Logic seminar: conservation of number, difference equations, and a technical problem in positive characteristic.

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(Slides not used in talk, but see IHP-VFA.)
1. Conservation of number. (Leibniz, Poncelet, Schubert, ...)
2. Number of points of a variety over a finite field.
3. From PF (pseudo-fintie fields) to FA_{fin} (finite-dimensional difference equations.)
4. Numbers (Grothendieck-Cavalieri ring)
5. Conservation of number (Weil-Cavalieri ring.)

1. Conservation of number. (Leibniz, Poncelet, Schubert, ...)
The number of solutions of a given algebro-geometric configuration, when it is finite, does not change upon a small perturbation of the parameters; this persists even upon specialisations that change the topology.

Intuition: let X_t be the solution set over t; we view it as varying over $t \in \mathbb{P}^1$. Then $X_t = \{a_1(t), \ldots, a_n(t)\}$, and each $a_i(t)$ is a continuous function of t with limit $a_i(0)$ in X_0. This gives an explicit map $X_{t_0} \to X_0$; need to justify why it exists at all, why the limit exists, why it is surjective, what to do when it is not 1-1.

Example: (Schubert) L_0, L_1, L_2(skew) lines in space (in $\mathbb{P}^3$), L_t a fourth line indexed by t. N_t = number of lines intersecting each of the four.
Easy to see that $N_t = 2$ if L_t, L_0 intersect. (Consider point of intersection, and plane through both).
When all four intersect, or two are equal, infinite answer.
When L_t, L_0, L_1, L_2 are skew, still $N_t = 2$.

Example: Bézout's theorem.

The problem consists in this: To establish rigorously and with an exact determination of the limits of their validity those geometrical numbers which Schubert especially has determined on the basis of the so-called principle of special position, or conservation of number, by means of the enumerative calculus developed by him.
Although the algebra of today guarantees, in principle, the possibility of carrying out the processes of elimination, yet for the proof of the theorems of enumerative geometry decidedly more is requisite, namely, the actual carrying out of the process of elimination in the case of equations of special form in such a way that the degree of the final equations and the multiplicity of their solutions may be foreseen.
(Hilbert 1900; problem 15).

See Fulton, Intersection Theory (1985) for answer, involving algebraically closed fields, flat and proper morphisms, intersection multiplicity.
A partial generalisation of this approach to (1993) Zariski geometries

was an important means for showing that they are, very nearly, ordinary algebraic geometry. For difference or differential equations, an ideal theoretic approach seems daunting.

2. Number of points of a variety over a finite field.

$$X(\mathbb{F}_q) = \Delta_X \cap \Phi_X$$

Δ, Φ the graphs of Id, ϕ_q on X.
Move Φ_q, or move Δ, and use conservation of number.
For fixed p, let

$$a_n = X(\mathbb{F}_{p^n})$$

W1 (Weil, Lang-Weil) $a_n = p^n + O(p^{n-1/2})$

W2 (Weil for curves, Grothendieck) The a_n satisfy a linear recurrence with integer coefficients.

In fact, for $n \geq$ some n_0 the sequence is an exponential sum

$$\sum_{i=1}^{b} \alpha_i c_i^n$$

for certain algebraic numbers c_i and integers α_i. [1]

- Deligne: each $|c_i| = p^{m/2}$ for some $m \leq n$ (note this includes W1). Though Deligne's theorem is used in the proof, it will not be relevant in this talk.)

The proof of (W2) uses cohomology, and does not exhibit an explicit connection among the point sets $X(\mathbb{F}_{p^n})$. For *curves*, proofs by Stepanov and by Kapranov do imply such a connection.

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[1]; in the generalization to difference definable sets, we allow the coefficients α_i to have denominator p^l.

3. From PF to FA_{fin}: asymptotic Frobenius.
Ax used (W1) to axiomatize the theory of pseudo-finite fields (ultraproducts of finite fields.)
Van den Dries: what about ultraproducts of $K_q = (F_p^{alg}, +, \cdot, \phi_q)$?
ϕ_q = the Frobenius automorphism $x \mapsto x^q$.
Φ_q = graph of ϕ_q.
Difference equations:
Can bring to form: $(x, \sigma(x)) \in S$. $S \leq X \times X^\sigma$.
FA_{fin}: the union of all solutions to nontrivial difference equations; an $\bigvee$-definable subset of a model of FA.
Includes the solutions to the equation $\sigma(x) = x$, or $S = \Delta_X$ - the pseudo-finite fields.
- $a_n = S \cap \Sigma_n$.

Theorem. $|S \cap \Phi_q| = aq^d + O(q^{d-\frac{1}{2}})$ *Thus any nonprincipal ultraproduct of K_q is a model of $ACFA = \widetilde{FA}$, the model companion of the theory FA of fields with automorphism.*

The estimate (and more) was conjectured by Deligne, and proved by Pink (under resolution) and Fujiwara 94. But these proofs assumed additional hypotheses on S, that appeared difficult to get around. (Difference equations do not naturally live on $X \times X$; they may live in X^n.) Finally in Varshavsky 2015 found a short cohomological proof that $|S \cap \Phi_q| > 0$ for large n, which suffices for the logical statement.

What about the linear recurrence, $W2$? How to state the conservation of number, for 0-dimensional difference varieties?

4. Numbers.

$\mathbb{N}$: Finite sets; up to bijective functions.

$+, \cdot$.

Build in $/, -$. $\mathbb{Q}^{>0}, \mathbb{Q}$.

(Cav) If $f : Y \to X$ is definable and $[Y_x] := [f^{-1}(x)] = c$ for all $x \in X$, then $[Y] = c[X]$.

Now take everything definable, in some theory T. This defines the Grothendieck-Cavaleri ring of T, $GC(T)$. (With $-$ only, the Grothendieck ring, $K(T)$.)

N.B. Kapranov showed that the recurrence relations for curves are valid already in the Grothendieck ring. This is not true in higher dimensions (Larsen-Luntz.) I would expect that a 'dynamic' (conservation of number) equivalence may be needed. How to formulate it?

5. Internality.
Baldwin-Lachlan example of 'no Vaughtian pairs over P, yet Q is not algebraic over P". E.g. a free $\mathbb{Z}/4\mathbb{Z}$-module B. $0 \to A \to B \to A \to 0$ exact, where $A = 2B$.
Zilber understood that this is an essential structural feature of $\aleph_1$-categorical structures; groups can be non-commutative; but the basic structure is $f : Y \to X$, $f^{-1}(x)$ a torsor for G_x, and in any case isomorphic to an X-definable set S_x (actually, groupoid.)
Note in this situation we have, when all are finite: $|Y| = |X||S_x|$.
Assumption reduced to X *stably embedded.* If T' is a multi-sorted theory, T some of the sorts, T stably embedded in T', and no Vaughtian pairs T'/T, then T' succumbs to an analysis by finitely many T-internal extensions as above.

Theorem. *Assume T' interprets T and is analyzable over T (with a hypothesis on liaison groups, e.g. solvable or linear.). Then the natural map $CG(T) \to CG(T')$ is an isomorphism.*

Example. *$T' = iVFA_{fin}$; so the residue field is a model of FA_{fin}; $T =$ theory of residue field.*

Here $iVFA$ is the theory of *valued* fields with an automorphism, with $v(\sigma(x)) > nv(x)$ whenever $v(x) > 0$ and $n \in \mathbb{N}$. This is valid in ultraproducts of valued fields L with the Frobenius automorphism ϕ_q.
Picture in one variable.

6. Conservation of number and the Cavalieri-Weil ring.
Let T, T' be pseudo-finite theories; T the asymptotic theory of K_q, T' of an expansion L_q; and let j_0, j_∞ be two interpretations of T in T', with T'/T analyzable.
We have two isomorphisms $j_0, j_\infty : CG(T) \to CG(T')$.
"Move" $[X]$ to $X' := j_\infty^{-1} j_0[X]$.
Then for almost all q, $X(K_q) = X'(K_q)$.
Cavalieri-Weil ring: factor out CG(T) by this new motivic equivalence.

7. But is iFA embedded, and stably embedded, in $iVFA$?
Is there a model companion $i\widetilde{VFA}_{fin}$, and is FA_{fin} embedded therein?

Theorem (Azgin 2007 thesis - char 0.). *FA_{fin} is embedded and stably embedded in VFA_{fin}*

But this relies on residue char. 0, and runs into the usual technical issues in char. $p > 0$.
Work with Yuval Dor.

theorem. *FA_{fin} is embedded and stably embedded in VFA_{fin}*

Theorem 1. *For any formula ϕ of FA_{fin}, (W2) holds for $a_n = |\phi(K_q)|$.*

Theorem 2. *For any formula ψ of VFA_{fin} there exists a formula ϕ of FA_{fin} with $|\psi(K_q) = |\phi(K_q)|$. In particular, (W1,W2) hold.*

Intertwined proofs. E.g. the proof for $ACFA_{fin}$ goes through VFA_{fin}. Also the proof that $\widetilde{FA_{fin}}$ is the asymptotic theory of Frobenius valued fields requires, for one axiom, the estimate (i) for some valued difference formulas.

A thoroughgoing analogy between difference geometry and algebraic geometry, mediated by asymptotic Frobenius. So far, barely sampled.

$\curvearrowright$: $\quad \sigma \mapsto q$

tr. deg. $\curvearrowright$ log_p degree

$\dim_{total}(X) \curvearrowright log_p|X|$

$tr.dim(X) \curvearrowright \dim(X)$

$\mathbb{Z}[\sigma] \overset{\curvearrowright}{\scriptstyle\sigma\mapsto q} \mathbb{Z}$

$k[X]_\sigma \curvearrowright k[X]$

σ-archimedean $\curvearrowright$ archimedean

σ-DVR 's : (—fair to work with algebraically closed fields, at first go.) $\curvearrowright$ DVR's.

σ-henselian $\curvearrowright$ henselian

Completion of $k(X)_\sigma \curvearrowright k((X))$ E.g. solution to $X^\sigma - X = t$.

Tame ramification (prime to σ) $\curvearrowright$ tame ramification (prime to p)

Abhyankar's lemma. Tame ramification can become unramified under base extension; but wild ramification persists and at the limit

turns into an immediate extension. E.g. solution to $X^\sigma - tX = 1$.
? $\curvearrowright$ Poineau's theorem ?
Axioms designed to make sense under this dictionary.

Question 1. *Let X be a strongly reduced affine difference variety, v a valuation on $k(X)$. Is it true (perhaps after an inseparable extension) that there exists a difference variety Y with a birational (or purely inseparable) dominant morphism $Y \to X$, such that v is centered on a smooth point of Y?*

To begin with we can ask the question for k of characteristic zero. This neither implies nor is implied by resolution or uniformization in positive characteristic, but seems a very interesting asympototic case. Of course one should begin with X of transformal dimension one over $k \models ACFA$; is the transformal normalization transformally finite over X? Is it smooth? What about the zero-dimensional case? Is there a way to ignore it?