A logic for global fields

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Abstract

I will discuss the notion of an *existentially closed structure*, and give classical examples, including the local fields $\mathbb{R}$, $\mathbb{Q}_p$, and $\mathrm{k}((t))$. I will then describe a language capable of capturing some global structure: roughly, the embedding of a field in its adeles, constrained by the product formula. It is

conjectured that $\mathbb{Q}^a$ and $\mathrm{k}(t)^a$ are existentially closed in this language. I will discuss the statement, and a proof in the function field case. There are connections to (distributional) Fekete-Szegő theorems, and to non-archimedean Calabi-Yau type theorems.

This is joint work with Itaï Ben Yaacov.

1. Existentially closed structures.
2. A language for global fields.
3. $k(t)^a$ is existentially closed.
4. Example: distributional Fekete-Szégo.
5. Example: a Lefschtez principle.
6. Proof: Reduction to a theorem in geometry.
7. Proof: positive intersection products, Legendre duality.
8. Non-archimedean Yau-type theorems (surjectivity statements for Monge-Ampère.)
9. Further conjectures. Example: Nevanlinna theory.

1 Existentially closed structures

Language.

Variables: $X = (X_1, \ldots, X_r)$, $Y = (Y_1, \ldots, Y_l)$.

Basic formulas: For L_{rings}: $p(X) = 0$, $p \in \mathbb{Z}[X]$; for L_{val}, also $|p(X)| < 1$.

We will also permit formulas ϕ taking real values, say in a closed interval I_ϕ, rather than truth values $\{0, 1\}$.

A universal theory T: a collection of sentences $(\forall X)(\phi_1(X), \ldots, \phi_n(X)) \in C$, where ϕ_k takes values in I_k, and C is a closed subset of $\Pi_{k=1}^n I_k$.

Definition. *M is an* existentially closed *model of T if for any structure $N \geq M$, with $N \models T$, basic formulas $\phi_i(X, Y) \in L$, $(i = 1, \ldots, l)$, $\epsilon > 0$, and any b from M and a from N, there exists a' from M with $|\phi_i(a, b) - \phi_i(a', b)| < \epsilon$, $i = 1, \ldots, l$.*

Examples

1. $\mathbb{C}$ is existentially closed for T=commutative domains. (Gauss, Hilbert).

   Lefschetz principle: a sentence true in almost all F_p^a is true in $\mathbb{C}$.

2. $\mathbb{R}$; T=real fields. (Descartes, Tarski.)

   Robinson's proof of Artin's theorem / Hilbert's 17th problem: if a rational function $f(t)$ is not a sum of squares, construct an ordering of $\mathbb{R}(t)$ where $f(t) < 0$; then by existential closure, $(\exists t' \in \mathbb{R}) f(t') < 0$.

3. (Ax-Kochen) The class of valued fields $\mathbb{Q}_p$, asymptotically. T=theory valued field with conditions on value group and residue field- including char. 0.

   "asymptotically" means that p should be large compared to the formula $\phi(X, Y)$. Equivalently, nonprincipal ultraproducts are existentially closed.

   Asymptotic Artin's conjecture /Ax-Kochen theorem follows from Chevalley-Tsen: if $Y \leq \mathbb{P}^n$ is a hypersurface of degree $\leq \sqrt{n}$, then $Y(\mathbb{F}_p((t)) \neq \emptyset$, so $Y(\mathbb{Q}_p((t))) \neq \emptyset$.

4. Finite fields (Cebotarev, Weil; Ax), the ring of algebraic integers (Rumely, Moret-Bailly; Van den Dries, Macintyre), ...

In each of these cases, the existential closure is a point in a much more extensive theory: quantifier elimination to a certain level, structure of definable sets, stability-type properties. Early consequences include:
—decidability.
—Lefschetz principle. (e.g. recently for motivic integration, (3).)
—precise statement of analogy $\mathbb{F}_p((t))$ and $\mathbb{Q}_p$ (isomorphism of ultrapowers.)

Thus far, no analogous results in global geometry. It is known (Gödel, J. Robinson,Matiyasevich.) that $\mathbb{Q}$ cannot directly be a model companion of any reasonable theory. We will try for $\mathbb{Q}^a$ and $k(t)^a$, viewed as limits of global fields, number or function fields. We take essentially the minimal reasonable language capable of expressing the product formula.

2 The language

The terms are polynomials over $\mathbb{Z}$; equality is a $\{0,1\}$-valued relation as usual.
Basic relations R_t: A symbol R_t for each *tropical term* t = term in the language $+, \min, 0, \alpha \cdot x$ of divisible ordered Abelian groups. to be interpreted as functions $(F^*)^n \to \mathbb{R}$.
Local interpretation of R_t Let (K, v) be a valued field, or a subfield of $\mathbb{C}$ with $v(x) = -\alpha \log |x|$. For x with $x_i \neq 0$, interpret $R_t{}^v(x)$ as $t(vx_1, \ldots, vx_n)$.
Global intended intepretation: We think of $R_t(x)$ as the *expected value* of $R_t{}^v(x)$ with respect to an implied measure on valuations. Write a basic formula

$$R_t(f_1(x), \ldots, f_n(x)) =: \int t(vf_1x, \ldots, vf_nx)dv$$

.

Among them, the height: $x^+ = -\min(-x, 0)$.
$ht(x) = R_t(x) = \int v(x)^+ dv$

Height has a structural role in the definition of quantifiers and limits of GVF's, but I will not go into this here.

Connectives $\min, \max, 0, +, \alpha \cdot x$.

Quantifiers The analogue of quantifiers in real-valued logic is inf and sup operators. Let $\psi_{n,\epsilon}(t)$ be 1 on $[-n, n]$, 0 on $|t| > n + \epsilon$, and a linear interpolation on $[n, n + \epsilon]$. Let $\phi(x, y)$ be a formula. Then so is $\sup_x \psi_{n,\epsilon}(ht(x))\phi(x, y)$.

We view this as a quantifier over x of height up to about n.

All formulas are preserved by ultrapowers.

It will turn out that the Weil projective height functions $(x_0, \ldots, x_n) \mapsto ht((x_0 : \cdots : x_n)) = -int \min_i v(x_i)dv$ suffices to generate the language, at least in the purely non-archimedean case. At any rate, it will suffice to remember that heights are given by a formula.

3 Universal axioms

Let LVF be the set of pairs (ϕ, t) of formulas $\phi(x_1, \ldots, x_n)$ in the language of rings implying $\Pi_i x_i \neq 0$, t a tropical term, such that the theory of valued fields / normed fields implies t is positive on the amoeba of ϕ:

$$VF \models (\forall x)(\phi(x) \implies t(v(x_1), \ldots, v(x_n)) \geq 0)$$

Axioms GVF *for globally valued fields:*

1. $(F, +, \cdot)$ is an integral domain.
2. The R_t are compatible with permutations of variables and dummy variables.
3. (Linearity:) $R_{t_1+t_2} = R_{t_1} + R_{t_2}$. $R_{\alpha t} = \alpha R_t$.
4. (Local-global positivity for amoebas) If $(\phi, t) \in LVF$ and $\phi(a_1, \ldots, a_n)$ then $\int t(v(a_1), \ldots, v(a_n))dv \geq 0$.
5. (Product formula) $\int v(x)dv = 0$

4 Classical structures

Number fields and function fields K have obvious GVF structures: assign masses m_v to the places v so that $(\forall x \in K^*) \sum_v m_v v(x) = 0$, and define

$$R_t(x_1, \ldots, x_n) = \sum_v m_v t(v(x_1), \ldots, v(x_n))$$

The assignment of masses m_v is unique up to a scalar multiple by Artin-Whaples, and in fact the GVF structure is similarly unique. In particular, we can unamiguously refer to K^a as a GVF.

Incidentally, this is not true for all GVF's K; but it is true for K such that K^a is e.c. There is always a unique Galois-invariant extension to K^a. .

Conjecture 4.1. *Let K be a global field. Then K^a is existentially closed in the GVF language.*

Theorem 1. *Conjecture 4.1 holds in the function field case. In fact for any field* k, $\mathrm{k}(t)^a$ *is existentially closed as a GVF.*

Explicitly: let $K = \mathrm{k}(C)$, C a curve over k. For any variety V over K, any finite number of GVF formulas $\phi_1, \ldots, \phi_r$ and potential values $\alpha_1, \ldots, \alpha_r$ formally consistent with the product formula, there exists a finite extension K' of K and $c' \in V(K)$ with $\phi_i(c')$ as close as desired to α_i.

Some standard corollaries:

Proposition. *Automatic effectiveness for existence questions that can be formulated in the GVF lanugage over function fields.*

Proposition. *A Lefschetz principle holds for such questions.*

Example (Bogomolov conjecture)**.** Given an Abelian variety A over a global field K with trace 0 to finite fields and (for simplicity) a curve C on A of genus > 1, it is conjectured that that for some $h > 0, b \in \mathbb{N}$, $C(K^a)$ has $\leq b$ points $a_1, \ldots, a_b$ of canonical height $\leq h$ on A.

Proved by Ullmo, Zhang in the number field case, Gubler, Yamaki, Cinkir in many cases for function fields.

Theorem 1 provides an algorithm guaranteed to produce h, b and (the degree of) these points p_i. ('search for $a_1, \ldots, a_b$ *and* a proof from GVF that no further solutions exist.')

Let $f : C \to U$ be a non-isotrivial, generically smooth family of projective curves of genus ≥ 2, say over $\mathbb{Q}$, and embed C_K canonically in its Jacobian A_K. Cinkir proves Bogomolov's conjecture is true in this case, over $K = \mathbb{Q}^a(U)$.

By Theorem 1, the same is true for $\mathbb{F}_p(U)$, for almost all primes p.

A similar uniformity result holds for families of rational curves:

There exists $b \in \mathbb{N}, \epsilon > 0$ such that if $t \in U(\mathbb{Q}^a)$ has large enough height h, $C = C_t$, then $C(\mathbb{Q}^a)$ has at most b points a with

$$ht_{can}(a) \leq \epsilon h$$

This follows from Theorem 1 and Cinkir's theorem, though neither mentions number fields at all. To prove it one uses the family of GVFs $\mathbb{Q}^a[r]$; they are just renormalizations of $\mathbb{Q}^a$:

$$R_t(f_1, \ldots, f_n)^{\mathbb{Q}^a[r]} := \frac{1}{r} R_t(f_1, \ldots, f_n)^{\mathbb{Q}^a}$$

In $\mathbb{Q}^a[r]$, the height of 2 is $\log(2)/r$; asymptotically, these are purely non-archimedean GVFs.

To prove the corollary, we suppose it is false; then there exists a sequence of curves $C_i = C_{t_i}$, $t_i \in U(\mathbb{Q}^a)$ of height r_i, and at least i distinct points $a_{i,j} \in C_i(\mathbb{Q}^a)$ with $ht_{can}(a_{i,j}) \leq \epsilon r_i$. Then in $\mathbb{Q}^a[r]$, a_i has height 1, and $ht_{can}(a_{i,j}) \leq \epsilon$. It follows that in the non-archimedean GVF in $K = \mathbb{Q}(x)^a$ there exists a curve $C = C_t$ with $t \in U(K)$ of height 1, so C_t is not isotrivial, and with as any points of $C(K)$ as desired, of arbitrarily small canonical height; this contradicts Cinkir's theorem. (In fact one can even find a sequence $a_j \in C(K)$, such that $ht_{can}(a_j) \to 0$, using transitivity of $Aut(K)$ on height-1 elements of $\mathbb{P}^1$)

Further examples of Lefschetz - intersection profiles of subvarieties with curves of high degree - will be mentioned later, if time permits.

Distributional Fekete-Szegő

Fekete-Szegő (1953) asked: *When does a compact subset C of $\mathbb{C}$ contain infinitely many Galois orbits of algebraic integers?* (Polya, Schur 1918 for intervals: iff length ≥ 4.)

This, they did not succeed in answering, but they gave a beautiful answer to a toplogical relaxation of the question: *There exists a sequence of Galois orbits, whose Hausdorff limit is an infinite subset of C*, if and only if C has *capacity* ≥ 1.

The *capacity* can be defined in several ways, including the Chebyshev number and the transfinite diameter.

The theory was generalized by Cantor to an adelic setting.

K a global field; S be a finite set of primes of K, including archimedean primes. Let $\bar{X} \leq \mathbb{P}^n$ be a normal projective variety over K, $X = \bar{X} \cap \mathbb{A}^n$ the corresponding affine variety.

Let X_p be the Berkovich space of X over K_p. An *adelic set:* $A = (A_p)_p$ with $A_p \subset_{compact} X_p$, and $A_p = X(\mathcal{O}_p)$ for $p \notin S$. [1]

The *Chebyshev constant* $\mathrm{Ch}(A)$ is defined by:

$$-\log \mathrm{Ch}(A) = \lim_{n\to\infty} \frac{1}{n} \sup_{\deg(f)\leq d} \Sigma_p \inf_{v\in A_p} v(f)$$

Rumely fully generalized the theory to *curves.* Several of the definitions of capacity have been generalized to higher dimensions (Chinburg 1991, ..., Chinburg-Moret-Bailly-Pappas-Taylor 2012), with some implications for Galois orbits, but no sharp characterization so far.

Here we will look at a further, measure-theoretic relaxation: roughly, we do not ask whether *all* points of the Galois orbit are in a neighborhood of C, but *almost all.* For this we obtain a sharp characterization in all dimensions.

[1] Actually Cantor used $X(\mathbb{C}_p)$, Chamber-Loir moved to X_p for similar questions.

Given a finite set S, the characteristic measure is the probability measure giving equal mass to each point of S.

Let K be a function field; so that K^a is existentially closed as a GVF. A an adelic set.

Say A *supports a Galois limit measure* iff there exists a sequence of Galois orbits O_i whose characteristic measures weakly converge to a measure on A, and in addition,

- (genericity) only finitely many O_i lie on any given proper K-subvariety of X.
- (generic integrality) For $x_i \in O_i$, as $i \to \infty$,

$$\sum_{p \notin S} \int_{v|p} \max_{i=1}^{n} v^{-}(x_i) \to 0$$

Theorem 2. *K a global field, with K^a existentially closed. Then A supports a Galois limit measure if and only if* $\mathrm{Ch}(A) \geq 1$.

Proof: the Chebyshev number condition allows a soft construction of a GVF extension $L \cong K(X)$, $a \in X(L)$, with $\int t(v(a))dv = 0$ for t supported away from A. By Theorem 1, there exist approximations a_i to a within K. Now integrating t over all valuations above a place p of K = fixing one v_p above p and summing over the Galois orbit.

Remark on the transfinite capacity.

Proof of Theorem 1

1. Reduction to equations *over the constant subfield.* This is a standard model-theoretic lemma, using the large automorphism group of K^*. *approximate* automorphisms of $K = k(x)^a$, e.g. with $x \mapsto x^{1+1/100}$, become automorphisms at the limit. This works also over $\bar{Q}[1]$, but is not known to me for $\bar{Q}[r]$ for nonconstant r.)

2. Geometric description of formulas.

X a normal projective variety over $k = k^a$. $K = k(X)$.

Given a very ample Cartier divisor H on X, consider the associated projective embedding; recall that the Weil height with respect to this embedding is given by a formula:

$$ht_H = \int -\min(v(s_0), \dots, v(s_m))dv$$

where $s_0, \dots, s_m \in K$ are a basis of the linear system of H $((s_i) + H \geq 0.)$

This gives a pairing:

(GVF structures on K/k) $\times$ (very ample divisors on X) $\to \mathbb{R}$.

For a fixed GVF structure p on K, $H \mapsto ht_H^p$, is a linear map on $Pic_{\mathbb{R}}(X)$, positive on the effective cone. From this it follows that it must vanish on the bounded subgroup $Pic^0(X)$.

Let $N_1(X)$ be the $\mathbb{R}$-space generated by the curves on X, up to numerical equivalence.

$$N_1^+(X) = \{c \in N_1(X) : (\forall \text{effective Cartier } H)(c, H) \geq 0\}$$

We have just described a map from GVF structures on K/k to $N_1^+(X)$.

3. Dually, for any H, $p \mapsto c(p) \cdot H$ is described by a formula, and if H is allowed to range over Cartier divisors of blowups, such formulas generate all. (Given the fixed field structure.)

$N^1(X) \rightsquigarrow \{formulas\}, \quad D \mapsto \phi_D$.

4. On the other hand, points of X in K are given by irreducible curves on X. To approximate a given structure, these curves need to avoid a hypersurface, and approach the class of the structural 1-cycle $c \in N_1^+(X)$. We are thus led to the following problem:

5. (*) Multiples of irreducible curves on Zariski open sets, are dense in the nef 1-cycles.

From this point on, the problem is purely geometric.

Example: smooth projective surfaces. (*) follows from: nef divisors are approximated by ample divisors. Here Nakai-Moishezon + Bertini irreducibility suffice. In the case of arithmetic surfaces, does an ample line bundle have a plurisection whose poles form a single Galois orbit?

Higher dimensional case: nef 1-cycles approximated by A^{n-1}, A ample on a blowup.

A theorem of Boucksom-Demailly-Pau-Peternell 2004 (in char. 0) asserts that the *convex hull* of such divisors is dense. A proof using positive intersection products is given in BFJ.

We use the same methods, along with Legendre duality, to obtain:

Theorem 3. *Let X be a normal variety over an algebraically closed field. Let c be in the open cone of curves dual to the pseudo-effective cone of X. Then c is the increasing limit of cycles A^{n-1}, A ample on a blowup of X.*

In fact $c = B^{<n-1>}$, B a big $\mathbb{R}$-Cartier divisor on X, $< n-1 >$ the 'positive intersection product'.

Along with Bertini, this gives (*).

Big divisors

X a normal projective variety over k, of dimension n.

We consider $\mathbb{R}$-Cartier divisors; $D = \sum \alpha_i D_i$, $\alpha_i \in \mathbb{R}$. If each $\alpha_i \geq 0$, write $D \geq 0$, D effective.

An rational function $f \in K = k(X)$ is a *section* of D if $(f) + D \geq 0$. $L(mD)$ is the space of sections of mD.

An effective divisor D is *big* if some $L(mD)$ has algebraically independent sections.

Positive intersection product

We define $< D >^k$ with $k = n$ or $k = n - 1$.

Let $O \subset X$ be Zariski open. Let $m \in \mathbb{N}$.

Let $s_1, \ldots, s_k$ be generic sections of $L(mD)$. Let Z be the Zariski closure of their common zero locus in O. (A curve if $k = n - 1$, a 0-dimensional scheme if $k = n$.) The class $[Z]$ in $N_{n-k}(X)$ does not depend on the generic choice, and decreases with O but stabilizes for small enough O. Define $< X >_m^{<k>} = \frac{[Z]}{m}$.

$$< D >^k := \lim_{m \to \infty} < X >_m^{<k>}$$

Remark: taking $k = n$, Demailly (1993) gave this definition of *volume*, i.e. the leading coefficient of the section growth function. BFJ showed vol is differentiable, and

$$\mathrm{vol}'(D) \cdot H = \lim_{t \to 0} \frac{\mathrm{vol}(D + tH) - \mathrm{vol}(D)}{t} = n < D >^{n-1} \cdot H$$

Also, $\mathrm{vol}^{1/n}$ is concave on the big cone.

This was initially in char. 0, but using Okounkov's methods, it is easy to obtain the same in all characteristics.

Theorem 2 follows using a version of Legendre duality, concerning the derivative of a concave function.

Non-archimedean Yau-type theorems

On a smooth Kähler variety, the Monge-Ampère operator takes metrized line bundles to volume forms.

A non-archimedean analogue was developed by Kontsevich-Tschinkel and Chambert-Loir.It maps metrized line bundles to measures on Berkovich space.

The general definition uses a limit procedure; here I will discuss only the purely geometric level, which is easily defined.

Let k be an algebraically closed field, U a smooth projective curve over k, $\pi : X \to U$ a normal projective variety over U, $\dim(X) = n + 1$. The divisors c lying above divisors of U are the *vertical* divisors of X/U, and a measure will just be a positive real-valued function on them. Each such c has a multiplicity m_c in its fiber.

Let L be an ample line bundle on X.

$$\mu_L(c) := m_c c \cdot L^n$$

We extend this to big divisors B using the positive intersection product:

$$B \mapsto \hat{\mu}_B$$

$$\hat{\mu}_B(c) = m_c c \cdot < B >^n$$

In the 1950's, Calabi proved injectivity of the Ampère-Monge operator on smooth Kähler metrizations of a given line bundle, (up to a scalar), and conjectured surjectivity to (appropriately normalized) volume forms; this was proved by Yau in 1977.

A non-archimedean version in characteristic zero appears in a recent theorem of Boucksom-Favre-Jonsson, with antecedents in Kontsevich-Tschinkel (2001, unpublished text). They obtain uniform convergence to general semi-positive metrized line bundles, but ask for additional information when beginning with a model measure.

With a little adjustment, Theorem 3 implies (but does not seem to follow from) a relative version:

Theorem 4. *Let L be an ample line bundle on X. Let μ be a nowhere vanishing positive measure on the vertical divisors, such that the total mass of each fiber X_t is $\mu(I_t) = \deg(L)$. Then there exists a big $\mathbb{R}$-Cartier divisor B on X with generic part L, $\hat{\mu}_B = \mu$.*

Remarks:

Assume X is smooth, and let v be a valuation of $K = k(U)$ over k. Then μ determines a measure on the Berkovich space of X over (K, v); the theorem implies that there exists a metrized line bundle, positive increasing limit of ones arising from nef models, whose Monge-Ampere measure is μ. The same follows for any measure on the Berkovich space with total mass prescribed as above, and with a certain non-vanishing condition.

B can be taken to be *quasi-free*, i.e. determined as a supremum of sections of mB.

One can add a multiple of a divisor arising from U (analogous to scalar multiples in Calabi.) I do not know other sources of non-uniqueness (for quasi-free B.)

Remark. The *topological* Fekete-Szegő theorem amounts to finding an irreducible curve, close to a given ray of nef 1-cycles, and at the same time orthogonal to a certain finite set of irreducible Weil divisors. In the case of curves over a function field, the set of divisors in question can be contracted in a morphism to an Artin algebraic space, probably giving another proof of Rumely's function-field Fekete-Szegő, and showing that the distributional and topological conditions coincide.

A curve selection theorem

Let X be a smooth projective variety over $\mathbb{Q}$; let Y, $Y_1, \ldots, Y_m$ be subschemes.

If L is a number field and $x \in X(L)$, let $\delta(x, Y)^L = \int \delta_v(x, Y) dv$ be the weighted sum of the local distances from x to Y, over all valuations (and $-\log |\cdot|$) v of L.

Note that $\delta(x, Y)^L$ is the L-value of a certain quantifier-free formula $\phi_Y(x)$ in the language of GVF's.

Proposition. *Assume* $a_n \in X(\mathbb{Q}^a)$*,* $ht(a_n) \to \infty$*, with* $\lim_{n\to\infty} \delta_{Y_k}(a_n)/ht(a_n) = e_k$*; let* $\epsilon > 0$*. Then there exists a curve* C *on* X *such that for any sequence* $a'_n \in C(\mathbb{Q}^a)$*,* $ht(a'_n) \to \infty$*, we have* $\lim_{n\to\infty} |\delta_{Y_k}(a'_n)/ht(a'_n) - e_k| < \epsilon$*.*

Proof. Choose $r_i = ht(2)/ht(a_i)$ so that $\mathbb{Q}^a[r_i]$ gives a_i height 1. Consider any non-principal ultrafilter u on the index set $\mathbb{N}$, and let (L, a) be the GVF ultraproduct of $(\mathbb{Q}^a[r_i], a_i)$. Then (L, a) is a purely non-archimedean GVF, and $\delta_{Y_k}(a) = \phi(a)^L = e$. There exists $a' \in K = k(t)^a$ with $e' = \phi(a')^K$

satisfying $|e' - e| < \epsilon$. In fact $a' \in k(C)$ for some curve C, so a' corresponds to a morphism $g : C \to X$. We may choose a' so that $g(C)$ avoids any given proper subvariety of X. By computing the meaning of ϕ in $k(t)^a$ we see that $\bar{i}(C, Y_k) = e'$.

Conversely, if C is a curve on X defined over $\mathbb{Q}^a$, then for any sequence of distinct $a_i \in C(\mathbb{Q}^a)$ of bounded degree over $\mathbb{Q}$, $\delta_Y(a) \to i_Y(C)$. This follows upon taking normalized ultraproducts as above, from the uniqueness of the GVF structure on $k(C)$. $\square$

In particular, there exists such a sequence a_i' of bounded degree over $\mathbb{Q}$.

5 Further conjectures

A theory T admits a *model companion* if the class of existentially closed models of T is axiomatizable.

Conjecture 5.1. *There exists a model companion for GVF.*

Corollary (assuming Conjecture 5.1 and existential closure)**.** $\mathbb{F}_p(t)^a$, $\mathbb{Q}[r]^a$ *have isomorphic ultraproducts.*

Moreover, both are isomorphic to an ultrapower of $\mathcal{M}[\eta]$, *the meromorphic functions of growth of order at most* η, *with certain GVF structure arising from Nevanlinna theory, described below; this assuming also existential closure of the latter.*

Conjecture 5.2. *The model companion for GVF is stable, at least at the qf level.*

Value distribution theory

Let $\mathcal{M}$ be the field of meromorphic functions. (Or a countably generated algebraically closed subfield.) Fix a function $\eta(r)$ (say $\log(r)$ or r^d), and also an ultrafilter u on $\mathbb{R}^{>0}$, avoiding finite measure sets.

Let μ_r be the measure space on $\{a : 0 < |a| \leq r\}$ giving mass $\log(r/a)/\eta(r)$ to each point $0 < |a| < r$, and the uniform measure of mass $1/\eta(r)$ to the circle $|t| = r$. Define

$$v_a(f) = ord_a f \text{ for } |a| < r, \quad v_t(f) = -\log|f(t)|$$

$$ht_{\eta,u}(f) = \lim_{r \to u} \max(v_a f, 0) d\mu_r a$$

$$\mathcal{M}[\eta, u] = \{f \in \mathcal{M} : ht_{\eta,u}(f) < \infty\}$$

$$R_t(f_1, \ldots, f_n) := \lim_{r \to u} \int t(v_a f_1, \ldots, v_a f_n) d\mu_r a$$

.

.

The product axiom is Jensen's formula:

$$\sum_{0<|a|<r} \log \frac{r}{a} ord_a(f) + \frac{1}{2\pi} \int_0^{2\pi} - \log |f(re^{i\theta})| \, d\theta = O(1)$$

The $O(1)$ error term, divided by $\eta(r)$, goes to 0 so that we have asymptotically purely non-archimedean GVF.

In GVF language, by Theorem 1, $\mathcal{M}[\eta]$, has the same *universal* theory as the ultraproduct of the $\mathbb{Q}^a[r]$, and also as $\mathbb{C}(t)^a[1]$. This formalizes a (small!) part of Vojta's dictionary between number theory and value distribution theory, and sets a goal of formalizing more.