A model theory of heights

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Abstract

I will describe a model theoretic setting that incorporates global phenomena, such as heights, and allows for transfer of certain statements between geometry and number theory. The extent of tameness of this theory remains open, but the outlines are becoming increasingly sharp; I will survey some of the methods and results. Joint work with Itai Ben Yaacov.

1. Speculative motivation from motivic integration

In various counting theorems, notably proofs of rationality of generating series, the key is to replace numbers by (equivalence classes of) definable sets in a tame theory T. The sequence of numbers a_n is then replaced by one definable set D, evaluated in a sequence of structures A_n.

Some examples:

- X a scheme over $\mathbb{Z}_p$. b_n = number of points of $X(\mathbb{Z}/p^n\mathbb{Z})$ lifting to $X(\mathbb{Z}_p)$. Denef. Implicitly,
  T theory of p-adic Henselian valued fields with distinguished generic value group element γ $\phi = X(\mathcal{O})/\gamma\mathcal{O}$; $b_n = \phi((\mathbb{Z}_p, n))$.
- Finite fields. $T = Th(\{\mathbb{F}_p^n : n = 1, 2, \cdots\}) = PFA_p$. $\phi(\mathbb{F}_q)$. Weil, Ax, Kieffe. Kapranov.

  Denef-Loeser: Grothendieck ring = rational Chow motives.
- motivic integration (Kontsevich, Denef-Loeser, Cluckers-Loeser.)

  Takes values in Grothendieck ring of varieties / definable sets of ACF.

  Actually two theories play a role: $Th(k((t))) \rightsquigarrow ACF$ as residue field of $k((t))$.

  Essential point: k is stably embedded; induced structure = field structure.

  Contrast: $(k[t], k)$. Finite sets become uniformly definable as $\{\alpha : (x - \alpha)|a\}$.
- $a_n = \{x \in X(\mathbb{F}_p^{alg}) : \phi(x, x^{p^n})\}$. (H.-Dor) (Theory: $ACFA_{fin}$, as residue of $iVFA_{fin}$; stably embedded with natural induced structure; though here for a different reason, 'preservation of number' arguments.)

What about counting of rational points (or curves), of bounded height (degree) r?

Chambert-Loir-Loeser study such sets motivically in function field case $k(t)$, but motivically in k rather than r.

Could there be a tame theory including *heights?* $\phi(\mathbb{Q}[r])$ = points on $X(\mathbb{Q})$ of height $\leq r$?

Function field case: $f : C \to X$. E.g. distribution of $[f] \in S(H_2(X)) = H_2(X)/\mathbb{R}^+$, as $\deg(f) \to \infty$.

In any case it appears interesting develop a tame theory dealing directly with heights.

The theory of globally valued fields

Technical note on the logic

Heights, degrees are naturally real numbers:
$ht : \mathbb{P}^N(K) \to \mathbb{R}$.
$\deg : \mathbb{P}^N(K) \to \mathbb{R}$.

One could visualize generalizing $\mathbb{R}$ to other value groups (as done by H-Loeser for the local theory of Berkovich spaces.) But for the present we stick to $\mathbb{R}$.

This requires the use of continuous logic, a well-understood extension of first-order logic allowing to restrict the target of the height map to be $\mathbb{R}$, without losing compactness for bounded height regions of K.

Standard consequences of a first order axiomatization - effectiveness, compactness, transfer principles - generalize smoothly to this setting, as does stability theory.

The language

Two sorts: a field sort F; and a value sort $\mathbb{R}$.

Usual field operations $+, -, \cdot$ and relations $=, \neq$ on F.

Basic symbols R_t : A symbol R_t for each *tropical term* t = continuous, positively homogeneous function on $\mathbb{R}^n$. [1] To be interpreted as functions $(F^*)^n \to \mathbb{R}$.

Local interpretation, for an absolute value $||$ on F, $x = (x_1, \dots, x_n)$, $v(x) = (-\log |x_1|, \dots, -\log |x_n|)$.

$$R_t{}^v(x) = t(v(x))$$

Global intended intepretation: We think of $R_t(x)$ as proportional to the *expected value* $\int R_t{}^v(x) dv$ of $R_t{}^v(x)$ with respect to an implied measure on absolute values.

[1] Or just a term in the language $+, \min, 0, \alpha \cdot x$ of divisible ordered Abelian groups.

The height:

$$ht := -R_{min}$$

$$ht(x_0 : \cdots : x_n) = R_{min(u_0,\ldots,u_n)}(x_0, \ldots, x_n)$$

$ht(x) := ht(x : 1)$

Express: 'for almost all v, $s \geq s'$ as $R_t = 0$, where $t = (s' - s)^+$.

Universal axioms

1. $(F, +, \cdot)$ is an integral domain.
2. $\{R_t\}$ Compatible with permutations of variables, dummy variables.
3. (Linearity:) $R_{t_1+t_2} = R_{t_1} + R_{t_2}$. $R_{\alpha t} = \alpha R_t$.
4. (Positivity) For an affine variety $X \subset \mathbb{A}^n$: If $t(v(x)) \geq 0$ for every absolute value and every $x \in X$, then $R_t(a) \geq 0$ for $a \in X$.
5. (Product formula) $R_t = 0$ where $t(u) = u$.

Measure theoretic presentation (Gubler's M-fields) For any (countable) $K \models GVF$, there exists a measure μ on the space of absolute values of K, $v(x) = -\log|x|$, such that $(v \mapsto v(a))$ is in $L^1(\mu)$, and

$$R_t(x_1, \ldots, x_n) = \int t(v(x_1), \ldots, v(x_n)) d\mu(v)$$

Write $\int t(v(x_1), \ldots, v(x_n))$ for R_t. In particular,

$$ht(x_0 : \cdots : x_n) = -\int \min(v(x_0), \ldots, v(x_n)))$$

We have the product formula, for $0 \neq a$:

$$\int v(a) = 0$$

μ is unique up to a renormalization:
(v with mass m) $\rightsquigarrow$ ($2v$ with mass $m/2$.)

GVF extensions

Given a GVF K, an extension of the structure to $K(X)$ is determined by

- For (a.e.) each nonzero valuation v of K, a (uniquely determined) probability measure μ_v on the Berkovich space $X^{an}_{K_v}$ (or $X(\mathbb{C})$).
- a measure up to renormalization on X^{an}_K, K viewed as trivially valued.

Geometric presentation, over a constant field k Let K be a finitely generated field extension of k. Let $\mathcal{X}_K$ be the family of normal projective variety X over k with $k(X) = K$. Then these data are equivalent:

- A GVF structure on K;
- A Zariski-generic element a of some $X \in \mathcal{X}_K$.
- A compatible family of homomorphisms (for $X \in \mathcal{X}_K$)

$$h_X : NS(X) \to \mathbb{R}$$

  positive on the effective cone.
- A compatible family of elements $h_X \in N_1^+(X) \subset N_1(X) \subset H_2(X; \mathbb{R})$

$$h_X([D]) = ht_D(a) = -\int \min_{s \in \mathcal{O}_X(D)} v(s)$$

for very ample D.

This is the quantifier-free GVF picture. Allowing algebraically bounded quantifiers (e.g. $\phi(\sqrt{x})$) amounts to closing $\mathcal{X}$ under finite covers, $f : \widetilde{X} \to X$.

. Basic examples

- Natural GVF structures on $K = k(C)$, $k = k^{alg}$, C a curve $/k$; valuations = points of $C(k)$: $K[r]$ = each point of $C(k)$ has mass r.
- Similarly $\mathbb{Q}[r]$: $\mathbb{Q}_p$ has weight $r \log(p)$, while $\mathbb{R}$ has weight r.
- Unique Galois-invariant extension to finite field extensions

2nd motivation, from compactness: passage from number fields to function fields.

$\mathbb{Q}[r] = \mathbb{Q}$, with p given weight $r\log(p)$, $r > 0$.

$\bar{\mathbb{Q}}[r]$, = unique GVF extension of $\mathbb{Q}[r]$.

Consider $\overline{\mathbb{Q}}[r^*]$ for a nonstandard, infinitesimal r^*:

– *We consider only elements a of height $\leq n$ for some $n \in \mathbb{N}$.* This assures that every formula will have a finite value in $\mathbb{R}^*$. *Take standard part to obtain real values for formulas.*

Note: each $a \in \mathbb{Q}$ becomes a constant, i.e. $\int |v(a)| = 0$, i.e. $v(a) = 0$ for (almost) all v. So $\bar{\mathbb{Q}}[r^*]$ resembles a function field. Indeed $\overline{\mathbb{Q}(t)}[1] \prec_1 \bar{\mathbb{Q}}[r^*]$.

Example of (Robinson) 'transfer' from function fields to number fields.

Let A be an Abelian variety over a function field $K = \mathbb{Q}^a(C)$, $\hat{h}$ a canonical height, C a curve over $\mathbb{Q}$.

$$A_\epsilon = \{a \in A : \hat{h}(a) < \epsilon\}$$

Let X be a subvariety of A containing no translates of positive dimensional group subvarieties. The 'geometric Bogomolov conjecture' asserts for small enough $\epsilon > 0$,

$$X \cap A_\epsilon$$

consists of finitely many torsion points. It is in fact known in 'most' cases [2], by work of Cinkir, Gubler, Yamaki.

[2] when each simple factor of A has a place of bad reduction

A Abelian variety over $K = \mathbb{Q}(C)$, canonical height $\hat{h}$, $X \leq A$

$$A_\epsilon = \{a \in A : \hat{h}(a) < \epsilon\}$$

Assume $A_\epsilon \cap X(K^{alg}) = \{\tau_1, \ldots, \tau_k\}$, torsion points.

Consider the family $A_t, t \in C(\mathbb{Q}^a)$.

Proposition. *For any B, for some N, for any $t \in C(\mathbb{Q}^a)$ of large enough height, $\{a \in X_t : ht(Na) \leq Bht(a)\}$ is finite, with a bound uniform in t.*

This is related to conjectures and results of David-Philippon; a linear bound as above is known for powers of elliptic curves.

ht refers to a Weil height; one can also write $(A_{t,\epsilon ht(t)} \cap X_t)$ for some $\epsilon \sim B/N > 0$.

In particular, for $t \in C(\mathbb{Q}^a)$ of large enough height, any point of $(A_t)_{tor} \cap X_t$ is a specialization of some τ_i. The case $\dim(A) = 2$ was proved by Masser-Zannier (2015) (including t of small height!).

Proof. For almost all t we have the torsion points $\tau_1(t), \ldots, \tau_k(t)$. Suppose there exist points $t \in C(\mathbb{Q}^{alg})$ of arbitrarily large height, with a $k+1$'st torsion point $a_t \in (A_t)_{tor} \cap X_t$. Taking ultraproducts of $(\mathbb{Q}^{alg}[r], t, a_t)$ (where $r = 1/ht(t)$) we find a GVF K with an element t of height 1 and with a $k+1$'st point of height zero. This point must in fact be in $\mathbb{Q}(t)^{alg}$. As $\mathbb{Q}(t)^{alg}$ has a unique GVF structure with constants $\mathbb{Q}$ and $ht(t) = 1$, we obtain a contradiction to the choice of k.

□

Some points of contact with geometry

1. Nakai-Moishezon, Kleiman, Bertini, theory of $N_1^+(X)$: geometric presenation of GVF structures.
2. Minkowski-Harder-Narasimhan, stability and semistability; model-theoretic algebraic closure.
3. Non-archimedean Calabi-Yau, positive intersections, Bertini density of curves.
4. Lefschetz hyperplane theorems for effective cone reduction to relative curves;
5. Albanese, canonical heights; GVF canonical base change.
6. Auxiliary polynomials; qf stable embeddedness of the constants.
7. Hodge index theorem; stable embeddedness of k for algebraically bounded quantifiers.

Minkowski functions

An *adelic valuation* on $E = K^n$ is a term $u(b, x)$ (with x a variable for E, and v a variable valuation) which defines everywhere locally a K_v-norm on E.

For instance, over $\mathbb{Q}$, one can construct a term that equals $\min_{i=1}^{n} v(x_i)$ for finite v and $-\log(Mx, Mx)$ on $\mathbb{R}$; where $M = M(b) \in GL(E)$.

We restrict to the non-archimedean case. Then u_v carries the same information as the $\mathcal{O}_v$-module

$$\Lambda(u) = \{x \in E : u_v(x) \geq 0\}$$

Hence functors on modules induce functors on (adelic) valuations. In particular, $\bigwedge^n u$ is defined on the one-dimensional determinant space $\bigwedge^n E$. We define

$$\mathrm{vol}(E, u) = \int \bigwedge^{n} u_v(a)$$

$$\mathrm{slope}(E, u) = \mathrm{vol}(E, u) / \dim(E)$$

Lemma (Harder-Narasimhan). *There exists a unique maximal subspace E_{max} of (E, u) of maximal slope.*

Now suppose $u(x, b)$ depends on b; we obtain a function

$$b \mapsto E_{max} \in Gr(K^n)$$

That we call the Minkowski function associated to u. [3]

(Geometry of numbers; for any lattice in $\mathbb{R}^n$ there exists a filtration U_i of $\mathbb{R}^n$ by subspaces, and $b_i \in \Lambda \cap U_i \setminus U_{i-1}$, with $d(b_i, U_{i-1}) \geq C_n |b_i|$. This translates precisely to the N-S filtration upon taking an ultraproduct.)

[3] There are also destabilization functions for asymmetric finite field extensions, probably redundant.

Let $\widetilde{GVF}$ be the theory of all existentially closed GVF's. (Analogue of ACF for fields.)

Conjecture. *These generate the model-theoretic definable closure. Moreover* $\widetilde{GVF}$ *admits quantifier-elimination in a language with Minkowski functions.*

The proof involves in particular a canonical GVF type on E_{max}, concentrating on the Berkovich generic points of $\Lambda_v \cap E_{max}$. Existence is is closely related to the preservation of semistability under symmetric powers.

Non-archimedean Yau-type theorems and universality of curves

On a smooth Kähler variety, the Calabi Monge-Ampère operator takes metrized line bundles to volume forms. A non-archimedean analogue was developed by Kontsevich-Tschinkel and Chambert-Loir.In the 1950's, Calabi proved injectivity of the Ampère-Monge operator on smooth Kähler metrizations of a given line bundle, (up to a scalar), and conjectured surjectivity to (appropriately normalized) volume forms; this was proved by Yau in 1977. A non-archimedean version in characteristic zero appears in a recent theorem of Boucksom-Favre-Jonsson, strengthened by by Gil-Gubler-Jell-Künnemann-Martin, with antecedents in Kontsevich-Tschinkel (2001, unpublished text).

Here is a purely algebro-geometric statement.

Let k be an algebraically closed field, U a smooth projective curve over k, $\pi : X \to U$ a normal projective variety over U, $\dim(X) = n + 1$. The divisors c lying above divisors of U are the *vertical* divisors of X/U, and a measure will just be a positive real-valued function on them. Each such c has a multiplicity m_c in its fiber.

Given an ample line bundle L on X.

$$\mu_L(c) := m_c c \cdot L^n$$

We extend this to big divisors B using the positive intersection product (Demailly, $\cdots$) (zeroes of n generically chosen sections of mL, away from the base locus; renormalized limit with m.)

Proposition. *Let L be an ample line bundle on X. Let μ be a nowhere vanishing positive measure on the vertical divisors, such that the total mass of each fiber X_t is $\mu(I_t) = \deg(L)$. Then there exists a big $\mathbb{R}$-Cartier divisor B on X with generic part L, $\hat{\mu}_B = \mu$.*

We actually need an 'absolute' version:

Proposition. *Let X be a normal variety over an algebraically closed field, of dimension $n+1$. Let c be a positive linear map on $NS(X)$. Then $c = B^{<n>}$, B a big $\mathbb{R}$-Cartier divisor on X.*

Any consistent GVF formula over a constant field k is finitely satisfiable in some $k(C)$.

(They thus play the role of finite fields for schemes, or finite fields with Frobenius for difference schemes.)

In the geometric description of GVF's, must go from a curve class [c] to an irreducible curve C on X.

Problem: The class of GVFs is convex; the irreducible curves are not.

Solved by the last Proposition, and Bertini.

Example: multiplicative height 0

For $x \in \mathbb{Q}^{alg}$, $ht(x) = 0$ iff x is an *algebraic integer* and every Galois conjugate lies on unit circle. This is iff x is root of unity. (Kronecker.)

Let $\mu = \mu_{G_m}$ be the $(\bigwedge)$-definable subset of G_m defined by $ht(x) = 0$.

Theorem.

1. *The induced qf structure on μ is that of a pure group; it is qf-stably embedded. (includes Lang, Bilu.)*
2. *In the purely non-archimedean case, the induced structure on μ is that of a pure field; it is stably embedded for formulas including bounded algebraic quantifiers.*

Hodge index theorem and stable embeddedness of k for algebraically bounded quantifiers.

(An indication of the connection.)

With algebraically bounded quantifiers, can describe not only

$$h_X : NS(X) \to \mathbb{R}$$

but also $h_{\widetilde{X}} : NS(\widetilde{X}) \to \mathbb{R}$, for any finite morphism $\widetilde{X} \to X$.

Base change to $k' \geq k$ is innocuous for h_X and $h_{\widetilde{X}}$.

However, we now have new finite morphisms $\pi : \widetilde{X}' \to X$.

Stability predicts a canonical extension of $h : NS(\widetilde{X}') \to \mathbb{R}$ to $\tilde{h} : \widetilde{X}' \to \mathbb{R}$.

Geometrically, say for a surface X, a canonical extension is guaranteed by the Hodge index theorem: the kernel of $\tilde{h}'$ should vanish on the orthogonal complement to $\pi^*(NS(X))$. This uses the fact that the pullback of an ample A is ample, and modulo A, intersection is negative definite.

Show that indeed any deviation from orthogonality of the kernel implies forking.

Auxiliary polynomials and qf stability of mass in high codimension

Consider a variety X over a GVF K carrying a GVF structure. There exists a canonical base change of X from K to $L \geq K$; it assigns zero mass to new subvarieties of codimension > 1. Stability predicts that any alternative extension should 'fork'. For instance when $L = K(\alpha)$ for a constant α, if the new structure X_α assigns mass to a codimension 2 subvariety of X, then an intersection of sufficiently many distinct X_α should be empty. This indeed follows from:

Lemma. *Let $\{U_t : t \in T\}$ be an algebraic family of subvarieties of $X \subset \mathbb{P}^m$ of codimension ≥ 2. Let n be large and let $U_1, \ldots, U_n$ be elements of the famliy. Then there exists a hypersurface H of X of degree $O(\sqrt{n})$ containing all n varieties U_i.*

This is part of the proof of stable embeddedness of the constant field for qf formulas; it constrains the ability to assign positive mass to many U_i; if H is a low degree polynomial vanishing on many U_i, the zeroes of H would outbalance the poles.

Question 1. *Does there exist a polynomial f on $(\mathbb{A}^2)^n$, of degree $o(n^2)$, vanishing on all diagonals Δ_{ij}, $i < j \leq n$?*

Explicit positive answer given by Karim Adiprasito.

Question 2. *Let X, Y be affine varieties, $\Delta \leq X \times Y$ any correspondence of codimension at least 2. Let $\pi_{ij} : X^n \times Y^n \to X \times Y$ be the (i, j)-projection, and $\Delta_{ij} = \pi_{ij}^{-1}(\Delta_{ij})$. Does there exist a polynomial f on $(X \times Y)^n$, vanishing on M of the Δ_{ij} (multiplicities taken into account), with $deg(f) = o(M)$?*

Positive answer if $\dim(X) = 2$ or if $\mathrm{codim}(\Delta) \geq 3$.
This would imply that all qf formulas are stable.

Model-theoretic conjectures

Let $\widetilde{GVF}$ be the theory of all existentially closed GVF's. (Analogue of ACF for fields.)

1. $\widetilde{GVF}$ is complete, after a specification of the prime field $\mathbb{Q}[r]$ ($r > 0$), $\mathbb{Q}$ or $\mathbb{F}_p$.
2. $\widetilde{GVF}$ admits quantifier elimination in a language with Minkowski functions.
3. The classical GVF's $\mathbb{Q}[r]^a, k(t)[1]^a$ are models of $\widetilde{GVF}$.
4. Every formula of the above language is stable.

Beyond current boundaries:

- 'splitting conditions'. (Rumely). Expansions of residue field with a (Frobenius) automorphism. (reciprocity maps).
- Generalizations to higher value groups (as H.-Loeser in local case). (Widely spaced heights, as in Roth's finiteness theorem.)
- L^∞ along with L^1. (as in abc, Vojta dictionary.)

Universality of curves in transcendence degree 2

Any consistent GVF formula over a constant field k is finitely satisfiable in some $k(C)$.

Proof when $tr.deg._k L = 2$:
$L = k(S)$, S a smooth projective surface.
$Div(S)$ = linear combinations of curves on S.
Recall the intersection pairing $D_1 \cdot D_2$ on $Div(S)$.

- The GVF structure gives a linear map $m : Div(S) \to \mathbb{R}$, vanishing on on divisors (f) of $f \in L$.
- We are looking for a curve C such that $C \cap D$ is close to $m(D)$ for a given finite number of D. (This requires some preparation, blowing up S.)
- m vanishes on divisors algebraically equivalent to 0; hence it is given by intersection with a divisor e.
- As $e \cdot [D] \geq 0$ for any irreducible curve D (nef), e can be approximated by $\frac{1}{m}a$ with a very ample, $m \in \mathbb{N}$. (Nakai-Moishezon, Kleiman). In particular a is represented by an irreducible curve A.
- Let C be a smooth curve, $f : C \to A \leq X$ an m-to-one morphism; this is the point we looked for.

In higher dimension we need a theorem for nef curves, not for nef divisors. Proof based on BDPP, and Legendre duality.

For curves over $\mathbb{Q}$, all the ingredients of this proof appear to be in place (last one by F. Charles.)

That would include Rumely's results on capacity on curves, at least in a measure-theoretic approximation.