping theorem alluded to earlier; but now the image of $\widetilde{\beta}$ is trivial.

5.26. **Approximate kernels of quasimorphisms are non-laminar.** The proposition below implies that an approximate kernel E of a nontrivial quasimorphism is associated to no $\bigwedge$-definable group, in any expansion of the language. In other words, there is no chain of approximate subgroups D_n commensurable to E, with $D_{n+1}^2 \subset D_n$. In particular, the two classes of approximate subgroups, arising from homomorphisms into Lie group or from (nontrivial) quasimorphisms into $\mathbb{R}$, have trivial intersection.

Proposition 5.27. *Suppose ω is a commensurability class of approximate subgroups, equal to $\chi(g)$ for some homogeneous quasimorphism $g : G \to \mathbb{R}^n$, and also to $\chi(f)$ for some homomorphism $f : G \to L$ into a connected locally compact group, with dense image. Then g is a homomorphism.*

Proof. Since $G = \cup_d g^{-1}[-d, d]^n$, it suffices to prove for each d that g is a homomorphism on the subgroup $< E >$ generated by $E := g^{-1}[-d, d]^n$. So we may assume $G = \langle E \rangle$. Factoring out a compact normal subgroup of G does not change $\chi(f)$; so we may assume L has h no compact normal subgroups. Being homogeneous, g is conjugation-invariant, so E is normal, and thus $f(E)$ is normalized by $f(G)$ and so $cl(f(E))$ is normal in L. But $f(E)$ is precompact, so $cl(f(E))$ is compact. We can now apply Proposition C.3 with $H = L$, $K = cl(f(E))$. We must have $H_1 = H = L$ and $N = 1$. Thus there exists a rigid $A \leq L$, with $\langle cl(f(E)) \rangle \subset A$. But $\langle f(E) \rangle = f(G)$ is dense in L; so $L = A$ is abelian, and $f(\langle [G, G] \rangle) = 1$. Since $\chi(f)$ and E are commensurable, $\langle [G, G] \rangle$ is contained in finitely many translates of E, so $g(\langle [G, G] \rangle)$ is bounded. By homogeneity, $g(x^n) = ng(x)$ for $x \in [G, G]$, so $\mathbb{Z}g(x)$ is bounded and hence $g(x) = 0$, i.e. g vanishes on $\langle [G, G] \rangle$. Now

let $x, y \in G$; then $x^n y^n (xy)^{-n} \in \langle [G, G] \rangle$, so $g(x^n y^n (xy)^{-n}) = 0$ and hence $g(x^n) + g(y^n) + g((xy)^{-n})$ is bounded independently of n. But $g(x^n) + g(y^n) + g((xy)^{-n}) = n(g(x) + g(y) - g(xy))$ so letting $n \to \infty$ we have $g(x) + g(y) = g(xy)$, i.e. g is a homomorphism. $\qquad \square$

If in Proposition 5.28 we remove the assumption that $cl(f(G))$ is connected, we still obtain that g is a homomorphism on some *****

5.28. By way of contrast, we note a family of cases where an approximate homomorphism f with central error - hence in particular a quasimorphism - can be replaced with an actual homomorphism. For instance this applies to a centerless finitely generated or linear group, or a subgroup of a stable group. Factoring out the center yields an actual homomorphism $\bar{f}$, whose pullbacks of compact sets are not much larger in some sense: any any group topology on G making the elements of $\chi(f)$ discrete, also must make the elements of $\chi\bar{f}$ discrete.

See §A.5 for the definition of discrete-on-compact.

Proposition 5.29. *Let G, H be topological groups, $\widetilde{\Lambda}$ a subgroup of G. Let K be a compact subset of the center Z of H. Assume:*

• *For some finite set $F \subset \widetilde{\Lambda}$, the centralizer $C_G(F)$ is finite.*

Let $f : \widetilde{\Lambda} \to \mathsf{H} : K$ a discrete-on-compact quasi-homomorphism. Then the composition $\widetilde{\Lambda} \to \mathsf{H}/Z$ is a discrete-on-compact homomorphism.

Proof. Let $\pi : \mathsf{H} \to \bar{\mathsf{H}} := \mathsf{H}/Z$ be the quotient map, and $\bar{f} = \pi \circ f : \widetilde{\Lambda} \to \bar{\mathsf{H}}$. Let $T_h : \mathsf{H} \to \mathsf{H}$ be the map: $x \mapsto x^{-1}hx$. Then T_h factors through a map $t_h : \bar{\mathsf{H}} \to \mathsf{H}$. By definition of the quotient topology, $t_h : \bar{\mathsf{H}} \to \mathsf{H}$ is continuous. Hence it maps (pre)compact sets to (pre)compact sets.

Let W be a subset of $\widetilde{\Lambda}$ with $\bar{f}(W)$ precompact. It follows for any $g \in \widetilde{\Lambda}$ that $t_{f(g)}(\bar{f}(W))$ is precompact. Now $t_{f(g)}(\bar{f}W) = T_{f(g)}(f(W))$ and $f(g^W) \subset t_{f(g)}(f(W)) K^4$ so $f(g^W)$ is also precompact, and hence g^W is discrete in G.

Suppose $w_i \in W$ and $w_i \to a \in G$. Then for each $g \in F$, $w_i^{-1} g w_i \to a^{-1} g a$. Since g^W is discrete in G, it follows that $w_i^{-1} g w_i = a^{-1} g a$ for large i. As this holds for all $g \in F$, it follows that all w_i (for large enough i) lie in the same coset of $C_G(F)$. But $C_G(F)$ is finite. Thus W has no accumulation points in G, so it is discrete in G. $\qquad \square$

Question 5.30. Can the 'no chain' statement alluded to in the first paragraph of §5.27 be made more effective? For example, it follows from this statement that there exists an approximate subgroup D commensurable to E, such that for no approximate subgroup D' do we have $(D')^8 \subset D^4$. How can such a D be found or recognised?

Example 5.31. All known nontrivial definable approximate subgroups appear to live in theories with the strict order property. This is related to longstanding questions regarding simple theories, and to new ones concerning the existence property in theories of Shelah's class NSOP1. It is interesting in this connection to examine a first example of an approximate subgroup arising from a quasimorphism. We take Brooks' quasimorphism on the free group, see [41], Example 1. Let F be the free group on two letters x, y, and w a nontrivial cyclically reduced word; for definiteness take $w = xy$. Let $A \subset F$ be the set of words containing as many nonoverlapping copies of w as of w^{-1}; A is an approximate subgroup. The group $x^{\mathbb{Z}} = \langle x \rangle$ is contained in A. Moreover $x^n y \in A$ for $n < 0$ but not for $n > 0$. Hence the semigroup $x^{\mathbb{N}} = A \cap C_F(x) \cap Ay^{-1}$ is definable with parameters in F, and even definable in A given with partial multiplication and conjugation by x, y. Hence $(F, A, \cdot)$ has the order property.

The following example is based on the theory of generic types of [57], extended to groups G given as the limit of a directed family of commensurable subsets. This extension is explained below with respect to the existence of generics, and I expect that modifying the other points is routine.

We need only the easier theory over a model, and the proof may apply to other theories with a similar behaviour of generics; the chain condition on forking is not required, only the consequences for amalgamation. The natural candidates would be theories with Shelah's property NSOP1 ([63], [33]).

Example 5.32. Every approximate subgroup defined in a simple theory is laminar.

Proof. Let $G = \cup_i G_i$ with G_i symmetric and commensurable to each other, and such that for any i, j, for some $k \geq i, j$ we hqve $G_i G_j \subset G_k$. The commensurator of an approximate subgroup admits such a presentation (Lemma 5.1).

We follow the definitions and proof of [57], over some base M_0; types and independence are over M_0 if not otherwise stated. p is *generic* if for any a, there exists $b \models p$ with ab, a independent. The immediate stabilizer $St_0(p, q)$ of one type p into another q is defined as the set of a such that for some $b \models p$ and $c \models q$, $ab = c$ and a, b and b, c are independent. We let $St(p)$ be the group generated by $St_0(p, p)$; it is $\bigwedge$-definable, and generated in two steps, indeed $St(p) \setminus St_0(p)$ is non-generic. It follows that $St_0(p, q)$ agrees, up to a non-generic set, with a coset of $St(p)$; we denote this coset by $St(p, q)$. We now follow [57] to prove existence of a generic type p of G, lying in any given definble

set G_i. Define $\Psi^i_\alpha(x)$ as in the proof of [57] Lemma 3.8, only starting with $G_i(x)$ at stage $\alpha = 0$. Since for any ϕ and k the rank $D(\Phi, \phi*, k)$ is translation-invariant, and the rank of a finite union is the maximum of the ranks, for fixed k_0, the number $n_0 := D(G_i, \phi*, k_0)$ independent of i. By the same argument, inductively, $D(\Psi^i_\alpha(x)) = D(\Psi^i_\alpha(x))$. This allows the proof of genericity of the type p constructed for G_i to go through.

Note that the immediate stabilizer of p will lie in G_i^2, and the stabilizer subgroup of p will be contained in G_i^4. This is always the same group $C = G_0^{00}$, the smallest $M_0 - \bigwedge$-definable subgroup of G of bounded index. For two generic types p, q within C, we have $St_0(p,q) \neq \emptyset$, $St(p,q) \subset C$, and $St(p,q)$ is a coset of C, so $St(p,q) = C$. Hence for any generic types p, q, r in C we can find $a, b, c \in C$ with $a \models p$, $b \models q$, $c \models r$ and and $ab = c$.

At this point an elementary but slighty less perspicacious argument shows that $pp^{-1}pp^{-1} = C$ and then using compactness, that any the image of any G_i modulo G^{00} has open interior, giving laminarity. However we invoke instead the picture given by Theorem 5.20. If laminarity fails, then according to this theorem, after perhaps replacing G by the group $\widetilde{\Lambda}$ generated by some definable set commensurable to G_i, there exists a continuous quasimorphism $\alpha : G \to \mathbb{R} : [-1, 1]$ with unbounded image. Continuity implies that $\alpha(x)$ depends only on $tp(x)$. Choosing p, q, r with $\alpha(r) > \alpha(p) + \alpha(q) + 1$, and elements a, b, c as above, we see that the inequality contradicts the quasimorphism property.

$\square$

6. APPROXIMATE LATTICES IN SEMISIMPLE GROUPS

Let G be a locally compact group. Recall the definition of an approximate lattice from the introduction, and Appendix A.

The definitions below parallel those in [5]; we allow ourselves to define *Meyer sets* for all lattices (in our sense), so that a Meyer set in the sense of [5] is what we call a *uniform Meyer set*. (The requirement that M, S be commensurable is prima facie stronger than that in [5], but they are equivalent, see Lemma A.4.)

When we have a product $G \times H$ of two groups, we will usually denote the projections by π_1, π_2.

Definition 6.1 ([5])**.** Let G be a locally compact group.

- A subset $S \subset G$ is a *(uniform) model set* if there exists a locally compact group H, a (co-compact) lattice $\Gamma \subset G \times H$, and a compact $C \subset H$ with nonempty interior, such that Γ has dense image under

the projection π_2, π_1 is injective on Γ, and

$$S = \pi_1(\Gamma \cap (G \times C))$$

S is also called a *cut-and-project set* for Γ.

- A *(uniform) Meyer set* is a subset M of a (uniform) model set S, commensurable to S.

This generalizes a construction of lattices, fundamental to the definition of an arithmetic lattice where C is taken to be a compact *subgroup*; see [46] p.1. Here it is replaced, in effect, by an approximate subgroup.

The injectivity and density assumptions in Definition 6.1 are inessential: we can replace H by the closure H' of the image of $\pi_2(\Lambda)$, then factor out the closed normal subgroup $\pi_2(\ker(\pi_1))$ from H', without changing the cut-and-project sets.

Finding a cut-and-project model $\Gamma \leq G \times H$ for an approximate sublattice Λ is equivalent to the existence of homomorphism $f : \widetilde{\Lambda} \to H$, viewed as a partial map $G \to H$, with the interesting property of interchanging compact sets in the image with discrete sets in the domain; the relation between them is that Γ is the graph of f.

The identity map on the ring $\mathbb{Z}[1/2]$, as a partial map from the 2-adic completion to the real one, or vice versa, is a suggestive example of this phenomenon.

Problem 6.2 (Björklund and Hartnick, [5] p. 2918). In what locally compact second countable groups G is every cocompact approximate lattice a Meyer set?

This is equivalent to the apparently weaker statement that every approximate lattice Λ be commensurable to a cut-and-project set:

Lemma 6.3. *Let G, H be locally compact groups, Γ a lattice in $G \times H$ projecting injectively to G, Λ an approximate subgroup of G commensurable to a cut-and-project set of Γ. Then there exists a locally compact H' and lattice $\Gamma' \leq G \times H'$, such that Λ is a Meyer set of Γ', i.e. commensurable to and contained in a cut-and-project set of Γ'.*

Proof. Let $\widetilde{\Lambda}$ be the projection of Γ to G, and $f : \widetilde{\Lambda} \to H$ the homomorphism whose graph is Γ. Let ω be the commensurator of Λ. By Lemma 5.4, there exists $f' : \omega \to H'$ such that $\chi(f') = \chi(f) = \omega$. Let $\Gamma' \leq G \times H'$ be the graph of f'. By Lemma 5.1, $\Lambda \subset \mathrm{Comm}\,(\omega)$. Since $\Lambda \sim \chi(f')$, there exists a compact $C \subset H'$ with $\Lambda \subset (f')^{-1}(C)$. Thus Λ is a Meyer set of Γ' $\qquad\qquad\square$

Note that this may require a change of target H; even if G, H are connected Lie groups, it may be necessary to augment H with totally

disconnected factors. This will be clear in the arithmetic setting below, where p-adic factors appear.

Existing results fall into the amenable class; see [44]. We will complement this with a positive answer, without the restriction to cocompact approximate lattices, for almost simple groups over local fields, and their finite products.

6.4. Abstractly semisimple groups.

Definition 6.5. A locally compact group G is *abstractly semisimple* if

(1) G has no nontrivial normal abelian subroups.
(2) Outside the center, G has no discrete conjugacy classes.

Remarks 6.6. (1) Abstract semisimplicity remains true upon passing to a finite index normal subgroup H of G.
(2) Abstract semisimplicity is preserved by finite products.
(3) The class is also closed under restricted infinite products $\Pi'_i G_i$ with respect to compact subgroups C_i (see Definition C.1 (14.)), provided each C_i has trivial centralizer in G_i.
(4) A semisimple algebraic group over a local field is abstractly semisimple, at least upon moving to a finite index subgroup, and factoring out a finite center. These operations will not alter the validity of the results we will prove.
(5) Let G be an abstractly semisimple locally compact group. Then for some compact normal subgroup N, G/N has a finite index subgroup of the form $G_1 \times G_2$ with G_1 a totally disconnected abstractly semisimple locally compact group, and G_2 a connected semisimple Lie group.

Proof. (1) If X is a discrete conjugacy class of H, then the union of G-conjugates of X is a conjugacy class of G and is still discrete (the union being finite), so trivial . If A is a normal abelian subgroup of H, let $A_1, \ldots, A_n$ be the conjugate subgroups. Then $\cap_{i=1}^n A_i$ is G-normal and abelian, so trivial. We show by reverse induction on $k = n, \cdots, 1$ that the intersections of any k of the groups A_i is trivial. Assume this true for $k+1$; we prove it for k. If A, B are each the intersection of k of the A_i and are distinct, then $A \cap B = (1)$; so the various k-intersections commute and so generate a commutative G-normal subgroup, hence trivial. For $k = 1$ this gives $A = (1)$.

(2) The statement on discrete conjugacy classes is clear, since a conjugacy class in the product is a product of conjugacy classes. If A is a normal abelian subgroup of G, the projection of A to each factor is normal and abelian so trivial, hence A is trivial.

(3) The proof that there are no normal abelian subgroups is the same as in (2). If X is a discrete conjugacy class of G, pick $x \in X$; then the centralizer $C_G(x)$ is an open subgroup of G; thus for some finite $I_0 \subset I$, $1_{I_0} \times \Pi_{i \in I \smallsetminus I_0} C_i$ is a subset of $C_G(x)$. From this it follows that $x \in G_0 := \Pi_{i \in I_0} G_i \times 1_{I \smallsetminus I_0}$. So the projection $\pi_0 X$ of X to G_0 must already be discrete. By (2), $\pi_0 X = (1)$. Since $x \in G_0$ we have $x = 1$.

(4) By factoring out the center and moving to a finite index subgroup, we may take G to be a finite product of simple groups. Then it is clear that G has no nontrivial normal abelian subgroups, nor open ones. If a^G is a discrete conjugacy class, then $C_G(a)$ is open, and using the chain condition on centralizers, $C_G(a^G) = C_G(F)$ for some finite set F of conjugates of a; so $C_G(a^G)$ is an open normal subgroup; hence $C_G(a^G) = G$ so a is central.

(5) If G is connected, this is Yamabe's theorem (with $G_1 = 1$). Moreover in this case, G has a unique maximal compact normal subgroup. In general, factoring out the maximal compact N of the connected component, G^0 becomes a semisimple Lie group. It has have only finitely many outer automorphisms, using the structure theory for semimple Lie groups, and Borel-Tits; thus up to finite index, G/N decomposes as a product of G^0 with the totally disconnected group $C_G(G^0)$.

□

As a consequence of (3), if G is a semisimple algebraic group over $\mathbb{Q}$, then $G(\mathbb{A}_{\mathbb{Q}})$ has a co-compact subgroup that is abstractly semisimple modulo a compact center. For instance $SL_2(\mathbb{A}_{\mathbb{Q}})/Z$ is abstractly semisimple, where Z is the center $\{\pm 1\}^{\{\infty,3,5,7,11\cdots\}}$.

Let G be an abstractly semisimple group, and let Λ be a discrete approximate subgroup. Theorem 5.15 explains Λ in terms of a quasi-homomorphism into a Lie group; we would like to show that an ordinary homomorphism will already work. In case the error set is central, this is already carried out in Proposition 5.30, assuming in a strong sense that Γ is centerless: the centrality of the error set is played against the semisimplicity of G, showing we can factor out the centre of L without harm. The general case, where the error set is contained in a rigid normal subgroup A, is more complicated; A, when factored out, may require taking with it a bigger normal subgroup, that needs to be shown to be solvable. We postpone the technical details to Proposition 6.12 at the end of the section, and prove the theorem assuming it.

Definition 6.7. Let G be a topological group, and ω a commensurability class of approximate subgroups. ω is *strictly dense* if it does not belong[6] to a closed subgroup of infinite index in G.

Note that lattices are never strictly dense; see also the Example below. In view of this we will also say that Λ is a *strictly approximate lattice* if the commensurability class of Λ is strictly dense.

In case $\Lambda \in \omega$ and $\langle \Lambda \rangle \in \langle \omega \rangle_{min}$ in the sense of Proposition 5.8, Λ is a fortiori strictly dense in G, of course.

Example 6.8. If $G = G_1 \times G_2$, Λ_1 an approximate lattice in G_1, Λ_2 a lattice in G_2, then $\Lambda_1 \times \Lambda_2$ is an approximate lattice in G, that is not dense; it belongs to the closed subgroup $G_1 \times \Lambda_2$.

We prove the first laminarity result assuming strict density. Later, in the classical setting at least, we will recognize Example 6.8 as the only obstruction to strict density, so that the theorem applies to all approximate lattices.

Theorem 6.9. *Let G be an abstractly semisimple locally compact group. Let Λ a discrete approximate subgroup of G, with strictly dense commensurability class. Then there exist a connected semisimple real Lie group L and a discrete subgroup $\Gamma \leq G \times \mathsf{L}$ with cut-and-project sets commensurably covering Λ.*
Supplements:
- L *can be taken to be centerless and to have no nontrivial compact normal subgroups.*
- *We can take the projection $\Gamma \to \mathsf{L}$ to be injective with dense image, and $\Gamma \to G$ to be injective.*
- *If Λ is an approximate lattice in G, then Γ is a lattice in $G \times \mathsf{L}$, and the cut-and-project sets of Γ are commensurable to Λ.*

Proof. By Theorem 5.15, there exists a subgroup $\widetilde{\Lambda} \leq \langle \Lambda \rangle$, a Lie group L with $L/Z(L)$ connected, a normal compact subset K of a rigid abelian subgroup $A \leq L$, and a quasi-homomorphism

$$f : \widetilde{\Lambda} \to L : K$$

with $f(\widetilde{\Lambda})A$ dense in L, such that $f^{-1}(C)$ commensurably covers Λ for some compact $C \subset L$ with nonempty interior, and in fact is commensurable with Λ. We use the commensurability here to conclude that $f^{-1}(C)$ is discrete in G; so f is discrete-on-compact.

By assumption, $cl(\widetilde{\Lambda})$ has finite index in G; so it is still abstractly semisimple (Remark 6.6 (1)). Thus we may assume $cl(\widetilde{\Lambda}) = G$; and

---

[6]as defined in the beginning of §5

(i),(ii) of Proposition 6.12 hold (with $H = \widetilde{\Lambda}$.) By Proposition 6.12, L has a normal subgroup R such that L/R is semisimple, centerless, without nontrivial normal compact subgroups; and the induced map $\bar{f} : \widetilde{\Lambda} \to L/R$ is injective and discrete-on-compact. Since $A \subset R$, $\bar{f}$ is a homomorphism, The cut-and-project sets of L/R can only be bigger than those of L. Let $\mathsf{L} = L/R$, and let Γ be the graph of the composed map $\bar{f} : \widetilde{\Lambda} \to L \to L/R = \mathsf{L}$. Since $\bar{f}$ is discrete-on-compact, Γ is discrete. The projection $\Gamma \to G$ is injective by definition, and we saw that $\Gamma \to L$ and hence $\Gamma \to \mathsf{L}$ have dense image. That the kernel of $\Gamma \to \mathsf{L}$ is trivial follows from injectivity of $\bar{f}$; the kernel on the left is trivial since $\bar{f}$ is a function.

Assume now that Λ is an approximate lattice. Let X be a cut-and-project set of Γ, commensurably covering Λ. As Γ is discrete, it follows that X intersects each compact of G in a finite set, so that X is discrete. Now X^4 is also contained in a cut-and-project set of Γ, so by the same argument X^4 is discrete. The commensurability of Λ and X now follows from Lemma A.4. The fact that Γ is a lattice, cocompact if Λ is, comes from Lemma A.8.

$\square$

Remark 6.10. Assume G has the centralizer chain condition (3) (as is automatic in a linear group). Then in Theorem 6.9, the strict density requirement can be replaced by the weaker conditions (1),(2). In particular, it suffices to assume that $< \Lambda >$ is dense, and that G has no closed normal subgroup N with G/N discrete.

(1) Λ is weakly Zariski dense. This refers to the topology on G whose basic closed sets are defined by equations $ax = xb$ or $bx^{-1}ax = x^{-1}axb$, with $a, b \in G$. Notably normalizers $N_G(C_G(X))$ of centralizer groups $C_G(X)$ are wz-closed.

(2) (Weak irreducibility) There is no closed subgroup H of G such that $\Lambda \subset N_G(H)$ and Λ/H is commensurably covered by an infinite discrete subgroup of $N_G(H)/H$.

(3) Any centralizer group $C_G(X)$ is the centralizer group $C_G(X_0)$ of a finite subset X_0 of X. (Or at least, if H is a closed subgroup and $C_G(x) \cap H$ is open in H, then the interesection of conjugates of $C_G(x) \cap H$ in H is still open in H.)

Proof. Let $\widetilde{\Lambda}$ be a subgroup as in the beginning of the proof of Theorem 6.9. It suffices to check that $cl(\widetilde{\Lambda})$ is abstractly semisimple (and then proceed with the proof of Theorem 6.9.) If A is an abelian subgroup of G normalized by $cl(\widetilde{\Lambda})$, then the centralizer group $C_G(C_G(A))$ is also normalized, hence by weak Zariski density it is normal in G,

and thus by abstract semi-simplicity of G it is trivial; so $A = 1$ also. Now let $a \in cl(\widetilde{\Lambda})$ and suppose the conjugacy class of a in $cl(\widetilde{\Lambda})$ is discrete. Then $cl(\widetilde{\Lambda}) \cap C_G(a)$ is an open subgroup of $cl(\widetilde{\Lambda})$. By (3), $cl(\widetilde{\Lambda}) \cap C_G(a^{cl(\widetilde{\Lambda})})$ is still an open subgroup H of $cl(\widetilde{\Lambda})$. Thus $cl(\widetilde{\Lambda})/H$ is discrete. As Λ is commensurably covered by $cl(\widetilde{\Lambda})$, by (2), it follows that $cl(\widetilde{\Lambda})/H$ is finite, i.e. a finite index subgroup of $cl(\widetilde{\Lambda})$ centralizes a. By weak Zariski density, a finite index subgroup of G centralizes a. So a^G is finite and hence trivial, and $a = 1$. $\qquad\square$

Corollary 6.11. *Let G be a finite product of non-compact, center-less, simple algebraic groups over local fields. Let Λ be an approximate lattice of G. Then there exists a decomposition $G = G_1 \times G_2$, Λ is commensurable to $\Lambda_1 \times \Lambda_2'$ where $\Lambda_1 \leq \langle \Lambda \rangle$ is an approximate lattice of G_1 and Λ_2' is a lattice of G_2, and there exists a semisimple real Lie group L and an approximate lattice $\Gamma \leq G_1 \times \mathsf{L}$ with cut-and-project sets commensurable to Λ_1.*

Supplements:
(1) *If Λ is a uniform approximate lattice, then Γ is a uniform lattice, as is Λ_2'.*
(2) *The projections $\Gamma \to G_1, \Gamma \to \mathsf{L}$ can be taken to be injective and a dense embedding, respectively.*
(3) *The projection of Γ to G_1 is a subgroup $\widetilde{\Lambda}_1$. One can choose Γ so that $\widetilde{\Lambda}_1$ is minimal, in the sense that no subgroup of infinite index commensurably contains Λ_1; and L connected and without normal compact subgroups. In this case, L is uniquely determined, and $\Gamma, \widetilde{\Lambda}_1$ are determined up to commensurability.*
(4) *There exists no nontrivial discrete subset of G_1 normalized by Λ_1; in particular, no nontrivial subset normalized by Λ_1 and commensurably covered by Λ_1.*

Proof. Let $\widetilde{\Lambda}, \mathsf{L}$ and $f : \widetilde{\Lambda} \to \mathsf{L} : K$ be as given by Theorem 5.15, applied to $\Lambda \subset \langle \Lambda \rangle$. We may at this point replace Λ by the commensurable $\Lambda^2 \cap \widetilde{\Lambda}$, and so assume $\Lambda \leq \widetilde{\Lambda}$. If $cl(\widetilde{\Lambda})$ is itself abstractly semisimple, we proceed as in Theorem 6.9, obtaining L and a discrete $\Gamma \leq G \times \mathsf{L}$ with a cut-and-project set X commensurably covering Λ. We repeat the argument that Γ is a lattice (in $G \times \mathsf{L}$ and not only in $cl(\widetilde{\Lambda}) \times \mathsf{L}$): since Γ is discrete, it follows that X intersects each compact of G in a finite set, so that X is discrete. Now X^4 is contained in a cut-and-project set of Γ, so by the same argument X^4 is discrete. The commensurability of Λ and X now follows from Lemma A.4. The fact that Γ is a lattice, cocompact if Λ is, comes from Lemma A.8. The statements on

injectivity and density are as in Theorem 6.9. Supplement (3) follows
from Proposition A.9.

We thus need to handle the case where $cl(\widetilde{\Lambda})$ is not abstractly
semisimple. In any case, Λ and thus $cl(\widetilde{\Lambda})$ are Zariski dense by
Proposition A.12. Let A be an abelian subgroup of G normalized
by Λ. To show A is trivial it suffices to show the triviality of every
projection to one of the simple algebraic factors of G. Let $\pi_i : G \to G_i$
be such a projection; then $\pi(A)$ is abelian and normalized by $\pi(\Lambda)$;
hence the double centralizer of $\pi(A)$, a Zariski closed group, is
normalized by $\pi(\Lambda)$; so by Zariski density of the latter, it is trivial.

In a similar way one can show that a centralizer subgroup of G
normalized by Λ is normal. Indeed let $C = C_G(Y)$; both C and $N_G(C)$
are defined by coordinatewise conditions, so that if $\pi_i(C)$ is normal
in G_i for each i, then C is normal in G. Now $\pi_i(C)$ is a centralizer
subgroup of G_i, hence Zariski closed; and thus so is $N_{G_i}(\pi_i(C))$; since
the latter contains the Zariski dense set $\pi_i(\Lambda)$, it must equal G_i.

The remaining issue is condition (ii) of abstract semisimplicity, for-
bidding discrete conjugacy classes. Let D be the set of all elements
$a \in cl(\widetilde{\Lambda})$ with $a^{cl(\widetilde{\Lambda})}$ discrete in G; and let $G_1 = C_G(D)$ be the cen-
tralizer. G_1 is clearly normalized by $cl(\widetilde{\Lambda})$ and hence by Λ, so by the
above, it is a closed normal subgroup of G.

Let $G_2 = C_G(G_1)$; as G is a product of simple groups, it is clear
that G_1 is a product of some of them, and G_2 of the others. We have
$G \cong G_1 \times G_2$ naturally; $G_2 = C_G(G_1)$ contains D by definition.

By linearity of the factors of the product group G, $G_1 = \cap_{a \in D} C_G(a)$
is a finite intersection, i.e. equals $C_G(D_0)$ for some finite $D_0 \subset D$.
For $a \in D_0$, let U be a neighborhood of a with $U \cap a^{cl(\widetilde{\Lambda})} = \{a\}$;
then $C_G(a) \cap cl(\widetilde{\Lambda}) = \{g \in cl(\widetilde{\Lambda}) : a^g \in U\}$, so it forms an open
subgroup of $cl(\widetilde{\Lambda})$; and hence $G_1 \cap cl(\widetilde{\Lambda})$ is open in $cl(\widetilde{\Lambda})$. It follows
that $cl(\widetilde{\Lambda})/(G_1 \cap cl(\widetilde{\Lambda})) \cong cl(\widetilde{\Lambda})G_1/G_1$ is discrete as a topological space,
and so also as a subgroup of $G/G_1 \cong G_2$. The image $\pi_2(cl(\widetilde{\Lambda})) = \pi_2(\widetilde{\Lambda})$
is thus a discrete, closed subgroup of G_2. A fortiori, $\pi_2(\Lambda)$ is discrete
in G_2. By Proposition A.14, $\Lambda_1 = \Lambda^2 \cap G_1$ is an approximate lattice
in G_1; $\Lambda_2 = \Lambda^2 \cap G_2$ is an approximate lattice in G_2; and $\Lambda_1 \times \Lambda_2$ is
commensurable with Λ. Since $\pi_2(\widetilde{\Lambda})$ is a discrete subgroup containing
Λ_2, it is commensurable with Λ_2 by Lemma A.4; and so $\Lambda_1 \times \pi_2(\widetilde{\Lambda})$ is
also commensurable with Λ; the second factor $\pi_2(\widetilde{\Lambda})$ is a lattice in G_2.
Let $\Lambda_2' = \pi_2(\widetilde{\Lambda})$.

Now G_1 has no discrete conjugacy classes and so is abstractly
semisimple.

$f(\Lambda_2)A$ is a normal subgroup, and contained in a compact set (since $f(\Lambda)$ is). Since $\Lambda_1 \times \Lambda_2$ is commensurable with Λ, $cl(f(\Lambda_1)A) = cl(f((\Lambda_1 \times \Lambda_2)A))$ is co-compact in L. Under the conditions of Theorem 6.9, that we have for L, this implies $cl(f(\Lambda_1)A) = \mathsf{L}$.

We are now in the situation of the first paragraph with respect to G_1. In particular, as shown there, (3) holds for G_1. Using induction on the number of simple factors of G, we obtain the main theorem and supplements (1-3).

It suffices now to prove supplement (4) under the additional assumption (*) that no subgroup of infinite index in $\widetilde{\Lambda}$ commensurably contains Λ. Suppose S is a discrete subset of G_1, normalized by Λ_1, and $1 \neq a \in S$. Let $\Lambda' = \Lambda_1 \times \Lambda_2$. Since Λ_1 normalizes S, as does Λ_2, so does Λ', and hence the group $\widetilde{\Lambda}'$ generated by Λ' normalizes S. Now S is closed in G, so $cl(\widetilde{\Lambda}')$ normalizes S too. By (*), $\widetilde{\Lambda}'$ has finite index in $\widetilde{\Lambda}$. Hence $cl(\widetilde{\Lambda}')$ has finite index in $cl(\widetilde{\Lambda})$. Thus $a^{cl(\widetilde{\Lambda})}$. In the notation within the proof, we have $a \in D$ and hence G_1 commutes with a. But $a \in G_1$ so a also commutes with G_2, and hence is central in $G_1 \times G_2 = G$, contradicting our assumptions. This proves (4) under (*), and hence as noted above in general.

$\square$

This solves Problem 1 of [5] for semisimple algebraic groups over local fields, and their finite products.

*

We are still in debt of a technical lemma. When applied to lattices in (real) Lie groups, it is a classical statement going back to Zassenhaus and Auslander on the interaction of a lattice with the solvable radical. For us the main import will be an ability to convert approximate homomorphisms with error in a basic abelian group, to actual homomorphisms into a quotient Lie group.

Proposition 6.12. *Let L be a Lie group, with $L/Z(L)$ connected, containing a normal closed abelian subgroup $A \trianglelefteq L$, and a compact normal symmetric $K \subset A$. Assume $A = VD$, where $V \cong \mathbb{R}^n$ is closed and normal in L, and D is closed and central in L. Further assume that either V is rigid, or $K = (1)$.*

Let G be a second countable locally compact group, with a subgroup H satisfying:

(i) G has no nontrivial abelian subgroups normalized by H.
(ii) H has no conjugacy classes that are discrete in G, other than (1).

Let $f : H \to L : K$ be a discrete-on-compact quasi-homomorphism, with $f(H)A$ dense in L.

Then f induces an injective, discrete-on-compact homomorphism $\bar{f} : H \to L/R$, where R is a closed normal solvable subgroup of L, and L/R is semisimple and centerless, and without nontrivial normal compact subgroups.

Proof. We first show:

Claim 1. $f^{-1}(A) = 1$.

Proof. If A is rigid, let $g \in f^{-1}(A)$, and let g^H be the set of H-conjugates of g. Then $f(g^H) \subset f(g)^L K^3$; since A is rigid, $f(g)^L$ and hence also $f(g)^L K^3$ are precompact. So $g^H \subset f^{-1}f(g^H)$ is discrete (in particular closed) in G. By (ii) we have $g = 1$, showing that $f^{-1}(A) = 1$.

Now assume instead of rigidity that $K = 1$. Then f is a homomorphism. Hence $f^{-1}(1)$ is a normal, discrete subset of H, and it follows directly from (ii) that $f^{-1}(1) = 1$. Thus f is an injective homomorphism; so $f^{-1}(A)$ is an abelian subgroup of G. It is normalized by H, hence trivial by (i).

$\square$

It is only in the above proof of Claim 1 that we use the rigidity of all of A, rather than just K.

Claim 2.   There exists a closed, normal, solvable subgroup I of L containing A such that the induced homomorphism $f_1 : H \to L/I$ is discrete-on-compact.

Proof. Any Lie group has a compact neighborhood U of 1, satisfying (*) $[U, U] \subset U$; if $\mathfrak{u}$ is any compact neighborhood of 0 in the Lie algebra, then for small enough $t > 0$, $exp(t\mathfrak{u})$ will do. Moreover, we can arrange topological nilpotence (**): letting $U_1 = U, U_{n+1} = [U, U_n]$, we have $\cap_n U_n = (1)$.

It follows that every locally compact group G has a neighborhood satisfying (*): by Yamabe's theorem, G has an open subgroup S containing a compact normal subgroup N, with S/N a Lie group; pull back from S/N any neighborhood with this property.

Fix a a compact neighborhood W of 1 in G with $[W, W] \subset W$, and a compact neighborhood U of 1 in L with (*) and (**). Writing $\bar{U}_n$ for the image of U_n modulo A, we can also ensure that $\cap_n \bar{U}_n = 1$.

By assumption, $A = VD$ is a product of a vector group V, normal in L, with a closed group D central in L.

Fix a Euclidean inner product on $V \cong \mathbb{R}^n$; let $B_r(V)$ be the ball around 0 of radius r; normalize it so that $K^3 \subset B_1(V) \times D$. Also a

compact $B_1(D) \subset D$, such that $K^3 \subset B_1(V) \times B_1(D)$. Let $B_r(D) = B_1(D)^{\lfloor r \rfloor}$, and let $A_r = B_r(V)B_r(D)$.

The inner product induces an operator norm on $End(V)$. For $g, x \in L$, we will write x^g for gxg^{-1}. Let $ad : L \to End(A)^{op}$ be the endomorphism mapping g to the endomorphism $x \mapsto x^g$ of A. Write $||x||$ for the operator norm of $ad(x) - 1$ when restricted to V. This is a continuous function on L.

We can shrink U so as to satisfy:

(2) $||g|| < 1/10$ for $g \in U^2$.

In fact, $ad([x, y]) = [ad(x), ad(y)]$ so it suffices to pull back a neighborhood Ω of 1 in $GL_n(V)$ with the analogous property, and such that $[\Omega, \Omega] \subseteq \Omega$ (so as not to lose this property for U.)

Claim 2.1.   Let $n \geq 1, r \geq 2$. We have

$$[UA_r, U_nA_r] \subset U_{n+1}A_{r/4}.$$

Proof. Consider elements $\gamma_i = u_i a_i$, $u_i \in U, a_i \in A$. Assume $u_1 \in U_1, u_2 \in U$ and $a_i \in A_r$ $r \geq 2$.

We compute

$$[u_1 a_1, u_2 a_2] = [u_1, u_2]([u_2^{-1}, a_1][a_1, a_2][a_2, u_1^{-1}])^{u_2 u_1}$$

since $[a_1, a_2] = 1$,

$$[\gamma_1, \gamma_2] = [u_1, u_2]ad_{u_2 u_1}(^{u_2^{-1}-1}a_1 +^{1-u_1^{-1}} a_2)$$

where multiplication inside the abelian group A is written additively. If $u_1, u_2 \in U_n$ then $[u_1, u_2] \in U_{n+1}$. And if $a_1, a_2 \in B_r(V) + D$, then $^{u_2^{-1}-1}a_1$ and $^{1-u_1^{-1}}a_2$ lie in $A_{r/10}$ by (2), so $(^{u_2^{-1}-1}a_1 +^{1-u_1^{-1}} a_2) \in A_{r/5}$, and again by (2), $ad(u_2 u_1)(^{u_2^{-1}-1}a_1 +^{1-u_1^{-1}} a_2) \in A_{r/4}$ □

(Actually since D is central we have the stronger statement $[UB_rD, U_nB_rD] \subset U_{n+1}B_{r/4}$.)

Claim 2.2.   For $r \geq 2$, the set $\Delta_r = UA_r \cap f(W \cap H)$ is finite.

Proof. Since f is discrete-on-compact, and UA_r is compact, $f^{-1}(UA_r)$ is discrete, so $W \cap f^{-1}(UA_r)$ is finite. Applying f, we see that $f(W \cap H) \cap UA_r$ is finite.   □

Claim 2.3.   For any $r \geq 2$, Δ_r generates a nilpotent group modulo A.

Proof. Let $v_1, v_2 \in \Delta_r$. So for some $u_1, u_2 \in U, a_1, a_2 \in A_r, w_i \in W \cap H$, we have $v_i = u_i a_i = f(w_i)$. By the proof of Claim 2.1, $[v_1, v_2] = [u_1 a_1, u_2 a_2] = [u_1, u_2]a$ with $a \in A_{r/4}$. On the other hand, $f([w_1, w_2]) = [f(w_1), f(w_2)]k$ for some $k \in K^3 \subset A_1$; so $[v_1, v_2] = [f(w_1), f(w_2)] =$

$f([w_1, w_2)])k^{-1}$. Hence $[v_1, v_2]k \in f(W \cap H) \cap UA_{r/4+1}$. Since $r \geq 2$ this shows that $[v_1, v_2]k \in \Delta_r$.

Let $\mathbf{\Delta_r} = \Delta_r/A$ be the image of the finite set Δ_r in L/A. We have just shown that $\mathbf{\Delta_r}$ is closed under the commutator. Let $F_1 = \mathbf{\Delta_r}$ and $F_{n+1} = [F_1, F_n]$. Then $F_1 \supset F_2 \supset \cdots$ and the sequence must stabilize at some F_k with $[F_1, F_k] = F_k$. Since $F_k \subset \bar{U}_1$, and $[\bar{U}_1, \bar{U}_n] \subset \bar{U}_{n+1}$, it follows inductively that $F_k \subset \bar{U}_n$ for each n. Thus $F_k \subset \cap_n \bar{U}_n = (1)$. This proves that $\mathbf{\Delta_r}$ generates a nilpotent group. $\qquad\square$

Write Γ for the graph of f, and (with slight abuse of notation)

$$\Gamma A := \Gamma(1 \times A) = \{(w, f(w)a) : w \in H, a \in A\}$$

While Γ may not be a subgroup of $G \times L$, since f is an A-quasi homomorphism, ΓA is a subgroup of $G \times L$; hence so is the closure $cl(\Gamma A)$ Let C be the identity component:

$$C := cl(\Gamma A)^0$$

As A is normal, C is normalized by Γ. Let $\pi_2 : G \times L \to L$ be the second projection.

By a theorem of Zassenhaus (see [17]), all soluble subgroups S of the linear group $L/Z(L)$ are soluble of some fixed degree; hence the same holds for L. Call this bound $\mathbf{b} = \mathbf{b}(L)$. [7]

Claim 2.4. $\Gamma A \cap (W \times U)$ generates a $\mathbf{b}$-step soluble[8] subgroup of $G \times L$.

Proof. An element of ΓA has the form $(w, f(w)a)$ with $w \in H$. Thus an element of $\Gamma A \cap (W \times U)$ has the form $(w, f(w)a)$ with $w \in H \cap W$ and $f(w)a \in U$. Letting π_2 denote the second projection $G \times L \to L$, we have $\pi_2(\Gamma A \cap (W \times U)) \subset (U \cap f(W \cap H)A)$. For a finite $F \subset \Gamma A \cap (W \times U)$, $\pi_2(F) \subset U \cap f(W \cap H)A_r$ for some $r \geq 2$. Now

$$\pi_2(F)/A \subset (U \cap f(W \cap H)A_r)/A = (UA_r \cap f(W \cap H))/A = \mathbf{\Delta_r}.$$

So $\pi_2(F)/A$ generates a nilpotent group, by Claim 2.3. Hence $\pi_2(F)$ generates a soluble group. By definition of $\mathbf{b}$, it is $\mathbf{b}$-step soluble.

Let z be an element of the $\mathbf{b}$-th derived group of $\langle F \rangle$. Write $z = (x, f(x)a)$ with $x \in H$. Then $\pi_2(z) = f(x)a \in A$ so $f(x) \in A$ and hence $x = 1$ by Claim 1. Thus $\langle F \rangle$ is $\mathbf{b}$-step soluble.

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[7]The theorem is easy to prove for characteristic zero linear groups, such as $L/Z(L)$: by passing to Zariski closure it suffices to consider Zariski closed subgroups H; the solvable degree of the connected part H^0 is clearly bounded by the dimension; while Jordan's theorem presents the quotient H/H^0 as a normal abelian subgroup by a bounded subgroup.

[8]In fact we can show here, using connectedness of C^0, that $\pi_2(C)/A$ is nilpotent.

Since this holds for all finite $F \subset \Gamma A \cap (W \times U)$, the group $\langle \Gamma A \cap (W \times U) \rangle$ is **b**-solvable. $\qquad\square$

It follows that $cl(\Gamma A \cap (W \times U))$ still generates a **b**-solvable subgroup S of $G \times L$. Now $cl(\Gamma A) \cap (W \times U) = cl(\Gamma A \cap (W \times U))$ so S is generated by an open subset of $cl(\Gamma A)$; hence S is an open subgroup of $cl(\Gamma A)$. In particular, $C = cl(\Gamma A)^0 = S^0 \leq S$ and so C is solvable.

In particular the first projection $\pi_1(C)$ is soluble. Since C is normalized by ΓA, $\pi_1(C)$ is normalized by H. By assumption (i), $\pi_1(C) = (1)$. Thus $C = (1) \times I$ for some I, which is clearly a closed subgroup of L. I is normalized by $\pi_2(\Gamma)Z(L)$, which is dense in L by assumption. So I is normal in L; and I is soluble.

<p style="text-align:center">*</p>

It remains only to prove that the induced homomorphism $f_1 : H \to L/I$ is discrete-on-compact. By Claim 1, $f^{-1}(I)$ is soluble, and is normalized by H; hence by assumption (i), $f^{-1}(I) = (1)$ and so f_1 is injective. So we are in position to use Lemma A.6 (3); we thus need to show, for one compact neighborhood $W' \subset G$, that $f_1(W' \cap H)$ is discrete in L/I.

Let us write also ΓI for $\Gamma(1 \times I)$. Since $cl(\Gamma A)$ contains $cl(\Gamma)$ and $(1 \times I)$, and is closed; and on the other hand $A \subset I$, we have $cl(\Gamma A) = cl(\Gamma I)$.

We saw that $cl(\Gamma A)^0 = 1 \times I$. The group $cl(\Gamma A)/(1 \times I)$ is thus totally disconnected. So it has a profinite open subgroup, that we can write as $P'/(1 \times I)$, with P' an open subgroup of $cl(\Gamma A)$. The image of P' in L/I, namely $\pi_2(P')/I$, is also profinite, in particular totally disconnected; but by the Von-Neumann-Cartan theorem, being a closed subgroup of L/I, it is itself a Lie group. Thus $\pi_2(P')/I$ is finite; so $\pi_2^{-1}(I) \cap P'$ has finite index in P'; being closed, it is an open subgroup, hence open in $cl(\Gamma A)$ too. Let $P = \{a : (a, 1) \in P'\}$; then $P \times I$ is a clopen subgroup of $cl(\Gamma I)$.

Let us now show that $f_1(W')$ is discrete. Otherwise, some subsequence of distinct points of $f_1(W')$ has a limit point in L/I; and it follows that some sequence of distinct points from $(W')(W')^{-1}$ converges to 1 in L/I. Refining further, using compactness of W' and of $W'' = (W')(W')^{-1}$, we find a sequence (a_i) of elements of $H \cap W''$, converging to some $a \in W''$, such that $f_1(a_i) \to 1$. Let U' be a precompact open neighborhood of 1 in L; then $U'I/I$ is an open subset of L/I, containing $\bar{b}$; so $f(a_i) \in U'I$ for almost all i; $f(a_i) = u_i c_i$, with $c_i \in I$ and $u_i \in U'$. We have $(a_i, u_i c_i) \in \Gamma$, so $(a_i, u_i) \in \Gamma I$. Since U' is precompact, we may assume $u_i \to u \in cl(U')$. Then $(a, u) \in cl(\Gamma I)$.

Since $P \times I$ is open in $cl(\Gamma I)$, almost all pairs (a_i, u_i) lie in a single coset of $P \times I$. In particular, the u_i lie eventually in a single coset of I; this implies that modulo I they are finite in number. This contradicts the assumption of distinctness of the elements $f_1(a_i) = f(a_i)I = u_i I$, and ends the proof of Claim 2.

$\square$

$*$

We return to the proof of the Proposition. The composed map $H \to L \to L/A$ is a group homomorphism, injective by Claim 1. Since I/A is solvable, $f^{-1}(I)$ is solvable. By (i) we have: $f^{-1}(I) = 1$. Let $A_1 = I$ and let $f_1 : H \to L/A_1$ be the composition $H \to L \to L/A_1$. So f_1 is injective. By Claim 2, f_1 is discrete-on-compact.

We have now achieved the main goal, of replacing the quasi-homomorphism f by an actual homomorphism f_1, still discrete-on-compact. Moreover f_1 is injective. We still wish to change it further so that the target Lie group becomes semisimple and centerless. To this end we repeat the procedure inductively, constructing a sequence $A_1 \leq A_2 \leq \cdots$ of closed normal subgroups of L, such that f induces a discrete-on-compact, injective homomorphism $f_k : G \to L/A_k$.

Assume A_k, f_k have been defined and satisfy these conditions. We proceed as follows. In case L/A_k has a nontrivial center, let A_{k+1}/A_k be the center of L/A_k. Then the composed homomorphism $f_{k+1} : H \to L/A_{k+1}$ is discrete-on-compact by Claim (2), applied to f_k.

The kernel of f_{k+1} is an abelian normal subgroup of H, hence trivial by assumption (i).

Assume then that A_{k+1}/A_k has trivial center. If L/A_k has a nontrivial normal compact subgroup, choose A_{k+1} to be such a subgroup. Then $f_{k+1} : H \to L/A_{k+1}$ remains discrete-on-compact trivially, since a pullback of a compact from L/A_{k+1} to L/A_k is compact. The kernel, being the pullback of a compact, is discrete in G, and normal in H, hence must by trivial by (ii). (Note f_k is an injective homomorphism at this point, i.e. $K = 1$.)

Assume therefore that L/A_k has no such normal subgroups.

If L/A_k has a nontrivial normal closed connected abelian subgroup, let A'/A_k be such a subgroup. Then A'/A_k is a connected locally compact abelian group so it is isomorphic to $\mathbb{R}^n \times B$ with B a maximal compact subgroup. If B' is a G-conjugate of B, then BB' is also compact, so $BB' = B$. It follows that B is normal in G, so should have been chosen in the previous step, unless $B = 1$. Thus $B = 1$ so A'/A_k is a vector group. In this case the hypotheses of the Proposition apply to

G, H with $f_k : H \to L/A_k$ and A'/A_k in place of f and A. By Claim 2 (now with $K = 1$), there exists a closed, solvable normal subgroup A_{k+1} of L containing A', such that the induced map $f_{k+1} : H \to L/A_{k+1}$ is discrete-on-compact. Again the kernel of f_{k+1} is soluble and so trivial by assumption (i).

Assume L/A_k has no normal subgroups of the types considered so far. If L/A_k has a nontrivial normal closed abelian subgroup A', the connected component of A' is trivial. Since A' is a Lie group, it is discrete, and so central, returning to a previously considered case.

The remaining possibility is that L/A_k has no nontrivial closed normal subgroups that are either abelian or compact. When this occurs the construction terminates.

Note that $\dim(L/A_k) \geq \dim(L/A_{k+1})$; equality holds only in case A_{k+1}/A_k is discrete; this cannot recur twice in a row, since if A_{k+2}/A_{k+1} and A_{k+1}/A_k are both discrete then so is A_{k+2}/A_k, and it is central in L/A_k by connectedness, so A_{k+2} would have been chosen already at stage k. Thus $\dim(L/A_k)$ descends and strictly so at every other step, so the construction proceeds at most $2 \dim(L)$ times.

If A_k is the last stage, then $f_k : H \to L/A_k$ is a discrete-on-compact homomorphism to a centerless Lie group, with no normal abelian subgroups, hence semisimple, and with no compact normal subgroups. $\square$

When Γ is a discrete subgroup of a connected Lie group G, Claim 2 of Proposition 6.12 is the "Zassenhaus Lemma" of [1]. Curiously, though Auslander explicitly invokes an effective version of (**), his proof appears to make no use of it; he uses rather the exponential shrinking within A due to (2). In the approximate situation, (2) cannot be used in this way since upon iterated approximate commutation, the A-component remains bounded but need not approach 0. We used instead, after some rearrangement, the convergence of the elements u_i towards 1.

7. ARITHMETICITY, RECOGNITION, EXTENSIONS

Let K be a global field. Let $\mathbb{A} = \mathbb{A}_K$ be the adele ring of K. If S is a finite set of places of K, we may write $\mathbb{A} = \mathbb{A}_S \times \mathbb{A}'_S$, where $\mathbb{A}_S = \Pi_{v \in S} K_v$, and $\mathbb{A}'_S$ is the restricted direct product of the $K_v, v \notin S$. Let $\pi_S : \mathbb{A} \to \mathbb{A}_S$ be the projection.

Definition 7.1 (Arithmetic approximate lattice). Let K be a global field, H a connected, almost simple algebraic K-subgroup of SL_n. Let C be a compact open neighborhood of 1 in $H(\mathbb{A}'_S)$, and

$$\Lambda = \pi_S(H(K) \cap (\mathbb{A}_S \times C)).$$

If $\psi : G \to H(\mathbb{A}_S)$ is a topological group isomorphism, we refer to $\psi^{-1}(\Lambda)$ as an *arithmetic approximate lattice*.

An *arithmetic Meyer set* (in $H(\mathbb{A}_S)$) is a subset of an arithmetic approximate lattice, commensurable to it.

Remark 7.2. If one demands that S contain all archimedean primes v where $H(K_v)$ is not compact, one may (without changing the commensurability class) choose C to be a subgroup; in that case one obtains the definition of an arithmetic lattice. ([46] p.1)

Remarks 7.3. (1) If interested only in the approximate subgroup Λ, one can take a finite set $S'' = S \sqcup S'$; let $\mathcal{O}_{K,S''}$ be the ring of S''-integers in K; let C be a compact neighborhood of 1 in $H(\mathbb{A}_{S'})$ and let $\Lambda = \pi_S(H(\mathcal{O}_{K,S''}) \cap (\mathbb{A}_S \times C))$. One can take S' to consist just of archimedean primes, if desired. Since K embeds into $\mathbb{A}_S$, and also into $\mathbb{A}_{S'}$, we have a map $f : H(\mathcal{O}_{K,S''}) \to H(\mathbb{A}'_S)$, showing incidentally that Λ is laminar.

(2) On the other hand, consider a pair $(\Lambda, \widetilde{\Lambda})$, where $\widetilde{\Lambda}$ is any subgroup of G commensurating Λ and containing Λ. If (G, Λ) is arithmetic, laminarity is also shown by a homomorphism $f : \widetilde{\Lambda} \to H(\mathbb{A}_K)$, with $\Lambda \in \chi(f)$, and $f(\widetilde{\Lambda}) \leq H(K)$. This is because $\widetilde{\Lambda} \leq H(K)$ (Lemma A.19), and by Definition 7.1, $\Lambda = j^{-1}(C)$ where $j : \widetilde{\Lambda} \to H(K)$ is the inclusion.

For this, if $\widetilde{\Lambda}$ is not finitely generated, a finite product would not suffice.

Note here that if C is chosen to contain $H(\mathcal{O}_{K_v})$ whenever v is non-archimedean, then $f(\widetilde{\Lambda})$ contains $H(\mathcal{O}_K)$.

Take the Weil restriction of scalars H' of H from $\mathcal{O}_K$ to $\mathbb{Z}$. [9] We obtain a homomorphism $h : \widetilde{\Lambda} \to H'(\mathbb{A}_\mathbb{Q})$ with $\Lambda \in \chi(f)$, and $H'(\mathbb{Z}) \leq h(\widetilde{\Lambda}) \leq H'(\mathbb{Q})$.

For simplicity we restrict to groups of adjoint type below. This allows us to use the simpler definition of 'arithmetic lattice' in [46], p.1; and it is convenient to have centerless groups. For classification up to commensurability this is not a real issue; If G is a group with finite center Z, $\pi : G \to G/Z$ the projection, then any $X \subset G$ is commensurable with $\pi^{-1}(\pi(X))$.

---

[9]$\mathcal{O}_K$ is isomorphic to $\mathbb{Z}^n$ as an abelian group, and so a choice of basis gives a polynomial interpretation of the ring $\mathcal{O}_K$ over the ring $\mathbb{Z}$, with universe $\mathbb{Z}^n$. Any affine scheme $W \subset \mathbb{A}^N$ over $\mathcal{O}_K$ becomes interpreted as $W^* \subset \mathbb{A}^{Nm}$ over $\mathbb{Z}$, in such a way that $W(\mathcal{O}_k)$ can be identified with $W^*(\mathbb{Z})$.

Theorem 7.4. *Let G be a finite product of noncompact, adjoint, simple algebraic groups over local fields. Then any approximate lattice Λ_0 in G is commensurable with a product Λ of lattices and arithmetic approximate lattices of direct factors of G.*

Supplements

- *We may choose $\Lambda \subset \langle \Lambda_0 \rangle$.*
- *In case no lattices appear in the product, Λ_0 is an arithmetic Meyer set.*

Proof. By Corollary 6.11, Λ_0 is commensurable to a cut-and-project set of a lattice $\Gamma \leq G \times \mathsf{L}$. Here L is a centerless semisimple Lie group, and can be viewed as the connected component (of finite index) in a semisimple adjoint algebraic group over $\mathbb{R}$.

Passing to a finite index sublattice of Γ if necessary, we can express Γ as a product of irreducible lattices $\Gamma_i \leq G_i \times \mathsf{L}_i$, where $G = \Pi_i G_i$ and $\mathsf{L} = \Pi_i \mathsf{L}_i$; in the sense of [46], p. 1. If U_i is a compact open neighborhood of 1 in L_i, the cut-and -project sets in G for Γ and $\Pi_i U_i$ is the product of the corresponding sets for Γ_i and U_i. Thus Λ_0 is commensurable to the cut-and-project set of Γ with respect to $\Pi_i U_i$. From this it is also clear that Λ_0 is commensurable to $\Pi_i(\pi_i(\Lambda_0))$; moreover evidently $\Gamma \subset \Pi_i(\pi_i(\Lambda_0))$. Thus proving the result for each Λ_i will suffice, so we may restrict attention to a single such factor; renaming, we take Γ to be an irreducible lattice in $G \times \mathsf{L}$.

In case L is trivial, Γ is a lattice in G, commensurable with Λ_0, and there is nothing further to show. Assume from now on that L (and G) are nontrivial. Then, $G \times \mathsf{L}$ has rank at least 2, and so Theorem 1 of [46], p. 2 applies: in the notation of Definition 7.1, we may identify G with $H(K_S)$, Γ with $H(K)$ as embedded diagonally in $H(K_S) \times H(\mathbb{A}'_S)$ and Λ with $\pi_1(\Gamma \cap (H(K_S) \times C))$, with C a compact open neighborhood of 1 in $H(\mathbb{A}'_S)$. This yields the arithmeticity of Γ.

It remains to show that Λ_0, is contained in a cut-and-project set Λ of Γ. We have in any case $\Lambda_0 \subset \Lambda F_0$ for some finite set F_0. Note that one can take $F_0 \subset \Lambda_0 \Lambda$. By Lemma 5.1, $F_0 \subset \Lambda_0 \Lambda \subset \mathrm{Comm}\,(\Lambda)$.

By Borel density (§A.12), $\pi_1(H(K) \cap (H(K_S) \times C))$ is Zariski dense in H. (It is here that we use the assumption that $G = H(K_S)$ has no compact factors.) By Lemma A.19, $\mathrm{Comm}\,(\Lambda) \leq H(K)$. So $F_0 \subset H(K)$. Now we may view F_0 as a subset of $H(\mathbb{A}'_S)$ as well; and it is clear that $F_0 C$ is compact and $\Lambda_0 \subset \pi(H(K) \cap (K_S \times F_0 C))$.

$\square$

Corollary 7.5. *Let G be a semisimple algebraic group over a local field of positive characteristic, or a finite product of such groups. Then every approximate lattice in G is commensurable with a lattice.*

Proof. By Theorem 7.4, any approximate lattice in G is commensurable with a product of lattices, and arithmetic approximate subgroups. In the arithmetic case, the field K will also be of positive characteristic (say by Borel-Tits), and so the complementary $H(\mathbb{A}'_S)$ will be totally disconnected. One can thus take the compact open neighborhood $C \subset H(\mathbb{A}'_S)$ to be a subgroup; and in this case, the cut-and-project set is a lattice. $\qquad\square$

7.6. **Recognition.** We will find that an abstractly semisimple locally compact group, possessing a strictly dense approximate lattice, is itself of adelic origin. The method of proof involves two uses of complementarity, relating two locally compact groups by bouncing off an approximate subgroup of a finite dimensional Lie group. Margulis arithmeticity is invoked, this time also via the commensurator criterion. Abstract semisimplicity is only used in ordre to invoke Theorem 7.4; laminarity of the approximate lattice could be assumed instead.

As mentioned in the introduction, Theorem 7.7 and Remark 7.8 essentially go over earlier results of Bader, Furman and Sauer [3]. They use lattices in place of approximate lattices, and make the semisimplicity assumption only on the archimedean factor, but assume the lattice embeds into that factor. Using Theorem 7.4, it is possible deduce a version of Theorem 7.7 from [3]. We retain it as an example of the flexibility of the approximate lattice framework.

Theorem 7.7. *Let G be an abstractly semisimple locally compact group. Let λ be a strictly dense commensurability class of approximate lattices in G. Then there exists a semisimple algebraic group H over $\mathbb{Q}$, and a subgroup M of $H(\mathbb{Q})$ containing $H(\mathbb{Z})$, such that G is compactly isogenous to the closure T of M in $H(\mathbb{A}_{\mathbb{Q}})$.*

Proof. By Theorem 6.9 there exists a connected semisimple real Lie group L and a lattice $\Gamma \le G \times \mathsf{L}$ such that $\pi_1 : \Gamma \to G$ is injective, $\pi_2 : \Gamma \to \mathsf{L}$ is injective with dense image; and with $\widetilde{\Lambda} = \pi_1(\Gamma)$ and $f = \pi_2 \circ \pi_1^{-1}$, we have $\lambda = \chi(f)$. Moreover L is centerless has no nontrivial compact normal subgroups.

By the strict density of λ, $\widetilde{\Lambda}$ is dense in G. Let $\widetilde{\Omega} = f(\widetilde{\Lambda})$. Then Γ is also the graph of a map $f^{-1} : \widetilde{\Omega} \to \widetilde{\Lambda} \subset G$.

Now let W be a compact neighborhood of 1 in G, and let $\Omega = f(W \cap \widetilde{\Lambda})$. By Lemma A.8, applied on the right this time, Ω is an approximate lattice in L.

We now apply Theorem 7.4 to (L, Ω). We find a direct product decomposition $\mathsf{L} = \Pi_{i=1}^k L_i$, and approximate lattices $\Omega_i \le L_i$ with $\Pi_i \Omega_i$

commensurable with Ω, such that each Ω_i is an irreducible lattice, or an arithmetic approximate lattice. Let $\pi_i : \mathsf{L} \to L_i$ be the projection.

Since f has dense image, so does $\pi_i f$ for each i. As Ω_i is discrete, $\pi_i \widetilde{\Omega}$ and Ω_i cannot be commensurable. Hence in case Ω_i is a lattice, while it may have rank one, it is still arithmetic, by the commensurator criterion of Theorem 1 of [46] (p.2; note finite generation is automatic in characteristic zero, according to point (b)).

The fundamental commensurability class of the locally compact group G is preserved under $\widetilde{\Lambda}$-conjugation; hence the commensurability class of Ω is preserved under $\widetilde{\Omega}$-conjugation. Thus $\widetilde{\Omega} \subset \mathrm{Comm}\,(\Pi_i \Omega_i)$. Let $\widetilde{\Omega}_i = \pi_i(\widetilde{\Omega})$. It follows that $\widetilde{\Omega}_i \leq \mathrm{Comm}\,(\Omega_i)$.

By Remark 7.3 (2), there exist homomorphisms $h_i : \widetilde{\Omega}_i \to H_i(\mathbb{A}_{\mathbb{Q}})$ with image between $H_i(\mathbb{Z})$ and $H_i(\mathbb{Q})$, and with $\Omega_i \in \chi(h_i)$. Let $H = \Pi_i H_i$ and $h = \Pi_i h_i$; then $\Omega \in \chi(h)$, and $H(\mathbb{Z}) \leq h(\widetilde{\Omega}) \leq H(\mathbb{Q})$. Let $M = h(\widetilde{\Omega})$ and $T = cl(M)$. Then $h : \widetilde{\Omega} \to T$ is a homomorphism with dense image and $\Omega \in \chi(h)$.

But we already know a homomorphism with dense image from $\widetilde{\Omega}$ to a locally compact group and canonical commensurability class Ω namely $f^{-1} : \widetilde{\Omega} \to G$. By Proposition 5.5, T and G are compactly isogenous, by an isogeny respecting $\widetilde{\Omega}$.

$\square$

$*$

Let us refer to abstractly semisimple locally compact groups T of the form appearing in Theorem 7.7 as *Margulis groups*.

Remark 7.8. Up to compact isogeny, and up to subgroups of profinite index, any Margulis group is itself a restricted product of semisimple groups over $\mathbb{R}$ and the $\mathbb{Q}_p$, with respect to the open subgroups $\mathbb{Z}_p$.

Proof. By the proof of Remark 6.6 (5), after factoring out a compact normal subgroup, we have $T = T_{na} \times T_{ar}$, where T_{ar} is a connected semisimple Lie group, and T_{na} is the closure of $\widetilde{\Lambda}$ in $H(\mathbb{A}_{fin})$, where $A_{fin} = \Pi'_p H(\mathbb{Q}_p)$ are the finite adeles. It suffices to prove the statement for T_{na}.

T_{na} contains the closure of $H(\mathbb{Z})$ in $H(\mathbb{A}_{fin})$, which in turn contains an open subgroup of $\Pi_p H(\mathbb{Z}_p)$. Hence T_{na} is open in $H(\mathbb{A}_{fin})$. We may take H to be centerless, and a product of groups $H_1, \ldots, H_m$ that are almost $\mathbb{Q}$-simple; rewrite the restricted product as $\Pi'_q H_q$, where $q = (p, j)$ and $H_q = H_j(\mathbb{Q}_p)$. Let $\pi_q : \mathbb{A} \to H_q$ be the projection. Let I be the set of q such that $\pi_q(T)$ is not compact. Let T' be the projection of T to $\Pi'_{q \in I} H_q$. The kernel of $T \to T'$ is compact. For $q = (p, j) \in I$,

the projection $\pi_q(T)$ is open and unbounded in $H_j(\mathbb{Q}_p)$. Hence by the theorem of Tits in the title of [58], Theorem T, $\pi_q(T)$ contains $H_j(\mathbb{Q}_p)^+$. Since the groups $H_j(\mathbb{Q}_p)^+$ are simple and non-isomorphic, the image of T' in any finite product $\Pi_{q \in I_0} H_q$ must contain $\Pi_{q \in I_0} H_q^+$. But T' is open, hence closed, and so it contains the full restricted product $\Pi'_{q \in I} H_q^+$. □

This gives in particular an analogue for approximate lattices of some of the results of [55].

As I learned from Emmanuel Breuillard, many examples of abstractly semisimple groups not embeddable in a restricted product of semisimple algebraic groups are known, including certain automorphism groups of trees and non-p-adic analytic pro-p groups. Such groups thus do not contain a strictly approximate lattice.

7.9. **A non-laminar approximate lattice.** Let $\Lambda = SL_2(\mathbb{Z}[1/p])$, a lattice in $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_p)$. We will define a central extension $\hat{G}$ of G by $\mathbb{Q}_p$ and a certain extension of Λ by a laminar approximate lattice in $\mathbb{Q}_p$, yielding an approximate lattice in $\hat{G}$ that is not laminar.

The group $SL_2(\mathbb{R})$ has a nontrivial central extension $\widetilde{SL_2(\mathbb{R})}$, determined by a bounded 2-cocycle β taking values in $\{-1, 0, 1\} \subset \mathbb{Z}$. (See e.g. [30] for an o-minimal approach; see also the general theorem on characteristic classes in [23] 1.3, p. 231.) We will at different points view β as taking values in $\mathbb{R}$, in $\mathbb{Q}_p$ and in $\mathbb{Z}[1/p]$.

Claim . For any index subgroup Λ' of Λ, β induces a nontrivial class in $H^2_{bdd}(\Lambda', \mathbb{Z}[1/p])$.

Proof. We will prove the stronger claim, that β induces a nontrivial class in $H^2_{bdd}(\Lambda', \mathbb{R})$.

Consider the exact sequence:

$$1 \to \mathbb{Z} \to \widetilde{SL_2(\mathbb{R})} \to SL_2(\mathbb{R}) \to 1$$

If we change coefficients from $\mathbb{Z}$ to $\mathbb{R}$, the extension

$$1 \to \mathbb{R} \to \mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})} \to SL_2(\mathbb{R}) \to 1$$

remains nonsplit: the commutator subgroup of $\widetilde{SL_2(\mathbb{R})}$ is the entire group so $\widetilde{SL_2(\mathbb{R})}$ admits no nontrivial homomorphisms to $\mathbb{R}$. Thus β represents a nonzero cohomology class in $H^2(SL_2(\mathbb{R}), \mathbb{R})$; and a fortiori in the bounded cohomology group $H^2_{bdd}(SL_2(\mathbb{R}), \mathbb{R})$.

Define β_1 on $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{Q}_p)$ by $\beta_1(x, y) = \beta(x)$. Then β_1 represents a nonsplit extension

$$1 \to \mathbb{R} \to \mathbb{R} \times_{\mathbb{Z}} \widetilde{SL_2(\mathbb{R})} \times SL_2(\mathbb{Q}_p) \to G \to 1$$

so β_1 has a nonzero class in $H^2_{bdd}(G, \mathbb{R})$. By Theorem 1.4 of [49], since $2 < 4 = 2rk(G)$, and Λ is an irreducible lattice, the restriction map

$$H^2_{bdd}(G, \mathbb{R}) \to H^2_{bdd}(\Lambda, \mathbb{R})$$

is an isomorphism. Hence the restriction of β to Λ remains nonzero in $H^2_{bdd}(\Lambda, \mathbb{R})$, and by the same token, in $H^2_{bdd}(\Lambda', \mathbb{R})$ for any finite index subgroup Λ' of Λ (which is also an irreducible lattice.) □

Let $\hat{\Lambda}$ be the central extension of Λ by $\mathbb{Z}[1/p]$ determined by β,

$$1 \leftarrow \Lambda \leftarrow \hat{\Lambda} \leftarrow \mathbb{Z}[1/p] \leftarrow 1$$

Concretely, we can let $\hat{\Lambda} = \Lambda \times \mathbb{Z}[1/p]$, with multiplication

$$(\lambda_1, a_1)(\lambda_2, a_2) = (\lambda_1 \lambda_2, a_1 + a_2 + \beta(\lambda_1, \lambda_2))$$

Let $\delta(\lambda, a) = -a$, where now $a \in \mathbb{Z}[1/p]$ is viewed as a real number. Then $\delta : \hat{\Lambda} \to \mathbb{R} : [-1, 1]$ is a quasimorphism. Let Δ be the (symmetric) approximate kernel $\Delta = \{x : -2 < \delta(x) < 2\}$. By Proposition 5.11, Δ is an approximate subgroup of $\hat{\Lambda}$. We have $\pi_1(\Delta) = \Lambda$, so if a subgroup S of $\hat{\Lambda}$ contains an approximate subgroup commensurable to Δ, then $\pi_1(S)$ has finite index in Λ, and thus ZS has finite index in $\hat{\Lambda}$ (where Z is the central image of $\mathbb{Z}[1/p]$ in $\hat{\Lambda}$). Let Λ' be the image of S in Λ; then Λ' has finite index, and we saw in the Claim that β is not trivial in $H^2_{bdd}(\Lambda')$.

Now if we had $\delta = b + h$ on S, where b is bounded and h is a homomorphism, then for $x, y \in \Lambda'$, choosing s, t with $(x, s), (y, t) \in S$, we would have: $\beta(x, y) = \delta((x, s)(y, t)) - \delta(x, s) - \delta(y, t) = b((x, s)(y, t)) - b(x, s) - b(y, t)$, so that $\beta = db = 0$ as a class in $H^2_{bdd}(\Lambda')$. Hence δ is not at bounded distance from a homomorphism on S.

By Proposition 5.28, the commensurability class of the approximate subgroup Δ of $\hat{\Lambda}$ is not laminar.

Now view β as a 2-cocycle with values in $\mathbb{Q}_p$, and construct in the same way a central extension

$$1 \to \mathbb{Q}_p \to \widetilde{SL_2(\mathbb{R})_{\mathbb{Q}_p}} \to SL_2(\mathbb{R}) \to 1.$$

Here $\widetilde{SL_2(\mathbb{R})_{\mathbb{Q}_p}}$ is obtained from the universal covering group of $SL_2(\mathbb{R})$ by taking a fiber product with $\mathbb{Q}_p$ over $\mathbb{Z}$. This also describes

the locally compact group topology on $\widehat{SL_2(\mathbb{R})_{\mathbb{Q}_p}}$; it has an open sub-group isomorphic to the inverse limit of coverings of $SL_2(\mathbb{R})$ of order p^n.

Taking products with $SL_2(\mathbb{Q}_p)$, we obtain

$$1 \to \mathbb{Q}_p \to \hat{G} \to G \to 1.$$

We can view the universe of $\hat{G}$ as being $G \times \mathbb{Q}_p$, with multiplication twisted by β as above. Then the product measure is a Haar measure; translation by an element of $\hat{G}$ involves a certain transvection on $\mathbb{Q}_p \times G$, and transvections are measure preserving. Thus the product measure is a Haar measure.

Δ is discrete in $\hat{G}$: suppose $(\lambda_i, a_i) \in \Delta$ and they approach a limit point $(\lambda, a) \in \hat{G}$. Them $\lambda_i \to \lambda \in G$; since Λ is discrete in G, we have $\lambda_i = \lambda \in \Lambda$ for large i; we may arrange by translation that $\lambda_i = \lambda = 1$. In this case we have $a_i \to a \in \mathbb{Q}_p$. But $\{a \in \mathbb{Z}[1/p] : |a|_{\mathbb{R}} < 2\}$ is discrete in $\mathbb{Q}_p$. Thus $a_i = a$ for large i.

Δ is a lattice in $\hat{G}$: if $C_1 \subset G$ is a Borel set of finite measure with $C_1 \Lambda = G$, then $(C_1 \times \mathbb{Z}_p)\hat{\Lambda} = \hat{G}$, and of course $\mathbb{Z}_p \subset \mathbb{Q}_p$ is compact and hence of finite measure.

Thus Δ is a non-laminar approximate subgroup of $\hat{G}$.

7.10. **Approximate lattice extensions.** Let G be a connected linear algebraic group over a local field k (the discussion extends naturally to a finite product of such groups.) Then G has soluble radical $R = R(G)$; it is a soluble Zariski closed normal subgroup, with $\mathbf{G} := G/R$ semisimple. One can either choose R connected, or such that G/R is centerless; the latter will be a little more convenient for us. We define $G_{ss} = G/R$.

Trying to reformulate the Björklund-Hartnick problem, given the amenable and semisimple answers, we may first ask:

Question 7.11. Is Λ/R above always an approximate lattice in G/R? For connected G at least, I am fairly sure, a slight modification (and simplification) of Lemma 6.12 gives a positive answer.

N.B.: Machado has answered this positively at least over characteristic zero local fields.

Let us consider the case where $\mathbf{\Lambda} := \Lambda/R$ is an approximate lattice; then we know it is a product of arithmetic approximate lattices, and lattices. In case $\langle \Lambda \rangle$ is dense in G (as is natural to assume), $\langle \mathbf{\Lambda} \rangle$ will be dense in G/R, and then all the components of $\mathbf{\Lambda}$ will be arithmetic. By Proposition 5.8, $\langle \mathbf{\Lambda} \rangle$ contains an element Γ of $\langle \bar{\omega} \rangle_{\min}$; and by pulling

back such an element, we may reduce to the case that $\langle \Lambda \rangle \in \langle \bar{\omega} \rangle_{\min}$. In this case, $\langle \Lambda \rangle$ is an S-arithmetic group (of course it is not a lattice of G/R, but of some other product of semisimple groups.) Now we may further ask:

Question 7.12. Let $\mathbf{G}$ be a product of semisimple groups over local fields, and $\mathbf{\Lambda}$ an approximate lattice in $\mathbf{G}$, with $\langle \mathbf{\Lambda} \rangle$ isomorphic to a given S-arithmetic group $\widetilde{\Lambda}$. When is it true that for all G with $G_{ss} \cong \mathbf{G}$, every approximate lattice Λ of G with $\Lambda/R = \mathbf{\Lambda}$ is laminar?

7.13. We sketch a possible argument that the answer to Question (7.12) is: just when:

($\diamond$) $H_b^2(<\mathbf{\Lambda}>, V) = (0)$ for all rigid actions of $<\mathbf{\Lambda}>$ on Euclidean space $V = \mathbb{R}^n$, induced from continuous action of G.

If $\diamond$ holds, taking into account that $H_b^2(<\Lambda>, V) = H_b^2(<\mathbf{\Lambda}>, V^R)$ (by the mapping theorem [23] p.40, extended to nonconstant coefficients, [56], cf. [49].), the laminarity follows from Proposition 5.25 (using Proposition 5.8 for H.)

Conversely if $H_b^2(\mathbf{\Lambda}, V) \neq (0)$, let β be a nontrivial bounded 2-cocycle, and use it to form an extension $\hat{\Lambda}$ of $\mathbf{\Lambda}$ by the module V. By the mapping theorem we have $H_b^2(\hat{\Lambda}) = H_b^2(\Lambda)$ canonically since V is amenable. Thus pulled back to $\hat{\Lambda}$, β still represents a nontrivial class in $H_b^2(\hat{\Lambda})$; but now it maps to 0 as a class in $H^2(\hat{\Lambda})$. Hence by the the discussion below 5.25, we find a nonlaminar approximate subgroup in a central extension of $\hat{\Lambda}$. It remains to embed this in an appropriate algebraic group, as in 7.9, so as to produce a negative answer for this instance of Question 7.12.

The bounded cohomology of S-arithmetic groups, as $\mathbf{\Lambda}$ must be here, is studied in depth in [11], [49], providing I believe both instances where $H_b^2(\mathbf{\Lambda}, V)$ is zero for all V, and otherwise.

APPENDIX A. APPROXIMATE LATTICES

In this Appendix, G, H denote second countable, locally compact groups; μ will denote a left invariant measure on G.

We develop the basics of approximate lattices under a slightly different (and more elementary) definition than given in [5]. In the case of strong or cocompact approximate lattices, these results appear in [5] and [44].

Definition A.1. An approximate subgroup $\Lambda \leq G$ is an *approximate lattice* in G if it is discrete, and the equivalent conditions (1)-(3) of Proposition A.2 hold.

Proposition A.2. *Let Λ be a discrete approximate subgroup. Then the following properties are equivalent:*

(1) $\Lambda D = G$ for some Borel set D of finite measure.
(2) There exists a finite bound on $\mu(C)$, over all Borel sets $C \subset G$ with $\Lambda^2 \cap CC^{-1} = (1)$; i.e. such that multiplication is 1-1 on $\Lambda \times C$.
(3) There is no Borel $C \subset G$ with $\mu(C) = \infty$, and $\Lambda^2 \cap CC^{-1} = (1)$.

If these conditions hold, then G is unimodular, i.e. μ is also right invariant; and so the three dual conditions are also equivalent.

Proof. Note that as G is second countable, the discrete subset Λ is countable.

Assume (1). Let C be a Borel set, with $\Lambda^2 \cap C^2 = (1)$. We have $C \subset \cup_{a \in \Lambda} C_a$, where $C_a = (aD \cap C)$. By the invariance assumption, $\mu(C_a) = \mu(a^{-1}C_a)$. The sets $a^{-1}C_a$ are disjoint Borel subsets of D; so

$$\mu(C) \leq \sum_{a \in \Lambda} \mu(C_a) = \sum_{a \in \Lambda} \mu(a^{-1}C_a) \leq \mu(D).$$

(2) trivially implies (3).

Now assume (3). By Lemma A.3 (applied to $X = \Lambda^2 \smallsetminus (1)$, $Y = G$, with opposite multiplication) there exists a Borel M with $MM^{-1} \cap \Lambda^2 = (1)$, and $G = \Lambda^2 M$. By (3), M has finite volume. But $\Lambda^2 \subset \Lambda F$ for some finite F, so $G = \Lambda FM$, and $FM = \cup_{f \in F} fM$ has finite volume.

To prove unimodularity, let $\rho(g)$ be the constant, given by the uniqueness of the Haar measure, such that for all Borel C we have $\mu(Cg) = \rho(g)\mu(C)$. Pick a symmetric compact open neighborhood C of 1, with $\Lambda^2 \cap C^2 = (1)$; this is possible since Λ^2 is discrete. Then for any $g \in G$, we have $(Cg)(Cg)^{-1} = CC^{-1}$, so $\Lambda^2 \cap (Cg)(Cg)^{-1} = (1)$. By (2) there is an absolute bound on $\mu(Cg)$ over all g. Hence ρ is bounded, and the image of ρ is a bounded subgroup of $\mathbb{R}^{>0}$, i.e. 1. $\square$

Unimodularity was also proved in [5] for strong approximate lattices, in the sense defined there.

The following statement on a Borel set of approximate representatives will be essential. See [39] for a study of much deeper questions along this line.

Lemma A.3. *Let G be a discrete or second countable locally compact group, Y a Borel subset, and X a discrete subset of G with $1 \notin X$, and $X = X^{-1}$. A subset $B \subset Y$ is called X-free if $B^{-1}B \cap X = \emptyset$, and maximally X-free if it is maximal with this property, i.e. $Y \subset BX$. Then there exists a Borel set $B \subset Y$ that is maximally X-free.*

Proof.

Claim 1. Assume Y is a countable union of Borel subsets Y_n, and for each n, any Borel $Y' \subset Y_n$ has a maximally X-free Borel set. Then we can solve the problem inductively: having defined $B_m \subset Y_m$, let $Y'_n = Y_n \setminus \cup_{m<n} B_m X$, and let B_n be a maximally X-free subset of Y'_n. Then $B = \cup B_n$ will be maximally X-free in Y.

In view of this, since G can be written as a countable union of compact neighborhoods, we may assume Y is precompact.

For any Borel $Z \subset Y$, $b^{-1}Z \cap X$ is finite, since X is discrete in G and Y precompact. For a given finite subset H of X, let

$$Z_H = \{b \in Z : b^{-1}Z \cap X = H\}.$$

Claim 2. For any Borel $Z \subset G$, and any finite $H \subset X$, we can find a maximally X-free Borel subset of Z_H, or of any given Borel subset of Z_H.

We prove Claim 2 by induction on $|H|$. For $|H| = 0$, Z_H is already X-free, so any $Z \subset Z_H$ is maximally X-free in Z. Assume $|H| = n$. Let $Z' = Z_H$. For $b \in Z'$, we have $b^{-1}Z' \cap X \subset H$. We can thus divide Z' into finitely many sets, according to the value of $b^{-1}Z' \cap X$. For any proper subset of H', the induction hypothesis applies to $Z'_{H'}$. Thus it suffices to deal with Z'_H. If $Z'_H = (Z_H)_H$ is empty, there is nothing to prove. Otherwise, H is a finite subgroup of G. To see this, let $a \in (Z_H)_H$ and let $x, y \in H$. Then $ax \in Z_H$, so $axy \in Z$. It follows that $xy \in a^{-1}Z \cap X$; since $a \in Z_H$, we have $xy \in H$. So Z_H is a union of H-orbits, $y^{-1}z \in X$ iff y, z lie in the the same H-orbit. So given a Borel $W \subset Z_H$, it suffices to find a set of representatives for the H-orbit equivalence relation restricted to W. (for instance we can fix a Borel linear ordering on G, and choose the least element of W in each H-orbit.) This proves the Claim for $(Z_H)_{H'}$ for any $H' \subset H$, and hence by Claim 1, it is valid for Z_H.

Now X, being a discrete subset of a second-countable space, is countable; so there are only countably many possibilities for the finite set $b^{-1}Y \cap X$. We thus have a partition of Y into countably many Borel sets Y_H. By Claim (2), any Borel subset of each of these has a Borel maximally free set, and hence by Claim 1, so does Y. $\qquad\square$

Lemma A.4. *If an approximate lattice is contained in a set Λ with Λ^4 discrete, they are commensurable, and Λ is an approximate lattice too.*

Proof. Let X be an approximate lattice contained in Λ. Let U be a symmetric compact neighborhood of 1, small enough so that $U^2 \cap \Lambda^4 = (1)$. Let $\{c_i : i \in I\}$ be a maximal collection of elements of Λ, such that the sets Xc_i are disjoint. If $x, x' \in X, i, i' \in I$, $u, u' \in U$ and

$xc_iu = x'c_{i'}u'$, then $u'u^{-1} \in \Lambda^4$ so $u = u'$; then by the disjointness of the sets Xc_i we have $i = i'$; so finally $x = x'$. In particular, the sets c_iU are disjoint; and multiplication is 1-1 on $X \times (\cup_{i \in I}c_iU)$. Hence, using Proposition A.2 (3),

$$|I|\operatorname{vol}(U) = \sum_{i \in I} \mu(c_iU) = \mu(\cup_{i \in I}c_iU) < \infty.$$

So I is finite. From maximality of I, it follows by the usual argument that $\Lambda \subset \cup_{i \in I}X^{-1}Xc_i$: if $d \in \Lambda$, then $Xc_i \cap Xd \neq \emptyset$ for some i, so $d \in X^{-1}Xc_i$. As we assumed X is an approximate subgroup, Λ and X are commensurable. Using condition Proposition A.2 (2) it is clear that Λ is an approximate lattice.

$\square$

A.5. **Complementarity.** Let $f{:}G \to H$ be a partial map between topological spaces, i.e. a map $f : X \to H$ where $X \subset G$. We will say that f is *discrete-on-compact* if the pullback of any (pre)compact set in H is discrete in G.

   f is *precompact-to-discrete* if whenever $C \subset G$ is compact, then $f(C \cap \widetilde{\Lambda})$ is discrete in H.

Lemma A.6. *Let G, H be locally compact groups, $\widetilde{\Lambda}$ a subgroup of G, and $f : \widetilde{\Lambda} \to H$ a group homomorphism.*

(1) If f is discrete-on-compact, and $X \subset H$ is compact, then $f^{-1}(X)$ is uniformly discrete.

(2) If f is discrete-on-compact, it is precompact-to-discrete.

(3) Assume G is second countable, and f has finite kernel. Then the converse to (2) holds: if f is precompact-to-discrete, then f is discrete-on-compact. In fact if $f(U \cap \widetilde{\Lambda})$ is discrete in H for one neighborhood U of 1 in G then f is discrete-on-compact.

Proof. (1) Clear since $f^{-1}(XX^{-1})$ is discrete.

   (2) In a locally compact space, to prove discreteness of $f(C \cap \widetilde{\Lambda})$, it suffices to show that $f(C \cap \widetilde{\Lambda}) \cap C'$ is finite for every compact C'; this is clear since $f^{-1}(C')$ is discrete in G, so $C \cap f^{-1}(C')$ is finite.

   (3) Reversing (2), given a compact $C' \subset H$, we must show that $f^{-1}(C')$ is discrete; it suffices to show that no 1-1 sequence $(c_i : i \in \mathbb{N})$ of elements of $f^{-1}(C')$ converges to any $c \in G$. Indeed if $c_i \to c$ then $\{c, c_i : i \in \mathbb{N}\}$ is compact, so $\{c_i : i \in \mathbb{N}\}$ is precompact, and $\{f(c_i) : i \in \mathbb{N}\}$ is discrete in H; hence intersects C' in a finite set; since f is finite-to-one, $\{c_i : i \in \mathbb{N}\}$ is finite, in contradiction to the assumption that the c_i are distinct.

To prove the stronger version, we may assume that $\widetilde{\Lambda}$ is dense in G. Let U be a nonempty open set with $f(U \cap \widetilde{\Lambda})$ discrete; we will show that $f(D \cap \widetilde{\Lambda})$ is discrete for any compact subset D of G. For each point $d \in D$, let $\gamma(d) \in \widetilde{\Lambda} \cap Ud^{-1}$. Then $D \subset \cup_{d \in D} \gamma(d)^{-1}U$; so a finite number of translates $\gamma_i U$ cover D, with $\gamma_i \in \widetilde{\Lambda}$. Hence $f(D \cap \widetilde{\Lambda}) \subset \cup_i f(\gamma_i)f(U \cap \widetilde{\Lambda})$; so $F(D \cap \widetilde{\Lambda})$ is a finite union of discrete subsets of H and hence discrete in H.

$\square$

A.7. Discrete-on-compact homomorphisms and Meyer sets.

Let G, H be locally compact groups; let $\pi_1 : G \times H \to G, \pi_2 : G \times H \to H$ be the projections. Given a discrete subgroup $\Gamma \leq G \times H$ such that π_1 is injective on Γ, let $\widetilde{\Lambda} = \pi_1(\Gamma)$ and $f = \pi_2 \circ (\pi_1|\Gamma)^{-1} : \Lambda \to H$. Then f is discrete-on-compact. Indeed if $C \subset H$ is compact, and U any compact set in G, then $f^{-1}(C) \cap U$ is finite since $(U \times C) \cap \Gamma$ is finite; hence $f^{-1}(C)$ is discrete in G. Conversely, if $f : \widetilde{\Lambda} \to H$ is discrete-on-compact, then the graph of f is a discrete subgroup of $G \times H$.

The uniform part of the proposition below is Prop. 2.13 of [5]. We do not assume in advance that H is unimodular, so that if G has an approximate lattice and f exists, then H is automatically unimodular.

Proposition A.8. *Let G, H be second countable locally compact groups, $\widetilde{\Lambda}$ be a subgroup of G, $f : \widetilde{\Lambda} \to H$ a discrete-on-compact homomorphism with dense image. Let Γ be the graph of f, viewed as a subset of $G \times H$. Let $\Lambda = f^{-1}(C)$, where C is a compact, symmetric subset of H with nonempty interior. Then:*

(1) Γ is a lattice iff Λ is an approximate lattice.
(2) Γ is a uniform lattice iff Λ is a uniform approximate lattice.

Proof. Assume Λ is an approximate lattice. Let U be a Borel subset of G of finite volume, with $\Lambda U = G$. We will show that $\Gamma(U \times C^2) = (G \times H)$. As $U \times C^2$ has finite volume (and is compact if U is), it follows that Γ is a lattice, and is uniform if U is compact.

To see that $(U \times C^2)\Gamma = (G \times H)$, let $(g, h) \in G \times H$. By density, $f(a) \in hC$ for some $a \in \widetilde{\Lambda}$. So $f(a)C \subset hC^2$. Now $a\Lambda \cap gU^{-1} \neq \emptyset$ since $g \in aG = a\Lambda U$. Let $b \in a\Lambda \cap gU^{-1}$. Then $g \in bU$; and $f(b) \in f(a)f(\Lambda) \subset hC^2$, so $h \in f(b)C^2$. Thus $(g, h) \in (b, f(b))(U \times C^2) \subset \Gamma(U \times C^2)$.

Now assume Λ is not a lattice. By Proposition A.2 (3), there exists a Borel $U \subset G$ with $\mu(U) = \infty$, and $\Lambda^2 \cap UU^{-1} = (1)$. In other words

$f^{-1}(C)^2 \cap UU^{-1} = (1)$, so $(U \times C)(U \times C)^{-1} \cap \Gamma = (1)$. Thus Γ is not a lattice.

In the uniform case, it remains to show that if Γ is a cocompact lattice, then there exists a compact set of the form $U \times C$, with $(U \times C)\Gamma = (G \times H)$; this is clear since any compact subset of $G \times H$ is contained in a compact rectangle. $\qquad\square$

Let us call a closed, symmetric neighborhood of 1 in a totally disconnected locally compact group H *sensible* if it is contained in some compact open subgroup; more generally for locally compact H, call C sensible if the image C/H^0 in H/H^0 is sensible. In particular if H is connected, then any symmetric compact neighborhood of 1 is sensible.

Proposition A.9. *Let G, H be locally compact, $\Gamma \leq G \times H$ a lattice projecting densely to H and 1-1 to G. Let $\widetilde{\Lambda} = \pi_1(\Gamma)$, $f = \pi_2 \circ \pi_1^{-1} : \widetilde{\Lambda} \to H$. Let C be a compact symmetric subset of H with nonempty interior, $\Lambda = f^{-1}(C)$, and $X \subset \widetilde{\Lambda}$ a symmetric set commensurable with $\Lambda := f^{-1}(C)$. Then $X \subset f^{-1}(C')$ for some compact $C' \subset H$.*

In case C is sensible, $\langle X \rangle$ contains a subgroup of $\langle \Lambda \rangle$ of finite index. Thus the commensurability class of Λ belongs to $\langle \Lambda \rangle$ but not to any infinite index subgroup.

In particular for any two sensible C, C', not only the approximate subgroups $f^{-1}(C), f^{-1}(C')$ but also the groups generated by them are commensurable.

Proof. We have $X \subset \cup_{i=1}^{m} \Lambda a_i$, for some $a_1, \ldots, a_m \in \widetilde{\Lambda}$; let $C' = \cup_{i=1}^{m} C f(a_i)$. Clearly $X \subset f^{-1}(C')$. At this point it is also clear that X is an approximate lattice in G. ($X^2 \subset f^{-1}((C')^2)$ and the later is commensurable with $f^{-1}(C)$, so X is an approximate subgroup. On the other hand $\Lambda \subset \cup_{i=1}^{l} X b_i$ so if $\Lambda M = G$ with M Borel of finite measure, then $X(\{b_1, \ldots, b_l\}M) = G$ and $(\{b_1, \ldots, b_l\}M)$ has finite measure.)

We have $\Lambda \subset \cup_{i=1}^{l} X b_i$. Since $C \subset \cup_{i=1}^{l} cl(fX)b_i$, it follows from Baire category that $cl(f(X))$ has nonempty interior. Thus $cl(f(X))^2$ contains a neighborhood of 1 in H.

Now assume H is connected. Then $H = \cup_n cl(f(X))^n$. So $f(\langle X \rangle)$ is dense in H. By Proposition A.8 (1), the graph Γ' of $f|\langle X \rangle$ is a lattice in $G \times H$. By the (easier lattice case of) Lemma A.4, Γ' has finite index in Γ. Projecting to G we see that $\langle X \rangle$ has finite index in $\widetilde{\Lambda}$.

Finally, assume more generally that C is sensible. We may replace X by the commensurable set $X^2 \cap \Lambda^2$ (5.7); so we may assume that $X \subset \langle \Lambda \rangle$. By assumption, C/H^0 is contained in an compact open subgroup O/H^0. For statements about $\Lambda = f^{-1}(C)$, and subsets of $\langle \Lambda \rangle$, we may as well replace H by O; so we may assume H/H^0 is

compact. In this case every open subgroup of H has finite index in H. In particular $H' := cl(f(\langle X \rangle))$ has finite index in H. As above, the graph Γ' of $f|\langle X \rangle$ is a lattice in $G \times H'$, hence also a lattice in $G \times H$; and we conclude as in the previous paragraph. $\qquad\square$

Remark A.10. It can also be shown that if $\Gamma \le G \times H$ is a lattice, with dense projection to H, then there exists a Borel $M \subset G$ of finite volume and a *compact* $C \subset H$ with $(M \times C)\Gamma = G \times H$. (The proof starts similarly to Lemma A.3, then invokes 4.2, 4.5 of [39].) Thus the lack of uniformity in an irreducible lattice of a product can be relegated to any one of the factors.

This gives an alternative proof of Proposition A.8 (2).

The same fact on representative sets is moreover true for approximate lattices.

A.11. **Borel-Wang density.** Borel density is addressed in [6] for strong and for uniform approximate lattices. We will give a direct proof for approximate lattices in the sense we use here.

If $G = \Pi_i G_i$ is a product or restricted product of algebraic groups over possibly different fields (viewed with this structure, disregarding possible isomorphisms between the fields), we will say that a subset $X \subset G$ is *Zariski dense* if the projection to each G_i is Zariski dense.

Proposition A.12. *Let G be an algebraically connected semisimple algebraic group with no anisotropic factors over a local field. Then every approximate lattice in G is Zariski dense.*

Likewise for finite or restricted products of such groups over possibly different local fields; an approximate lattice is not commensurably contained in a product of Zariski closed subgroups, other than the entire ambient group.

Proof. Let Λ be an approximate lattice in G, and let C be the Zariski closure. We have $1 \in \Lambda = \Lambda^{-1}$ so $1 \in C = C^{-1}$. We have $\Lambda^2 \subset \Lambda F$ for some finite set F, so $C^2 \subset CF$. Let C_1 be an irreducible component of C of maximal dimension; then $C_1^2 \subset C_2 F$ for some component C_2 of C, and it follows that $C_1^2 = C_2 f$ for some $f \in F$; from this it follows easily that C is contained in a finite union of cosets Hg_i of a Zariski closed subgroup H of G, $g_i \in N_G(H)$, and that Λ is commensurable to H.

If Λ is a closed approximate subgroup of a locally compact group G, say Λ has *property S* if for every symmetric neighborhood U of the identity in G and every element $g \in G$, there is a positive integer n such that $g^n \in U\Lambda^2 U$. Equivalently, the sets $g^n U\Lambda$ are not pairwise disjoint. For closed subgroups, this agrees with the definition in [69].

Now if Λ is an approximate lattice, (S) holds. To see this take U symmetric, with $U^2 \cap \Lambda^2 = (1)$. If the sets $g^n U$ are not pairwise disjoint, the condition holds trivially (using only $1 \in \Lambda$). Thus we may assume they are pairwise disjoint, so $\cup g^n U$ is a set of infinite measure. By Proposition A.2 (3), $(\cup g^n U)^{-1} (\cup g^n U) \cap \Lambda^2 \neq (1)$; so for some $n \geq m$, $(g^n U)^{-1} g^m U \cap \Lambda^2 \neq (1)$. We have $n > m$ since $U^2 \cap \Lambda^2 = (1)$. Thus $U g^{n-m} U \cap \Lambda^2 \neq (1)$ and $g^{n-m} \in U \Lambda^2 U$, proving (S).

Of course this implies that (S) holds for any subgroup H containing Λ. According to the Borel-Wang density theorem for groups, [69] Corollary (1.4), if H is Zariski closed this implies that $H = G$.

In case G is a finite or restricted product of such groups G_i, in a product $\Pi_i G_i$, since Λ has property (S), it is clear that each projection $\pi_i(S)$ has property (S) too; it follows by definition that Λ is Zariski dense. $\qquad\square$

A.13. Product decomposition.

The hypotheses of the following Proposition hold notably (using Proposition A.12) when G_1, G_2 are centerless, semisimple groups with no anisotropic factors over a local field.

Proposition A.14. *Let G_1, G_2 be centerless locally compact groups, both satisfying the chain condition for centralizers, and such that an approximate lattice is not contained in any proper centralizer subgroup. Let Λ be an approximate lattice in $G = G_1 \times G_2$. Let $\pi_i : G \to G_i$ be the projection, and let $\Lambda_i = G_i \cap \Lambda^2$ (we identify G_1 with $G_1 \times 1$, notationally.) Then the following conditions are equivalent:*

(1) $\pi_2(\Lambda)$ is discrete in G_2.
(2) Λ_1 is a approximate lattice in G_1.
(3) $C_{G_1}(\Lambda_1) = 1$.
(1') $\pi_1(\Lambda)$ is discrete in G_1.
(4) Λ is commensurable with a product of an approximate lattice in G_1 and an approximate lattice in G_2.

Proof. $(1 \to 2)$ Certainly Λ_1 is discrete in G_1, since Λ is discrete. It is an approximate subgroup of G_1 by §5.7. To show that it is an approximate lattice, it remains to show that conditions (2) or (3) of Lemma A.2 hold, i.e. that Λ_1 has finite covolume. We will check this at the same time for $\pi_2(\Lambda)$. Let μ_1, μ_2 be Haar measures on G_1, G_2, and let $\mu = \mu_1 \times \mu_2$ be the product Haar measure on G.

Let E_i be any Borel subset of G_i, with $E_1^2 \cap \Lambda_1 = (1)$, $E_2^2 \cap (\pi_2 \Lambda)^2 = (1)$. Then $(E_1 E_2)^2 \cap \Lambda^2 = (1)$: if $e_i \in E_i^2$ and $e_1 e_2 \in \Lambda^2$, then $e_2 \in E_2^2 \cap (\pi_2 \Lambda)^2$; by assumption $e_2 = 1$, so $e_1 \in \Lambda^2 \cap G_1 = \Lambda_1$, hence $e_1 = 1$. Thus $E_1 E_2$ has finite (indeed bounded) measure. As $\mu(E_1 E_2) =$

$\mu_1(E_1)\mu_2(E_2)$, by fixing some small open neighborhood E_2 of 1, we see that there is an absolute bound on $\mu_1(E_1)$ and in any case it cannot be infinite. Hence Λ_1 is an approximate lattice in G_1.

$(2 \rightarrow 3)$ Assume (2). By assumption, if $a \in G_1$ and $C_{G_1}(a) \neq G_1$, then $C_{G_1}(a)$ cannot contain Λ_1; so $a \notin C_{G_1}(\Lambda_1)$. Thus $C_{G_1}(\Lambda_1)$ consists only of central elements of G_1, hence equals (1).

$(3 \rightarrow 1')$ By the chain condition on centralizers in G_1 and (3), there exists a finite $F \subset \Lambda_1$ with $C_{G_1}(F) = (1)$. We now use the argument of Lemma 5.30. Let $a_i \in \pi_1(\Lambda)$, and suppose $a_i \rightarrow a \in G_1$, and let $f \in F$. We have $a_i f a_i^{-1} \in \Lambda^4$, and $a_i f a_i^{-1} \rightarrow a f a^{-1}$. Now Λ^4 is discrete in G, so $a_i f a_i^{-1} = a f a^{-1}$ for large i. Thus $a^{-1} a_i \in Z_G(f)$ for each $f \in F$. So $a = a_i$ for large i. This proves the discreteness of $\pi_1(\Lambda)$.

By repeating the circle, it is now clear that (1-3) are equivalent, and also equivalent to their duals (1'-3') replacing G_1, G_2. Thus Λ_1, Λ_2 are both approximate lattices, hence so is $\Lambda_1 \times \Lambda_2$. Clearly $\Lambda_1 \times \Lambda_2 \subset \Lambda^2$. Since Λ^8 is discrete, by Lemma A.4, $\Lambda_1 \times \Lambda_2$ is commensurable with Λ. (As is $\pi_1\Lambda \times \pi_2\Lambda$, incidentally.)

This gives (4). The implication $(4 \rightarrow 1)$ is clear. $\qquad\square$

By Proposition A.12, the assumption that proper centralizer subgroups not contain a lattice is true for groups G_1, G_2 satisfying the hypothesis there.

In case only G_1 satisfies the chain condition on centralizers, we still have the implications $1 \rightarrow 2 \rightarrow 3 \rightarrow 1' \rightarrow 2' \rightarrow 3'$, and $2 + 2' = 4$. (So $2 \rightarrow 4$.)

The Proposition justifies a definition. Let G be a connected, centerless, semisimple groups with no anisotropic factors over a local field.

Definition A.15. An approximate lattice Λ of G is *irreducible* if there is no closed normal subgroup N of G such that the image of Λ in G/N is discrete.

Corollary A.16. *Let G be a finite product of noncompact simple algebraic groups over local fields. Then every approximate lattice of G is commensurable to a product $\Pi_{i=1}^k \Lambda_i$, where $G = \Pi_{i=1}^k G_i$ and Λ_i is an irreducible approximate lattice in G_i.*

A.17. Commensurator in algebraic groups.

Lemma A.18. *Let L be a field, K a perfect subfield, H a centerless, connected algebraic group over K. Let $g \in H(L)$ and assume $g^{-1}H(K)g \cap H(K)$ is Zariski dense in $H(K)$. Then $g \in H(K)$.*

Proof. Let $D := g^{-1}H(K)g \cap H(K)$. As D is Zariski dense, $C_H(D) = (1)$; hence $C_H(F) = (1)$ for some finite set $\{d_1, \ldots, d_k\} \subset D$; we have

$gd_ig^{-1} = b_i$ with $d_i, b_i \in H(K)$; this uniquely determines g in $H(L^{alg})$, so g is definable in ACF_K and hence lies in $H(K)$ $\qquad\square$

Lemma A.19. *Let H be a connected algebraic group over a perfect field K. Let Λ be an approximate lattice, Zariski dense in $H(K)$. Let L be a field extension, and let $C = \mathrm{Comm}\,(\Lambda)$ be the commensurator of Λ in $H(L)$. Then $C \leq H(K)$.*

Proof. As noted in §5.7, if A, B are commensurable approximate subgroups, they are both commensurable to $AA \cap BB$, hence using connectedness of H, $AA \cap BB$ is Zariski dense in H. Let $g \in \mathrm{Comm}\,(\Lambda)$. Let $A = \Lambda$, and $B = g^{-1}\Lambda g$; since $A \subset H(K)$ we have $A^2 \subset H(K)$, and similarly $B^2 \subset g^{-1}H(K)g$; so if $g \in H(L)$ commensurates Λ, then $H(K) \cap gH(K)g^{-1}$ is Zariski dense in H. By Lemma A.18, this implies that $g \in H(K)$. $\qquad\square$

Question A.20. If $G_1 \leq G_2$ are locally compact groups, and Λ is an approximate lattice of G_2 contained in G_1, is G_1 unimodular? Equivalently, is Λ an approximate lattice of G_1? The case $G_1 \trianglelefteq G_2$ is easy (in this case, carring a finite Haar measure, G_2/G_1 is compact). But the general case seems to require a little care (if not a surprise).

Question A.21. Formulate an adelic version of Corollary 6.11.

Question A.22. Clarify the relation to the Björklund - Hartnick definitions of approximate lattice.

   N.B.: Simon Machado has shown that a strong approximate lattice is also a finite covolume approximate lattice; and in turn a finite covolume approximate lattice is an approximate lattice in the sense of [5]. It follows from [5] and the results here that in semisimple groups, the notions of strong and finite covolume approximate lattice coincode; but the general case remains open.

Question A.23. Let Λ be an approximate lattice in a non-amenable locally compact group. Could Λ be definably amenable in a reasonable language, or even for that matter outright amenable in the sense of [44]? For co-compact Λ, [5] answer negatively for metric amenability.

APPENDIX B. CATEGORIES OF APPROXIMATE SPACES.

   We will use no category theory in this paper, at least not explicitly. Nevertheless it seems useful to describe a number of categories that clarify the difference between the automorphism group of the core, and the two Lascar groups.

B.1. **The categories CorSp and ApprSp.** Let X, Y denote locally compact, Hausdorff topological spaces. A *proper correspondence* is a closed subspace $S \leq X \times Y$ such that the projections $\mathrm{pr}_{S,X} : S \to X$, $\mathrm{pr}_{S,Y} : S \to Y$ are proper maps. If $S \subset X \times Y$ and $T \subset Y \times Z$ are proper correspondences, we let $S^{tr} = \{(y, x) : (x, y) \in S\}$, $T \circ S = \{(x, z) \in X \times Z : (\exists y \in Y)(x, y) \in S \wedge (y, z) \in Z\}$. In case $X = Y = Z$ and $S = T$, we write this as $S^{\circ 2}$, and similarly define higher *iterates*.

The objects of CorSp are pairs (X, A_X) with X a locally compact space, and $A_X \subset X^2$ a symmetric, reflexive proper correspondence. A *morphism* $(X, A_X) \to (Y, A_Y)$ is a proper correspondence $S \leq X \times Y$ such that $\mathrm{pr}_{S,X} = X$ and $S \circ A_X \circ S^{tr} \subset A_Y$.

B.2. **The category ApprSp.** In ApprSp, the objects are the same as in CorSp; the second condition on a morphism is weakened to: $S \circ A_X \circ S^{tr} \subset A_Y^{\circ n}$ for some n; and two morphisms $S, T : (X, A_X) \to (Y, A_Y)$ are *identified* if $S \subset A_Y^m \circ T$ for some m, and vice versa.

The map $X \mapsto (X, Id_X)$ extends to a fully faithful functor of the usual category of locally compact topological spaces, into ApprSp. The more general objects of ApprSp, of the form (X, S), can be viewed as the space X where the points $x \in X$ are only known with some 'experimental error'; if x is the result of a measurement, the actual point may be any x' with $(x, x') \in S$. The traditional approach in model theory, in this situation, was to factor out the equivalence relation generated by S, losing track of the topology; or the smallest closed equivalence relation containing S, retaining a locally compact space but usually a much smaller one. Viewing (X, S) as an object in ApprSp is an alternative way of 'factoring out S'; it preserves, in general, much more information. Only in case $S^{\circ m}$ is an equivalence relation E for some m, does the object (X, S) of ApprSp reduce to the quotient space X/E.

In some cases we can do better, obtaining an object of CorSp canonically.

B.3. **The categories CoGr and CApprGr.** Here the objects are pairs (G, K) where X is a locally compact topological group and K is a compact, normal, symmetric subset of G. A morphism $(G, K) \to (H, L)$ is a subset $M \leq G \times H$, with $(1, 1) \in M$, $M = M^{-1}$ and $M^2 \subset M(K \times L)$.

We have a natural functor CoGr $\to$ CorSp, mapping (G, K) to (G, A_K) where $A_K = \{(x, y) \in G^2 : xy^{-1} \in K\}$.

The category CApprGr is obtained from this in a similar way to ApprSp.

An example is the object (T, K) of CApprGr, where $T = \mathbb{R}^2/\Lambda$ is a torus, $\Lambda \cong \mathbb{Z}^2$ a torus in $\mathbb{R}^2$, and $K = [0, 1] \times \{0\}$ is a one-dimensional interval. Typically, K generates a dense subgroup of T, and so $K/\langle K \rangle$ is an indiscrete space, and $K/cl(\langle K \rangle)$ is a point. On the other hand (T, K) retains considerable information, in fact T may be recovered from it.

Appendix C. Group-theoretic terminology

We collect here a few group-theoretic definitions and statements that we will need.

Definition C.1. Let G be a topological group.

(1) We denote the connected component of the identity in G by G^0. $[x, y]$ is the commutator $xyx^{-1}y^{-1}$. For $X, Y \subset G$, the commutator *set* $[X, Y]$ is defined as $\{[x, y] : x \in X, y \in Y\}$. The group generated by X is denoted $\langle X \rangle$. (The commutator subgroup is thus $\langle [G, G] \rangle$.) $C_G(A)$ denotes the centralizer of A in G, $C_G(A) := \{g \in G : (\forall a \in A)(ga = ag)\}$. $Z(G) := C_G(G)$. A *centralizer subgroup* is one of the form $C_G(X)$ for some X; equivalently, a subgroup H with $C_G(C_G(H)) = H$. We say that a subset of a group is *normal* if it is conjugation invariant.

(2) We will denote the image of a subset X of G modulo a normal subgroup N by $X/N \subset G/N$.

(3) The *commensurator* $\mathrm{Comm}(\Lambda)$ of a symmetric set $\Lambda \subset G$ is the group of elements $g \in G$ with $g\Lambda g^{-1}$ commensurable to X. When Λ is an approximate lattice, the commensurator always contains the group $\tilde{\Lambda}$ generated by Λ.

(4) A *vector group* is a topological group isomorphic to $\mathbb{R}^n$ for some n. A *Lie group* is a group with a real analytic structure. (We will consider p-adic groups too, as algebraic groups with their natural p-adic topology, but will not use the term Lie groups for them.)

(5) By a *basic abelian group* we mean a compactly generated locally compact Abelian group with no nonzero compact subgroups. Such a group has the form $A = \mathbb{Z}^l \times \mathbb{R}^n$, $l, n \in \mathbb{N}$.

(6) A normal subgroup N of G will be called *rigid* if it is closed in G, and can be written $N = DE$ with $D \cong \mathbb{Z}^n$, D central in G, and E a Euclidean space on which G acts by rigid motions; thus the action of G by conjugation factors through $O_n = Aut(E_n)$.

(7) A precompact subset of G will be called *rigid* if it is contained in some rigid normal subgroup.

(8) G is *compactly generated* if some compact $X \subset G$ generates G as a pure group. Given such a compact generating set X, let $f :$

$Free(X) \to G$ be the natural map from the free group generated by X to G; then G is *compactly presented* (via X) if ker f is generated (algebraically) by words in X of some bounded length r.

(9) $cl(S)$ dentoes the closure of S, when S is a subset of a given topological space.

(10) We will say that a set $W \subset G$ is *discrete* in G if it has no accumulation points in G. W is *uniformly discrete* if 1 is not an accumulation point of $W^{-1}W$.

(11) A subset $W \subset G$ is *precompact* if the closure of W is compact.

(12) Let a group G act on a metric space X by isometries. We say that a subset W of G is *bounded* (with respect to this action) if for some $a \in X$, $Wa := \{w(a) : w \in W\}$ is contained in a ball of finite radius. Equivalently, for *all* $a \in G$, Wa has finite radius. (Since $d(a, b) = d(ga, gb)$, if $d(a, ga) \leq \beta$ we have $d(b, gb) \leq d(b, a) + d(a, ga) + d(ga, gb) \leq \beta + 2d(a, b)$.)

(13) A quasimorphism is an quasi-homomorphism $f : G \to \mathbb{R} : K$ for some compact $K \subset \mathbb{R}$. It is *homogeneous* if $f(x^n) = nf(x)$. Any quasimorphism is uniquely a sum of a bounded function into $\mathbb{R}$ and a homogeneous quasimorphism ([41].) Note that a homogeneous quasimorphism f is symmetric, i.e. $f(x^{-1}) = -f(x)$, since $f(x) + f(x^{-1})$ is also homogeneous, and bounded. We also have $f(y^{-1}xy) = f(x)$, since $|f(y^{-1}x^ny) - f(x^n)|$ is bounded.

(14) Let G_i be a topological group, for $i \in I$. Let C_i be an open subgroup of G_i. The restricted product $\Pi'G_i$ is defined to be the subgroup G of Π_iG_i consisting of elements $(a_i)_{i \in I}$ with $a_i \in C_i$ for all but finitely many $i \in I$. If each G_i is locally compact and almost every C_i is compact, then G is locally compact.

Lemma C.2. *Let A be a basic abelian group, and H a group of automorphisms of A. Let D be a dense subset of A, and assume Hd is precompact, for each $d \in D$. Then H preserves an innder product on A^0; and one can write A as a direct sum $A = A^0 \oplus B$, with $B \cong \mathbb{Z}^k$ fixed pointwise by a finite index subgroup H_1 of H.*

Proof. We have $A^0 \cong \mathbb{R}^n$. Since D is contained in no proper subspace of A^0, one can find a basis $c_1, \ldots, c_n \in D$ for $A^0 = \mathbb{R}^n$. So $Hc_1, \cdots, Hc_n$ are precompact. Let G be the subgroup of $GL_n(\mathbb{R}) = End(A^0)$ preserving $cl(c_i^H)$ for each i. Then it is clear that G is a compact subgroup of $GL_n(\mathbb{R})$; hence G preserves an inner product on $\mathbb{R}^n$ (namely the integral over G of the G-conjugates of the standard inner product). Thus H too preserves this inner product; and so every H-orbit on A^0 is precompact.

Next consider the action on $A/A^0 \cong \mathbb{Z}^k$. Since A/A^0 is discrete, the image of D is all of A/A^0. Let $e_1, \ldots, e_k$ be a $\mathbb{Z}$-basis for A/A^0. Then the orbit E_i of e_i is finite; we have restriction maps $H \to Sym(E_i)$; the common kernel of these maps is a finite index subgroup H_1 of H, fixing each e_i and hence fixing A/A^0. Lift e_i to a point d_i of D.

Claim . Some point b_i of $d_i + A^0$ is fixed by H_1.

Proof. Since H_1 fixes A/A^0, it leaves invariant the A^0-coset $d_i + A^0$. Also, $H_1 d_i$ is precompact. The closed convex hull of $H_1 d_i$ is a compact, convex, H_1-invariant subset of $d_i + A^0$. Since the Euclidean distance is preserved, so is Lebesgue measure, and so also the center of mass of this convex set; it is a fixed point for the action of H_1. $\square$

Let $B = \langle b_1, \ldots, b_k \rangle$. Since the image of B modulo A^0 is $\mathbb{Z}^k$, we also have $B \cong \mathbb{Z}^k$. The decomposition we promised is achieved.

$\square$

Discrete groups whose every conjugacy class is finite (called FC-groups) were studied by I. Schur, R. Baer, and B.H. Neumann, see [54]. This was later extended to locally compact groups, and precompact conjugacy classes, by Ushakov and Tits [65], with emphasis on the connected case, and by [24], [67] in general. Roughly speaking, such groups are extensions of abelian groups by compact ones. We require a version describing a single precompact conjugacy class. Thanks to Yves de Cornulier for explaining the relevant literature; previously, the deduction of this local version was a little more elementary but considerably more elaborate.

Proposition C.3. *Let H be a locally compact topological group. Let K be a normal precompact subset of H. Then there exist closed normal subgroups $N \leq A \leq H_1 \leq H$ with H/H_1 finite, N compact, with A/N basic abelian and rigid in H_1/N, such that $\langle K \rangle \cap H_1 \leq A$. Moreover, $N, A \leq cl(\langle K \rangle)$.*

Proof. Let $G = cl(<K>)$ be the closed subgroup of H generated by K. Then G is normal in H.

By [15] 2.C.6, a locally compact group is compactly generated iff every open subgroup U generates it, once it is augmented by some finite set. This condition clearly holds for G, since the compact set $cl(K)$ generates a dense subgroup; it suffices to add to U a finite set F such that $cl(K) \subset FH$. Thus G is compactly generated.

Let $B(G)$ be the union of all precompact conjugacy classes of G. Then $B(G)$ contains $<K>$, so it is dense. By [16], Theorem 2.8, B(G) is closed; hence must equal G.

Now [24], Theorem 3.20, applies; by this theorem, G has a characteristic compact normal subgroup P, with G/P basic abelian. Now H acts on G/P by conjugation. Let D be the image of $< K >$ in G/P; any $d \in D$ has pre-compact orbit under the action of H. By Lemma C.2, a subgroup H_1 of finite index in H acts rigidly on G/P. Let $A = G \cap H_1$, $N = P \cap H_1$; then all the statements are clear. $\square$

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