Specialization of difference equations in positive characteristic

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Abstract

A difference equation over an increasing transformal valued field is known to be analyzable over the residue field. This leads to a dynamical theory of equivalence of finite dimensional difference varieties, provided one knows that the residue field is stably embedded as a pure difference field.

This talk will be devoted to that latter problem.

- Joint work (nearing completion) with Yuval Dor.
- useful discussions with Zoé Chatzidakis.
- related results by Martin Hils and Gönenç Onay.
- characteristic zero settled by Salih Durhan in [Azgin10].

Transformally valued fields

$$(K,+,\cdot,\sigma);\ (\Gamma,+,<,\sigma);(k,+,\cdot,\sigma)$$

$$\text{val}: K^{\cdot} \to \Gamma \cup \{\infty\}$$

$$res: K \to k \cup \{\infty\}$$

$$v(\sigma(x)) = \sigma(v(x)),\ \text{res}(\sigma(x)) = \sigma(\text{res}(x))$$

Frobenius (valued) (fields:

$$\sigma(x) = x^q$$

$iVFA$: (increasing valued fields with automorphism)

$$\gamma > 0 \implies \sigma(\gamma) > n\gamma$$

Finite dimension

Let k_0 be a difference field, K_0 a valued difference field of transformal dimension 1 over k_0; e.g. $k_0(C)_\sigma$, C a curve.

FA_{fin/k_0} is the many sorted theory, whose sorts correspond to finite order difference equations over k_0. A model of the model companion $\widetilde{FA_{fin}}$ can be identified with a model of ACFA, truncated to the FA_{fin}-sorts. $FA_{fin} = FA_{fin/\mathbb{F}_p}$.

The theory of pseudo-finite fields is present as the sort $\sigma(X) = X$.

Drinfeld modules.

$iVFA_{fin}$ has sorts as FA_{fin/K_0}, but with the valuative structure as well; a model of $iVFA_{fin}$ is a model of $iVFA$, truncated to the FA_{fin}-sorts.

The additional structure is 'scattered'; for each sort S and any difference polynomial F, $\mathrm{val}F(X_1, \ldots, X_n)$ can take only finitely many values on S^n.

Theorem 1.

- *$iVFA_{fin}$ admits a model companion $\widetilde{iVFA}_{fin}$, axiomatized by ACFA in the residue field, and Newton polygon axioms.*
- *It eliminates quantifiers if one adds function symbols for definable functions of ACVF (=henselization) and ACFA (typical: $A_\sigma A_p^{-1}$ where $A_p(x) = x^p - x, A_\sigma(x) = x^\sigma - x$.) (Amalgamation over algebraically closed difference subfields.)*
- *$\widetilde{iVFA}_{fin}$ is the asymptotic theory of models of $ACVF_p$ with Frobenius automorphisms $x \mapsto x^q$.*
- *The residue field is fully embedded in $iVFA_{fin}$; the image under* res *of a definable subset of K^n is defined purely using difference equations.*

This makes possible a *dynamic theory of equivalence* for $iVFA_{fin}$ (refining, conjecturally nontrivially, the scissors equivalence of the Grothendeick group of algebraic varieties.) To be discussed elsewhere, but here is an application. Fix a prime p, and a difference variety X of finite total dimension over $\mathbb{F}_p$. Recall $K_{p^n} = (F_p^{alg}, +, \cdot, x \mapsto x^{p^n})$.

Theorem (Rationality)**.**

$$|X(K_{p^n})| = \sum_{i=1}^{b} \alpha_i c_i^n$$

for some $c_1, \ldots, c_m, \alpha_1, \ldots, \alpha_m \in \mathbb{Q}^{alg}$*, and large enough* $n \in \mathbb{N}$*.*

Proved by moving X to a formula where Grothendieck's cohomological representation is available.

In fact the theorem remains true when X is definable using $\{+, \cdot, \sigma, \mathrm{val}\}$, for $K_{p^n} = (F_p(t)^{alg}, +, \cdot, x \mapsto x^{p^n}, v)$.

I'll try to bring out three aspects of the proof of Theorem 1.

- The use of stable independence / base change for stably dominated types (HHM; HL).
- Lattice limits.
- Uniformization (used for the stable embeddedness. We use a version for transformal curves, after modification of the function field.)

Theorem 2.

- *$iVFA$ admits a model companion $\widetilde{iVFA}$, with natural axioms.*
- *Amalgamation over inversive, transformally henselian, algebraically closed difference subfields; equivalently, $\widetilde{iVFA}$ eliminates quantifiers if one adds function symbols for definable functions of ACFA, and for transformal henselization.*

Stable amalgamation - valued fields

$K \models ACVF$.

$p = L/K$ an extension ($L = K(a)$), with value group $\Gamma(K) = \Gamma(L)$.

L_d a finite dimensional K-subspace of L. (image of polynomials of degree $\leq d$ in a.)

$$J_d(L/K) = \{f \in L_d : \mathrm{val} f \geq 0\}$$

Assume each J_d is a *finitely generated* $\mathcal{O}$-module. (Lattice). Then L/K is *stably dominated*, controlled by an element of $\varprojlim_d Hom(J_d, k)$.

Conversely, given a compatible sequence p of lattices $\ulcorner \Lambda_d \urcorner \in S_d(K) = GL_d(K)/GL_d(\mathcal{O}_K)$ over a base A, given any $M \geq A$, define canonically an extension $p|M$ of M with $J_d(p|M/M) = \Lambda_d$. $M \mapsto p|M$ is a *definable type* p over A.

Here A may be a base structure compromising imaginaries; e.g. generic type of $\{x : \mathrm{val}(x) = \alpha\}$.

When the extensions KM/M are Abhyankar, the sequence (J_d) is determined by finite data. In this case we say we have a *strongly stably dominated type*. These form a union of definable families; [H-Loeser], cf. Jerôme Poineau's talk.

Background: asymptotic Frobenius

A third bridge from difference geometry to algebraic geometry.

$\curvearrowright$: $\sigma \mapsto q$

Replace $\sigma(x)$ by x^q in all equations. (Formally a functor from difference schemes to sequences of schemes; extending the usual functor from a scheme S over $\mathbb{Z}$ to the sequence $S \otimes \mathbb{F}_p$.)

A rough dictionary:

tr. deg. $\curvearrowright$ log_p degree.

Finite total dimension $\curvearrowright$ finite.

$\dim_{total}(X) \curvearrowright log_p|X|$

transformal dimension $\curvearrowright$ dimension

$\mathbb{Z}[\sigma] \overset{\curvearrowright}{_{\sigma \mapsto q}} \mathbb{Z}$

$k[X]_\sigma \curvearrowright k[X]$

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Analyzability liaison groups $\curvearrowright$ Galois theory, higher ramification groups.

Many notions of algebraic geometry readily lift to one of difference algebra, guided by compatibility with the $\curvearrowright M_q$

Transformally algebraic: satisfies a nontrivial difference equation.

Derivatives: $(X^\sigma)' = 0$.

Transformal Hensel lemma. (For a complete, σ-archimedean K, if $F \in K[X]_\sigma$, $\mathrm{val} F(a) > 2\mathrm{val} F'(a)$ then F has a root near a.)

Transformally henselian field: transformal Hensel's lemma holds. [1]

Newton polygon of $F(X) = \sum a_\nu X^\nu$:

lower convex hull of the set of points $(\nu, \mathrm{val}(a_\nu))$, in the plane over the ordered field $\mathbb{Q}(\sigma)$.

[1]Warning: Urbana notation differs on this point. A beautiful theory of Hensel-Newton approximations is developed in [Azgin-Van-den-Dries09], [Azgin10], and called ϕ-henselian. They are designed *not* to specialize to 'henselian' but to give an account of immediate extensions. The ϕ-henselization in this sense is not in dcl. We suggest calling these *surhenselian* and will continue with the terminology of [H04].

σ-archimedean : for $x \in \Gamma_{>0}$, $\sigma^{-n}(x)$ and $\sigma^{n}(x)$ are cofinal in $\Gamma_{>0}$.

Axioms for iVFA designed to make sense under this dictionary.

Newton polygon axioms

An: Let F be a difference polynomial, and α a slope of the Newton polygon of F. Then there exists a with $\mathrm{val}(a) = \alpha$, $F(a) = 0$.

This captures all *one variable* axioms. In particular, it implies transformal henselianity.

Obviously true in Frobenius ultrapowers; this can be used to show that they are existentially closed and universal, and thus (An) holds in existentially closed models of $iVFA$.

Stable corespondences axioms

As: Let $q(x,y)$ be a strongly stably dominated definable type in $2n$ variables x,y. Assume $q|y = (q|x)^\sigma$. Then there exist $(a,b) \models q$ with $\sigma(a) = b$.

Remarks:

- Using a Bertini principle from [H-Loeser], can restrict to the case: $\dim(p) = \dim(p_{|x}) = \dim(p_{|y})$.
- True in existentially closed models - generalizes same proof for ACFA.
- (Ar): ACFA in residue field - a special case of (As).
- *A posteriori*, for $iVFA_{fin}$, (Ar)+(An) imply (As). But (As) are considerably more flexible to work with. In particular,

Amalgamation for iVFA

Let $K = K^{alg}$, $K \leq L, M \models iVFA$.

Induction on σ-archimedean rank. In higher rank, assume K is transformally henselian. Consider σ-archimedean case: if $0 < \alpha, \beta \in \Gamma$ then $\beta < \sigma^n(\alpha)$ for some n.

The functorial nature of stable amalgamation for VF immediately implies amalgamation for Abhyankar $iVFA$ extensions; the automorphisms must respect the canonical valued field amalgam LM; and $\Gamma(LM) = \Gamma(M)$.

Usual induction on $tr.deg._K L$.

Reduce to wildly ramified / immediate case.

Transformal wild ramification

$K = k(t)_\sigma^{alg}$, $K_n = K(\sigma^{-n}(t))$, $K^{inv} = \cup K_n$

$$\sigma(x) - tx = 1$$

Root:

$$a = t^{1/\sigma} + t^{1/\sigma + 1/\sigma^2} + t^{1/\sigma+1/\sigma^2+1/\sigma^3} + \cdots$$

a/K_n is ramfied; order σ; generic in a ball of vradius $1/\sigma + 1/\sigma^2 + \cdots + 1/\sigma^n$

a/K^{inv} is generic in a properly infinite intersection of balls; 'imperfect', boojum, type IV.

Way out

$\lim = 1/(\sigma - 1)$ is σ-rational.

At least within σ-archimedean models, can treat the intersection of balls with rational limit as a new, slightly infinitary operation; from this point of view, the ball b around a of vradius $1/(\sigma - 1)$ is definable over the base.

Now, $tp(a/K^{inv}, b)$ is stably dominated.

Remark. Poineau defined a canonical amalgamation over any ACVF with value group $\mathbb{R}$. The above can be used to interpret Poineau's amalgamation as a stable amalgamation.

Here, we transpose from $\mathbb{R}$ to $\mathbb{Q}(\sigma)$. But then the existence - and rationality - of a limit needs to be proved.

$$0 < \cdots < \mathbb{Q}\sigma^{-2} < \mathbb{Q}\sigma^{-1} < \mathbb{Q} \cdot 1 < \mathbb{Q}\sigma < \mathbb{Q}\sigma^{2} < \cdots$$

In general, say that a lattice Λ is the *limit* of a family Λ_i of lattices, if the associated (valuative) norms converge pointwise on the vector space.

$$v_\Lambda(a) = \mathrm{val}(c) \iff c^{-1}a \in \Lambda, (mc)^{-1}a \notin \Lambda (m \in \mathcal{M}).$$

Equivalently when Λ_i are increasing, the volume of Λ_i approaches the volume of Λ.

Proposition. *$J_d(L/K)$, while not a lattice, has a unique limit lattice $\widehat{J_d}(L/K)$. It can be used to define a canonical extension, to any $M \geq K$ in which* $\mathrm{val}(K)$ *is cofinal towards 0^+.*

Some open questions

1. Is the residue field stably embedded $iVFA$?
2. Uniqueness of the transformal henselization? (True in σ-archimedean case.)
3. Is $\widetilde{iVFA}$ true asymptotically in the Frobenius valued fields K_q?

A positive answer would imply QE for $iVFA$ in the same language as for $iVFA_{fin}$. All follow from a concrete question in σ-archimedean rank 1:

Question. *Let $L \models iVFA$, with field of representatives F, and $F \leq K \models FA$. Let $M = (LK)^h$. Can M have a proper, σ-invariant finite field extension?*

In transformal dimension one, there can be no such extension; this is proved using:

Proposition. *(a certain uniformization for transformal curves, [H04].) Let $L = L^{alg} \models iVFA$ have transformal* *transcendence dimension* 1 *over a difference field* F*; assume* F *maps bijectively to* $\mathrm{res}(L)$*. Then* $\widehat{L} \cong \widehat{F(t)_\sigma^{alg}}$*, and similarly for the transformal henselization.*

Question. *Uniformization in higher dimension?*