DEFINABILITY PATTERNS AND THEIR SYMMETRIES

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Abstract. We identify a canonical structure Core (T) associated to any first-order theory T, reflecting patterns of partial definability of types, uniformly in any type space over a model of T. The core generalizes the imaginary algebraic closure in a stable theory and the hyperimaginary bounded closure in simple theories. The core admits a compact topology, not necessarily Hausdorff, but the Hausdorff part can already be bigger than the Kim-Pillay space of T, and in fact accounts for the general Lascar group.

Using the core structure, we obtain simple proofs of a number of results previously obtained using topological dynamics, but working one power set level lower. The Lascar neighbour relation is represented by a canonical relation L_1 on Core (T); the general Lascar group G_{Las} of T can be read off this compact structure. This gives concrete form to results of Krupiński, Newelski, Pillay, Rzepecki and Simon, who used topological dynamics applied to large models of T to show the existence of compact groups mapping onto G_{Las}.

In an appendix, we show that a construction analogous to the above but using infinitary patterns recovers the Ellis group of [33], and use this to sharpen the cardinality bound for their Ellis group from $\beth_5$ to $\beth_3$, showing the latter is optimal.

There is also a close connection to another school of topological dynamics, set theory and model theory, centered around the Kechris-Pestov-Todorčević correspondence. We define the Ramsey property for a first order theory, and show - as a simple application of the core construction applied to an auxiliary theory - that any theory T admits a canonical minimal Ramsey expansion T^{ram}. This was envisaged and proved for certain Fraissé classes, first by Kechris-Pestov-Todorčević for expansions by orderings, then by Melleray, Nguyen Van Thé, Tsankov and Zucker for more general expansions. We also show that for a complete theory T in a countable language with prime model M, the universal minimal flow F of $\mathrm{Aut}(M)$ can described as the space of expansions of M to a model of T^{ram}.

1. Introduction

Among the gems uncovered in Shelah's work on stable theories, but applicable to all first order theories, not least was Galois theory for imaginary algebraic elements ([37]). Following the introduction of imaginaries - quotients by definable equivalence relations - there is a duality between the definable closures of algebraic elements, and closed subgroups of finite index of a certain profinite group

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Gal_{sh}, the Galois group of the theory. For Shelah this served as background to a fundamental result, the finite equivalence relation theorem, that will be recalled below.

Extending the independence theorem for general simple theories, Kim and Pillay [22] required quotients by equivalence relations that are not definable, but only intersections of definable relations. This again led to a beautiful Galois correspondence in any first order theory between bounded hyperimaginaries and subgroups of a compact Hausdorff group Gal_{KP}. It was later incorporated into continuous logic [7], and used in relating finite combinatorial structures with Lie groups.

Kim and Pillay were guided by work of Lascar, who studied small quotients of the automorphism group of large saturated models; he showed the existence of a maximal such quotient group G_{Las}. Lascar denoted his group by G, writing in parentheses: 'G for Galois' and asking whether it coincides with the compact group Gal_{KP}. The latter question was answered negatively by Ziegler. The Galois nature of Gal_{KP} has been clearly demonstrated; closed subgroups correspond to definably closed subsets of the bounded (a.k.a. compact, a.k.a. algebraic) closure of $\emptyset$. But no evidence of the Galois nature of the full group G_{Las} has emerged; it does not take part in a meaningful Galois correspondence, and is not the automorphism group of any known structure associated with T.

One can of course define, on each sort V of a model M of T, a *set* V^M_{Las} of Lascar types as a quotient of the type space of M; but as no topology or algebraic structure is defined on it. Given T alone, only the cardinality (or 'Borel cardinality', see [19]) of this set is actually well defined; and certainly G_{Las} cannot be recovered from it. Moreover, the Lascar group may leave no trace on any sort belonging to finitely many variables.

Our aim is to find an alternative Galois group canonically associated with T, that incorporates G_{Las} within it.

Krupiński, Pillay and Rzepecki, [31], [34], [32] showed intriguingly that G_{Las} is (in many ways) a quotient group of a compact Hausdorff topological group; this suggests that V_{Las} too is a quotient of some canonical space, carrying some structure deduced from T. In [33] a cardinality bound was found on the cardinality of the Ellis groups and thus in principle provided a canonical compact cover of G_{Las}, though at some remove from definable sets of T. ([33] found a cardinality bound of $\beth_5$; we will show that the correct bound for their group is $\beth_3$.) This followed a program initiated by Newelski and bringing to work the Ellis groups of topological dynamics, going through certain semigroups.

Here we start over, in a sense going back to the original setting of the finite equivalence relation theorem. Morley realized that type spaces *over models* carry information about a theory that goes far beyond the space of types over $\emptyset$. The difficulty, of course, is to extract model-independent information; Morley introduced a topology, with properties independent of the (sufficiently saturated) base model, that led quickly to the notion of ω-stability and Morley rank. Shelah saw that one can work with local type spaces: for a finite set γ of formulas and a distinguished variable y, consider the Boolean algebra of formulas $\phi(x,b)$ with b from M, $\phi \in \gamma$, and the Stone space $S_\gamma(M)$ [1]. This led to stability. In both cases, the ranks essentially exhaust the information available from the topology alone.

We will enrich the topology to a relational structure on these type spaces, in a certain language $\mathcal{L}$, the language of definable patterns of T. We will then find a canonical structure Core(T) - the universal pattern space of T - organizing this information, depending on the theory alone. Core(T) is embeddable in the type space of any model, hence of cardinality $\leq 2^{|T|}$. The automorphism group G of Core(T) thus acts on a geometry directly constructed from T.

$\mathcal{J} =$ Core(T) is compact, but not necessarily Hausdorff; however each complete type of the core has a canonical compact Hausdorff quotient structure by a quantifier-free definable equivalence relation, and can be viewed as an imaginary sort. There is a compact Galois duality between these sorts and their automorphism groups.

Let $\mathfrak{g}(T)$ be the group of *infinitesimal* elements of Aut(Core(T)) in its action on Core(T), namely those that stabilize the closure of any open set. Then $\mathcal{G} = G/\mathfrak{g}$ is compact Hausdorff quotient group of Aut(Core(T)). As Core(T) is homogeneous for atomic types and may have a continuum of pairwise orthogonal ones, Aut(Core(T)) can have cardinality $\beth_2(|T|)$; but in any case we have $|\mathcal{G}| \leq 2^{|T|}$ (Corollary 3.26).

We now come to the Lascar neighbour relation L_1; it holds between two elements of a sufficiently saturated model if they have the same type over some elementary submodel, see 4.1. The pattern language $\mathcal{L}$ can define it on $S(M)$; it is represented on $\mathcal{J}$ by the same formulas. L_1 further induces a relation L_1 on the Hausdorff part $\mathcal{J}_h$. This also determines a distinguished compact subset L_1 of $\mathcal{G}$, namely the automorphisms that move elements no further than to their Lascar neighbours. The general Lascar group can then be interpreted as $\mathcal{G}/\langle L_1\rangle$. Of course at this point one may prefer *not* to factor out $\langle L_1\rangle$, but treat and $(\mathcal{G}, L_1)$ as the right invariant of T in the world of compact topological groups. As a check, while the Lascar group and Lascar strong types may not be visible at the level of finitely many variables, $(\mathcal{G}, L_1)$ behaves in the model-theoretically expected way, reducing as a projective limit of automorphism groups of the finitary spaces.

1.1. **Definability patterns.** To explain the relational structure on $S_\gamma(M)$, recall the 'fundamental order' of the Paris school presentation of stability theory [36], and specifically the maximal classes of this order. For each type $p(x)$ over a model M, and formula $\phi(x,y)$, we let $(d_p x)\phi(x,y)$ denote the set of $b \in M^y$ with $\phi(x,b) \in p$. This is simply a subset of M. In some cases it is 0-definable, so that

[1] more properly denoted $S_{\gamma;x}$ to point out the distinguished variables; but we usually view this data as embedded in γ.

$(d_p x)\phi(x,y) = \theta(y)$; equivalently, the formulas $\phi(x,y) \& \neg\theta(y)$ and $\neg\phi(x,y)\&\theta(y)$ are *omitted* or *not represented* in p, meaning that no substitution instance lies in p. A type $p \in S(M)$ is *maximal* in the fundamental order if no type represents a strictly smaller set of formulas.

More generally, given a k-tuple $(p_1,\ldots,p_k)$ of types, a k-tuple $(\phi_1,\ldots,\phi_k)$ of formulas in matching variables, and a formula α, we can say that $t = (\phi_1,\ldots,\phi_k;\alpha)$ is represented in $(p_1,\ldots,p_k)$ if for some $b \in \alpha(M)$ we have $\phi_1(x,b) \in p_1, \cdots, \phi_k(b) \in p_k$. We let $\mathcal{R}_t$ be a relation symbol, asserting that t is *not* represented. This we view as a k-ary relation on any type space $S(M)$; forming a language $\mathcal{L}$. Let $\mathcal{T}$ be the universal theory of $S(M)$ with this structure; it does not depend on the choice of M.

Theorem 1.2. *$\mathcal{T}$ has a unique universal existentially closed model $\mathcal{J}$. It has cardinality at most λ_T, the number of finitary types of T. The automorphism group* $\mathrm{Aut}(\mathcal{J})$ *has a canonical compact topological group quotient $\mathcal{G} = \mathrm{Aut}(\mathcal{J}_h)$, where $\mathcal{J}_h$ is the union of the Hausdorff imaginary sorts of $\mathcal{J}$. There exists a canonical surjective homomorphism $\mathcal{G} \to L$, where L is the Lascar group of T, with compactly generated kernel.*

We will call $\mathcal{J}$ the *core* of T; though the essential point is that it is a relational structure, rather than simply a topological one. The conjugacy class of a tuple in $\mathcal{J}$ is determined by the atomic type in $\mathcal{L}$; such types will be called *pattern types*. They will be defined more syntactically below.

It may happen that an atomic 2-type of $\mathcal{L}$ restricts to the same 1- type ρ in each coordinate, but nevertheless includes partial definability relations that rule out equality of the two 1-types over a model. In this case, two copies of ρ must be included in $\mathcal{J}$. It is the symmetry between them that the group $G = \mathrm{Aut}(\mathcal{J})$ expresses. In particular when $G = 1$ (and only then), $\mathrm{Aut}(\mathcal{J}) = 1$, $\mathcal{J}$ reduces precisely to the fundamental order. In general the maximal elements of the fundamental order on a given sort can be viewed as the type space of the corresponding sort of $\mathcal{J}$.

As an example, consider an antireflexive relation $R(x;y,z)$. Let T be the model completion (the only rule is $\neg R(x,y,y)$.) We consider type spaces in this single relation, with distinguished variable x (the case of complete types is not really different.) Then 1-types $tp(a/M)$ over a model M describes a directed graph on M, defined by $R(a,y,z)$. Here $\mathcal{J}$ will have four elements, corresponding to the empty graph, the complete graph, and two copies of a "linear ordering" [3]. Taken individually there is nothing more to say about the linear orderings, but taken as a pair, $\mathcal{J}$ asserts that one is precisely the opposite ordering of the other. The evident symmetry of the two orderings is in this case the automorphism group of $\mathcal{J}$.

[3](the pattern type represents the axioms of linear orderings, rather than an ordering on any specific set.)

Evidently $\mathfrak{J}$ knows about Ramsey's theorem. Ziegler's examples alluded to above yield other examples of finite $\mathfrak{J}$, permuted by other cyclic groups. This connects to the work of another large school connecting model theory with topological dynamics, around the Kechris-Pestov-Todorčević correspondence [21]. There has been relatively limited interaction between them so far; notably [20] show that $\aleph_0$-categorical structures with the Ramsey property admit a functorial joint embedding property, and in particular have trivial Lascar group. We return to this below, extending the connection to arbitrary first-order theories. As in [17], a weaker property than Ramsey suffices, namely total definability of $\mathfrak{J}$ for T itself rather than the 'second-order' expansion T^* that forms the substrate for Ramsey theory.

1.3. First order logic will be used in this paper at three levels of generality. The most basic is a complete first order theory T. It will be convenient to Morleyize it, i.e declare all formulas to be atomic. This done, T becomes the model completion of the universal part $T_\forall$ of T, and thus carries the same information.

More general is the setting where we are given only a universal theory T; we will be interested especially in the class of existentially closed models, that may or may not be elementary. If the class of models of T has the joint embedding property, we will say it is *irreducible*. (Equivalently, T is the universal theory of some structure; or the universal theory of a class of models with the joint embedding property.) If $Mod(T)$ admits amalgamation, we will say it is a Robinson theory.

On the other hand given a complete first order theory, we may Morleyize it by adding names for each definable set; it becomes a relational language with quantifier-elimination. A theory with quantifier elimination is completely determined by the universal part.

Thirdly, we will consider *primitive universal theories*, described in the next section; where closure under negation is not assumed. Our main task is to describe algebraic closure in the positive setting; it goes a little beyond the bounded closure of [22] or the (compact) algebraic closure of [7].

The word 'definable', unqualified, will always mean: definable without parameters.

1.4. **Patterns and definable types.** It is also possible to introduce the core as a relational structure by means of a direct description of its types spaces.

Let T be an irreducible universal theory, with a distinguished sort V. We also fix a 'parameter sort' P, and assume γ is a pattern sort, i.e. set of formulas ϕ on $P \times V$, including all formulas on V alone. [4]

[4] The focus on $(P, V; \gamma)$ allows a more elementary description, but does not lose any generality, since we allow V to be a projective limit of sorts; we could deal with any family of sorts by taking products.

Let $L_V(X)$ be the language L of T, and augmented with some additional predicates $X_1, \ldots, X_m$, standing for subsets of V. We will write $X = (X_1, \ldots, X_m)$.

Assume given a universal theory T_{ext} of $L_V(X)$, restricting to T on L. A *(maximal) pattern type* for this data is a maximal universal theory p for $L_V(X)$, containing T_{ext}.

The case we will be concerned with in practice is the theory T_{ext} of γ-externally definable sets relative to T; this will correspond to a type of an element of the core at V. Assume here that $\gamma = \{\gamma_1, \ldots, \gamma_n\}$, with variable x and parameter sort V; and let T^{ext}_γ be the $L_V(X)$-theory, whose models are the structures $(M, A_1, \ldots, A_n)$ such that there exist $M \leq N \models T$ and $c \in P(N)$ with $A_i = \{v \in V(M) : N \models \gamma_i(c, v)\}$.

Let $M \models T_V, A \subset V(M)$. An L_V-universal theory p is *finitely satisfiable* in (M, A) if for any existential sentence ψ true in M, there exists a (finite) substructure M_0 of M with $M_0 \models \psi$, and such that $(M_0, A) \models p$.

Equivalently, there exists an elementary extension (M^*, X^*) of (M, X) and an embedding $f : M \to M^*$, such that $(M, f^{-1}(X^*)) \models p$.

Let us say that two universal sentences $(\forall x)\psi(x), (\forall y)\phi(y)$ of $L(X)$ are *incompatible* if along with T_{ext} they jointly imply a universal sentence of L, not already in T.

If p is a type in the language of patterns, maximality amounts to this: for any quantifier-free formula $\phi(x_1, \ldots, x_n)$ of $L(X)$, *either* $(\forall x)\phi(x) \in p$, *or* some incompatible universal sentence $(\forall y)\psi(y)$ lies in p.

Lemma 1.5. *Let $M \models T$, and let $A \subset V(M)$ be externally γ-definable. Let p_0 be a universal theory of $L(X)$, with $(M, A) \models p_0$. Then some γ- pattern type $p \supseteq p_0$ is finitely satisfiable in (M, A).*

Proof. By Zorn's lemma, there exists a universal theory $p \supseteq p_0$ of $L(X)$ that is finitely satisfiable in A, and is maximal with this property. We have to show that p is a pattern type. Let $\phi(x_1, \ldots, x_k)$ be a quantifier-free formula of $L(X)$. Then $p \cup \{(\forall x)\phi\}$ is either equal to p, or is no longer finitely satisfiable in (M, A). In the latter case, by definition, there exists an L-formula $\theta(x_1, \ldots, x_m)$ consistent with T, such that if $M \models \theta(a_1, \ldots, a_m)$ and $M_0 = \{a_1, \ldots, a_m\}$ then $\neg\phi$ is realized in (M_0, A). Let

$$\psi(x_1, \ldots, x_m) = \theta(x_1, \ldots, x_m) \to \bigvee_{y_1, \ldots, y_k} \neg\phi(y_1, \ldots, y_k)$$

where $y_1, \ldots, y_k$ range over k-tuples from among $x_1, \ldots, x_m$. Then $p \models \psi$, and $(\forall x)\psi$, $(\forall x)\phi$ are incompatible. $\square$

A *definable* pattern type is one that simply asserts that X coincides with some 0-definable set of T (by a qf formula without parameters). Maximality is then clear. We say T is *has definable patterns* (for a given pattern sort γ) if every maximal pattern type (in the sort γ) is definable.

Theorem 1.6. *Let T be an irreducible universal theory. There exists a unique minimal expansion of T to a universal theory T^{def} that has definable patterns. The self-interpretations of T^{def} over T form a group, isomorphic to* $\mathrm{Aut}(\mathfrak{J})$.

If T is stable, then T^{def} simply names the imaginary algebraic constants, and so amounts to 'working over $\mathrm{acl}^{eq}(\emptyset)$' in the sense of Shelah. If T is NIP, then T^{def} is NIP.

The proof, and precise definition of minimality and uniqueness, will be given at the end of § 3 (Proposition 3.17.)

1.7. **Elementary Ramsey theory.** Structural Ramsey theory is usually defined in terms of isomorphism types of substructures, or complete types. See [49], [38], [15] and references there. [6] However it is also very natural in a first-order setting, using formulas in place of complete types. This extends to continuous logic, and brings out the unity in instances of structural Ramsey theory such as for affine spaces over a finite field, Dvoretzky-Milman for Hilbert spaces, and Van den Waerden for arithmetic progressions.

In Ramsey's theorem, one is given a set M. We then consider $D = M^n$ or $D = M^{[n]}$, and an *arbitrary* subset A of D. The desired outcome is a 'large' subset M_0 of M, such that A, restricted to $D(M_0)$, has a simple, explicitly described structure.

In the structural generalization, $M \models T$ is a structure. D is a sort (or a definable set of M^n). Again A is an arbitrary subset of $D(M)$. We seek a large M_0 such that A has as regular a structure as possible on $D(M_0)$. Here 'large' means: a finite substructure of M realizing a prescribed existential sentence of T. Equivalently, M_0 can be taken to be a copy of M in some ultrapower (M^*, A^*) of (M, A).

While Ramsey theory appears to be second-order, considering arbitrary colorings on D, a simple device does present it as a special case of the theory of patterns: we simply introduce a new sort P and a relation that allows P to parameterize colorings on D, with no constraints. A pattern type associated to the new theory will be referred to as a free pattern type for the original one. Thus a free pattern type is simply a maximal universal theory for $L_V(X)$, whose restriction to L is T.

Now Lemma 1.5 tells us immediately what 'as regular as possible' can mean. We cannot do better than a free pattern type, and some free pattern type can always be achieved. Thus the basic structural Ramsey question for a theory T changes from a yes/no question to a more qualitative and functorial one: describe the free pattern types of T. The simplest case is that every free pattern type is *definable*; in this case we will call the theory *Ramsey*. We will see that every

[6]Krzysztof Krupiński informed me that with Anand Pillay, he developed an extension of [21] to such a setting.

universal theory has a canonical Ramsey expansion, whose automorphism group is an interesting invariant of T.

Expand the language by an an additional sorts P and binary relation $R \subset P \times V$. However we keep the same universal theory $T_\forall$, adding no axioms on R; we denote it T^*_V. Clearly T^*_V is an irreducible universal theory. We define $\mathrm{Core}^{\,ram}(T) := \mathfrak{J}^* := \mathrm{Core}_{\,R}(T^*_V)$, i.e. the parameter sorts of the core of the derived theory T^*_R. We also write $\mathcal{L}^{ram}(T)$ for the language of $\mathrm{Core}^{\,ram}(T)$.

Let $\gamma = \gamma^{ram}(T)$ be generated by finitely many formulas $R(x_i, y)$ and arbitrary formulas of L. A γ-pattern type for T^* will be called a *free pattern type* for T at V.

Definition 1.8. We say that a theory T is a *Ramsey theory* at V (or has the Ramsey property at V) if all free pattern types for T on V are definable. It is *everywhere Ramsey* if it is Ramsey at V for all sorts of T.

In view of Lemma 1.5, an equivalent form:

T is Ramsey at V iff for every $M \models T$ and every $A \subset V(M)$, there exists a formula $\theta(x)$ such that for every formula $\phi(x_1, \ldots, x_n)$ consistent with T, there exist $a_1, \ldots, a_n \in M$ with $M \models \phi(a_1, \ldots, a_n)$, and

$$M \models \theta(a_i) \iff a_i \in A$$

Equivalently, some elementary extension $(M*, A*)$ of (M, A) contains a copy M' of M such that $A * \cap M'$ is a 0-definable subset of M'.

Remark 1.9. Assume T is $\aleph_0$-categorical, with quantifier-elimination in a relational language, and with a single sort V; let $M \models T$. The above definition relates to the terminology of [21], [15] in the following way. Let $\mathcal{A}$ be the class of finite models of $T_\forall$. For any $A \in \mathcal{A}$, we have an imaginary sort V^A coding embeddings of A into V; so $V^A(M)$ can be canonically identified with $\mathrm{Hom}(A, V(M))$. We also have $V^{[A]} = V^A/Aut(A)$, the set of substructures of V isomorphic to A. Now T (or rather the class of finite models of T) has the Ramsey property in the sense of [21], [15] iff T has the Ramsey property at $V^{[A]}$ for all $A \in \mathcal{A}$. Also by the KPT correspondence, $\mathrm{Aut}(M)$ is extremely amenable iff T has the Ramsey property at V^A for all A (or equivalently at V^n for all n.) Note that $V^A, V^{[A]}$ are complete types; so definability of the coloring amounts to constancy.

Theorem 1.10. *Let T be a complete theory. There exists a unique minimal everywhere Ramsey expansion T^{ram}.*

The self-interpretations of T^{ram} over T form a group, $G^{ram}(T)$.

See Proposition 5.4 for a precise definition of minimality and uniqueness.

In case T is the theory of pure equality, T^{ram} will be the theory of dense linear orderings (up to a strong bi-interpretability). In general, T^{ram} is is a complete first order theory in a bigger language, whose additional relations are indexed by the elements of the dual sorts of the theory T^*. The proof and various examples are in § 5.

In Appendix B, we will show also that when T has countable language and a prime model M , $G = \mathrm{Aut}(M)$, the space of expansions of M to T^{ram} is the universal minimal flow U of G, i.e. it has no closed G-invariant subspaces, and admits a continuous G-invariant map into any other compact space with this property. We can also write: $\mathrm{Hom}(J^*, S(M)) = U$. Assuming L is countable, J can have cardinality continuum, but the statement nonetheless has a taming effect on U; whereas a priori we know only that $|U| \leq \beth_2$, J is parameterized by a compact structure of cardinality continuum, uniquely determined by its own theory, which in turn is closely controlled by T.

Theorem 1.10, and the result of the Appendix, generalize a line of theorems, beginning with the theory of pure equality by Glasner and Weiss; then for a wide class of $\aleph_0$-categorical theories by [21], explaining the connection to structural Ramsey; the restriction to $\aleph_0$-categorical theories, or Fraissé classes, was due to the approach using topological dynamics of the automorphism group G. The result was extended to all $\aleph_0$-categorical T provided the universal minimal flow of G is metrizable, in [25],[52], [6]. In these cases, T^{ram} has locally finite language relative to T (see Remark B.7.) [15] showed that the hypothesis is not always valid.

1.11. **The Ellis group.** In Appendix A we consider a type-definability analogue $\bar{\mathfrak{J}}$ of $\mathfrak{J}$. It relates to the notion of *content* of [33], as $\mathcal{L}$ relates to the fundamental order of [36]. We show that $\mathrm{Aut}(\bar{\mathfrak{J}})$ is isomorphic to the Newelski-Ellis group. This presents the Ellis group as the automorphism group of a natural structure, and leads immediately to a bound of $\beth_3$, improving the bound $\beth_5$ of [33]. We show by example that $\beth_3$ is in fact optimal for the Ellis group.

1.12. **Open problems.** Many directions are left open; here are a few.

(1) Develop a relative theory, with $\mathrm{Core}\,(T_a)$ parameterized over a definable set.
(2) Develop the Galois correspondence. See 3.18, but also:
(3) There exists a 1-1 correspondence between a family of subgroups of $\mathrm{Aut}(\mathfrak{J})$, and a family of reducts of T^{ram} containing T; describe the closed subgroups and the closed expansions of T
(4) Under what circumstances, apart from Proposition 5.2, does the definable-pattern expansion T^{def} have a model completion?
(5) Connection with NIP and with honest definitions. Characterize the pattern types.
(6) Let D be strictly minimal. Is the canonical Ramsey expansion at D, a NIP theory?
(7) Develop $\mathrm{Core}\,(T)$ for a primitive universal theory T.

(8) Explore further the duality between the parameter and variable sorts.
(9) Develop the continuous logic generalization of [18], i.e. generalized finite imaginaries to generalized compact Hausdorff imaginaries. Does the Hausdorff part of $\mathfrak{J}$ comprise the entire generalized algebraic closure in this sense?
(10) The same for positive logic. Is $\mathrm{Core}\,(T)$ the entire absolute algebraic closure?
(11) The definable group analog; a more concrete description of canonical compact covers of G/G^{000} for a definable or ind-definable group G should become accessible. This may shed light on the Massicot-Wagner problem of describing general approximate subgroups.
(12) Investigate the degree of effectiveness of $\mathrm{Core}\,(T)$ or rather of the pattern type spaces that determine it. (See Remark 2.9.)
(13) We have not ruled out that $G = \mathrm{Aut}(\mathfrak{J})$ is outright Hausdorff, i.e. $\mathfrak{g}_! = 1$; but see Example 3.36. Nov. 2021 update: this is now shown in Example 5.9. For any discrete group Γ we let T_Γ be the theory of free Γ-actions. We show that the universal minimal flow of any discrete group Γ is dual to $\mathfrak{J}(T_G)$, and use this to give an example of a pp type with non-Hausdorff automorphism group. Many interesting questions on this connection with topological dynamics remain; see Question 5.10.

The constructions in the body of the paper are entirely self-contained; beyond some elementary lemmas on Hausdorff quotients (Appendix C), no topological dynamics is used. Only in the appendices, where we describe the universal minimal flow and the Ellis group of the type space flow in our terms, do we assume knowledge of the definitions of these objects.

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2. Existentially closed models

Following work of Shelah, Pillay, and Ben-Yaacov [41], [40],[3], the setting of existentially closed structures of universal theories, and of positive logic, has come to be viewed as a natural and mild generalization of the usual first order context. In particular basic stablity and sometimes simplicity was thus generalized by these three authors. For the more basic theory of saturated models this was carried out earlier by Mycielski, Ryll-Nardzewski and Taylor [29], [30], [43], soon after the work in the first-order case by Jónsson, Keisler, and Morley-Vaught (see [26]). In this section we give a self-contained treatment of the facts we need, in current terminology.

Let $\mathcal{L}$ be a (many-sorted) language. [9] A *positive primitive* (pp) formula is one of the form $(\exists x_1, \dots, x_k) \bigwedge_{j=1}^{l} \phi_j(x)$, with ϕ_j atomic. We regard pp formulas as the fundamental ones for $\mathcal{L}$, though occasionally we will consider slightly higher ones. A theory axiomatized by negations of pp sentences will be called *primitive-universal.* The set of such sentences true in a structure M is denoted $Th_{p\forall}(M)$. By Lemma 2.3, for existentially closed models, the full universal theory (in the usual sense) is completely determined by the primitive universal theory.

A primitive universal theory $\mathcal{T}$ of $\mathcal{L}$ is called *irreducible* if it is the primitive universal theory of some model. Thus if $\mathcal{T} \models \alpha \vee \beta$, with α, β primitive universal, then $\mathcal{T} \models \alpha$ or $\mathcal{T} \models \beta$. Equivalently, any two models of $\mathcal{T}$ admit homomorphisms into some third model of $\mathcal{T}$. Note that since α, β are primitive universal, this can be a weaker condition than irreducibility as a universal theory.

We will consider irreducible theories $\mathcal{T}$, and will be interested only in such models. In other words we are really concerned with $\mathcal{T}^{\pm} := \mathcal{T} \cup \{\psi : \mathcal{T} \not\models \neg\psi\}$ where ψ ranges over pp sentences.

Definition 2.1. A model A of $\mathcal{T}$ is *existentially closed* (abbreviated e.c.) if for every homomorphism $f : A \to B$, where $B \models \mathcal{T}$, and any $\mathcal{L}_A$-pp sentence ϕ allowing equality, if $B_A \models \phi$ then $A_A \models \phi$. Here $\mathcal{L}_A$ is $\mathcal{L}$ expanded by constants for the elements $a \in A$; they are interpreted as a in A_A and as $f(a)$ in B_A. [10]

The usual direct limit construction shows that any model A of $\mathcal{T}$ admits a homomorphism $f : A \to B$ into an existentially closed model B of $\mathcal{T}$, with $|B| \leq |\mathcal{L}_A| + \aleph_0$. Any existentially closed model of $\mathcal{T}$ is a model of $\mathcal{T}^{\pm}$.

Remark 2.2.
1. Any homomorphism from an e.c. model of $\mathcal{T}$ to a model of $\mathcal{T}$ must be injective, and indeed an embedding, i.e. an isomorphism onto the image.
2. Let E be a conjunction of pp-formulas. Assume it is a strong congruence: in any model A of $\mathcal{T}$, and for any non-logical symbol $R(x, x_1, \dots, x_n)$ in the language, if $a, b \in A$ and $A \models E(a, b) \wedge \phi(a, a_1, \cdots, a_n)$ then $A \models \phi(b, a_1, \cdots, a_n)$. Then E coincides with equality in any existentially closed model of $\mathcal{T}$. (Such an E, a conjunction of basic formulas in fact, will exist in the theories of interest to us. and will save us the need to consider separately atomic formulas built from the equality symbol.)

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[9] We may or may not wish to allow a logical equality symbol in general; but we will use it in the definition of an existentially closed structure. In practice, Remark 2.2 (2) will allow us to restrict attention to formulas constructed without the equality sign, in particular in the definition of the pp topology on an e.c. model.

[10] If we begin with a logic without equality, allowing equality in ϕ has the effect of considering only e.c. models where two elements with the same atomic type over the entire model are equal; this is needed for a reasonable definition of the cardinality of the model. Any model can of course be collapsed to one with this ('Barcan') property, losing no meaningful information.

Let $\mathcal{T}$ be a primitive-universal theory. For two pp formulas $\phi(x), \psi(x)$, write $\phi \perp \psi$ if $\mathcal{T} \models (\neg \exists x)(\phi \wedge \psi)$. Part (1) of the following syntactical lemma is the substitute for the law of excluded middle. Part (2) refers briefly to possibly infinitary sentences, beyond the pp level. It follows from (2) that any two e.c. models of $\mathcal{T}$ share the same universal theory, and the same universal quantifications of Boolean combinations of pp formulas.

Lemma 2.3. *Let $\mathcal{T}$ be the primitive universal theory of M, and let E be an e.c. model of $\mathcal{T}$. Let $\phi, A_i (i \in I), B_j (j \in J)$ be pp formulas of $\mathcal{L}$. Assume I, J are finite, or more generally that any set of cardinality $|I \cup J|$ of pp formulas that is finitely satisfiable in M is satisfiable in M. Then:*

(1) If $a \in E^x$, then either $E \models \phi(a)$ or $E \models \phi'(a)$ for some $\phi' \perp \phi$.

(2) Let ψ be the (possibly infinitary) sentence:

$$(\forall x)(\bigwedge_{i \in I} A_i \to \bigvee_{j \in J} B_j(x))$$

If $M \models \psi$, then $E \models \psi$.

(3) If M is $|E|^+$-pp-saturated, and $M \models (\forall x)\theta$ where θ is any (possibly infinitary) Boolean combination of pp formulas, then $E \models (\forall x)\theta$.

(4) If θ is any finite Boolean combination of pp formulas and $M \models (\forall x)\theta$, then $E \models (\forall x)\theta$.

Proof. (1) Say $\phi(x) = (\exists y)\psi(x, y)$, with ψ atomic. Let Δ be the atomic diagram of E, [12] and $\Delta' = \Delta \cup \{\psi(x, c)\} \cup \mathcal{T}$. If $\phi(a)$ fails, then Δ' is inconsistent since E is e.c.. So some finite conjunction $\psi'(a, d)$ is inconsistent with $\{\psi(x, c)\} \cup \mathcal{T}$. Let $\phi'(x) = (\exists u)\psi'(x, u)$. It follows that $\phi \perp \phi'$ and $E \models \phi'(a)$.

(2) We prove the contrapositive. Suppose that there exists c with $E \models A_i(c)$ for each i, but $E \models \neg B_j(c)$ for each j. Since E is e.c. and $B_j(c)$ fails, there must exist a pp formula $B'_j \perp B_j$ such that $E \models B'_j(c)$. Then for any finite $I_0 \subset I, J_0 \subset J$, $\{A_i(x), B'_j(x) : i \in I_0, j \in J_0\}$ is satisfiable in E, so $(\neg \exists x) \bigvee_{i \in I_0} A_i \vee \bigvee_{j \in J_0} B'_j$ is *not* a sentence of $\mathcal{T}$. Hence $\{A_i : i \in I\} \cup \{B'_j : j \in J\}$ is finitely satisfiable and hence satisfiable in M. But this means the implication does not hold in M.

(3) Under the assumptions, there exists an embedding $h : E \to M$; pp formulas and their negations are preserved by h since E is e.c.; hence arbitrary Boolean combinations are preserved; and universal quantifiers descend to substructures, as usual.

(4) Let M^* be an $|E|^+$-saturated extension of M. Then the pp-theory of M^* is $\mathcal{T}$ and $M^* \models (\forall x)\theta$; so (3) applies, and $E \models (\forall x)\theta$.

$\square$

[12] the diagram consists of atomic sentences of L_E true in E.

In particular if a pp formula R has k distinct solutions in some e.c. $M \models \mathfrak{T}^{\pm}$, then R has k distinct solutions in any $M \models \mathfrak{T}^{\pm}$. Thus the number of solutions of R is any e.c. model (viewed as a finite number or ∞) is the same.

2.4. **Morleyzation.** It is sometimes desirable to modify the language by a definitional expansion, so that every pp formula becomes atomic.[13] This can be done without changing the category of e.c. models.

Let us see this for the existential quantifier. Let $\phi(x, y)$ be atomic in the language $\mathcal{L}$, and let $\mathcal{L}^+ = \mathcal{L} \cup \{\Phi(x)\}$, where $\Phi(x)$ intended to stand for $(\exists y)\phi(x, y)$. Let $\mathfrak{T}^+$ be the theory consisting of all sentences $\neg(\exists x_1, \cdots, x_n) \bigwedge_j Q_j$, where each Q_j is a symbol of $\mathcal{L}^+$, and where if Q_j is replaced by $(\exists y)\phi$ in each occurrence, we obtain a consequence of $\mathfrak{T}$. For any $M \models \mathfrak{T}$, define a $\mathcal{L}$-structure M^+ by interpreting Φ as $(\exists y)\phi(x, y)$.

Claim . $M \mapsto M^+$, $N \mapsto N|\mathcal{L}$ define a 1-1 correspondence between e.c. models of $\mathfrak{T}$ and of $\mathfrak{T}^+$.

Proof. Let N be an e.c. model of $\mathfrak{T}^+$, $M = N|\mathcal{L}$, and let $a \in \Phi(N)$. Then the definition of $\mathfrak{T}^+$ implies that $(\exists y)\phi(x, y)$ is consistent with $\mathfrak{T}$ along with any pp formula true of a. Since N is e.c., we have $N \models (\exists y)\phi(x, y)$. Conversely, if $N \models (\exists y)\phi(x, y)$, then the axioms $\mathfrak{T}^+$ continue to hold if we modify N by setting $\Phi(a)$ to be true (since in any potential counterexample to an axiom, replacing each occurence of $\Phi(a)$ by $(\exists y)\phi(x, y)$ would yield a counterexample in M to an axiom of $\mathfrak{T}$.) Again since N is e.c., we have $\Phi(a)$. So we have $N = M^+$.

Next let us see that M is e.c. Let $f : M \to M'$ be a homomorphism. Then f extends to a homomorphism $f : M^+ \to (M')^+$, and the existential closudness of $M^+ = N$ immediately implies the same for M.

Conversely, assume M is e.c. Then $M = (M^+)|L$. It remains to show that M^+ is e.c. Let $g : M^+ \to N$ be a $\mathcal{L}^+$- homomorphism. To prove the Tarski-Vaught property, i.e. existential closedness of M^+ with respect to this map, we may compose g with any homomorphism $N \to N'$. So we may assume N is e.c.; and thus by the above, $N \models \Phi \iff (\exists y)\phi$. This easily implies the existential closedness of M^+. □

In particular, $\mathfrak{T}^+$ is irreducible if $\mathfrak{T}$ is; and $\mathfrak{T}^+$ eliminates the quantifier in $(\exists y)\phi(x, y)$.

One could similarly deal with finite disjunctions. If P is added to stand for $P_1 \vee P_2$, the axioms would be $\neg\exists x \bigwedge_j Q_j$, where each Q_j is P or an existing symbol, such that replacing each P with P_1 or P_2 (chosen arbitrarily) yields a consequence of $\mathfrak{T}$. For $M \models \mathfrak{T}$ we define M^+ naturally, and show as above that an e.c. model N of $\mathfrak{T}^+$ has the form M^+, with M an e.c. model M of $\mathfrak{T}$. Conversely if $M \models \mathfrak{T}$

[13] In the present paper, this section is used only in § 2.10 and in the footnote of Appendix A, but not in any of the main results.

is e.c., then M^+ is e.c.; for if $f : M^+ \to N$ is a homomorphism, $Th_{p\forall}(N) = \mathcal{T}^+$, we may assume N is e.c., etc.

In the setting of $|L|^+$-pp-saturated e.c. models one can even eliminate an *infinite* conjunction, $\bigwedge_i P_i$, by introducing a symbol P for it, obtaining a language $\mathcal{L}^+$. (For simplicity we consider a single conjunction, but any family can be handled in the same way.) We let

$$\mathcal{T}^+ = \{\neg(\exists x_1, \cdots, x_n) \bigwedge_j Q'_j : \mathcal{T} \models \neg(\exists x_1, \cdots, x_n) \bigwedge_j Q'_j\}$$

where $Q'_j = Q_j$ if Q_j is not one of the P_i, and $Q'_j \in \{P_i, P\}$ if $Q_j = P_i$. Note that $\mathcal{T}^+$ contains $\mathcal{T}$, and that in any model N of $\mathcal{T}^+$, if we re-interpret P_i by as $P_i(N) \cup P(N)$, we obtain again a model of $\mathcal{T}^+$.

Any model M of $\mathcal{T}$ expands canonically to a model M^+ of $\mathcal{T}^+$, with P interpreted as $\cap_i P_i$. (If M is $|\mathcal{L}|^+$- saturated, and then the universal primitive theory of M is precisely $\mathcal{T}^+$, showing that $\mathcal{T}^+$ is irreducible if $\mathcal{T}$ is.)

In case M is e.c., this is the largest possible interpretation of P: if P' is another, then P' implies $\neg Q$ for every $Q \perp P_i$, so P' implies P_i for each i, and hence P' implies P. Moreover M^+ retains the property that every homomorphism on M^+ is an embedding: if $f : M^+ \to N$ is a homomorphism, then f is an L-embedding on M, and $P' = f^{-1}(P(N))$ is a possible alternative interpretation of P, containing $P(N^+)$, so equal to it; hence f is an $L \cup \{P\}$-embedding. In particular, endomorphisms of M^+ are automorphisms. (However, we do not necessarily have $M^+ \models (\mathcal{T}^+)'$, if M is not sufficiently saturated; and in particular M^+ may not be e.c. This issue disappears if the conjunction is finite.)

Conversely, let N is an e.c. model of $\mathcal{T}^+$, M the reduct to a model of $\mathcal{T}$. As noted above, reinterpreting P_i by as $P_i(N) \cup P(N)$ results in another model N' of $\mathcal{T}^+$ with the identity map a homomorphism $N \to N'$; so we must have $N = N'$, i.e. P implies P_i in N. It follows that if $f : M \to M'$ is a homomorphism, then it is also a homomorphism $N \to (M')^+$; since N is e.c., the Tarski-Vaught property holds for $N \to (M')^+$ and in particular for $M \to M'$. Thus M is an e.c. model of $\mathcal{T}$. Now $N \to M^+$ is a homomorphism, so as N is e.c., $N = M^+$.

This shows in particular that the e.c. models of $\mathcal{T}$ and of $\mathcal{T}^+$ can be canonically identified, when a finite conjunction is eliminated. In case $\mathcal{T}$ is p.p. bounded, the universal (and thus pp-saturated) e.c. model does not change; except that $\cap_i P_i$ is now also named by P.

Of course, even if each P_i admits a complement, P may not; thus naming a type here does not have the effect it does in [28].

2.5. **Saturated models and bounded theories.** The category of e.c. models with embeddings admits amalgamation: if $f_i : A \to B_i$, we may embed each B_i in an ultrapower A^* of A, then compose with a homomorphism to an e.c. model.

Since homomorphisms need not be injective, there may be an upper bound θ on the cardinality of existentially closed models. Call $\mathfrak{T}$ *ec-bounded* in this case. This is indeed the case that concerns us; *assume from now on that* $\mathfrak{T}$ *is ec-bounded.*

An e.c. model M is called κ*-saturated* if for any e.c. $A \leq M$ and any embedding $f : A \to B \models \mathfrak{T}$ with $|B| < \kappa$, there exists a homomorphism $g : B \to M$ with $g \circ f = Id_A$. The usual existence theorem for κ-saturated models remains valid: for any cardinal $\kappa \geq |L|$, there exists a κ^+-saturated e.c. model (of cardinality $\leq 2^\kappa$). Thus there exists a κ-saturated e.c. model $\mathfrak{U}$ of $\mathfrak{T}$ of cardinality $\leq \theta$, which is κ-saturated for all κ. In particular, the irreducibility assumption on $\mathfrak{T}$ implies that $\mathfrak{U}$ is *universal* in the sense that any model N of $\mathfrak{T}$ admits a homomorphism into $\mathfrak{U}$; if N is e.c., N embeds into $\mathfrak{U}$. Note that $\mathfrak{U}$ is homogeneous for pp types.

Proposition 2.6. *Assume* $\mathfrak{T}$ *is ec-bounded. Then it has a unique universal e.c. model* $\mathfrak{U}$ *(up to isomorphism.) Any homomorphism on* $\mathfrak{U}$ *is an embedding, and any endomorphism* $f : \mathfrak{U} \to \mathfrak{U}$ *is an isomorphism. If* $\mathfrak{U} \leq V \models \mathfrak{T}^{\pm}$ *then there exists a homomorphism* $r : V \to \mathfrak{U}$ *with* $r|\mathfrak{U} = Id_{\mathfrak{U}}$.

Proof. Existence of a saturated (in any cardinality) $\mathfrak{U}$ was seen above; it is in particular universal. We also noted that homomorphisms on $\mathfrak{U}$ are embeddings. Let $f : \mathfrak{U} \to \mathfrak{U}$ be an endomorphism, with image U'. Then $f^{-1} : U' \to \mathfrak{U}$ is an embedding, that extends (by the $|\mathfrak{U}|^+$- saturation of $\mathfrak{U}$) to an embedding $g : \mathfrak{U} \to \mathfrak{U}$. Since g is injective, while $g|U'$ is surjective, we must have $\mathfrak{U} = U'$. For the last statement, as $\mathfrak{U}$ is universal there exists a homomorphism $f : V \to \mathfrak{U}$; on $\mathfrak{U}$ it induces an isomorphism g; so $r = g^{-1} \circ f : V \to \mathfrak{U}$ is as required.

It remains to show that any universal e.c. model U is isomorphic to the saturated $\mathfrak{U}$. Since U is universal, there exists a homomorphism $f : U \to \mathfrak{U}$, which must be an embedding; so we may assume $U \leq \mathfrak{U}$. Then there exists a retraction $r : \mathfrak{U} \to U$. But endomorphisms of $\mathfrak{U}$ are isomorphisms, so $\mathfrak{U} \cong U$. ☐

Remark 2.7. We are dealing here with the analogue of *finite* structures in first-order logic, or *compact* ones in continuous logic. This is the basic material of algebraic closure in positive logic. In [43], the universal e.c. structure of a pp-bounded theory is studied under the name of *minimum compactness.*

We observe (though we will make no use of the fact) that an ec-bounded theory is *equational*, in particular stable, in the following sense:

(E) If $p(x, y), q(x, y)$ are $\mathfrak{T}$-contradictory pp partial types, there is a finite bound on the length of a sequence (a_i, b_j) with $p(a_i, b_j)$ for $i < j$ and $q(a_i, b_i)$.

Otherwise, in some $M \models \mathfrak{T}$ there will be a long chain of such elements (a_i, b_i). By homomorphically mapping into an e.c. model, we may assume M is e.c. For $i < j$ we have $p(a_i, b_j)$ and so not $q(a_i, b_j)$, yet we do have $q(a_i, b_i)$; so $b_i \neq b_j$. This contradicts the bound on the size of e.c. models.

2.8. **The pp topology on** $\mathfrak{U}$ **and** $\mathrm{Aut}(\mathfrak{U})$**.** Let us topologize $\mathfrak{U}$, taking as a pre-basis the complements of sets of the form $\{x : R(x, c_1, \ldots, c_k)\}$, with R pp and $c_1, \dots, c_k \in \mathcal{U}$. Under this topology, $\mathcal{U}$ *is T1:* if $a \neq b \in \mathcal{U}$, there exists a pp R such that $\mathcal{U} \models R(a,b) \;\wedge\; (\forall x)\neg R(x,x)$. [15] Then $\neg R(x,b)$ is an open set including b, but not a; so $b \notin cl(a)$. As this holds for all $b \neq a$ we have $cl(a) = a$.

Also, $\mathcal{U}$ *is compact*: consider a family F_i of basic closed sets with the finite intersection property. F_i is defined by $R_i(x, c_i)$ with R_i pp. In an elementary extension U' of $\mathcal{U}$, one can find d' with $\mathcal{R}(d', c_i)$ holding for all i. By Proposition 2.6 there exists $r : U' \to \mathcal{U}$, $r|\mathcal{U} = Id_{\mathcal{U}}$. Let $d = r(d')$. Then $R_i(d, c_i)$ holds for each i, so $d \in \cap_i F_i$ and $\cap_i F_i \neq \emptyset$.

Let $G = \mathrm{Aut}(\mathcal{U})$. Since $\mathcal{U}$ is many-sorted, a function $f : \mathcal{U} \to \mathcal{U}$ is actually a sequence of functions $f_S : S(\mathcal{U}) \to S(\mathcal{U})$, indexed by the sorts S; we take the sorts to be closed under finite products. Each $f : \mathcal{U} \to \mathcal{U}$ can be viewed as a certain function on the disjoint union of sorts, respecting the projection maps from products to factors. We give G the topology of pointwise convergence, induced from the space of functions $\mathcal{U} \to \mathcal{U}$. Thus if $a_1, \dots, a_n, b_1, \dots, b_m \in \mathcal{U}$ and R is pp, then

$$\{g : \neg R(ga_1, \dots, ga_n, b_1, \dots, b_m)\}$$

is a pre-basic open set. As $\mathcal{U}$ is T1, so is G. Let us see that G *is compact.* Let u be an ultrafilter on a set I, and let $g_i \in G$, $i \in I$; we need to find a limit point g of $(g_i)_i$ along u. Let $\mathcal{U}^*$ be the ultrapower of $\mathcal{U}$ along u, and let $g_* : \mathcal{U} \to \mathcal{U}^*$ be the ultraproduct of the maps $g_i : \mathcal{U} \to \mathcal{U}^*$; let j be the diagonal embedding $\mathcal{U} \to \mathcal{U}^*$, ultrapower of $Id : \mathcal{U} \to \mathcal{U}$. As $\mathcal{U}^* \models \mathcal{T}^{\pm}$, Proposition 2.6 provides a homomorphism $r : \mathcal{U}^* \to \mathcal{U}$ with $r \circ j = Id_{\mathcal{U}}$. Let $g = r \circ g_*$. Then $g \in End(\mathcal{U}) = \mathrm{Aut}(\mathcal{U})$. If $R(g_i a_1, \dots, g_i a_n, b_1, \dots, b_m)$ holds for u-almost all $i \in I$, then $\mathcal{U}^* \models R(g_* a_1, \dots, g_* a_n, jb_1, \dots, jb_m)$ so $\mathcal{U} \models R(ga_1, \dots, ga_n, b_1, \dots, b_m)$. Hence g is indeed a limit point of $(g_i)_i$ along u.

left and right translation are continuous. Indeed a pre-basic open set has the form $B = \{g : g(p) \in W\}$ where W is a pre-basic open subset of $\mathcal{U}$, and $p \in \mathcal{U}$. For $a \in G$ we have $aB = \{h : h(p) \in W'\}$ and $Ba = \{g : g(p') \in W\}$ where $W' = aW$ and $p' = a(p')$. These are also pre-basic open. Further, *inversion on G is continuous.* Indeed a pre-basic closed subset of $\mathcal{U}$ has the form $\{z : (z, q) \in R\}$ where R is a basic relation, and q is a tuple from $\mathcal{U}$. Thus a pre-basic closed subset of G has the form $W = \{g \in G : (g(p), q) \in R\}$ where p is a tuple of elements of $\mathcal{U}$. So $W^{-1} = \{g \in G : (p, g(q)) \in R\}$, another pre-basic closed set of G, with the parameters and test points interchanged.

Let $\mathcal{U}_h$ denote the union of all $P(\mathcal{U})$, with P a pp partial type that is Hausdorff in the pp topology. (Including imaginary sorts, defined below.) We will see in the

[15] Let E be the congruence generated by (a,b), e.g. if there are no function symbols E is the equivalence relation identifying a, b only. Let $\mathcal{U}' = \mathcal{U}/E$ so that $\mathcal{U} \to \mathcal{U}'$ is a homomorphism. As $\mathcal{U}$ is e.c., $\mathcal{U}'$ cannot be a model of $\mathcal{T}$, so that $\mathcal{T} \models \neg S(z,x,x)$ and $\mathcal{U} \models S(c,a,b)$ for some conjunction S of atomic formulas. Let $R(x,y) = (\exists z) S(z,x,y)$.

case of interest to us that the restriction $\mathrm{Aut}(\mathcal{U}) \to \mathrm{Aut}(\mathcal{U}_h)$ is surjective. In any case, with the topology described above, it is clear that $\mathrm{Aut}(\mathcal{U}_h)$ *is Hausdorff.*

At this level of generality, it follows from Ellis' joint continuity theorem [12] (relying on a Baire category argument) that $\mathrm{Aut}(\mathcal{U}_h)$ is a compact Hausdorff topological group, acting continuously on $\mathcal{U}_h$. In our setting, with $\mathcal{U} = \mathrm{Core}\,(T)$ the pattern space of a theory T, we can easily see this directly; the compact-open and finite-open topologies coincide on G by Remark 3.27.

2.9. **Logical complexity.** Assume $\mathcal{L}$ is countable. What is the logical complexity of the above construction; for instance of determining, given $\mathcal{T}$, whether $\mathrm{Aut}(\mathcal{U}) = 1$? We have $\mathrm{Aut}(\mathcal{U}) \neq (1)$ iff there exist conjugate but distinct elements in $\mathcal{U}$; this is iff there exists a maximal pp type $p(x, y)$ with (a) equal restrictions to x, y, and (b) $p(x, y)$ guaranteeing distinctness of x, y. Now (b) holds iff for some pp $\theta(x, y)$ in p, $\mathcal{T} \models \neg(\exists x)\theta(x, x)$. On the other hand, (a) holds iff for all $\phi(x)$, and ϕ' orthogonal to ϕ, $p(x, y)$ contains a formula orthogonal to $\phi(x)\&\phi'(y)$. This is (at worst) an analytic (Σ^1_1) condition on $\mathcal{T}$.

Likewise for existence of a homomorphism into a fixed finite group or compact Lie group; also for $\mathrm{Aut}(\mathcal{U}_h)$.

It is also worth nothing that if $\mathrm{Aut}(\mathcal{U}) = 1$, then $\mathcal{U}$ admits a Borel structure. The natural map taking an element of the core to a pattern type is in this case 1-1, and the image of $\mathcal{U}$, as well as of the relations $\mathcal{R}_t$, is Borel.

2.10. **Imaginary quotients.** Assume $\mathcal{U}$ admits $\bigwedge, \exists$-elimination, in the sense that a conjunction of finitely many atomic formulas is atomic, and a pp relation is also atomic. This can be achieved by an appropriate Morleyzation, see § 2.4.

Let E be a closed equivalence relation on some sort Σ of $\mathcal{U}$; i.e. E is an intersection of pp-definable subsets E_n of Σ^2, and is an equivalence relation on Σ. For simplicity, we will consider only the sort Σ; we will write E and E_n also for the diagonal relations on Σ^k, i.e. $(x_1, \ldots, x_k)E(y_1, \ldots, y_k)$ iff $\bigwedge_{i=1}^k x_i E y_i$. We can add an additional sort $\bar{\Sigma} = \Sigma/E$; with the natural map $\pi : \Sigma \to \bar{\Sigma}$. But we are interested at the moment in $\bar{\Sigma}$ on its own right. We let $\mathcal{U}'$ be the $\mathcal{L}$-structure with universe $\bar{\Sigma}$, and with $R(\mathcal{U}') := \pi R(\mathcal{U})$ for every n-ary atomic relation R. *Note that a sentence $R = R' \cap R''$, true in $\mathcal{U}$, need not remain true in $\bar{\Sigma}$. Thus $\mathcal{U}'$ may not admit $\bigwedge$-elimination.* Let $\mathcal{T}'$ be the primitive universal theory of $\mathcal{U}'$. What are the axioms of $\mathcal{T}'$? For a single atomic relation R, the sentence $\neg\exists x R$ will be in $\mathcal{T}'$ if and only if it is in $\mathcal{T}$. But for a conjunction, say of two conjuncts R, R', we have:

$$(*)\ \mathcal{T}' \models \neg\exists x(R(x) \wedge R'(x)) \text{ iff for some } n, \quad \mathcal{T} \models \neg\exists x, x'(R(x) \wedge R'(x') \wedge E_n(x, x'))$$

Lemma 2.11. *$\mathcal{U}'$ is the universal e.c. model of $\mathcal{T}'$. Any endomorphism of $\mathcal{U}'$ lifts to an automorphism of $\mathcal{U}$.*

Proof. We first check that $\mathcal{U}$ is universal. Let $A \models \mathcal{T}'$, and let $(a_i)_{i \in I}$ enumerate the universe of A. We introduce variables $(x_i : i \in I)$. Also for each instance of an atomic k-place relation $R(a_{i_1}, \ldots, a_{i_k})$ valid in A, we introduce new variables $y_1, \ldots, y_k$ especially for this instance of R, and let

$$\Gamma_R = \{R(y_1, \ldots, y_k) \wedge y_\nu E_n x_{i_\nu} : \nu = 1, \ldots, k\}.$$

This collection of formulas can be realized in $\mathcal{U}$, using the saturation of $\mathcal{U}$ and the description (*) of $\mathcal{T}'$ above. Such a realization defines a map $f : A \to \mathcal{U}$, mapping a_i to the realization of x_i, such that the composition $\pi \circ f : A \to \mathcal{U}'$ is a homomorphism. This proves universality of $\mathcal{U}'$.

Next let $f : \mathcal{U}' \to \mathcal{U}'$ be an endomorphism. Let $a = (a_i)_{i \in I}$ enumerate the universe of $\mathcal{U}$. Choose $c_i \in \mathcal{U}$ with $\pi(c_i) = f(\pi(a_i))$. Let $\Gamma(x)$ be the atomic type of a in appropriate variables $x = (x_i)_{i \in I}$. We seek b realizing

$$\Gamma'(y) = \Gamma(y) \cup \{y_i E c_i : i \in I\}$$

By saturation of $\mathcal{U}$, it suffices to prove consistency; so consider finitely many formulas of Γ'; for instance $R_1(y_1, y_2) \wedge R_2(y_1, y_2) \wedge y_1 E_n c_1 \wedge y_2 E_n c_2$. By the $\bigwedge$-elimination assumption about $\mathcal{L}$, there exists an atomic R with

$$\mathcal{U} \models R \iff (R_1 \wedge R_2).$$

Thus we reduce to the case of a single R: we have to solve $R(y_1, y_2) \wedge y_1 E_n c_1 \wedge y_2 E_n c_2$. Existence of such y_1, y_2 follows from (*) and the fact that $\mathcal{U}' \models R(\pi(a_1), \pi(a_2))$ and hence, f being a homomorphism, $\mathcal{U}' \models R(\pi(c_1), \pi(c_2))$. Thus there exists b with $\mathcal{U} \models \Gamma'(b)$. Define $F(a_i) = b_i$; then $F : \mathcal{U} \to \mathcal{U}$ is an endomorphism (and hence an automorphism) of $\mathcal{U}$, and $\pi \circ F = f \circ \pi$.

To see that $\mathcal{U}'$ is e.c., let $f : \mathcal{U}' \to A$ be a homomorphism into a model of $\mathcal{T}'$. Compose f with some homomorphism $A \to \mathcal{U}'$. To show that f is an embedding, it suffices to show the same of the composition; so we may assume $f : \mathcal{U}' \to \mathcal{U}'$. In this case we saw that there exists $g : \mathcal{U} \to \mathcal{U}$ inducing f. As g is an automorphism, so must be f.

□

2.12. **Type spaces.** For a model M of $T^{\pm}$, and a finite set of variables y of L, we let M^y be the set y-tuples of elements of M, i.e. the set of functions from y to M preserving sorts. Given a set γ of formulas of L along with a distinguished set x of variables, a γ-type p over M is a set of formulas of the form

$$p = tp_\gamma(a/M) = \{\phi(x, b) : \phi(x, y) \in \gamma, b \in M^y, N \models \phi(a, i(b))\}$$

where $i : M \to N$ is a homomorphism, $a \in N^x$, $N \geq M$. The set $S_\gamma(M)$ of γ types over M has a natural compact topology, with basic open sets of the form $\{p : \phi(x, b) \in p\}$. The subspace of *maximal* types is Hausdorff. When γ includes all formulas with distinguished variables x, we write $S_x(M)$. We will assume that γ is closed under negations (at least in the sense that any $\phi(x, a)$ with $\phi \in \gamma$ is equivalent to the negation of some $\phi'(x, b)$ with $\phi' \in \gamma$.)

Type spaces will be treated, notationally, as simplicial spaces ([27]), meaning that we can write $S(M)$ for the data associating to any γ the space $S_\gamma(M)$. For infinite sets of formulas Γ, S_Γ can be defined in the same way, or equivalently as the inverse limit of S_γ over all finite $\gamma \subset \Gamma$.

Remark 2.13 (Interpolation)**.** Let $\mathcal{L} \subset \mathcal{L}'$ be relational languages, $\mathfrak{T}'$ a primitive universal theory, $\mathfrak{T} = \mathfrak{T}'|\mathcal{L}$. Let N' be an e.c. model for $\mathfrak{T}'$, $N := N'_L$ the reduct of N' to L. Assume interpolation holds in this form: if $R' \in \mathcal{L}'$, $R \in \mathcal{L}$ are pp formulas, $\mathfrak{T}' \models \neg(R \wedge R')$, and $N' \models R'(a)$, then for some pp $S \in \mathcal{L}$ we have $N' \models S(a)$ and $\mathfrak{T} \models \neg(R \wedge S)$. Then N is an e.c. model of $\mathfrak{T}$. If N' is universal e.c., so is N.

In particular, this is the case if $\mathcal{L}'$ is obtained from $\mathcal{L}$ by adding constant symbols.

Proof. Let $M \models \mathfrak{T}$, and let $f : N \to M$ be a homomorphism. Expand M to an $\mathcal{L}'$-structure M' by interpreting any basic relation symbol R' of $\mathcal{L}$ as $(R')(M') = f(R'(N))$; thus $f : N' \to M'$ is a $\mathcal{L}'$- homomorphism. It is easy to see from the assumed interpolation property that $N' \models \mathfrak{T}'$. Given this, the e.c. property of N' with respect to f includes the same for N. Hence N is e.c. If N' is universal e.c., let $M \models \mathfrak{T}$; then the diagram of M is consistent with $\mathfrak{T}$ and hence with $\mathfrak{T}'$; so there exists a $\mathcal{L}$-homomorphism $g : M \to M'$ into a model of $\mathfrak{T}'$. By universality of N', there exists $h : M' \to N'$, and hence by composing we have a homomorphism $M \to N$. □

3. A relational structure on type spaces

Let T be a universal theory. We assume that any two models of T can be embedded into a single model (joint embedding property). We allow T to be many-sorted, and sometimes refer to a product of sorts, or a definable subset, as itself a sort.[18] We take a fixed countable set of variables for each sort. $|L|$ is the number of formulas of L. *Unless otherwise stated, we consider only quantifier-free formula in this section.* [19] Let

$$T^{\pm} = T_\forall \cup \{\neg\phi : \ \phi \text{ universal }, \phi \notin T_\forall\}$$

We aim to associate with T a *language* $\mathcal{L}$ (the *pattern language*), a canonical irreducible primitive universal theory $\mathfrak{T}$ of $\mathcal{L}$, and a canonical model $\mathfrak{J} :=: \mathrm{Core}\,(T)$ of $\mathfrak{T}$, the *core* of T.

and an enrichment of the type spaces of models of T to models of this theory.

[18] Formally, these are indeed imaginary sorts.

[19] Let us say that T is *QEble* if there exists a complete theory T_1 with quantifier elimination, whose universal part is T. If we *begin* with a complete first-order theory T', we first Morley-ize to obtain a thery T_1 with QE , then let $T = (T_1)_\forall$ be the the universal part, and apply the theory below to T in order to obtain results about T'.

The language $\mathcal{L}$ has the same sorts as the type spaces of T, i.e. a sort for each set of formulas γ along with a set of distinguished variables x. For an $\mathcal{L}$-structure A, this sort will be denoted by S_γ. [20]

Let x_i be an n- tuple of variables, for $i = 1, \dots, n$; they will be referred to as the distinguished variables. Let y be an additional tuple of variables (the *parameter variables.*) Let $t = (\phi_1, \dots, \phi_n; \alpha)$ be an n-tuple of formulas $\phi_i(x_i, y)$ of γ, and let $\alpha(y)$ be a formula.

To each such t, α we associate a relation symbol $\mathcal{R}_t$ of $\mathcal{L}$, taking variables $(x_1, \dots, x_n)$.

For any $M \models T^{\pm}$, we define an $\mathcal{L}$-structure whose sorts are $S_\gamma(M)$ for the various sorts γ. When $t = (\phi_1, \dots, \phi_n; \alpha)$ and $\phi_i \in \gamma_i$ we define $\mathcal{R}_t$ on $S = S_{\gamma_1} \times \cdots \times S_{\gamma_n}$ thus:

$$\mathcal{R}^S_{t;\alpha} = \{(p_1, \dots, p_n) \in S : \neg(\exists a \in \alpha(M)) \bigwedge_{i \leq n} (\phi_i(x_i, a) \in p_i)\}$$

We omit α from the notation in case α is universally true, i.e. $\alpha(M) = M^y$. If γ_i is closed under conjunctions with the formula $\alpha(y)$, then $\mathcal{R}_{t;\alpha} \equiv \mathcal{R}_{t'}$ where $t' = (\phi_1(x,y) \wedge \alpha(y), \cdots, \phi_k(x,y) \wedge \alpha(y))$.

Example 3.1. If $\phi(x)$ has no parameter variables, then $\phi \notin p$ iff $S \models \mathcal{R}_\phi(p)$. Thus the atomic type of p in $S_x(M)$ determines the restriction of p to $S_\gamma(\emptyset)$.

Example 3.2. $\mathcal{R}_{\phi(x,y) \iff \theta(y)}$ captures the set of types $p(x)$, admitting $\neg\theta(y)$ as a ϕ-definition.

Example 3.3. Let $\phi(x, y) \in \gamma$. In $S_\gamma(M)$, the relation

$$E_\phi :\equiv \mathcal{R}_{(\phi, \neg\phi)} \wedge \mathcal{R}_{(\neg\phi, \phi)}$$

holds of a pair p, p' iff they restrict to the same ϕ-type over M. In any $S_\gamma(M)$ and also in any e.c. model, E_ϕ is an equivalence relation, and the intersection of all E_ϕ is the diagonal. Similarly, for a finite set of formulas γ, equality is definable by a pp formula, as is more generally the restriction map $S_\gamma \to S_{\gamma'}$ for $\gamma' \subset \gamma$.

Example 3.4. Assume $\phi(x, y)$ is *free* in the sense that for any distinct $b_1, \dots, b_n \in M^y$, there exists a such that $\phi(a, b_i)$ iff $i \leq n$ is odd. (A strong negation of NIP.) Then $S_\phi(M)$ carries a Boolean algebra structure: for any $p, q \in S_\phi(M)$ there exists a unique $r \in S_\phi(M)$ with $\phi(x, b) \in r$ iff $\phi(x, b) \in p \& \phi(x, b) \in q$, and likewise for the other Boolean connectives; they are all described by basic $\mathcal{L}$-formulas; these formulas will define a Boolean algebra structure on any e.c.

[20] We take only *finite* sets of formulas γ for the official sorts. Still for infinite Γ, we can define S_Γ as the projective limit of S_γ over all finite $\gamma \subset \Gamma$. This will be compatible with definitions below .In particular a homomorphism defined on the official sorts extends uniquely to the derived infinite ones.

model of the universal primitive theory of $S_\phi(M)$. Any *compact* model for the pp topology (such as Core (T) below) will in fact be a complete Boolean algebra.

Lemma 3.5. *(1) If $M, N \models T$, $M \leq N$, the restriction map $r_{N,M} : S(N) \to S(M)$ is an $\mathcal{L}$- homomorphism.*
(2) Let u be an ultrafilter, M^u the ultrapower of M. There exists a canonical ultrapower map $j_u : S(M) \to S(M^u)$; it is an $\mathcal{L}$-embedding.
(3) If $M, N \models T^\pm$, then in (1), $r_{N,M}$ admits a section, i.e. a homomorphism $j : S(M) \to S(N)$ with $r \circ j = Id_{S(M)}$.

Proof. (1) is clear.

So is (2): if $M \prec M', b \in M'$ and $p = tp(b/M)$, let $j_u(p) = tp(b/M^u)$ where b is identified diagonally with its image in the ultrapower $(M')^u$. Note that the relations $\mathcal{R}_t(tp(b_1/M), \cdots, tp(b_k/M))$ are first-order definable in the pair (M', M), hence persist.

For (3) let $i : N \to M^u$ be an embedding over M, let i_* be the pullback by i of types over M^u to types over N, and let $j = i_* \circ j_u$. Then $r \circ j = d_* \circ j_u = Id_{S(M)}$. □

Corollary 3.6. *The primitive universal theory of $S(M)$ does not depend on the choice of model $M \models T^\pm$.*

Proof. Let $M, N \models T^\pm$. Then N embeds into an ultrapower M^* of M. By Lemma 3.5 (1,2) we have $Th_{p\forall}(N) \subset Th_{p\forall}(M^*) = Th_{p\forall}(M)$. □

One can also see this directly - if $\mathcal{T} \models \neg(\exists x) \bigwedge_{i=1}^m \mathcal{R}_t(x)$ then (as this is due to a finite inconsistency) the same is true in $S(M)$. If restricting to a set of formulas γ, it suffices to have $Th_\forall(M) = T_\forall$ in the parameter sorts of γ.

Definition 3.7. The *theory of T-patterns* is the common primitive universal theory of all type spaces $S(M)$. It will be denoted by $\mathcal{T}$.

It is easy to write down the axioms of $\mathcal{T}$ explicitly. For instance, $(\forall \xi)\neg\mathcal{R}_\phi(\xi)$ will be an axiom of $\mathcal{T}$ iff for some $\theta(u_1, \ldots, u_n)$,

$$T^\pm \models (\exists u_1, \ldots, u_l)\theta \& (\forall x)(\forall u_1, \ldots, u_n)(\theta \implies \bigvee_j \phi(x, u_j))$$

In other words, the definable partial type $\{\phi(x, u) : u \in \alpha(M)\}$ is inconsistent, for any model $M \models T$.

Remark 3.8. $\mathcal{T} = \mathcal{T}(T)$ varies continuously with $T^\pm$, in the sense that if $\sigma \in \mathcal{T}(T)$ then there exists a finite $T_0 \subset T^\pm$ such that for all T' with $T_0 \subset (T')^\pm$ we have $\sigma \in \mathcal{T}(T')$.

Lemma 3.9. *Let $A \models T^\pm$. Then any model of $\mathcal{T}$ admits a homomorphism into $S(A)$. In particular if E is an e.c. model of $\mathcal{T}$, then E admits an embedding into $S(A)$.*

Proof. Let $S = S(A)$, made into an $\mathcal{L}$-structure by the natural interpretation of $\mathcal{R}_t$. Consider the space of sort-preserving functions $E \to S$, with pointwise convergence topology, relative to the topology of S. Since S is (on each sort) compact, the space of functions is compact. The subspace of functions preserving finitely many given instances of the relations $\mathcal{R}_t$ is closed, and non-empty since any pp sentence true in E is is true in S. Hence a map exists preserving all instances of all relations $\mathcal{R}_t$. This is a homomorphism, and in case E is e.c. it must be an embedding. □

The argument of Lemma 3.9 was given in [29] for general compact topological algebras, generalizing earlier results in the theory of modules.

It follows from Lemma 3.9 that $\mathcal{T}$ is ec-bounded. Thus by the results of § 2.5, a unique universal e.c. model of $\mathcal{T}$ exists.

Definition 3.10. The core of T, Core (T) is the universal e.c. model of $\mathcal{T}$.

When T is fixed, Core (T) will be denoted by $\mathcal{J}$.

We view $\mathcal{J}$ as an $\mathcal{L}$-structure; it is thus endowed also with the pp topology. Likewise we give $G = \mathrm{Aut}(\mathcal{J})$ the topology described in § 2.8. Thus $\mathcal{J}$ and G are compact T1 spaces.

Let $\mathfrak{g} = \mathfrak{g}_{\mathcal{J}}$ be the normal subgroup of $G = \mathrm{Aut}(\mathcal{J})$ described in § C. Let $E_{\mathfrak{g}}$ be the equivalence relation on (each sort of) $\mathcal{J}$ given by $\mathfrak{g}$-conjugacy. $E_{\mathfrak{g}}$ appears in general to be a complicated equivalence relation on $\mathcal{J}$; the visible complexity upper bound, when $|L|$ is countable, is: no worse than analytic. But on each atomic type we will see that it is closed.

The following proposition can be regarded as a form of quantifier elimination. It implies, in particular, that it is possible to compute the core separately for each sort. Let us call a set $\gamma(x; y)$ of formulas *full* if it includes all formulas $\theta(y)$ in (any) parameter variables alone.

Remark 3.11. At the level of generality we are working with, of irreducible universal theories, the fulness assumption can easily be relativized. Suppose γ is a finite set of quantifier-free formulas. By a slight Morleyzation we can take them to be atomic; let L_γ be the sublanguage of L generated by γ. Let T_γ be the given universal theory T, restricted to L_γ. Then T_γ is itself an irreducible universal theory, and fulness of γ is now tautological.

Proposition 3.12. *Let T be an irreducible universal theory, and consider the sort γ of $\mathcal{L}, \mathcal{T}$ for any full γ. Then:*

(1) *For any pp formula $A(\mu)$ there exist atomic formulas $\Xi_k(\mu)$ such that in in $\mathcal{J}_\gamma$, as well as in $S_\gamma(M)$ for any e.c. model M of T,*

$$A \iff \bigwedge_k \Xi_k$$

Here k ranges over an index set K of cardinality at most $|L|$.

(2) *If T is the universal part of a complete first-order theory with QE, K can be taken countable, and the fulness assumption on γ can be restricted to any given (finite) family of parameters sorts.*
(3) *$\mathfrak{J}$ is homogeneous for atomic types.*
(4) *An atomic type of $\mathfrak{T}$ is the type of an element of $\mathfrak{J}$ if and only if is maximal.*
(5) *$\mathfrak{T}$ admits elimination of finite conjunctions, at least if models of $T^{\pm}$ have more than one element.*
(6) *If γ is the Boolean closure of a finite set with a set formulas in the parameter variables alone, then equality can be defined in terms of the basic symbols $\mathcal{R}_t$.*

Proof. Let us first consider the the case of $S_\gamma(M)$ where γ is a finite set of formulas, along with formulas in the parameter variables alone. For simplicity (and using a standard trick, without loss of generality) assume $\gamma(x,u)$ is a single formula.

We can write $A(\mu)$ in 'normal form' as

$$A(\mu) \equiv (\exists \xi)(\mathcal{R}_\phi(\xi) \wedge \pi(\xi) = \mu),$$

where π is the coordinate projection $S_{xy} \to S_x$, and $\mathcal{R}_\phi$ asserts that $\phi(xy,u)$ is not represented in ξ. (See (5).) Now given $p(x)$, there exists $q \in S_\gamma \cup \phi(M)$ extending $p(x)$ and omitting $\phi(xy,u)$ *unless* $p(x) \cup \{\neg\phi(xy,b) : b\}$ is inconsistent with T and the quantifier-free diagram of M; i.e. for some $c = (c_1,\ldots,c_m)$, $b = (b_1,\ldots,b_l)$,e from M and $\theta \in L$, we have $\gamma(x,c_i) \in p$ for each i, $M \models \theta(c,b,e)$ and

$$T \models (\forall x,y,v,u,w)\neg(\bigwedge_{i=1}^{m} \gamma(x,v_i) \wedge \bigwedge_{j=1}^{l} \neg\phi(xy,u_j) \wedge \theta(u,v,w))$$

Then $S(M) \models (\exists \xi)(\mathcal{R}_t(\xi) \wedge \pi(\xi) = p)$ iff for each such θ,

$$\gamma(x,v_1) \wedge \cdots \wedge \gamma(x,v_m) \wedge \theta(u,v,w)$$

is omitted in p. Let $\Xi_{m,\theta} = \mathcal{R}_{\gamma,\cdots,\gamma;\theta}$.

Then we have shown that

$$S(M) \models A(\mu) \iff \bigwedge_m \Xi_m(\mu)$$

In case T is the universal part of a complete first order theory with QE, θ may be taken to be a quantifier-free formula equivalent to $(\forall x)(\forall y)\neg(\bigwedge_{i=1}^{m} \gamma(x,v_i) \wedge \bigwedge_{j=1}^{l} \neg\phi(xy,u_j))$; thus the $\bigwedge$ above can be taken to range over a countable set, regardless of the cardinality of the language.

This concludes (1,2) in the case of finite γ, for $S_\gamma(M)$.

The case of an arbitrary γ follows, since we will have $(\exists \xi)(\mathcal{R}_t(\xi) \wedge \pi(\xi) = \mu)$ iff for every finite $\gamma' \leq \gamma$, letting $\mu_{\gamma'}$ be the restriction of μ to γ', $(\exists \xi)(\mathcal{R}_t(\xi) \wedge \pi(\xi) = \mu_{\gamma'})$.

Using the compactness of $S(M)$, Lemma 2.3 (2) shows that the infinitary equivalence above is also valid in $\mathfrak{J}$.

It was already noted in § 2.5 (just above Proposition 2.6) that $\mathfrak{J}$ is homogeneous for pp types. Thus (1) implies (3).

Next we show maximality of the atomic types of elements of $\mathfrak{J}$. Let P be the atomic type of a in $\mathfrak{J}$. Consider any pp formula ψ not true of a in $\mathfrak{J}$. Then (since $\mathfrak{J}$ is e.c.) some pp formula ϕ is true of a and contradicts ψ in models of $\mathfrak{T}$. By the above, ϕ is equivalent to $\bigwedge \Xi_k$, with Ξ_k atomic. Each Ξ_k must be in P; and some finite conjunction Ξ of the Ξ_k must contradict ψ (otherwise realize $\psi \wedge \bigwedge \Xi_k$ in some elementary extension, and retract to $\mathfrak{J}$.) So $\neg\psi$ follows from P. Hence P is maximal; no ψ can be properly added to it. The converse, that a maximal atomic type is represented in j, is clear since it is realized by some tuple a in some $A \models \mathfrak{T}$ and there exists a homomorphism $A \to \mathfrak{J}$, which must by maximality be an embedding on a. Thus (4).

(5) E.g. (p, p') omits $\psi(x, u) \& \psi'(x', u)$ *and* omits $\phi(x, u) \& \phi'(x', u)$ iff (p, p') omits $\theta(x, u, v, v') \& \theta'(x, u, v, v')$, where v, v' are additional variables, and θ agrees with ϕ if $v = v'$, with ψ if $v \neq v'$; and similarly θ'.

(6) If γ is generated by the single formula γ along with parameter formulas, as we may assume, then $p = q$ in any type space, and hence in the core, if and only if $\gamma(x, a) \in p \& \neg\gamma(x, a)$ is omitted, and dualy. ☐

3.13. **Duality.** Let T be a universal theory, and M a universal domain, i.e. a highly qf- saturated and qf-homogenous model of $T^{\pm}$. Existence of M is equivalent to T being Robinson, i.e. $Mod(T)$ admitting amalgamation under embeddings. Types over M will be referred to as global types, and types means: qf types.

Proposition 3.14. *Let $A \leq M$ and let $B \models T$. Let b enumerate B. There is a canonical 1-1 correspondence between:*

- *$\mathcal{L}$-homomorphisms $h : S(A) \to S(B)$.*
- *Extensions of $tp(b/\emptyset)$ to a global type, finitely satisfiable in A.*

This is also valid locally for γ-types, with $\gamma(x, y)$ closed under Boolean combinations, and $\mathcal{L}$ restricted to the formulas $\mathcal{R}_t$ with $t = (\phi_1, \ldots, \phi_n)$, $\phi_i \in \gamma$.

Proof. Let $h : S(A) \to S(B)$ be an $\mathcal{L}$-homomorphism. Define a global type $p(y)$:

$$\phi(a, y) \in p(y) \iff \phi(x, b) \in h(tp(a/A))$$

If $\phi_i(a_i, y) \in p(y)$ for $i = 1, \ldots, n$, let $q_i = tp(a_i/A)$, and let $t = (\phi_1, \ldots, \phi_n)$; suppose $\bigwedge_i \phi_i(a_i, y)$ is not satisfiable in A; then $S(A) \models \mathcal{R}_t(q_1, \ldots, q_n)$; so $S(B) \models \mathcal{R}_t(hq_1, \ldots, hq_n)$; but $\phi_i(x, b) \in hq_i$ for each i, a contradiction.

Conversely, let $p(y)$ be an extension of $tp(b/\emptyset)$ to a global type, finitely satisfiable in A. In particular, when a, a' realize the same type q over A,

$\phi(a,y) \& \neg\phi(a',y)$ cannot be in p; so we can define $(d_q x)\phi(x,y) \in p$ to hold iff $\phi(a,y) \in p$ for some/all $a \models q$. Define $h : S(A) \to S(B)$ by:

$$h(q) = \{\phi(x,b) : (d_q x)\phi(x,y) \in p(y)\}$$

If $S(A) \models \mathcal{R}_t(q_1,\ldots,q_n;\alpha)$, $q_i = tp(a_i/A)$, $t = (\phi_1(x_1,y),\cdots,\phi_n(x_n,y))$, then there is no $b \in \alpha(A)$ with $\phi_i(a_i,b)$. As p is finitely satisfiable in A, it is not the case that each $\phi(a_i,y)$ is in p, where y is a variable corresponding to a finite tuple b_1 of coordinates of b, with $\alpha(b_1)$. Thus $S(B) \models \mathcal{R}_t(h(q_1),\ldots,h(q_n))$.

□

Remark 3.15. Composition of homomorphisms corresponds by duality to an operation on invariant types, related to tensor product. Consider a b-invariant type p_b, and an a-invariant type q_a. We define a third a-invariant type r_a. Namely let $a \subset E$; to define $r_a|E$, let $b \models q_a|E$, and $c \models p_b|E \cup \{b\}$; let $r_a|E = tp(c/E)$. When p_b is finitely satisfiable in b and q_a in a, it is easy to see that r_a is also finitely satisfiable in a.

In terms of this product, one can characterize minimal retractions $S(M) \to S(M)$, and so carry out the whole theory on the dual level.

3.16. **Expansion with definable patterns.** We repeat the statement of Proposition 1.6 in more detail.

Let T be an irreducible universal theory, V a distinguished sort and γ a set of formulas on $V \times P$, for various parameter sorts P, closed under negations. We view products of parameters sorts as parameter sorts themselves.

We consider irreducible universal theories T' expanding T by new relations on the parameters sorts. By an *interpretation of T' in T'' over T*, we mean here a map α of quantifier-free formulas of T' on parameter sorts, into quantifier-free formulas in the same variables for T'', compatible with change of variables and finite Boolean combinations, and such that $T' \models (\forall u)\psi$ iff $T'' \models (\forall u)\alpha(\psi)$; and $\alpha(\phi) = \phi$ for any quantifier-free formulas ψ of L' and ϕ of L. The notion of composition of interpretations over T is clear; we thus have a category $\mathcal{C}_T$. In this setting, a bi-interpretation is simply an interpretation with a 2-sided inverse.

Proposition 3.17. *(=Proposition 1.6.) There exists a unique minimal expansion T^{def}_γ of T that has definable patterns at γ.*

More precisely, let $\mathcal{C}^{def}_T$ be the full subcategory of of $\mathcal{C}_T$ consisting of those T' that have definable patterns at γ. Then $\mathcal{C}^{def}_T$ has an object T^{def} that maps into any other; and T^{def} is unique up to bi-interpretation. The bi-interpretation is unique up to composition with a self-interpretation of T^{def} over T. The self-interpretations of T^{def} over T form a group, isomorphic to $\mathrm{Aut}(\mathrm{Core}\,(T))$.

Any model of $T^{\pm}$ expands to a model of T^{def}.

Proof. We may assume γ includes all qf formulas on P alone, and is closed under Boolean combinations. The language $\widetilde{L}$ consists of the language of T, along with new relations $Q_{\phi;a} \subset P$ for each $a \in J := \text{Core}_\gamma(T)$ and $\phi \in \gamma$. The theory T^{def} is read tautologically off the pattern types, so that if $J \models \mathcal{R}_{\phi_1,\ldots,\phi_n;\alpha}(a_1,\ldots,a_n)$ then T^{def} includes

$$(\forall x_1,\cdots,x_n)\neg(\alpha(x_1,\ldots,x_n) \wedge \bigwedge Q_{\phi_i;a_i}(x_i))$$

as well as

$$Q_{\neg\phi;a} \vee Q_{\phi;a}$$

To expand a model M of T to a model of $\mathfrak{T}$ thus amounts to specifying an $\mathcal{L}$-homomorphism $h : J \to S_\gamma(M)$, and interpreting $Q_{\phi,a}$ as the ϕ-definition of $h(a)$.

$$(1) \qquad Mod(T^{def}) = \{(M,h) : M \in Mod(T), h : J = \text{Core}_\gamma(T) \to S_\gamma(M)\}$$

Claim . T^{def} is irreducible.

Proof. Note that up to T^{def}-equivalence, the $Q_{\phi;a}$ are closed under Boolean combinations (for instance $Q_{\neg\phi;a} = \neg Q_{\phi;a}$); and include the qf formulas on P alone. Thus to see that T^{def} is irreducible, we have to show that if x, y are disjoint tuples of variables and

$$T^{def} \models (\forall x)Q_{\phi;a}(x) \vee (\forall y)Q_{\phi';a'}(y) \qquad (*)$$

$T^{def} \models (\forall x)Q_{\phi;a}(x)$ or $T^{def} \models (\forall y)Q_{\phi';a'}(y)$.

Indeed assume (*). Let $M \models T$ be existentially closed. Let $j : J \to S_\gamma(M)$ be an embedding. Note that $j(a)$ and $j(a')$ are γ-types over M, finitely satisfiable in M by the existential closedness of M. Let M' be an $|M|^+$-saturated elementary extension of M. Then $j(a)$ and $j(a')$ are realized in M'. Let $c \models j(a)$ and let $c' \models j(a')$. Then $\neg\phi(c,m)\&\neg\phi'(c',m')$ is not represented by m, m' from M; so one of them is not represented, say the former; reading back this implies that $S_\gamma(M) \models \mathcal{R}_{\neg\phi}(j(a))$; as j is an embedding, $J \models \mathcal{R}_{\neg\phi}(a)$, so $T^{def} \models (\forall x)\neg Q_{\neg\phi;a}(x)$. Hence $Tg \models (\forall x)Q_{\phi;a}(x)$, as required. ☐

Let T' be any expansion of T, having definable patterns at γ; we will compare T^{def} with T'. Let γ' consist of γ along with all T'-qf-0-definable subsets of the parameter sorts (close under Boolean combinations.) Let $J = \text{Core}_\gamma(T)$, $J' = \text{Core}_{\gamma'}(T')$, $\mathcal{L}, \mathcal{L}'$ their languages. Let $M' \models (T')^{\pm}$, $M = M'|L$ the restriction to the language of T, $S' = S_{\gamma'}(M')$, $S = S_\gamma(M)$. We have a natural restriction map $r : S' \to S$. Then r is an $\mathcal{L}$-homomorphism, and any section $s : S \to S'$ (i.e. map satisfying $r \circ s = Id_S$, in this case unique) is also an $\mathcal{L}$-homomorphism. Let $j : J \to S$ be an $\mathcal{L}$-homomorphism, and $j' : J' \to S'$ an $\mathcal{L}'$-homomorphism; use j' to identify J' with an $\mathcal{L}'$-substructure of S'. We also have an $\mathcal{L}'$-retraction $\rho : S' \to J'$. Then $\rho \circ s \circ j : J \to J'$ is an $\mathcal{L}$-homomorphism. By (1), this corresponds to an enrichment of M to a model of T^{def}. But by total definability, for $q \in J'$ and $\phi \in \gamma$, the q-definition of ϕ is qf definable in T'. This gives a map of $\widetilde{L}$ to L' over L, mapping $Q_{\phi;a}$ to the $\rho \circ s \circ j(a)$-definition of ϕ. It is clear that the pullback of T' is precisely T^{def}. Thus we have interpreted T^{def} in T', fixing T.

We now check that T^{def} has definable patterns at γ. Let $T' = T^{def}$, and let notation $(\gamma', J', \mathcal{L}', M', S', r : S' \to S, s : S \to S', j' : J' \to S', \rho : S' \to J')$. In particular $M' \models T^{def}$, and the $\mathcal{L}'$-structure on it is given by a homomorphism $j : J \to M = M'|_L$. Thus the p-definition of any $\phi \in j(J)$ is definable in $\mathcal{L}'$. Now as we saw above, $\rho \circ s : j(J) \to J'$ is an $\mathcal{L}$-isomorphism. Thus if $q = \rho(s(p))$ then the q-definition of ϕ equals to p-definition of ϕ. Now $\rho \circ s(J) = J'$ (since $r \circ \rho \circ s$ is an isomorphism on $j(J)$ into $r \circ \rho \circ s \circ j(J)$, and r is 1-1.) Thus every element of J' is $\mathcal{L}'$-definable, as asserted.

Uniqueness is proved as in the first paragraph: let T'' be another universal theory expanding T, with definable patterns at γ, and minimal. We have found an interpretation f of T^{def} in T'' over T. By the assumed minimality of T'', we also have an interpretation g of T' in T^{def} over T. The composition $g \circ f$yields a self-interpretations of T^{def}. That corresponds to endomorphisms of J; but we know that endomorphisms of J are automorphisms; equivalently self-interpretations of T^{def} over T. We may assume, by twisting with such a self-interpretation, that $g \circ f = Id$. On the other hand, the interpretation of T'' in T^{def} must be 1-1 (if two qf formulas are intepreted by the same relation of T^{def}, they are equal.) Hence from $g \circ f \circ g = g$ we obtain $f \circ g = Id$ also.

Thus the two self-interpretations amount to a renaming of the new predicates indexed by J (by an automorphism of J), showing that T^{def}, T'' agree after a bijective matching of their new predicate symbols.

The last statement comes from (1) and Lemma 3.9. □

Proposition 3.18. *(T QEable). T has definable patterns iff T^{def} is an expansion by definition of T iff $|\mathrm{Hom}(J, S(M))| = 1$ for all $M \models T$ (or for some sufficiently saturated M.) More generally, let $J_0 \subset J$; if for all M, the restriction map $\mathrm{Hom}(J, S(M)) \to \mathrm{Hom}(J_0, S(M))$ is injective, then T^{def} is an expansion by definition of T along with the predicates of T^{def} corresponding to J_0.*

Proof. Let $h \in \mathrm{Hom}(J, S(M))$. Let $j \in J$, and consider a typical predicate of T^{def} corresponding to $q = h(j)$, namely $d_q x\phi(x, y)$ for some ϕ. The fact that h is a homomorphism is equivalent to implicit definability constraints on such predicates. The assumption is that these definability constraints determine the interpretation of the predicate uniquely (given the interpretation for $q' \in J_0$.) By Beth's theorem, $d_q x\phi(x, y)$ is definable (relative to similar predicates for J_0.) □

Remark 3.19. If T has definable patterns then every type over $\emptyset$ has an extension to a definable type over $\emptyset$; and also, by Proposition 3.14, to invariant type that is co-definable over $\emptyset$.

3.20. **Topology of** $\mathrm{Core}\,(T)$**.** *For the rest of the section, simply because the proofs were written with this assumption, we assume T is QEble; or at least, where indicated, Robinson. It is likely that much can be generalized.*

For stable theories, Shelah's finite equivalence relation theorem can be read as saying that distinct elements of $\mathcal{J}$ are separated by a finite definable partition. Here we will consider 0-definable family $\bar{E} = (E_d : d \in D)$ of (parameterically) definable m-partitions. The condition that two types over M are separated by E_d for *any* $d \in D(M)$ can be formulated as a basic formula $\Xi'_{\bar{E}}$ of $\mathcal{L}$, namely $\mathcal{R}_{E_u(x,x');D(u)}$.

Lemma 3.21. *Let $p, p' \in \mathcal{J}$ be distinct.*

(1) There exists a formula Ξ, finite conjunction of atomic formulas, with $\mathcal{J} \models \Xi(p,p')$, such that $\mathcal{T} \models \neg(\exists \xi)\Xi(\xi,\xi)$.
(2) (QEble case.) Let Ξ be as in (1). Then there exists m and a nonempty 0-definable family $\bar{E} = (E_d : d \in D)$ of m-partitions, so that $\mathcal{T} \models \Xi \to \Xi'_{\bar{E}}$.

Proof. (1) By the maximality of atomic types realized in $\mathcal{J}$, Lemma 3.12 (4), applied to the type of (p,p').
(2) Let $M \models T$. Ξ is a finite conjunction of basic formulas $\mathcal{R}_{\psi,\psi'}$; we consider a single one for simplicity (or using the elimination of finite conjunctions.) Since $\mathcal{T} \models \neg(\exists \xi)\Xi(\xi,\xi)$, there is no type $p(x)$ satisfying $\Xi(p,p)$. So $tp(x/M) = tp(x'/M)$ is inconsistent with the conjunction C of all $\neg(\psi(x,c) \wedge \psi'(x',c))$. By compactness, there exists a finite $C_0 \subset C$ and a finite M-definable partition of M into definable sets $\phi_1(x,d), \dots, \phi_n(x,d)$ such that for each i, each $\phi_i(x,d) \wedge \phi_i(x',d)$ is already inconsistent with C_0. Let $E_d(x,x')$ be the equivalence relation:

$$\bigwedge_i (\phi_i(x,d) \iff \phi_i(x',d))$$

Then E_d is part of a 0-definable family of definable equivalence relations with $\leq 2^n$ classes, and each having the required property (i.e. no pair of equivalent elements can satisfy C_0.) □

Similarly:

Remark 3.22. Let γ, γ' be two sets of formulas. Assume: whenever $\phi(x,y) \in \gamma$, there exist variables x', y' with $\phi'(x',y') \in \gamma'$, and vice versa. Let $J = \mathcal{J}_\gamma$, $J' = \mathcal{J}_{\gamma'}$, and let M be any model of $T^{\pm}$. Then there exists a canonical bijection between $\mathrm{Hom}(J, S(M))$ and $\mathrm{Hom}(J', S(M))$.

Proof. We may assume that $\gamma \subset \gamma'$, by comparing both to $\gamma \cup \gamma'$. In this case it suffices to show that if p is a γ'-type, then $h(p) = q$ iff $h(p|\gamma) = q|\gamma$. This is clear since any homomorphism h must preserve the 'change of variable' relations $\mathcal{R}_t$ with $t = (\phi(x,y), \neg\phi'(x',y'))$ and $t = (\neg\phi(x,y), \phi(x',y'))$.

□

Lemma 3.23. *Let $a \in \mathfrak{J}$.*

- *We have $Ga = P(\mathfrak{J})$ where P is the $\mathcal{L}$-atomic type of a.*
- *The map $e_a : G \to \mathfrak{J}$, $g \mapsto ga := g(a)$ is continuous and and closed.*
- *G-conjugacy is an intersection of $|L|$ open relations on $\mathfrak{J}$.*

Proof. (1) This is the homogeneity for atomic types, Lemma 3.12.

(2) A basic closed subset of G has the form

$$F = \{g : (gb, c) \in P\}$$

where $P \subset \mathfrak{J}^n$ is pp definable. This makes continuity evident.

Since the basic closed sets are closed under finite intersections, and $\mathfrak{J}$ is compact, it suffices for closedness to prove that the image of a basic closed set F is closed; this is a set of the form

$$e_a(F) = \{ga : (\exists g \in G)(gb, c) \in P\} = \{a' : (\exists b')\, P(b', c) \wedge (a, b) \sim (a', b')\}$$

where $\sim$ denotes G-conjugacy. Let Q be the maximal atomic type of (a, b); then

$$e_a(F) = \{a' : (\exists b')P(b', c) \wedge Q(a', b')\} = \{a' : R(a', c)\}$$

where $R(x, z)$ is the pp formula $(\exists y)(P(y, z) \wedge Q(x, y))$; it is closed by definition.

(3) (a, b) are G-conjugate iff for all atomic Q, Q' such that $\mathfrak{T} \models \neg(\exists x)(Q(x) \wedge Q'(x))$, we have $\neg(Q(a) \wedge Q'(b))$.

□

Let $\mathfrak{g}_!$ be the set of elements of G that act infinitesimally on each type of $\mathfrak{J}$; this may be smaller than $\mathfrak{g}$. In the notation of § C, $\mathfrak{g}_! = \mathfrak{g}_X$ where X is the disjoint union of all maximal atomic types of $\mathfrak{J}$. $X/\mathfrak{g}_!$ denotes the orbit space, i.e. the quotient of X under $\mathfrak{g}_!$-conjugacy.

Proposition 3.24. *Let P be a maximal atomic type of $\mathfrak{J}$.*

(1) $P_h := P/\mathfrak{g}_!$ is Hausdorff

(2) If L is countable, $P/\mathfrak{g}_!$ is metrizable.

Proof. (1) Let $N = \mathfrak{g}_! = \mathfrak{g}_X$, with X as above. By Lemma C.1, N is a closed normal subgroup of G, and G/N is Hausdorff. Moreover for $c \in P$, the map $G \to P$, $g \mapsto gc$ is closed, by Lemma 3.23. By Lemma C.2 (1) and Remark C.3, $\mathfrak{g}_!$-conjugacy coincides on P with $\mathfrak{g}_P$-conjugacy, and $P_h = P/\mathfrak{g}_!$ is Hausdorff.

Since P_h is Hausdorff, the diagonal of P_h is closed, so pulling back to P we see that the graph $E_{\mathfrak{g}_!}$ of $\mathfrak{g}_!$-conjugacy on P is pp-closed. Moreover the pp topology on $(P_h)^2$ coincides with the product topology.

(2) Since $E_{\mathfrak{g}_!}$ is pp-closed, it follows from the pp-homogeneity of $\mathfrak{J}$ (quantifying out any parameters) that $E_{\mathfrak{g}_!}$ is $\bigwedge$-pp-definable. As $\mathfrak{J}$ is e.c., if $(a, b) \notin E_{\mathfrak{g}_!}$ then there exists a pp-definable C with $(a, b) \in C$ and $C \cap E_{\mathfrak{g}_!} = \emptyset$. Thus $E_{\mathfrak{g}_!}$ is the intersection of $|L|$ open sets, namely the complements of these sets C. Now metrizability of the quotient follows from Lemma C.4.

□

Proposition 3.25. *Any $\bigwedge$-pp definable subset P of $\mathfrak{J}$ has a dense subset of cardinality $\leq |L|$. Hence if P is a maximal atomic type, then $|Aut(P_h)| \leq 2^{|L|}$.*

Proof. Denote an image of $\mathfrak{J}$ in $S(M)$ by J; we identify $\mathfrak{J}$ with J and P with $P(J)$. Three topologies are visible on P: the intrinsic pp topology $\mathfrak{t}_p$; the topology $\mathfrak{t}_{p,ext}$ induced from the pp topology on $S(M)$; and the topology induced from the usual logic topology $\mathfrak{t}_l$ on $S(M)$, where a clopen set corresponds to a formula of $L(M)$; this last topology has basis B_l with $|B_l| \leq |L|$. We have $\mathfrak{t}_p \subseteq \mathfrak{t}_{p,ext} \subseteq \mathfrak{t}_l$. For each $u \in B_l$ with $u \cap P \neq \emptyset$, pick $j_u \in u \cap P$, and let $D := \{j_u : u \in B_l, u \cap P \neq \emptyset\}$. Then D is $\mathfrak{t}_{p,ext}$-dense in P: if $U \in \mathfrak{t}_{p,ext}$ and $U \cap P \neq \emptyset$, then $u \cap P \neq \emptyset$ for some $u \in B_l, u \subset U$. Hence $j_u \in U \cap D$. It follows in particular that D is $\mathfrak{t}_p$-dense in P.

Thus the image D_h of D in P_h is dense in P_h. Since P_h is Hausdorff, an automorphism fixing a dense set is the identity; so any automorphism σ of P_h is determined by $\sigma|D_h$. Thus $|Aut(P_h)| \leq 2^{|L|}$.

□

Corollary 3.26. *(1)* $|\mathfrak{J}| \leq 2^{|L|}$
(2) $|\mathrm{Aut}(\mathfrak{J})| \leq 2^{2^{|L|}}$
(3) $|\mathcal{G}| \leq 2^{|L|}$.

Proof. Let $M \models T$, $|M| \leq |L|$. By Lemma 3.9 $\mathfrak{J}$ embeds into $S(M)$; thus $|\mathfrak{J}| \leq 2^{|L|}$; and so $|\mathrm{Aut}(\mathfrak{J})| \leq 2^{2^{|L|}}$.

By Proposition 3.25, $\mathfrak{J}$ has a dense set D of size $\leq |L|$, Any automorphism σ of $\mathfrak{J}$ fixing D (pointwise) has the property that for a nonempty open U, $\sigma(U) \cap U \neq \emptyset$; i.e. $\sigma \in \mathfrak{g}$. Thus the restriction $\sigma|D$ determines σ modulo $\mathfrak{g}$. Since $|\mathfrak{J}| \leq 2^{|L|}$, we have $|\mathfrak{J}^D| \leq 2^{|L|}$. Thus $|\mathcal{G}| = |G/\mathfrak{g}| \leq 2^{|L|}$.

□

The third item is similar, but not quite comparable, to the statement in Proposition 3.25 that the automorphism group of any Hausdorff type of $\mathfrak{J}$ has cardinality $\leq 2^{|L|}$. Example 3.36 shows a Hausdorff $\mathrm{Aut}(\mathfrak{J})$ of cardinality $2^{2^{\aleph_0}}$ is possible.

Remark 3.27. $\{\sigma \in Aut(\mathfrak{J}) : \sigma(F) \subset U\}$ is open, for any closed F and open U.

Proof. F is an intersection of a family of basic closed sets F_i, that we may take to be closed under finite intersections. By compactness, $\sigma(F) \subset U$ iff $\sigma(F_i) \subset U$ for some i. So we may assume F is basic-closed; similarly we may assume U is basic open. Say $F = \{p : \mathcal{R}_{\phi,\phi'}(p,q)\}$, $U = \{p' : \neg\mathcal{R}_{\psi,\psi'}(p,q)\}$. So $\sigma(p) \in U$ iff $\neg\mathcal{R}_{\psi,\psi'}(p,q')$, with $q' = \sigma^{-1}(q)$. Then $\sigma(F) \subset U$ iff there is no p with $\mathcal{R}_{\phi,\phi'}(p,q) \wedge \mathcal{R}_{\psi,\psi'}(p,q')$. Now embed $\mathfrak{J}$ in $S(M)$, with image J. Then such a p exists in J iff it exists in $S(M)$. In $S(M)$, the existence of such a p is a consistency question that amounts to $\mathcal{R}_{\theta,\theta';\alpha}(q,\sigma^{-1}(q))$ for a certain family of θ, θ', α. Hence the set of pairs $(q, \sigma^{-1}(q))$ for which a p exists is $\bigwedge$-pp; the set of pairs for which it does not is pp-open. Hence the condition on σ is pp-open too. □

Remark 3.28. The natural map $Aut_L(M) \to Aut_{\mathcal{L}}(S(M))$ is an isomorphism. Injectivity is clear using the embedding $i : M \to S(M)$, mapping m to the algebraic type $x = m$. The image of i (in any given sort) is the complement of a basic relation of $\mathcal{L}$, since it is precisely the set of types representing the formula $x = m$. Any definable relation $\alpha(x_1, \dots, x_n)$ on M is mapped by i to a $\mathcal{L}$-definable relation on $i(M)$, namely the negation of $\mathcal{R}_{x_1=y_1,\cdots,x_n=y_n;\alpha}$. Thus any $\mathcal{L}$-automorphism of $S(M)$ induces an automorphism of M on the copy $i(M)$. Finally $\phi(x, c) \in p$ iff $\mathcal{R}_{\phi(x,y),x'=y';\phi(y,y')}$ does not hold of the pair $(p, i(c))$; so a $\mathcal{L}$-automorphism fixing $i(M)$ is trivial.

3.29. **Examples.**

Example 3.30. For finite γ, any 0-definable γ-type is represented by a unique element of $\mathcal{J}_\gamma$; it is uniquely characterized by an atomic formula of $\mathcal{L}$ as in Example 3.2, and so is fixed by any retraction.

More generally almost 0-definable γ-types, i.e. definable types whose canonical definitions are imaginary elements algebraic over $\emptyset$, can only be permuted among themselves by an $\mathcal{L}$-retraction, and so are present in $\mathcal{J}$.

When γ consists of stable formulas, $\mathcal{J}_\gamma$ is the discrete finite space of γ-types definable almost over $\emptyset$; equivalently definable over $\mathrm{acl}^{eq}(\emptyset)$. It was here that Shelah introduced imaginaries, and algebraic closure.

A slightly larger class are the *densely definable* pattern types: p is *densely definable* if for any consistent ϕ, for some consistent ϕ' implying ϕ, and some ψ, p implies that $X = \psi$ on ϕ'. (Again one can check that this implies maximality of p.) When the underlying sort forms a complete type of T, this is the same is definability. Any densely definable is represented by an element of the core. Moreover if p, p' are densely definable and densely equal, i.e. for any consistent ϕ, for some consistent ϕ' implying ϕ, p, p' have he same definition on ϕ', then they are necessarily represented by the same element.

Example 3.31. For the random graph, in the home sort, $\mathcal{J}$ has two elements, corresponding to the two definable types (adjacence to all or to none.) Similarly for DLO. For the triangle-free graph, it is the unique definable type.

Example 3.32. Assume T has a model M whose every element is definable. Then the underlying space of $\mathcal{J} = \mathrm{Core}\,(T)$ is nothing more than the type space over $\emptyset$. Indeed we have as usual an $\mathcal{L}$-embedding $\mathcal{J} \to S(M)$, commuting with the two maps into $S(\emptyset)$; since $S(M) \to S(\emptyset)$ is an isomorphism, the map $\mathcal{J} \to S(M)$ must be surjective.

This remains true for γ-types with distinguished variables x and parameter variables y, i.e. $\mathrm{Core}_\gamma \cong S_\gamma(\emptyset)$, provided $(\exists y)\phi \in \gamma$ for all $\phi \in \gamma$. For this homeomorphism to hold, it suffices that every element of M^y be definable.

Nevertheless, the associated expansion may not be trivial; see for instance Example A.5.

Similarly, returning for simplicity to complete types, if every element of M is algebraic, $\mathcal{J}$ can be identified as a space with the Shelah strong types.

Example 3.33 (cf [39])**.** Consider the basic ingredient of Ziegler's example of a non-G-compact theory: an oriented circle with $\mathbb{Z}/n\mathbb{Z}$ action. Or a relational variant, taking a random dense subset of the circle $\mathbb{R}/\mathbb{Z}$, with the relation $y < x < y + 1/n$. In either case, $\mathcal{J}$ is finite but nontrivial; it is essentially $\mathbb{Z}/n\mathbb{Z}$ with the regular $\mathbb{Z}/n\mathbb{Z}$-action.

Example 3.34 (Connected Lie groups)**.** For the circle $x^2 + y^2 = 1$ in RCF with the rotation-invariant semi-algebraic relations (Poizat's example), or for the oriented circle as in Example 3.33 but with the action of an irrational rotation, $\mathcal{J}$ is the standard circle. The embedding to $S_x(M)$ for a model M can be taken to be via the Lebesgue-weakly random types. The retraction takes a type over M to the unique coset of the infinitesimal subgroup containing it.

Example 3.35. Countable theories with $\mathcal{J}$ Hausdorff, of cardinality $2^{\aleph_0}$, $|\mathcal{G}| = 2^{\aleph_0}$.

(1) Consider the model completion of the theory of graphs with infinitely many disjoint unary predicates P_n. We consider the sort S_γ where γ is the graph adjacency formula (considering S_x would make no difference.) Let M be a countable model. There are $2^{\aleph_0}$ maximal definability patterns of 1-types over M; one can choose $\gamma(x, u)$ to hold for all $u \in P_n$, or for none; and this, independently of n. These are the maximal atomic types of $\mathcal{T}$. They must all be represented in $\mathcal{J}$, hence $|\mathcal{J}| = 2^{\aleph_0}$. $\mathcal{J}$ is Hausdorff; if $p \neq q$, say $(d_p x)(\gamma(x, u) \wedge P_1(u)) = P(u)$ while $(d_q x)(\gamma(x, u) \wedge P_1(u)) = \perp$. Let R be the atomic formula asserting that $\gamma(x, u) \wedge P_1(u)$ is omitted, and R' the atomic formula asserting that $\neg\gamma(x, u) \wedge P_1(u)$ is omitted. Then $\neg R, \neg R'$ are disjoint open sets separating p, q. We have $\mathrm{Aut}(\mathcal{J}) = 1$.

(2) Let L have a ternary relation $\gamma(x; y, y')$; we will concentrate on the sort S_γ (with distinguished variable x.) In addition, as above, L has infinitely many disjoint unary predicates $P_n(y)$. T states that each$\gamma(a; y, y')$ is a tournament: $(\forall x, y, y')\neg(\gamma(x; y, y') \wedge \gamma(x; y', y))$, $(\forall x, y, y')\gamma(x, y, y') \vee \gamma(x, y', y) \vee y = y'$. Further, for each $m < n$, $T \models (\forall x, y, y')(P_m(y) \wedge P_n(y') \rightarrow \gamma(x, y, y'))$. Let $p \in \mathcal{J}$. Then $(dp_x)\gamma(x, y, y')$ defines a linear ordering, with P_m earlier to P_n if $m < n$. For any subset W of ω, there exists an automorphism σ_W of $\mathcal{J}$, flipping the ordering on P_n for some n, such that $\sigma_W(p), p$ agree above P_n iff $\alpha \in W$. Thus $\mathcal{G} \cong \mathbb{Z}/2\mathbb{Z}^{\mathbb{N}}$ and $|\mathcal{G}| = 2^{\aleph_0}$.

An example with $\mathfrak{g}_{\mathcal{J}} = \mathrm{Aut}(\mathcal{J})$ and $|\mathrm{Aut}(\mathcal{J})| = \beth_2$:

Example 3.36. Topological dynamics comes back into the picture if both some set theory, and a group action, are built into the theory T. In our approach, the topological dynamics arises as an example via a specific theory; in [33],[34], [21], by contrast, it is the first-order theory that is treated as an example of a topological dynamics, via the type spaces of saturated models. In Example 5.8 we will see how that universal minimal flow of any discrete group Γ is dual to $\mathfrak{J}(T)$ for an appropriate theory $T = T_\Gamma$. Here we give a hands-on treatment of the case of $\mathbb{Z}$.

Let T be the model completion of a bipartite graph $R \subset P \times Q$, with an invertible map $s : Q \to Q$ generating a $\mathbb{Z}$-action on Q. We are interested in $\mathfrak{J}_\gamma$ where $\gamma(x, y) = \{R(x, y)\}$, with distinguished variable x of sort P.

When $\mathfrak{J}$ is embedded into $S(M)$, we can identify an element p of $\mathfrak{J}$ with a subset $(d_p x)R(x, y)$ of $Q(M)$. We take M so that $Q(M)$ is a single $\mathbb{Z}$-orbit; if we pick momentarily a point of the $\mathbb{Z}$-orbit, we can view $\mathfrak{J}$ as a set of subsets of $\mathbb{Z}$. We have:

- $\mathfrak{J}$ is translation invariant. Indeed there exists a basic relation $R(p, q)$ asserting that $(x \in p, \sigma(x) \notin q) \vee (x \notin p, \sigma(x) \in q)$ is omitted. Then $S(M) \models (\forall p)(\exists q)R(p, q)$, so this must be true in $\mathfrak{J}$. In particular the family of subsets of $\mathbb{Z}$ corresponding to $\mathfrak{J}$ does not depend on the choice of point.
- $\mathfrak{J}$ contains all periodic sets. Indeed the elements of $S(M)$ of order m are captured by a basic relation; $S(M)$ contains 2^m such sets, so all of them must be in $\mathfrak{J}$. In particular $\mathfrak{J}$ is dense in $2^{\mathbb{Z}}$ with the topology of pointwise convergence.
- For any $p_1, \dots, p_k \in \mathfrak{J}$, any configuration that occurs on some interval of length m (i.e. a k-tuple of subsets of $[b, \dots, b+m]$) recurs infinitely often on other intervals. (Otherwise we could get rid of this configuration by an ultrapower and restriction to a $\mathbb{Z}$-orbit, finding a homomorphism that is not an embedding on $\{p_1, \dots, p_k\}$.)
- Let $m \in \mathbb{N}$, and let $a_0, \dots, a_k$ be subsets of $\{0, \dots, m\}$; $\bar{a} := (a_0, \dots, a_k)$. Let $W_{\bar{a}} = \{(p_0, \dots, p_k) : (\exists b)(p_i|[b, \cdots, b+m] = b + a_i)\}$ Given $p = (p_1, \dots, p_k)$, let $W_{\bar{a}}(p) = \{p_0 : (p_0, p) \in W_{\bar{a}}\}$. Then the $W_{\bar{a}}(p)$ form a basis for the pp-topology. We have $W_{\bar{a}}(p) \neq \emptyset$ provided $p \in W_{(a_1, \dots, a_k)}$.
- If $W_{\bar{a}}(p) \neq \emptyset$, and $W_{\bar{a}'}(p') \neq \emptyset$, then their intersection in $S(M)$ is nonempty, $W_{\bar{a}}(p) \cap W_{\bar{a}'}(p') \neq \emptyset$ and even includes periodic sets; these are necessarily in $\mathfrak{J}$.

  Thus any two nonempty open sets in $\mathfrak{J}$ have a nonempty intersection. It follows that $\mathfrak{g} = G$. Any continuous map on $\mathfrak{J}$ into a Hausdorff topological space is constant.
- We have $|G| = 2^{2^{\aleph_0}}$; moreover, unlike $\mathfrak{J}$, G has a Hausdorff quotient of that cardinality. To see this let I be a subset of the interval $(0, 1)$ such that $I \cup \{1\}$ is a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space. The dynamical system $(\mathbb{R}/\mathbb{Z})^I$, with transformation $(a_i) \mapsto (a_i + i)$, is a minimal system.[28] For $i \in I$, we define an

[28] To see this, reduce to finite linearly independent J; if Y is a closed invariant subset of $(\mathbb{R}/\mathbb{Z})^J$, translate so that $0 \in Y$; then Y is the closure of the subgroup generated by the element $(j)_j$; so Y is itself a closed subgroup; so some rational linear relation holds along it; in particular $\sum m_i \alpha_i = 0$, contradiction.

element c_i of $S(M)$, namely the type corresponding to the set of $n \in \mathbb{Z}$ such that ni lies in $[0,1/2)/\mathbb{Z}$. Let p_i be the atomic $\mathcal{L}$-type of c_i. Then p_i is a maximal atomic type, and the realization set of p_i in $\mathfrak{J}$ forms a copy of $\mathbb{R}/\mathbb{Z}$, on which $\mathbb{R}/\mathbb{Z}$ acts $\mathcal{L}$- automorphically. (It is easier, and sufficient for our purposes, to find a copy of $\mathbb{Z}$ (appearing as the image of $\mathbb{Z}i$ in $\mathbb{R}/\mathbb{Z}$), on which $\mathbb{Z}$ acts by $\mathcal{L}$-automorphisms; the sets are shifts of each other, and the $\mathcal{L}$-2-type sees this.) Moreover, for distinct elements $i_1, \ldots, i_k$ of I, these 1-types are almost orthogonal - they generate complete k-types. This demonstrates an action of $(\mathbb{R}/\mathbb{Z})^I$ (or $\mathbb{Z}^I$ in the easier version) on a subset of $\mathfrak{J}$; by Lemma 3.12, it follows that $\mathrm{Aut}(\mathfrak{J})$ has $(\mathbb{R}/\mathbb{Z})^I$ as a homomorphic image.

- We saw that $\mathfrak{g} = G$, while a large Hausdorff quotient exists: we have $\mathfrak{g}_q = 1$ for certain complete pp types q, arising from homomorphisms of G into the circle group. We will see in Example 5.9 the existence of other complete pp type $p \subset \mathfrak{J}$, such that G induces a countably infinite group G_p of automorphisms of p. As G_p is quasi-compact it cannot be Hausdorff.

4. Automorphisms of the core, and the Lascar group

We return to the setting of a complete first order theory.

4.1. **Lascar distance.** Let N be a model of T. We will call two elements a, a' of the same sort in N *Lascar neighbors* if for every 0-definable family $\bar{E} = (D, E_d)_{d \in D}$ of finite partitions, $N \models (\exists d) a E_d a'$. Equivalently, for any formula $\phi(u)$ consistent with T, and finite set γ of formulas, there exists $b \in \Phi(N)$ with $tp_\gamma(a/b) = tp_\gamma(a'/b)$. The Lascar neighboring pairs are the solution set of a partial type $L_1^2(x, x')$.

For a type $q(x, x')$ let us write $L_1^2(q)$ if $q(a, a')$ implies $L_1^2(a, a')$. For a pair of 1-types $p, p' \in S_\gamma(M)$, we define:

$$L_1(p, p') \iff (\exists q)(L_1^2(q) \wedge \pi_1(q) = p \wedge \pi_2(q) = p')$$

So L_1 is a binary $\bigwedge$-pp-definable relation of $\mathcal{L}$. In particular, L_1 is also defined on $\mathfrak{J}$.

If $L_1(p, p')$ holds, we say that p, p' are Lascar neighbors, or have *Lascar distance at most* 1. We define, in any $S(M)$ or in $\mathfrak{J}$, the symmetric relations L_n of Lascar distance at most n:

$$L_n(x, y) \iff (\exists x = x_1, \ldots, x_n = y) \bigwedge_{i<n} L_1(x_i, x_{i+1})$$

and the Lascar equivalence relation

$$L_\infty = \cup_n L_n$$

Call p_1, p_2 *close neighbors* if for every definable family of finite local partitions $(E_b)_{b \in B}$, for some $b \in B(N)$ and $d' \in D(N)$, $x_i E_b d' \in p_i$. This implies that $p_1 \cup p_2 \models L_1^2(x_1, x_2)$, and in particular p_1, p_2 are neighbors.

In any $\aleph_0$- saturated model N, we have $|N^x/L_\infty| \leq 2^{|L|}$.

By Lemma 3.12, each L_n can also be written as a conjunction of atomic formulas of $\mathcal{L}$.

We remark that for $p, p' \in S(M)$, we have pL_1p' iff for any consistent ϕ, two realizations of p, p' can have the same type over *some* realization of ϕ, *not necessarily in M*. Strengthening the requirement to ask for a witness in M leads, in J, to the equality relation $p = p'$; see Lemma 3.21.

Let $\mathsf{Las}_{\mathfrak{J}} := \mathfrak{J}/L_\infty$, and $\mathsf{Las}_M = S(M)/L_\infty$. (Sort by sort.)

Proposition 4.2. *Let $j : \mathfrak{J} \to S(M)$ be an $\mathcal{L}$-homomorphism. Then (sort for sort) j induces a bijection $j_* : \mathsf{Las}_{\mathfrak{J}} \to \mathsf{Las}_M$.*

Proof. We can find a homomorphism $r : S(M) \to J := j\mathfrak{J}$ with $r|J = Id_J$. (Proposition 2.6). Let $g = j^{-1} \circ r$. Since r, j are homomorphisms, for $a, b \in \mathfrak{J}$ we have $\mathfrak{J} \models L_n(a, b)$ iff $S(M) \models L_n(ja, jb)$. Thus j induces an injective map $\mathsf{Las}_{\mathfrak{J}} \to \mathsf{Las}_M$. It remains to show that it is surjective; it suffices to show that $r : S(M) \to J$ preserves Lascar types. We have this is a strong form:

Claim A. For all $p \in S(M)$ we have $L_1(p, r(p))$; in fact for *any* $a \models p$ and $b \models r(p)$ we have aL_1b.

Indeed let $p \in S_x(M)$, $p' = r(p)$. Since $p' \in J$ we have $r(p') = p'$. Let $q \in S_{x,x'}(M)$ be any type extending $p(x) \cup p'(x')$, and let $q' = r(q)$. Then q' extends $p'(x) \cup p'(x')$, so $q' \vdash (xL_1x')$ (witness: M.) But by 3.1, $q|\emptyset = q'|\emptyset$. Thus $q \vdash (xL_1x')$. This proves the claim and the proposition. $\square$

We would of course prefer to say that $j_* : \mathsf{Las}_{\mathfrak{J}} \to \mathsf{Las}_M$ is an isomorphism, not just a bijection. However Las_M does not classically carry any structure, beyond that of a set acted on by $\mathrm{Aut}(M)$. We can thus do not better than compare the two as permutation groups.

Let $G = \mathrm{Aut}(\mathfrak{J})$, with the topology described in § 2.8, and let $\mathfrak{g} = \mathfrak{g}_{\mathfrak{J}}$ be the subgroup of infinitesimal autormophisms with respect to the action of G on $\mathfrak{J}$ (Appendix C). Let $\mathcal{G} = G/\mathfrak{g}$; so $\mathcal{G}$ is a compact Hausdorff topological group. As in Proposition 3.24, we also let $\mathfrak{g}_P$ denote the infinitesimal subgroup with respect to the action of G on P, and let $\mathfrak{g}_! := \cap_P \mathfrak{g}_P$ be the intersection of $\mathfrak{g}_P$ over all maximal atomic types P of $\mathfrak{J}$. So $\mathfrak{g}_! \leq \mathfrak{g}$.

Fix a homomorphism $j : \mathfrak{J} \to S(M)$. Then j_* identifies $\mathsf{Las}_{\mathfrak{J}}$ with Las_M, and induces a homomorphic embedding of $G = \mathrm{Aut}(\mathfrak{J})$ into $Sym(\mathsf{Las}_M)$ (we will also denote it j_*.)

Define the Lascar group $G_{\mathsf{Las};\gamma}$ as the image of $\mathrm{Aut}(M^*)$ in the group of permutations of $\mathsf{Las}_{M;\gamma} := S_\gamma(M)/L_\infty$; where $M^* \succ M$ is a sufficiently saturated extension, and Las_M is identified with Las_{M^*} via the restriction map on types.

Lemma 4.3. *(1) The image $j_*(G) \leq Sym(\mathsf{Las}_{M;\gamma})$ is precisely the Lascar group $G_{\mathsf{Las};\gamma}$.*

(2) *(Taking γ rich enough). If $g \in \mathfrak{g}$ then $j_*(g)$ is the identity on $\mathsf{Las}_{\mathfrak{J}}$. In fact, for $g \in \gamma$ and $p \in \mathfrak{J}$ we have $pL_2g(p)$.*

Proof. (1) We first show that $j_*(G)$ falls into the Lascar group Las_S. $g \in Aut(J)$. Let $M^* \succ M$ be a highly saturated and homogeneous extension of M Let $p \in J$ be $Aut(J)$-conjugate, $q = g(p)$, and let $a, b \in M^*$, $a \models j(p), b \models j(q)$. Then in particular $tp(a/\emptyset) = tp(b/\emptyset)$ (Example 3.1). So there exists $\gamma \in Aut(M^*)$ with $\gamma(a) = b$. It follows that γ maps the Lascar type of a (and of p) to the Lascar type of b (and of q). This applies to $*$-types too, and we can take p to be rich enough to enumerate all Lascar types of elements of S. This will show that the permutation of Las_S induced by g and by γ coincide.

In the converse direction, it suffices to show (for any M) that for any $\sigma \in \mathrm{Aut}(S(M))$, the permutation induced by σ in Las_M lies in the image of j_*. Let $r : S(M) \to J$ be a retraction. Then $r \circ \sigma$ defines an automorphism of J. Since we saw that r preserves Lascar types (Claim A of Proposition 4.2), $r \circ \sigma$ induces on J/L_∞ (identified with Las_M) the same permutation as σ. This shows that j_* is surjective.

(2) Let us identify $\mathfrak{J}$ with $J = j(\mathfrak{J})$. Let $g \in \mathfrak{g}$, $p \in J$, $p' = g(p)$. We will show that $L_2(p, p')$ holds (in $S(M)$; equivalently in J.) Let $\bar{E}$ be a definable family of finite partitions, as in 4.1. Let $\Xi' = \Xi'_{\bar{E}}$, as defined above Lemma 3.21; so $\Xi'(\eta, \eta')$ implies that η, η' are not close Lascar neighbors; in particular we have $\neg\Xi'(p, p)$. So $\neg\Xi'(p, \eta)$ defines an open neighborhood of p. Likewise $\neg\Xi'(p', \eta)$ defines an open neighborhood of p'. Since g is an infinitesimal automorphism, the intersection of these g-conjugate open sets is nonempty, so for some $q \in J$ we have $\neg\Xi'(p, q) \wedge \neg\Xi'(p', q)$. As $J \subset S(M)$ we can view p, p', q as types over M; let a, a', c be realizations; then there exist $d, d' \in D(M)$ with $aE_dcE_{d'}a'$. Since this holds for all $\bar{E}$, and any finite number of $\bar{E}$ have a common refinement, it follows that in some elementary extension M^* of M there exists c^* such that for any $\bar{E} = (D, E_d)_{d \in D}$, for some $d, d' \in D(M^*)$, $aE_dc^*E_{d'}a'$. Now by the definition of L_1^2 (§ 4.1), it follows that $L_1^2(a, c^*)$ and $L_1^2(c^*, a')$; so with $q^* = tp(c^*/M)$ we have $L_1(p, q^*)$ and $L_1(q^*, p')$, hence $L_2(p, p')$.

□

To define the full Lascar group G_{Las}, we take γ to be the set of all formulas, in countably many variables in each sort (both distinguished and parameter variables.) Las_T is *not* in general the inverse limit of $Las_{\gamma'}$ for finite $\gamma' \subset \gamma$. Let $G = \mathrm{Aut}(\mathfrak{J}) = \mathrm{Aut}(\mathfrak{J}_\gamma)$. Define a subset L_1 of G: $g \in L_1^G$ iff $(p, g(p)) \in L_1$ for all $p \in \mathfrak{J}$. Since L_1 is a closed relation on $\mathfrak{J}$, L_1^G is a closed subset of G in the pp topology. Also denote by $L_1^{\mathcal{G}}$ the image of L_1^G in $\mathcal{G}$. We will potentially just write L_1 for any of these. Note that L_1 is a closed, conjugation-invariant subset of G, hence this is also the case for $\mathcal{G}$.

Let M be a sufficiently saturated model of T, $j : \mathfrak{J} \to S(M)$ an $\mathcal{L}$-embedding, $J = j(\mathfrak{J})$, $r : S(M) \to J$ a retraction. We have a map: $\sigma \mapsto \alpha(\sigma) := r \circ \sigma | J$ from

$\mathrm{Aut}(M)$ to $\mathrm{Aut}(J)$. Now by Claim A, $r\sigma^{-1}(p)L_1\sigma^{-1}(p)$ for any $p \in S(M)$; so $\sigma r\sigma^{-1}(p)L_1p$; or $\sigma rqL_1\sigma(q)$, for $q \in S(M)$; thus $r\sigma r\tau(p)L_1r\sigma\tau(p)$; so α induces a homomorphism $\mathrm{Aut}(M) \to \mathrm{Aut}(J)/\langle L_1\rangle$, where $\langle L_1\rangle$ is the group generated in $\mathrm{Aut}(J)$ by the closed normal set L_1.

Any automorphism fixing a model satisfies $\sigma(p)L_1p$ and thus $r\sigma(p)L_1rp$, since r respects L_1; for $p \in J$ this reads $\alpha(\sigma)(p)L_1p$. Thus the group of strong Lascar automorphisms (generated, by definition, by automorphisms fixing a model) maps to the identity, and α induces a homomorphism $\mathrm{Aut}(G)/\mathrm{Aut}_L(G) \to \mathrm{Aut}(J)/\langle L_1\rangle$. The kernel of this homomorphism maps to the identity on $\mathrm{Aut}(J)$ and since $J/L_\infty = S(M)/L_\infty$, fixes all Lascar types. Thus the kernel is the identity, i.e. α induces an isomorphism $\mathrm{Aut}(G)/\mathrm{Aut}_L(G) \to \mathrm{Aut}(J)/\langle L_1\rangle$. By Lemma 4.3 (2), $\mathfrak{g} \subseteq \langle L_1\rangle$. Hence:

Proposition 4.4. $G_{\mathsf{Las}} = \mathrm{Aut}(M)/\mathrm{Aut}_L(M) \cong \mathcal{G}/\langle L_1\rangle$.

We end this section with an example (very similar to one used by Pillay) showing that L_1, restricted to a sort S, can depend on the full ambient structure in other sorts, and not only on the induced structure on S. Let S carry the structure of a free $\mathbb{Z}/2\mathbb{Z}$-action, written $x \mapsto \pm x$, and no additional structure. Let S' be another sort, and let $P \subset S\times S'$ be a 'random' relation with the property that $P(x,y) \iff \neg P(-x,y)$. Then on S as a structure we have $L_1 = S^2$. On the other hand on S as part of (S,S_1) we have: aL_1b iff it is not the case that $b=-a$. .

5. Elementary Ramsey theory

Recall the notion of the Ramsey property from the introduction:

Definition 5.1. A complete first order theory T is said to be *Ramsey* at a given sort S if any completion T' of T in the language L_P with a unary predicate $P \subset S$ adjoined has a model $N' = (N,A)$ ($N \models T$, $A = P^{N'} \subset N$) with an elementary submodel M of N, such that $P \cap M$ is a 0-definable predicate on M.

On the other hand, if T is an irreducible universal theory, we say that T is Ramsey (at V) if any irreducible universal T' in L_P has a model $N' = (N,A)$ ($N \models T$, $A = P^{N'} \subset N$) with an existentially closed substructure M of N, such that $P \cap M$ is a qf 0-definable predicate on M. Here P is a unary predicate of sort V.

T is *everywhere Ramsey* if it is Ramsey at S for all S. If $T = T_0^{eq}$ for a theory T_0 with single sort S, it suffices to check Ramseyness at S_0^n for each n.

Equivalently, for any $M \models T$ and any sufficiently saturated $N' = (N,A) \models T'$, there exists an elementary embedding $f : M \to N$ with $f^{-1}(A)$ 0-definable in M.

If given two (or more) predictes $P, P', \cdots$, we can first move to a model where P is definable, then to another where P' is definable; so the definition would not change if we allow a finite coloring (to be made definable), or several predicates P (or for that matter even an infinite number).

Lemma 5.2. *Assume $T_\forall = Th_\forall(M)$ is a Ramsey universal theory at V, M existentially closed. Then $Th(M)$ eliminates quantifiers for formulas on V.*

Proof. Consider a formula $\psi(x) = (\exists y)\phi(x,y)$, with ϕ quantifier-free. Let $A = \psi^M$. By Lemma 1.5, some pattern type q containing T is dense in (M, A). By the Ramsey property, q is definable by some quantifier-free definable D. If $T \models (\exists x, y)(\phi(x,y) \wedge \neg D(x))$, they by density there exists a finite $M_0 \subset M$ with $M_0 \models (\exists x, y)(\phi(x,y) \wedge \neg D(x))$, and $(M_0, A) \models q$, i.e. $A \cap M_0 = \psi(M_0) = D(M_0)$; this is clearly impossible. Thus $\psi(M) \subset D(M)$. Conversely, if $a \in D(M)$ and $M \models \neg\psi(a)$, then since M is e.c. there exists a pp formula ψ' incompatible with ψ, such that $M \models \psi'(a)$. Again by density there exists a finite M_0 with $\psi(M_0) = D(M_0)$, and such that some $a' \in D(M_0)$ satisfies $\psi'(a')$; this contradicts the incompatibility of ψ, ψ'.

□

Lemma 5.2 implies in particular that the class of finite models of $T_\forall$ has the amalgamation property, a theorem of [38].

Proposition 5.3. *Let T be an irreducible universal theory, with a distinguished sort or family of sorts V. There exists a unique minimal expansion T_V^{ram} to an irreducible universal theory that is Ramsey at V. Any M with $Th_\forall(M) = T$ admits an expansion to a model of T_V^{ram}; the space of expansions is just $Hom(J, S_\gamma(M))$.*

Proof. For simplicity we consider one sort V; form T_V^* as above Definition 1.8. Let γ denote the new 'second-order' relations introduced in the * operation. Apply Proposition 3.17 to T_V^* to obtain an irreducible universal theory $\widetilde{T}$ that has definable patterns at γ. Return now to the original sorts; call the result $T_\forall^{ram}$. Note that $\widetilde{T} = (T_\forall^{ram})^*$: the axioms of $\widetilde{T}$ are explicit and concern the new relations on the parameter sorts of γ, so they are visible already for $T_\forall^{ram}$. It follows that $T_\forall^{ram}$ is Ramsey. If T' is an expansion of T to an irreducible universal theory that is Ramsey, then $(T')^*$ has definable patterns with respect to the 'second-order' relations introduced in the * operation, and so interprets $\widetilde{T}$ (in the quantifier-free way described above Proposition 1.6). It follows that T' interprets $T_\forall^{ram}$. □

When V consists of all sorts, it follows from Lemma 5.2 that $T^{ram} := T_V^{ram}$ admits quantifier-elimination. We can apply these results to the Morleyzation of a complete first-order T. Taking into account the uniqueness in Proposition 3.17, we obtain Theorem 1.10, that we repeat below in a little more detail as Corollary 5.4.

Recall that a pair $T_1 \leq T_2$ of universal theories satisfies *interpolation* if whenever $R(x,y) \to S_2(y) \in T_2$ with $R \in L_1, S_2 \in L_2$ then for some $S_1 \in L_1$, $R(x,y) \to S_1(y) \in T_1$ and $S_1(y) \to S_2(y)$ in T_2.

It is easy to see that T, T' are complete theories with quantifier elimination in languages L, L' with $L \subset L'$, $T_\forall = T'_\forall|L$, and interpolation holds for the pair $T_\forall, T'_\forall|L$, then $T = T'|L$. (This is a special case of Lemma 2.13.)

If T admits quantifier-elimination, $T_1 = T_\forall$ is the universal part of T, and T_2 is the universal theory of some expansion of a model of T, then it is clear that interpolation holds (with S_1 an L-formula equivalent in T to $(\exists x)R(x,y)$.) In particular, interpolation holds between $T_\forall$ and the canonical Ramsey expansion of $T_\forall$.

Corollary 5.4. *(=Theorem 1.10)Let T be a complete theory. There exists an everywhere Ramsey expansion T^{ram} with this property: if T' is an everywhere Ramsey expansion of T and $N' \models T'$, then there exists an L- embedding $j : N' \to N$ with $N \models T^{ram}$, and so that the pullback of any definable subset of N is definable in N'.*

T^{ram} is unique up to bi-interpretability over T. The self-interpretations of T^{ram} over T form a group, $G^{ram}(T)$.

Proof. Here we may Morley-ize and so assume T admits quantifier elimination. The universal theory $T_\forall$ admits a canonical Ramsey expansion $T\forall^{ram}$ as a universal theory; this by Proposition 5.3. Let M be an existentially closed model of $T\forall^{ram}$. Then by Lemma 5.2, $Th(M)$ eliminates quantifiers. Let $T^{ram} = Th(M)$. It is uniquely determined by the universal part of $Th(M)$ which is just $T_\forall^{ram}$.

Minimality of T^{ram}, as well as the fact that $T \subset T^{ram}$, follows from the minimality of $T_\forall^{ram}$ given by Proposition 3.17, taking into account the above remarks about interpolation.

Let T' be an everywhere Ramsey expansion of T; again we may assume T' eliminates quantifiers. Let $N' \models T'$. Then we can expand $N'|_L$ to a model N'' of $T_\forall^{ram}$ (choosing a homomorphism $J \to S(N')$, so that each basic definable set of N'' is also N'-definable. Now N'' embeds into some $N \models T^{ram}$ as it has the correct universal theory, giving the minimality statement.

Conversely, if T' has the same minimality property, we may again assume T' eliminates quantifiers to prove first-order bi-interpretability with T^{ram}. The minimality property shows that $T'_\forall$ is minimal in the sense of universal theories, so in any case $T'_\forall$ and $T\forall^{ram} = T^{ram}_\forall$ are qf bi-interpretable over T. We may assume $T'_\forall = T^{ram}_\forall$. As T', T^{ram} admit QE, and have the same universal theory, they are now equal.

Since T^{ram} admits QE, any self-interpretation of $T^{ram}_\forall$ over T as a universal theory, extends uniquely to a bi-interpretation of T^{ram} over T as a 1st-order theory.

□

5.5. **Continuous logic version.** To see how this unifies Ramsey-type phenomena, we also formulate the continuous logic version.

In continuous logic, as presented e.g. in [5], V comes with a distinguished metric. An n-place predicate P on V is interpreted as a bounded real-valued function on V^n, uniformly continuous with respect to the metric. A *universal theory* is a a family of assertions that the values of a finite number of predicates $P_1, \dots, P_k$ lies in a given compact subset C of $\mathbb{R}^k$: $(\forall x)((P_1(x), \dots, P_k(x)) \in C)$. A *free pattern type* is a maximal universal theory in $L(X)$ whose restriction to L is $T_\forall$. p is *finitely satisfiable* in (M, X) if for any quantifier-free ϕ in $L(X)$ any $\epsilon > 0$, and any $a \in M^k$ there exists a (finite) $M_0 \subset M$ such that $(M_0, X|M_0) \models p$, and b from M_0^k with $|X(a) - X(b)| < \epsilon$. (Similarly for pattern types for externally definable sets.)

Equivalently, there exists an elementary extension (M^*, X^*) of (M, X) and an embedding $f : M \to M^*$, such that $(M, f^{-1}(X^*)) \models p$. Lemma 1.5 remain unchanged. We say that a theory T is a *Ramsey theory* at V (or has the Ramsey property at V) if all free pattern types for T on V are definable. This is also equivalent to the definition given at the beginning of the section (taken verbatim, with P interpreted as usual as real-valued.)

5.6. **Examples.**

Example 5.7. (1) Let T be the theory of infinite sets Ω, in the language of pure equality. Then $T^{ram}_{all} = DLO$.

If $J = \mathrm{Core}\, T$, H a finite group acting on a sort V, we have in general $J_{V/H} = J_V^H$ (the H-fixed points of J_V.)

If V is the sort of ordered pairs in T, and $U = V/Sym(2)$ the sort of unordered pairs, then J_V is the two-atom Boolean algebra, and $J_U = J_V^{Sym(2)} = \{0, 1\}$.

(2) Infinite affine spaces V over a finite field. Then T is a Ramsey theory at V, and also at the sort $V^{[n]}$ of n-element subspaces of V; this is the affine space Ramsey theorem, see [45]. A similar picture holds for projective spaces.

To study the sorts V^n, we may as well pass to the theory $Vect_{\mathbb{F}}$ of vector spaces over $\mathbb{F}$. Then T is not Ramsey at the main sort V. Indeed

T^{ram} is bi-interpretable with the theory of linearly ordered $\mathbb{F}$-spaces, such that each finite-dimensional vector space is lexicographically ordered with respect to some basis. (Note that this is not to be the same as a 'random' linear ordering adjoined to T, that makes an appearance in [21].)

(3) Affine spaces V over $\mathbb{Q}$ form a Ramsey theory at V; the only maximal patterns in $L[X]$ are the ones asserting $X = \emptyset$, or $X = V$. This is essentially equivalent to Van den Waerden's theorem on arithmetic progressions [44], [45]. Any consistent formula $\theta(x_1, \ldots, x_n)$ is implied by another of the form: $\bigwedge_{i\geq 2}(x_i - x_0) = \alpha_i(x_1 - x_0)$. And this formula is realized in any sufficiently long arithmetic progression $v_0, v_0 + v, \cdots, v_0 + m$. By Van den Waerden, for any set $A \subset V$, θ is realized either in A or in $V \smallsetminus A$; i.e. we can find an arbitrarily good approximation M_0 to a model, such that $(M_0, A) \models (\forall x)(x \in A)$ or $(M_0, A) \models (\forall x)(x \notin A)$. (Conversely, given a coloring of arbitrarily long intervals in c colors, with no monochromatic arithmetic progression of length l, a compactness argument gives a coloring of $\mathbb{Q}$ with no such arithmetic progresssion; but a model does contain a long arithmetic progression.)

(4) Let T be the theory of $\mathbb{Q}$-vector spaces V. Then T^{ram} includes the theory of ordered $\mathbb{Q}$-vector spaces. By contrast with e.g. [16], it cannot be interpreted in the the random linear ordering expansion of T. It would be good to determine T^{ram}; is it generated by DOAG along with the unary sets of the Ramsey expansion associated with the $\mathbb{Q}^*$-action, as in Example 5.8?

(5) Let V be an irreducible variety defined over a field K, and admitting a transitive action of an algebraic group G. Consider the invariant Zariski structure on V: a basic m-ary relation is a G-invariant K-Zariski closed subset of V^m.

For $V = \mathbb{A}^1$, G the two-dimensional group of affine transformations, this theory is Ramsey at V. This can be shown as a consequence of the generalized polynomial van der Waerden Theorem of [8], though it uses only a small part of the strength of that theorem. This is because for any formula $\phi(x_1, \ldots, x_n)$ consistent with the theory, there exist $\alpha_2, \ldots, \alpha_n \in K^{alg}$ such that for any $a \in V$ and $d \in K \smallsetminus (0)$, $V \models \phi(a, a + d, a + \alpha_2 d, \cdots, a + \alpha_n d)$; and using van der Waerden over $K(\alpha_2, \cdots, \alpha_n)$ to find $a \in V, d \in K^*$ such that $\phi(a, a+d, a+\alpha_2 d, \cdots, a+\alpha_n d)$ is monochromatic.

In particular, it follows that affine spaces V over an arbitrary infinite field K are Ramsey.

(6) Hilbert spaces (restricted to unit ball). Here a unary predicate X is interpreted not as a subset, but as a uniformly continuous function on the unit ball. The basic definable predicate here is the norm $X(v) = |v|$. Any continuous function $f(|v|)$ of the norm is definable, hence determines a pattern type. One may guess that these are the only pattern types,

and indeed this is a central theorem of Dvoretzky-Milman [46] (see [47], Theorem 1.2).

(7) Let $T = \widetilde{BA}$ be the theory of atomless Boolean algebras; the main sort will be denoted B, and we will also consider B^n for $n = 1, 2, \cdots$. Let $B_n \subset B^n$ denote the n-tuples of pairwise disjoint nonzero elements, whose sum is 1; then $B^{[n]} = B_n/Sym(n)$ is the sort of n-partitions of 1, or equivalently the sort coding subalgebras of B of size 2^n. The dual Ramsey Theorem of [48] states precisely that T is a Ramsey theory in the sorts $B^{[n]}$.

Let us compute T^{ram} in full. If B is a Boolean algebra with n atoms, and a linear ordering $a_1 < \cdots < a_n$ on these atoms. Then an element of B can be identified with a subset of $\{a_1, \ldots, a_n\}$ or equivalently an n-string of zeroes and ones; viewed this way, we have the reverse lexicographic ordering on B, which agrees with the given ordering on the atoms. An ordering of B obtained in this way will be called an Rlex ordering. This gives a 1-1 correspondence between finite linear orderings, and rlex-ordered finite Boolean algebras; it extends to an equivalence between the category BAO of rlex-ordered finite Boolean algebras, with injective, order-preserving Boolean homomorphisms, and the category of finite linear orderings with surjective maps f such that $a < b$ iff $f^{-1}(a) <_{rlex} f^{-1}(b)$. This makes it easy to see that BAO admits amalgamation. A subalgebra of an Rlex-ordered boolean algebra is also Rlex-ordered, as one can check, with respect to its own atoms. It follows that BAO is a Fraissé class, with an $\aleph_0$-categorical amalgamation limit $\widetilde{BAO}$. Note that B_n splits into $n!$ types in $\widetilde{BAO}$, differing only by rearrangement of the variables. Using this one sees easily that a substructure of a model of $\widetilde{BAO}$ realizing all $\widetilde{BA}$ types, also realizes all $\widetilde{BAO}$-types. This will be useful for checking the Ramsey property.

Now (up to bi-interpretability over $\widetilde{BA}$) we have

$$T^{ram}_{all} = \widetilde{BAO}.$$

This is easy to deduce from the previous statement. Viewed as a definable set in $\widetilde{BAO}$, B_n is definably isomorphic to $B^{[n]} \times Sym(n)$ (map $(a_1, \ldots, a_n) \in B_n$ to the pair (b, σ), where b is the image of $(a_1, \ldots, a_n)$ in $B^{[n]}$ and $\sigma(a_i) < \sigma(a_j)$ iff $i < j$.) It follows that the sort B_n is Ramsey, and B^n similarly admits a 0-definable embedding into a product of $\cup_{k \leq n} B_k$ times a finite set. (describing each a_i as a word in the linearly ordered atoms of the algebra generated by $a_1, \ldots, a_n$.)

At this point the Hales-Jewett theorem becomes visible too, as a consequence of Ramseyness of $\widetilde{BAO}$. We may think of the Boolean algebra of all subsets of $\{1, \ldots, N\}$; then a word in n letters $1, \ldots, n$ of length N can be presented as a n-tuple of disjoint elements of B, with sum 1. Let $B^*_n = \{(v_1, \ldots, v_n) \in B_n : v_1 < \cdots < v_n\}$. This is a complete type of

$\widetilde{BAO}$. Let c be a finite coloring of B_n. Then c (lifted to an elementary extension, then restricted) is definable on some elementary submodel M of $\widetilde{BAO}$; hence in particular on a set of the form

$$\{(v_0 \cup v_1, v_2, \cdots, v_n), (v_1, v_0 \cup v_2, v_3, \cdots, v_n), \cdots (v_1, \cdots, v_0 \cup v_n)\}$$

where $(v_0, \ldots, v_n) \in B_{n+1}$ and $v_0 < \cdots < v_n$. Since B_n^* is a complete type, c must be constant on this n-element set (called a combinatorial line.)

The strongly minimal theories in (1-6) have a small T^{ram}. At the other extreme we have disintegrated strongly minimal sets, specifically free group actions. Here the canonical Ramsey expansion is essentially the same construction - up to Stone duality - as the universal minimal flow of topological dynamics. From this point of view, the canonical Ramsey expansion can perhaps be viewed as a relational generalization of the universal minimal flow.

Example 5.8. Let Γ be a group, and T the theory of free Γ-actions on a set V. Here Γ is viewed as discrete, and we assume for the sake of the exposition that Γ is infinite, though the same will hold in the case of finite Γ. Form T^* as above Definition 1.8, and let $J := \mathfrak{J}^{ram}(T)_V = \mathrm{Core}\,(T_V^*)$, $\mathcal{L}^{ram} = \mathcal{L}(T_V^*)$. Then J is a Boolean algebra B with Γ-action, and no additional structure. The Stone space S of this algebra is a compact space with continuous G action. We will now show that it is the universal minimal flow of Γ.

For the Boolean algebra structure, see Example 3.4. The natural Γ action on types is clearly definable in $\mathcal{L}^{ram}$: $\gamma \cdot p = q$ if $R(a, x) \iff \neg R(\gamma(a), y)$ is omitted in (p, q). Using the quantifier elimination enjoyed by T, it is easy to see that this generates all of $\mathcal{L}^{ram}$. For instance, when $\Gamma = \mathbb{Z}$ with generator s, p omits the pattern of three consecutive elements iff $p \cap (s \cdot p) \cap (s^2 \cdot p) = 0$ (in the Boolean algebra.)

Minimality: suppose S' is a closed nonempty Γ-invariant subspace of S. Let B' be the Boolean algebra of clopen subsets. We then have a surjective Γ-Boolean algebra homomorphism $B \to B'$. It must be an isomorphism, since J is e.c. But then $S' = S$.

Universality: let S' be any minimal flow of Γ. We must find a Γ-invariant continuous map $S \to S'$.

First note that any minimal flow S' is covered by a totally disconnected minimal Γ-flow S^*, on which Γ acts without fixed points; namely any minimal subglow of the Γ-flow of ultrafilters on Γ. Indeed if we fix $s_0 \in S'$, the map $\gamma \mapsto gs_0$ extends (uniquely) to a continuous map $f : \Gamma^* \to S'$, where Γ^* is the space of ultrafilters on Γ; and f is Γ-invariant. Given $1 \neq g \in \Gamma$, it is easy to partition any $g^{\mathbb{Z}}$-orbit on S' into two or three disjoint subsets x such that $x \cap gx = \emptyset$; putting these partitions together we find a partition of Γ into at most three sets x with the same property. Thus no ultrafilter on Γ is fixed by g.

Hence, ignoring the topology, S^* is a model of T. It can be viewed as a parameter sort in a model of T^*; the R-type space over U identifies with the Boolean algebra of all subsets of S'. So there exists a homomorphism from this Γ-algebra to J; it restricts to a homomorphism from the algebra of clopen subsets of S^* to J; dually we find a Γ-invariant continuous map $S \to S'$. It follows that S is a universal minimal flow for Γ.

See Proposition B.5 and Remark B.8 for an alternative approach.

Example 5.9. As promised earlier, we prove the the existence of a complete pp type $p \subset J = Core(T_V^*)$, such that G induces a countably infinite group of automorphisms of p. Since $Aut(p)$ is quasi-compact, this implies that $Aut(p)$ cannot be Hausdorff.

Let Y be a totally disconnected compact flow of Γ, such $Aut_\Gamma(Y)$ (the group of homoeomorphisms of Y commuting with Γ) is countable, and any closed subflow of $Y \times Y$ projecting onto Y in either direction is either all of $Y \times Y$ or a finite union of automorphisms of Y. The Chacon example described in [11] is an instance; more generally, with $\Gamma = \mathrm{Aut}_\Gamma(Y) = \mathbb{Z}$, the totally disconnected graphic minimal sets of [2].

Let S be the universal minimal flow of Γ. If $f, g : S \to \Gamma$ are two surjective Γ-morphisms, then the image of S in $Y \times Y$ under (f, g) is a minimal subflow of $Y \times Y$, hence it must be the graph of an element α of $\mathrm{Aut}_\Gamma(Y)$. Hence $g = \alpha \circ f$, so the kernels of f and g coincide; so we have a closed, Γ-invariant equivalence relation E on S such that S/E is an isomorphic copy of Y; we rename it as Y; this incarnation of Y comes with a canonical quotient map $\pi : S \to Y$.

Let U be a clopen subset of Y, and let $u = \pi^{-1}(U)$. Then u is an element of the Boolean algebra B of clopen subsets of S, that we have identified with $J = Core(T_V^*)$. If $u' = g(u)$ for some $g \in Aut(J)$, then g induces an automorphism of S; it respects E and thus induces an automorphism g_Y of Y; conversely $g|p$ is determined by g_Y. This shows that p and G_p are countable.

Question 5.10.
1. Investigate further the connection of the topological dynamics of a group G to the model theory of $T = T_G$. Compare the theory of joinings of dynamical systems to the theory of orthogonality of pp types in $\mathfrak{J}$. It seems plausible that the maximal Hausdorff quotient of $\mathfrak{J}$ corresponds to the distal flows.
2. Computing the canonical Ramsey expansion at other sorts remains interesting; for pairs, we certainly find a linear ordering and hence many other linear orders obtained by Boolean combinations with unary sets; I am not sure if these are all, and if anything further is needed at the ternary level and above.
3. Presumably, the model completion of a single unary function behaves similarly, with 'colorings' that on the tree of ancestors of a given element

a depend only on the distance from a, periodically or 'almost periodically' as above.

Example 5.11. Let D be a non-Zilberian strictly minimal set, $T = Th(D)$. Assume more specifically that the language of D is generated by a symmetric ternary relation R, that we view as a set of unordered triples. Further assume R occurs for at most $n-2$ unordered triples from any n-element subset of D ($n \geq 2$.) In this situation we encounter the striking orientation construction of [14]. Namely, by [9] Theorem 2.3, for any model M there exist partial functions f, g such that

$$R(M) = \{(x, f(x), g(x)) : x \in D(M)\}$$

(let $f(a), g(a)$ be the second and third elements of the orientation if a is the first element of a triple $\{a, b, c\} \in R$, under the given orientation, and $f(a) = g(a) = a$ otherwise.) Then T^{ram} at the sort D^2 must include such partial functions (f, g). We can extend them to total functions, setting $f(a) = a$ or $g(b) = b$ where undefined (possibly they are globally defined in one / all e.c. models of T^{ram}.) In any case we obtain an action of the free semigroup on two elements, giving a theory T_e interpretable in T^{ram}, and with $T_e^{ram} = T^{ram}$.

Appendix A. Infinitary definablity patterns and the Ellis group

There is a standard parallel between definable types and invariant types in model theory; in the latter, $(d_p x)\phi(x, y)$ is not definable, but rather a union of type-definable sets. We consider now a richer language $\bar{\mathcal{L}}$ reflecting partial infinitary definability of this kind.[35]

The sorts of $\bar{\mathcal{L}}$ are the same as those of $\mathcal{L}$, i.e. indexed by a set γ of formulas of L, and a distinguished set of variables. We restrict γ to have at most countably many variables of each sort of L.

$\bar{\mathcal{L}}$ contains in particular a relation symbol $\mathcal{R}_t$ for each tuple $t = (\phi_1, \ldots, \phi_n; \alpha)$, where $\phi_1, \ldots, \phi_n$ are as before formulas $\phi_i(x, y)$, but now $\alpha(y)$ is a complete type (for a given ϕ_i, we take y to be a finite set of variables, while all but finitely many variables of x are treated as dummy in ϕ_i.)

The interpretation of $\mathcal{R}_t$ in a type space $S = S_\gamma M$ will be

$$\mathcal{R}_t^S = \{(p_1, \ldots, p_n) \in S^n : \neg(\exists a \in \alpha(M)) \bigwedge_{i \leq n} (\phi_i(x, a) \in p_i)$$

[35] While we will present it directly, it can also be treated as a special case of the construction of $\mathcal{J}$, applied to an infinitary Morleyzation $\mathbf{T}$ of T, obtained by adding a predicate symbol for *every complete type* r, and axioms $r \to \alpha$ for each $\alpha \in r$. This is a primitive universal theory, whose e.c. models are precisely the models of T realizing all types over $\emptyset$, with the expected interpretation of r. We can form Core $(\bar{T})$; it is equivalent to $\bar{\mathcal{J}}$ as defined below. Any relation of Core $(\bar{T})$ is easily seen to be equivalent to a conjunction of ones of the form $\mathcal{R}_t$ considered below. This requires extending the $\mathcal{J}$ construction to primitive universal theories.

This defines a closed subset of S^n.

It is clear that the set of true pp sentences is the same for all models M of T that realize all finitary types over $\emptyset$. This determines an irreducible primitive universal theory $\bar{\mathcal{T}}$. The earlier considerations go through: $\bar{\mathcal{T}}$ has a compact topological model, hence it is ec-bounded, hence it has a unique universal e.c. model $\bar{\mathfrak{J}}$.

Let λ_T be the number of finitary types of T over $\emptyset$. A model M realizing all types of cardinality λ_T exists, and thus $|\bar{\mathfrak{J}}| \leq 2^{\lambda_T}$.

Lemma A.1. *(1) Let A be a substructure of $S = S_x(M)$. The $\bar{\mathcal{L}}$-homomorphisms $A \to S$ form a closed set $\mathrm{Hom}_{\bar{\mathcal{L}}}(A,S) \subset S^A$, containing the image of $\mathrm{Aut}(M)$ under $\sigma \mapsto \sigma_*|A$.*

(2) Let $A \subset S$. Assume M is $\aleph_0$-homogeneous. Then the image of $\mathrm{Aut}(M)$ is dense in $\mathrm{Hom}_{\bar{\mathcal{L}}}(A,S)$.

Proof. (1) is clear from the definitions.

(2) Given finitely many types $p_1, \ldots, p_m \in A$, let $q_i = f(p_i)$, and consider any neighborhood U_i of q_i in S. We have to find $\sigma \in \mathrm{Aut}(M)$ with $\sigma_*(p_i) \in U_i$. We can find c from M and formulas $\phi_i(x,y)$ such that U_i is defined by $\phi_i(x,c)$. Let $r = tp(c)$. Then $S \models \neg \mathcal{R}_{\phi_1,\ldots,\phi_m;r}(q_1,\ldots,q_m)$. Since f is an $\mathcal{L}$-homomorphism, $S \models \neg\mathcal{R}_{\phi_1,\ldots,\phi_m;r}(p_1,\ldots,p_m)$. By definition of this symbol, there exists c' in M with $r(c')$ and $(d_{p_i}x)\phi_i(x,c')$ for each i. Let $\sigma \in \mathrm{Aut}(M)$ satisfy $\sigma(c') = c$ (using the $\aleph_0$-homogeneity of M.) Since $\phi_i(x,c') \in p_i$, we have $\phi_i(x,c) \in \sigma_*(p_i)$. Thus $\sigma_*(p_i) \in U_i$, as required.

$\square$

Let $\bar{G} = \mathrm{Aut}(\bar{\mathfrak{J}})$, $\bar{\mathfrak{g}} = \{g \in \bar{G} : (\forall U \in \mathfrak{t})(gU \cap U \neq \emptyset)\}$, and $\bar{\mathcal{G}} = \bar{G}/\bar{\mathfrak{g}}$.

We record the anaolog Lemma 3.26, moving up one power set:

Lemma A.2. *Let $\lambda = \lambda_T$, the number of types of T over $\emptyset$ in finitely many variables.*

(1) $|\bar{\mathfrak{J}}| \leq 2^{\lambda}$.
(2) $|\bar{G}| \leq \beth_2(\lambda)$
(3) $|\bar{\mathcal{G}}| \leq 2^{\lambda}$

Proof. (1) was already observed; (2) is an immediate consequence. (3) is proved as in Lemma 3.26.

$\square$

Remark A.3. (1) Any model A of $\mathcal{T}$ has a canonical 'minimal' expansion A_{min} to $\bar{\mathcal{L}}$, where

$$\mathcal{R}_{\phi;r} \iff \bigvee_{\alpha \in r} \mathcal{R}_{\phi,\alpha}$$

We have $A_{min} \models \bar{\mathcal{T}}$, since if a pp sentence α holds in A_{min}, say witnessed by $a_1, \ldots, a_n$, then any instance of $\mathcal{R}_{\phi;r}(a)$ holds only because some stronger statement $\mathcal{R}_{\phi,\alpha}$ holds; if $\bar{\mathfrak{T}}$ rules out α it certainly rules out the stronger version, but that involves only $\mathcal{L}$, whereas $\bar{\mathfrak{T}}|\mathcal{L} = \mathfrak{T}$.

(2) In particular there exists a homomorphism $\bar{\iota} : \mathfrak{J}_{min} \to \bar{\mathfrak{J}}$. It restricts to a homomorphism $\iota : \mathfrak{J} \to \bar{\mathfrak{J}}|\mathcal{L}$; this must be an embedding since $\mathfrak{J}$ is e.c.

(3) The embeddings $\mathfrak{J} \to \bar{\mathfrak{J}} \to S(M)$ induce maps $\mathfrak{J}/L_\infty \to \bar{\mathfrak{J}}/L_\infty \to S(M)/L_\infty$; since the composition $\mathfrak{J}/L_\infty \to S(M)/L_\infty$ is bijective, the two intermediate maps must be too.

(4) If ι is bijective, then every invariant type p of T is definable (p is represented by an element of $\bar{\mathfrak{J}}$; since $\bar{\iota}$ is surjective, it must be represented in $\mathfrak{J}_{min}$, which means that if a type $q(y)$ is contained in $(d_p x)\phi(x,y)$, then so is some formula containing q; then use compactness.)

(5) Thus in general ι is not bijective. By Proposition 2.5, it follows in this case that $\mathfrak{J} \not\cong \bar{\mathfrak{J}}|\mathcal{L}$ and in fact there is no embedding $\mathfrak{J} \to \bar{\mathfrak{J}}|\mathcal{L}$.

(6) Remaining in the case that $\mathfrak{J} \not\cong \bar{\mathfrak{J}}|\mathcal{L}$, let M be an $\aleph_0$-saturated, $\aleph_0$-homogeneous model of T, and let J be a copy of $\mathfrak{J}$ in $S(M)$. Then a retraction $r : S(M) \to J$ cannot be approximated by automorphisms of M. (Otherwise by Lemma A.1(1) it would be a $\bar{\mathcal{L}}$-homomorphism; restricted to some image of $\bar{\mathfrak{J}}$ in $S(M)$ it must be an $\bar{\mathcal{L}}$-embedding, yielding in particular an embedding of $\bar{\mathfrak{J}}|\mathcal{L}$ into $\mathfrak{J}$.) This contrasts with the retraction of $S(M)$ to the image of $\bar{\mathfrak{J}}$, and appears to indicate that $\mathfrak{J}$ cannot be constructed purely using the topological dynamics of $\mathrm{Aut}(M)$ acting on $S(M)$.

Example A.4. There are countable theories with $|\bar{\mathfrak{J}}| = \beth_1$, $|\bar{\mathfrak{G}}| = \beth_2$. (Compare Example 3.35.)

(1) Take the model completion of the theory of graphs with infinitely many unary predicates. Let M be $\aleph_0$-saturated of cardinality continuum. We see that there are $\beth_2$ invariant types over M, with a choice of 0/1 over each of the continuum many types over $\emptyset$. So $|\bar{\mathfrak{J}}| = \beth_2$.

(2) To see that one can have $|\bar{\mathfrak{G}}| \geq \beth_2$, let L have two sorts A, B, and infinitely many independent unary predicates P_n on B. A basic relation $R \leq A \times B^2$ is given, and $T_\forall$ asserts that for any $a \in A$, $R(a)$ is a tournament on B; further, $R(a)$ respects the lexicographic order:

$$(\forall x, y, y') \bigwedge_{i<n} (P_i(y) \iff P_i(y')) \wedge \neg P_n(y) \wedge P_n(y') \to R(x, y, y')$$

For $\alpha \in 2^\omega$, let $Q_\alpha = \cap_n P_n^{\alpha(n)}$, so that the Q_α are the complete types with respect to the unary predicates. Let $\bar{J}$ be an embedded image of $\bar{\mathfrak{J}}$ in $S(M)$. For any $p \in \bar{J}$, $(dp_x)R(x,y,y')$ defines a linear ordering on the sort B, so that $Q_\alpha < Q_\beta$ if α is lexicographically strictly below β. For any subset W of 2^ω, there exists an automorphism σ_W of $\bar{\mathfrak{J}}$, such that $\sigma_W(p), p$

agree above Q_α iff $\alpha \in W$. This is a copy of the Hausdorff compact (and separable) group $(\mathbb{Z}/2\mathbb{Z})^{2^{\aleph_0}}$, and shows that $|\bar{\mathcal{G}}| \geq \beth_2$.

Here is an example where $\mathcal{G}, \bar{\mathcal{G}}$ differ.

Example A.5. Let L have two sorts A, B, and infinitely many constants $b_1, b_2, \cdots$ in B. A basic relation $R \leq A \times B^2$ is given, and $T_\forall$ asserts that for any $a \in A$, $R(a)$ is a tournament on B; i.e. for $R(a,b,b)$ never holds, and for $b \neq b' \in B$ precisely one of $R(a,b,b')$ and $R(a,b',b)$ hold. Further, $R(a,x,b_j)$ holds iff $x = b_i$ for some $i < j$. T is the model completion. Let x, y be variables of sorts A, B respectively, and consider the x-sort of $\mathcal{J}$ and $\bar{\mathcal{J}}$. Then $\mathcal{J}_x$ reduces to a single point p; where $(dp_x)R(y,y')$ defines a linear order. On the other hand $\bar{\mathcal{J}}_x$ has two points p, q; $(d_p x)R(y,y')$ and $d_q x R(y,y')$ are both linear orderings, opposing on the generic type of T (i.e. on nonconstant elements.) Thus $|G| = 1$, $|\bar{G}| = |\bar{\mathcal{G}}| = 2$.

A.6. **The Ellis group.** In order to compare with definitions of the Ellis group in the literature (see [33]), we consider a sort $\mathcal{J}_x$ of $\mathcal{J}$, corresponding to the set γ_x of all formulas with distinguished variable x (and some countable set of parameter variables for each sort.)

Corollary A.7. *Assume M is an $\aleph_0$-saturated, $\aleph_0$-homogeneous model of T. Let E_M be the Ellis group associated with the action of* $\mathrm{Aut}(M)$ *on $S := S_x(M)$. Then $E_M \cong$* $\mathrm{Aut}(\bar{\mathcal{J}})$.

Proof. Let $j : \bar{\mathcal{J}} \to S$ be an $\bar{\mathcal{L}}$-embedding, $\bar{J} = j(\bar{\mathcal{J}})$. Let $r : S \to \bar{J}$ be a retraction. So $r \circ r = r$. By Lemma A.1(2), for any finite $F \subset S$, $r|F$ can be approximated in S^F by automorphisms of M. Thus r lies in the Ellis semigroup ES_M, and is idempotent. If $b \in ES_M$, then $r \circ b|\bar{J}$ is a homomorphism $\bar{J} \to \bar{J}$ hence an isomorphism (Proposition 2.6); let s be the inverse isomorphism; then the homomorphism $(s \circ r) : S \to \bar{J}$ is again in ES_M by Lemma A.1, and $(s \circ r)(br) = r$. This shows that any element br of $ES_M r$ generates $ES_M r$ as a left ideal, so ES_M is a minimal left ideal. The Ellis group E_M can be taken to be the subsemigroup rEr, under composition, with identity element r. The set $\bar{J}$ is preserved under the elements of rEr, defining an action of rEr on $\bar{J}$. Each element of rEr induces an $\bar{\mathcal{L}}$-homomorphism of $\bar{J}$, and so an isomorphism. Conversely by Lemma A.1, any $\bar{\mathcal{L}}$-automorphism of $\bar{J}$ is obtained in this way. We thus have a surjective homomorphism $E_M \to \mathrm{Aut}(\bar{J})$. It is injective since if $h \in rEr$ is the identity on $\bar{J}$, then $hr = r$, but $h = hr$ since $h \in rEr$ and $r^2 = r$. So $E_m \cong \mathrm{Aut}(\bar{J}) \cong \mathrm{Aut}(\bar{\mathcal{J}})$. □

Corollary A.8. *Let M be an $\aleph_0$-universal, $\aleph_0$-homogeneous model of T. Then E_M has cardinality at most $\beth_2(\lambda_T) \leq \beth_3(|L|)$.*

Proof. When x consists of countably many variables, this is immediate from Lemma A.7 and Lemma A.2. Note that if we take another copy x' of x, and let γ'' consist of Boolean combinations of $\gamma_x \cup \gamma_{x'}$, then $\mathfrak{J}_{\gamma''} = \mathfrak{J}_x \times \mathfrak{J}_{x'}$, and the diagonal is $\bigwedge$-pp-definable, namely it is the relation of omitting $\phi(x,y)\&\neg\phi(x',y)$ for each ϕ. Thus $\mathrm{Aut}(\mathfrak{J}_{\gamma'})$ projects bijectively to $\mathrm{Aut}(\mathfrak{J}_x)$ and to $\mathrm{Aut}(\mathfrak{J}_{x'})$. It follows that even if x is allowed to be a large list of variables, $\mathrm{Aut}(\mathfrak{J}_x)$ projects bijectively to the projective limit of $\mathrm{Aut}(\mathfrak{J}_u)$ with u ranging over finite subsets of some fixed countable set of variables. So we are reduced to that case. □

Remark A.9. Corollary A.8 is in fact valid for any $\aleph_0$-homogeneous model M ($\aleph_0$-saturated or not); the proof is the same, except that the Ellis group will be isomorphic to the automorphism group of a universal e.c. model of an appropriately stronger primitive universal theory than $\bar{\mathfrak{T}}$, ruling out types not realized in M.

(Incidentally, computing $\mathrm{Aut}(\bar{\mathfrak{J}})$ for Example 3.36 gives an example where the Ellis group for homogeneous models can look bigger than for the saturated model. For the saturated model of T, with infinitely many orbits, $\bar{\mathfrak{J}}$ will be isomorphic to $\mathfrak{J}$ (a diagonal copy in each orbit of $\mathbb{Z}$.) In particular $\mathrm{Aut}(\bar{\mathfrak{J}}) = \mathrm{Aut}(\mathfrak{J})$. But if we use a homogeneous model with m orbits of $\mathbb{Z}$, $\bar{\mathfrak{J}}$ will be the independent product of a copy of $\mathfrak{J}$ in each orbit, and $\mathrm{Aut}(\bar{\mathfrak{J}})$ will be the wreath product of $Sym(m)$ with $\mathrm{Aut}(\mathfrak{J})$.)

Here is an example of a countable theory whose $|\mathbf{G}|$, and thus the Ellis group, have cardinality $\beth_3$; compare 3.36.

Example A.10. The theory T will again include a bipartite graph $R \subset P \times Q$. On Q there are $\aleph_0$ independent equivalence relations E_n with two classes each; they can be viewed as giving a map p from Q to a torsor A over the group $\mathbf{2}^{\mathbb{N}}$ (where $\mathbf{2} = \mathbb{Z}/\mathbf{2}\mathbb{Z}$). There are also commuting definable maps $s_i : Q \to Q$, satisfying $s_i(s_i(x)) = x$; so that s_i preserves the classes of E_j for $i \neq j$, and flips the two classes of E_i. Thus p is a homomorphism; $p(s_i(x)) = s_i \cdot p(x)$ (where s_i is identified with the element of $\mathbf{2}^{\mathbb{N}}$ having a 1 just in the i'th position.) T is model complete, with universal theory as described above.

The sort Q is stable, though not stably embedded. But using Lemma 3.12, mutatis mutandis, $\bar{\mathfrak{J}}$ can be computed autonomously on this sort; this implies that in the sort Q, $\bar{\mathfrak{J}}$ has a single element in each class of the intersection $\cap_n E_n$; i.e. $Q(\bar{\mathfrak{J}})$ is a torsor for $\mathbf{2}^{\mathbb{N}}$, and p induces a bijection $Q(\bar{\mathfrak{J}}) \to A$. While Q has a unique 1-type, it has continuum many 2-types; namely for each $g \in \mathbf{2}^{\mathbb{N}}$ the type $q_g(x,y)$ asserting that $g \cdot p(x) = p(y)$. These types restricted to $\bar{\mathfrak{J}}$ are the graphs of bijections $Q(\bar{\mathfrak{J}}) \to Q(\bar{\mathfrak{J}})$, defining again an action of $\mathbf{2}^{\mathbb{N}}$ on $Q(\bar{\mathfrak{J}})$, compatible with the others.

Now each $a \in P$ defines a subset $R(a)$ of Q; we prefer to think of it as a function from Q to $\mathbf{2}$. Let $h : \mathbf{2}^{\mathbb{N}} \to \mathbf{2}$ be a homomorphism (not necessarily continuous.) We define an atomic type of $\mathcal{L}$ in the sort P, describing a function $f : Q \to \mathbf{2}$ such that on the 2-type $q_g(x,y)$ we have $f(x) = h(g) + f(y)$. For each h there are precisely two such functions f, f' with the same maximal atomic type, but with $f' = 1 - f$. Both are represented in $\bar{\mathfrak{J}}$, and each one is atomically $\mathcal{L}$-definable over the other.

Given $h_1, \ldots, h_k \in \mathrm{Hom}(\mathbf{2}^{\mathbb{N}}, \mathbf{2})$ linearly independent over the 2-element field, one sees easily that $q_{h_1}, \ldots, q_{h_k}$ are orthogonal in $\bar{\mathfrak{J}}$, i.e. the atomic k-type is determined by the 1-types. Choose a $GF(2)$-basis $(h_i)_{i \in I}$ for $\mathrm{Hom}(\mathbf{2}^{\mathbb{N}}, \mathbf{2})$. Let $a_i \in \bar{\mathfrak{J}}$ represent q_{h_i}, and let $b_i = 1 - a_i$. Then for any subset $C \subset I$, the function exchanging a_i, b_i for $i \in C$ and fixing a_i, b_i for $i \notin C$ preserves all atomic relations $\mathcal{R}_t$ of $\bar{\mathfrak{J}}$, and thus extends to an automorphism of $\bar{\mathfrak{J}}$. It follows that $\mathbf{2}^I$ is a homomorphic image of $\mathrm{Aut}(\bar{\mathfrak{J}})$, which thus has cardinality $2^{2^{2^{\aleph_0}}}$.

We conclude the appendix with a more general version of Lemma A.1 (see Remark A.12.

Let $M \models T$, $S = S_\gamma(M)$. View S as a compact Hausdorff space under the usual logic topology; and as as an $\mathcal{L}$-structure. For $\sigma \in \mathrm{Aut}(M)$, let σ_* denote the induced $\mathcal{L}$-automorphism of S. Let $A \subset S$. The set S^A of functions $A \to S$ will be considered as a compact topological space, with the topology of pointwise convergence.

Let M_A denote the structure M expanded with ϕ-definitions $(d_p x)\phi$ for each $\phi \in L$ and each $p \in A$. By a qf type of M_A over $\emptyset$, we mean a finitely satisfiable collection of formulas of the form $(d_p x)\phi$. (In (2) below, we really just need finitely many such formulas, along with a set of L-formulas.) The hypothesis on realizing types in (2,3) below is thus true whenever M_A is either saturated, or a qf-saturated existentially closed model of the universal theory of M_A.

Lemma A.11. *(1) Let A be a substructure of $S(M)$. The $\mathcal{L}$-homomorphisms $A \to S(M)$ form a closed set $\mathrm{Hom}_{\mathcal{L}}(A, S) \subset S^A$, containing the image of $\mathrm{Aut}(M)$ under $\sigma \mapsto \sigma_*|A$.*

(2) Let $A \subset S$. Assume M is $\aleph_0$-homogeneous, and M_A realizes all qf types over $\emptyset$. Then the image of $\mathrm{Aut}(M)$ is dense in $\mathrm{Hom}_{\mathcal{L}}(A, S)$.

(3) Assume in (2) that M is λ-homogeneous and M_A realizes all qf types in λ variables over $\emptyset$, where $\lambda \geq |A| + |L| + |M_0|$, $M_0 \leq M$. Let $f : A \to S(M)$ be an $\mathcal{L}$-homomorphism. Then there exists $\sigma \in \mathrm{Aut}(M)$ such that for all $p \in A$, $\sigma(p)|M_0 = f(p)|M_0$.

Proof. (1) is clear from the definitions.

(2) Let $f : A \to S_\gamma(M)$ be an $\mathcal{L}$-homomorphism. Given finitely many types $p_1, \ldots, p_m \in A$, let $q_i = f(p_i)$, and consider any neighborhood U_i of q_i in $S_\gamma(M)$. We have to find $\sigma \in \mathrm{Aut}(M)$ with $\sigma_*(p_i) \in U_i$. We can find c from M and formulas $\phi_i(x, y) \in \gamma$ such that U_i is defined by $\phi_i(x, c)$. Let $r = tp(c)$. For any $\alpha \in r$, $S(M) \models \neg\mathcal{R}_{\phi_1, \ldots, \phi_m; \alpha}(q_1, \ldots, q_m)$. Since f is an $\mathcal{L}$-homomorphism, $S(M) \models \neg\mathcal{R}_{\phi_1, \ldots, \phi_m; \alpha}(p_1, \ldots, p_m)$. Hence for some c_α with $\alpha(c_\alpha)$ we have $\alpha(c_\alpha) \wedge (d_{p_i} x)\phi_i(x, c_\alpha)$ for each i. As a consequence of $\aleph_0$-saturation,

there exists c' with $r(c')$ and $(d_{p_i}x)\phi_i(x,c')$ for each $i \leq m$. Let $\sigma \in \mathrm{Aut}(M)$ satisfy $\sigma(c') = c$ (using the $\aleph_0$-homogeneity of M.) Since $\phi_i(x,c') \in p_i$, we have $\phi_i(x,c) \in \sigma_*(p_i)$. Thus $\sigma_*(p_i) \in U_i$, as required.

(3) The proof is similar to (2), except that we consider all $p \in A$ and all neighborhoods U of $q = f(p)$ defined by some $\phi(x,c)$ with c from M_0 (allow c to be a λ-tuple enumerating M_0.) □

Remark A.12. On Lemma A.11 (2).

(1) Assume: for every tuple c from M, there exists a formula $\alpha \in tp(c)$ such that $\mathrm{Aut}(M)$ is transitive on $\alpha(M)$. Then the saturation assumption on M_A is not needed in the proof of Lemma A.11(2).
(2) Assume every every element of $\mathfrak{J}$ is represented by a definable type in $S(M)$. Then $M_A = M$, and the hypothesis of Lemma A.11 (2) is simply that M is $\aleph_0$-homogeneous and $\aleph_0$-saturated.

Appendix B. Universal minimal flow

B.1. We recall some definitions from topological dynamics. For any topological group G, a *flow* is a compact Hausdorff space X along with a continuous G-action on X; a morphism of G-flows is a continuous G-equivariant map. If G has a dense subset of size κ then so does X, so $|X| \leq 2^{2^\kappa}$. The flow is *minimal* if every G-orbit is dense. It is *universal minimal* if it admits a morphism into any other flow Y. All endomorphisms of a universal minimal flow M are bijective[39] It follows that M is unique up to a isomorphism. The same discussion can be carried out for *pointed minimal flows*, where morphisms, if they exist, are unique; in this case the universal one is easily unique, up to a unique isomorphism. Any minimal subflow Y_0 of the universal pointed minimal flow is a universal minimal flow (any G-map from F to a minimal flow Y must restrict to a map $Y_0 \to Y$.)

Recall the space of ultrafilters $\beta Z = \mathrm{Hom}(2^Z, 2)$ on a set Z; it is topologized as a closed subspace of 2^{2^Z}, and thus compact and Hausdorff. For a discrete group G, it is easy to see that $(\beta G, 1)$ is the universal minimal pointed flow of G

B.2. Let L be a countable language, M be a countable atomic structure, prime model of $Th(M)$, $G = \mathrm{Aut}(M)$. We view G as a topological group, by taking M to be discrete and giving G the pointwise convergence topology. We assume (for

[39] A fact due to Ellis; here is a possibly different proof: if $f : M \to M$ is an endomorphism, $f(M)$ is a subflow so $f(M) = M$. Suppose for contradiction that f is not injective. Construct an inverse system $(M_\alpha, f_{\alpha,\beta} : \beta \leq \alpha \leq \lambda)$, where $\lambda = |M|^+$, each $M_\alpha = M$, and each $f_{\alpha,\alpha+1} = f$. We set $f_{\alpha,\alpha} = Id_M$ and define $f_{\alpha,\beta}$ for $\beta < \alpha$ by induction on α. At successor stages, let $f_{\alpha+1,\beta} = f_{\alpha,\beta} \circ f$. At limit stages α, we must define a map $f_\alpha : M \to \varprojlim_{\beta<\alpha} M_\beta$. Such a map exists by universality of M. Thus M_λ can be constructed. But clearly $|M_\lambda| \geq \lambda > |M|$, a contradiction.

simplicity) $Th(M)$ has quantifier elimination, and let T be the universal theory of M.

Let x_m be a variable for each $m \in M$, as one does in the definition of 'diagrams' in elementary logic. We have the tautological assignment $\mathbf{a} : x_m \to m$ for these variables. Let I be the set of finite sets of these variables. For any $i \in I$, let a_i be the restriction of $\mathbf{a}$ to i, and let ϕ_i be a formula (in variables i) isolating $tp(a_i)$; so $V_i = \phi_i(M)$ is the G-orbit of a_i. (Note that we treat a_i not as an $|i|$-tuple, i.e. a function $|i| \to M$, but rather as an i-tuple, i.e. a function $i \to M$.)

Let $F_i = \beta V_i$, and let $a_i \in F_i$ denote the principal ultrafilter on a_i. If $i \subset i'$, we have a natural projection $\pi_{i',i} : F_{i'} \to F_i$.

Viewing I as an index set, partially ordered by inclusion, let $(V, \mathbf{a})$ be the inverse limit of all the $(V_i, a_i : i \in I)$, and let let $(F, \mathbf{a})$ be the inverse limit of the spaces F_i.

Note that G acts on I naturally; and if $g(i) = i'$, we have a natural bijection $V_i \to V_{i'}$ (change of variable according to g), and hence also $\iota_{i;g} : F_i \to F_{i'}$. We thus obtain an action of G on V and hence on F.

$$g(\iota_{i;g}(a_i)) = a_{i'} \tag{2}$$

(Here $\iota_{i;g}$ acts on the domain of the tuple a_i, within the set of variables, and then g acts on the image of a_i within M.)

By [52] Prop. 6.3, $(F, \mathbf{a})$ is the universal minimal pointed flow of G. A morphism $Y \to F$ being the same as a coherent family of morphisms $Y \to F_i$, we obtain:

Lemma B.3. *A minimal flow Y of $G = \mathrm{Aut}(M)$ is universal iff there exist continuous maps $\alpha_i : Y \to \beta V_i$, with $\pi_{i',i} \circ \alpha_i = \alpha_{i'}$, and $\alpha_i(gy) = \iota_{g^{-1}(i);g} \alpha_{g^{-1}(i)}(y)$.*

B.4. Recall the construction T^*_V, that renders each subset of each V_i externally definable: we add a sort V^*_i and new relation symbols $R_i \subset V^*_i \times V_i$ to obtain a bigger language L^*_V, with no new axioms.

Let J denote $\mathrm{Core}\,(T^*_V)$ restricted to sorts corresponding to R_i-types on V^*_i; thus V_i are parameter sorts. Likewise let S denote the space of types of T^* over M in variable sorts V^*_i (for some i.)

Let T^{ram} be the universal part of the minimal Ramsey expansion of T (Theorem 1.10).

Proposition B.5. *Let M be a countable atomic model. Then the space of expansions of M to a model of T^{ram}, is the universal minimal flow of $\mathrm{Aut}(M)$.*

Proof. Let J, S be as above. Recall (Proposition 3.17, (1)) that the space of expansions of M to a model of T^{ram} is isomorphic to $\mathrm{Hom}(J, S)$. We thus have to show that $\mathrm{Hom}(J, S)$ is the universal minimal flow of $G = \mathrm{Aut}(M)$.

Minimality of $\mathrm{Hom}(J, S)$ as a G-flow, i.e. the fact that every orbit is dense, follows from Lemma A.1 (2).

By Lemma B.3, it suffices to find continuous maps $\alpha_i : \mathrm{Hom}(J,S) \to \beta V_i$, functorial in i and compatible with the G-action.

We will use Lemma 3.14, for the theory T_V^*, specifically for $\gamma = \{R_i\}$ the relations connecting V_i with V_i^*; with $A = B = M$ there, and $N \geq M$ a large model of T^*. But first, fix a homomorphism $\rho : S \to J = \mathrm{Core}\,(T_V^*)$. Then any homomorphism $h : J \to S$ yields an endomorphism $h \circ \rho$ of S. By Lemma 3.14 we obtain an extension r_h of $tp(\mathbf{a})$ to a global γ-type, finitely satisfiable in M; in particular, we can restrict attention to the coordinates i, obtaining a global R_i-type in V_i, finitely satisfiable in M. Such a type corresponds precisely to an ultrafilter on V_i. Indeed, for $d \in V_i^*(N)$, let $s(d/M) := s(tp(d/M)) := \{c \in V_i(M) : aR_ic\}$; this subset of $V_i(M)$ has the same information as $qftp(d/V_i(M))$. Let $\alpha_i(h) = \{s(a/M) : aR_iy_i \in r_h\}$. Then $\alpha_i(h)$ is an ultrafilter on $V_i(M)$; for instance if $s(a/M) \subset s(a'/M)$, then $R_i(a,y_i)\&\neg R_i(a',y_i)$ is not satisfied by any element of M, so it is not in r_h; hence if $s(a/M) \in \alpha_i(h)$ then $s(a'/M) \in \alpha_i(h)$, so that $\alpha_i(h)$ is upwards closed. A similar argument shows that $\alpha_i(h)$ is closed under intersections, contains each set or its complement, and that for $i \subset i'$, $\alpha_{i'}(h)$ projects to $\alpha_i(h)$.

Continuity of α_i: Fix $d \in V_i^*$ and let $j = \rho(tp(d/M))$. Then $s(d/M) \in \alpha_i(h)$ iff $dR_iy_i \in r_h$ iff $x_iR_ia_i \in h(j)$; the set of h with this property is open by definition of the pointwise convergence topology on $\mathrm{Hom}(J, S(M))$.

It remains to compare the G-actions . Let $g \in G = \mathrm{Aut}(M)$, $h \in \mathrm{Hom}(J,S)$, $g^{-1}h := g^{-1} \circ h$. Fix i and let $i' = g^{-1}(i)$. Write $\iota = \iota_{i';g}$. Let $w \subset V_i(M)$; we will show that

$$w \in \alpha_i(h) \iff w \in \iota\alpha_{i'}(g^{-1}h)$$

Let p be a type in V_i^* over M with $s(p) = w$, and $p'(x')$ a type over M of elements of $V_{i'}^*$, differing from p only in the change of variable $i' \mapsto i$ determined by g, so that $s(p') = \iota^{-1}(w)$. Since this change of variable is expressible via an $\mathcal{R}_t$-relation between p and p', it remains true of $h\rho(p), h\rho(p')$. In particular,

$$w \in \alpha_i(h) \iff a_i \in s(h\rho(p)) \iff \iota^{-1}(a_i) \in s(h\rho(p'))$$

By (2) of § B.2, this is iff $g(a_{i'}) \in s(h\rho(p'))$ iff $a_{i'} \in s(g^{-1}h\rho(p'))$ iff $\iota^{-1}(w) \in \alpha_{i'}(g^{-1}h)$.

□

Remark B.6. (1) Let M be any countable structure. Then Proposition B.5 gives a description of the universal minimal flow of $Aut(M)$ in terms of expansions to T^{ram}, where T is the theory of M expanded by a relation for each $Aut(M)$-orbit on M^n. Alternativey one can use the infinitary pattern space of Appendix A.

(2) In the case of continuous logic, V^* should be replaced by the ind-sort of uniformly continuous maps $V(M) \to \mathbb{R}$, and R_i by evaluation. Presumably, a similar comparison to the Weil-Samuel compactification of G should work, but I have not checked any of the details.

Remark B.7. The results of [52], [6] (for the discrete logic case) read in this light as a dichotomy: J is sortwise finite or uncountable.

Indeed by Example 3.4, J carries a complete Boolean algebra structure on each sort V_i^*. Boolean algebras are always either finite, or admit an infinite set I of pairwise disjoint elements. A complete Boolean algebra of the latter kind must have cardinality at least continuum, since the sums of two distinct subsets of I are never equal.

If J is sortwise finite, then T^{ram} is a sortwise finite expansion of T, hence it has finitely many qf types of each sort extending any given type of T. In this case the model completion $\hat{T}$ of T^{ram} is $\aleph_0$-categorical if T is, and in any case has dense isolated types, as T does; and the space of expansion of M to a model of T^{ram} has a comeager G_δ orbit, namely the expansions to an atomic model of $\hat{T}$.

On the other hand if J is uncountable, then $\mathrm{Hom}(J, S(M))$ cannot be metrizable, indeed cannot admit a countable basis for the topology. For suppose is countable basis; an element $b \in B$ can be taken to be of the form $\{h \in \mathrm{Hom}(J, S) : (h(j_1), \ldots, h(j_n)) \in D\}$, with $j_i \in J$, and D an open set in $S := S(M)$. Let J_0 be the countable set of all j_i occurring in B. Now if $h_1 \neq h_2 \in \mathrm{Hom}(J, S)$, there exists b_i with $h_1 \in b_i$ and $h_2 \notin b_i$, and it follows that $h_1(j_i) \neq h_2(j_i)$. Thus each h is determined by $h(j), j \in J_0$. But now by Lemma 3.18, each h is definable from finitely many $h(j)$, so $|J| \leq \aleph_0$.

Remark B.8. Let us give another proof of Example 5.8 in light of Proposition B.5. Γ is an infinite group, T the theory of free Γ-actions. We give a proof of Let M be the prime model of T (a single Γ orbit.) Then $Aut(M)$ is another copy of Γ (acting on the right), with the discrete topology. Let $J = \mathfrak{J}^{ram}(T)$; it is a Boolean algebra with Γ-action, and write 2 for the 2-element Boolean algebra. We thus have two descriptions of the universal minimal flow U of Γ: by Proposition B.5, U is the space $Hom(J, S(M^*))$ of expansions of M to a model of T^{ram}, so we can write:

$$U = Hom_{\mathcal{L}^{ram}}(J, S(M^*)) = Hom_{Bool,\Gamma}(J, {}^M 2)$$

while by Example 5.8, U is the Stone dual to J, i.e.

$$U = Hom_{Bool}(J, 2)$$

These are compatible by a duality analogous to Frobenius reciprocity. A Γ-equivariant Boolean homomorphism from J to the algebra of functions from M into 2 can be viewed as a Γ-invariant function $J \times M \to 2$, where Γ acts trivially on 2. Thus

$$Hom_{Bool,\Gamma}(J,^M 2) = Hom_{Bool,\Gamma}(J \times M, 2)$$

By picking a point $m_0 \in M$ and evaluating there, we obtain a map $Hom_{Bool,\Gamma}(J \times M, 2) \to Hom_{Bool}(J, 2)$ which can easily be seen to be a bijection and a homeomorphism:

$$Hom_{Bool,\Gamma}(J \times M, 2) = Hom_{Bool}(J, 2)$$

Compatibility with the right Γ-action is also easy to check.

Appendix C. Hausdorff quotients

We include here some elementary statements on Hausdorff quotients of topological spaces. Any topological space X has a universal Hausdorff quotient, namely X/E for E the smallest closed equivalence relation on X. In the homogeneous case, one can describe E more effectively.

In this subsection, all quotients are given the quotient topology, i.e. the open sets are those whose pullback is open.

Let $(X, \mathfrak{t})$ be a topological space, G a group acting on X by homeomorphisms.

For $W \subset X$ and $x \in X$, let $Wx^{-1} := \{g \in G : gx \in W\}$. Also for $U \subset X$, write $WU^{-1} := \cup_{u \in U} Wu^{-1} = \{g \in G : gU \cap W \neq \emptyset\}$.

Define the infinitesimal elements of G (acting on X) to be

$$\mathfrak{g}_X = \cap_{\emptyset \neq U \in \mathfrak{t}} UU^{-1} = \{g \in G : \forall U \in \mathfrak{t}\ (U \neq \emptyset \to gU \cap U \neq \emptyset)\}$$

$\mathfrak{g}_X$ is a clearly a subgroup of G, invariant under automorphisms of $(G, X, \mathfrak{t})$ and in particular normal.

We can also write:

$$\mathfrak{g}_X = \{g \in G : (\forall U \in \mathfrak{t})\ g \cdot U \subseteq \mathrm{cl}(U)\} = \cap_{U \in \mathfrak{t}, u \in U} \mathrm{cl}(U)u^{-1} \tag{3}$$

Indeed if $gU \subset \mathrm{cl}(U)$, since U is dense in $\mathrm{cl}(U)$, U is not disjoint from the open set gU. Hence if $gU \subset \mathrm{cl}(U)$ for all U, then $g \in \mathfrak{g}_X$. Conversely assume $g \in \mathfrak{g}_X$. Let V be the open set $U \smallsetminus g^{-1}\mathrm{cl}(U)$. Then $gV \cap V = \emptyset$, so $V = \emptyset$, i.e. $gU \subseteq \mathrm{cl}(U)$.

$$\mathfrak{g}_X = \{g \in G : (\forall U \in \mathfrak{t})\ g \cdot \mathrm{cl}(U) = \mathrm{cl}(U)\} \tag{4}$$

Let $g \in \mathfrak{g}_X$. By equation (3), $gU \subseteq \mathrm{cl}(U)$. By continuity of g we have $g\,\mathrm{cl}(U) \subseteq \mathrm{cl}(U)$. Applying this to g^{-1}, we have $g^{-1}\,\mathrm{cl}(U) \subseteq \mathrm{cl}(U)$, equivalently $\mathrm{cl}(U) \subseteq g\mathrm{cl}(U)$. Thus $g\mathrm{cl}(U) = \mathrm{cl}(U)$.

For a final characterization of $\mathfrak{g}_X$, recall the Boolean algebra $ro(X)$ of regular open subsets of X. $U \subset X$ is *regular open* if $U = int(cl(U))$. Complementation in this Boolean algebra takes the form $U \mapsto int(cl(X \smallsetminus U))$. If $g \in \mathfrak{g}_X$ then $g(cl(U)) = cl(U)$ so if U is regular open, then $g(U) = g(int(cl(U))) = int(g(cl(U))) = int(cl(U)) = U$. Thus $\mathfrak{g}_X$ fixes $r.o.(X)$ pointwise. Conversely if g fixes acts trivially on $ro(X)$, then for any open U we have $g(U) \subset g(int(cl(U))) = int(cl(U))$ or $g(U) \subset cl(U)$, and so $g \in \mathfrak{g}_X$.

Lemma C.1. *Let $G \times X \to X$ be a group action, and assume G, X are endowed with a topology so that the action $G \times X \to X$ is continuous in each variable. Then $\mathfrak{g}_X$ is closed, and $G/\mathfrak{g}_X$ (with the quotient topology) is Hausdorff.*

Proof. To prove $G/\mathfrak{g}_X$ is Hausdorff, it suffices to show that if $g_1, g_2 \in G$ and $g := g_2^{-1} g_1 \notin \mathfrak{g}_X$, then g_1, g_2 are separated by disjoint open $\mathfrak{g}_X$-invariant sets. Since $g \notin \mathfrak{g}_X$, by equation (3), for some $u \in X$ and $u \in U \in \mathfrak{t}$ we have $g \notin cl(U)u^{-1}$, i.e. $gu \notin cl(U)$. By equation (4), $\mathfrak{g}_X$ stabilizes $cl(U)$, hence also $cl(X \smallsetminus cl(U))$ and the complement $int(cl(U))$. Thus $X \smallsetminus cl(U)$ and $int(cl(U))$ are disjoint $\mathfrak{g}_X$-invariant open subsets of X; we have $u \in int(cl(U))$ and $gu \in X \smallsetminus cl(U)$. So $g_2 u \in int(cl(g_2 U))$ and $g_1 u = g_2 g u \in X \smallsetminus cl(g_2 u)$. Since $h : G \to X$ defined by $h(x) = x \cdot u$ is continuous, and $\mathfrak{g}$ is normal, $h^{-1}(int(cl(g_2 U))$ and $X \smallsetminus int(cl(g_2 U))$ are disjoint $\mathfrak{g}_X$-invariant open subsets of G, separating g_1 from g_2.

□

Lemma C.2. *Assume $g \mapsto gx_0$ is a closed, surjective, continuous map, for any $x_0 \in X$. Assume also that X is T1 and G is compact as a topological space (or just that the stabilizer of a point of X is compact.) Then*

(1) If N is any normal subgroup of G with G/N Hausdorff, then X/N is Hausdorff.
(2) $X/\mathfrak{g}_X$ is the universal Hausdorff quotient space X_h of X: any continuous map from X to a Hausdorff space factors (uniquely) through $X/\mathfrak{g}_X$.

Proof. (1) Fix $x_0 \in X$, and let $H = \{g \in G : gx_0 = x_0\}$ be the stabilizer. The topology on X is just the quotient topology on G/H, since the map $g \mapsto gx_0$ is continuous and closed. Since X is T1, H is closed, hence compact. So the image $\bar{H}$ of H in G/N is compact, hence (as G/N is Hausdorff) closed; and hence $(G/N)/\bar{H} = G/(HN)$ is Hausdorff. But this is the same space as $(G/H)/N = X/N$.

(2) Let $f : X \to Y$ be a continuous map into a Hausdorff space. Then for $g \in \mathfrak{g}_X$ we have $f(gx) = f(x)$, since $f(gx), f(x)$ cannot be separated by disjoint open sets. Thus f factors through $f' : X/\mathfrak{g}_X \to Y$, which is continuous by definition of the quotient topology. □

Remark C.3. It follows from Lemma C.2 (1) and (2) that if $N \leq \mathfrak{g}_X$ and G/N is Hausdorff, then the same equivalence relation is induced on X by $\mathfrak{g}_X$ and by N.

Lemma C.4. *Let $(X, \mathfrak{t}_1)$ be a Hausdorff space. Let $\mathfrak{t}_2$ be a topology on X^2 containing the product topology $\mathfrak{t}_1^2$, with $(X^2, \mathfrak{t}_2)$ compact. Let $\Delta = \{(x,x) : x \in X\}$ be the diagonal, and assume $\Delta = \cap_{i \in I} G_i$ with G_i open, $\lambda = |I| + \aleph_0$. Then $\mathfrak{t}_1^2 = \mathfrak{t}_2$, and $\mathfrak{t}_1$ admits a basis of cardinality λ (and hence is metrizable, if $\lambda = \aleph_0$).*

Proof. As the compact $\mathfrak{t}_2$ contains $\mathfrak{t}_1^2$, which is Hausdorff, they are equal, and so $\mathfrak{t}_1$ is compact too. We have $\Delta = \cap_{n \in \mathbb{N}} G_n$ with G_n open. If $(a, b) \notin G_n$, find disjoint open U, V with $a \in U, b \in V$. By compactness, $X^2 \setminus G_n$ is covered by finitely many such $U \times V$. Thus in all, $X^2 \setminus \Delta$ is covered by λ open $U_i \times V_i$, with U_i, V_i disjoint. Let $\mathfrak{t}_0$ be the topology generated by these λ sets U_i, V_i. Then $(X, \mathfrak{t}_0)$ is Hausdorff, and $\mathfrak{t}_0$ contained in the compact $\mathfrak{t}_1$, so they are also equal. $\square$

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