Approximate equivalence relations

and approximate homogeneous spaces

Ehud Hrushovski

Words and Groups
Jerusalem, June 2012

Occasional equivalence relations

and approximate homogeneous spaces

Ehud Hrushovski

Words and Groups
Jerusalem, June 2012

A graph is a set Ω with a symmetric binary relation R; $R(a) := \{b : R(a, b)\}$ will be assumed finite.
We define a metric
$d_R(x, y) = \min n.\ (\exists x = x_1, \ldots, x_n = y, R(x_i, x_{i+1}))$.
Say d, d' are commensurable at scale α if an α-ball of d' is contained in finitely many α- balls of d, and vice versa.
A metric space is k-doubling at scale α if $d, (1/2)d$ are commensurable at scale α. It is *k-doubling* if $d, 2d$ are k-commensurable at all scales.
Subspaces of Euclidean space are doubling.
R is a k-approximate equivalence relation if $|R(a)| \leq k|R(b)|$ and d_R is k-doubling at scale 1: every 2-ball is a union of k 1-balls.
A Riemannian homogeneous space is a Riemannian manifold, with transitive isometry group, and - for our purposes - compact point stabilizer. (Classified e.g. when the stabilizer acts irreducibly on tangent space, Wolf.)

Background

Theorem (Szemerédi's lemma)

Let $\epsilon > 0$. There exists $M \in \mathbb{N}$ such that if G is a graph, $|G| > 1/\epsilon$, there exists a partition of $(1-\epsilon)|G|$ of the points of G into $k \leq M$ sets S_i of equal size, such that for $(1-\epsilon)k^2$ pairs (i,j) , (S_i, S_j) is ϵ-regular.

The regular pairs approximate a random bipartite graph.

Theme 1: Partition so as to get highly *approximately* symmetric graphs.

Theme 2: at the infinite limit, a large automorphism group; while a random bipartite graph on n elements has a trivial automorphism group.

Szemerédi's regular graphs have diameter ≤ 1. On the other extreme, there is much work about graphs of large diameter but *bounded degree*.

We will be interested in an intermediate regime: graphs of large finite degree, and large (≥ 5) or infinite diameter. Our main results will be *modulo* bounded degree graphs.

On the other hand, given a *bounded degree* graph (Ω, S), for $1 << d << diam(X)$, have a graph (Ω, R) where $R = S^{\circ d} =$ distance d for d_S.

Theme 3 (Under a doubling assumption): recover S from R. Or at least, $S^{d'}$ for $d' << d$.

Comparison to the theory of approximate subgroups

- A large approximate group, seen from a medium distance, looks like a neighborhood of the identity in a Lie group. $X/\Gamma \subset \widetilde{G}/\Gamma$.
- A first corollary is the existence of approximate subgroups at much finer scales, with uniform doubling constant.
- Breuillard-Green-Tao (BGT) saw further, a *discrete* structure inside the limiting Lie group. $\widetilde{G}/\Gamma$. Namely there is a canonical maximal length scale; if λ is of that scale,

$$\{b\Gamma : (\exists a) a^{\lambda} = b\}$$

  is a *lattice* inside a nontrivial central subgroup of $\widetilde{G}/\Gamma$.
- BGT conclude essential *nilpotence* of $\widetilde{G}$. Thus only rare Lie groups can be seen as the symmetries of finite approximate structures *in this sense.*
- For approximate equivalence relations we will find generalizations of (1,2) and show that they are sharp; (4) is certainly not the case; existence of possible analogs of (3) is left open.

Elementary remarks

- Given a subset X of a group G, define $R_X(x, y) \iff xy^{-1} \in X$. Then X is an *approximate subgroup* of G iff R_X is an approximate equiv. relation.
- *canonical* statements about R tend to translate to statements about X. So the results we will mention generalize the corresponding ones for approx. groups.
- We will assume the class sizes $|R(a)|$ are of a fixed order of magnitude ($|R(a)| \leq k|R(b)|$), or even nearly equal for simplicity.
- An ϵ- *slice* is a set Z such that $|Z \cap R(a)| \leq \epsilon$ for any a (We are willing to ignore such a set if ϵ is small.)

The 99% - theory

- An essentially equivalent notion is of a near - equivalence relation: $|R^{\circ 3}(a)| \leq k|R(b)|$. (Then $R^{\circ 2}$ is an approx. eq. rel'n. This is the the measure-theoretic vs. purely metric definition of a *doubling condition at one scale*.)
- The case $k = 1.01$ is very easy: the usual theory of *classes* shows that $R(a), R(b)$ are almost equal, or almost disjoint: if $z \in R(x), R(y)$ then $R(x), R(y) \subset R^{\circ 2}(z)$, so $|R(x) \cap R(y)| \geq .98|R(z)|$. This shows that $E(x, y) = (\exists z)(R(x, z) \& R(y, z))$ is an equivalence relation, statistically close to the relation R.

Examples

Let $\Omega = \Omega_n$ be n points of S^1, or $S^2, \cdots$ the sphere of radius 20, chosen at random. Let the graph structure $R(x, y)$ be "$d(x, y) \leq 1$".

If $a \in \Omega$, the ball $B_1(a)$ of d-radius 1 has around $n\mathrm{vol}(B_1)/\mathrm{vol}(S^2)$ points. Similarly for B_3. So R is an approximate equivalence relation.

The automorphism group of Ω_n is trivial.

But clearly Ω_n is a highly symmetric graph.

More generally, have highly symmetric finite approximations to any homogeneous Riemannian space G/K, G a Lie group, K compact. Choose n_i points on $\Lambda_i \backslash G/K$, where $\Lambda_i \to (1)$ is a lattice, n_i large. The "distance ≤ 1" relation is an approximate equivalence relation.

Sharpening the focus

A metric $d : \Omega^2 \to \mathbb{N}$ *admits a fine structure of dimension* e, *scale* s, *distortion* c if there exists a metric [1] $d' : \Omega^2 \to 2^{-s}\mathbb{N}$, such that

- The 2^e-doubling condition holds at every scale $2^{-s}, \ldots, 1$.
- d, d' are c-commensurable, up to a $1/c$-slice.

In the S^1 example, it is easy to reconstruct a fine-scale structure (distance $1/100$ say): $B_1(a) \cap B_1(b)$ large.

[1] actually we allow $d(x, y) = 0$ without $x = y$; in other words we factor out a (precise) equivalence relation, contained entirely in $R^{\circ 4}$.

Stabilizer Theorem

Theorem

Let R be a k- approximate equivalence relation. Then there exists a graph S on the same set of vertices, such that $S^{\circ 8} \subset R^{\circ 4}$, and for all $a \in \Omega$ outside an ϵ-slice U, $|S(a)| \geq O_k(1)|R(a)|$.
Moreover S is 0-definable, uniformly in (Ω, R), in an appropriate logic; in particular $Aut(\Omega, R)$ leaves U, S invariant.

Corollary (H.,Sanders/Breuillard-Green-Tao)

Let X be a k-approximate group. Then there exists Y with $Y^{\cdot 8} \subset X^{\cdot 4}$, X contained in boundedly many cosets of Y.

Zilber, H., Pillay, Ben-Yaacov, $\cdots$, H;
Balog-Szemerédi, $\cdots$, Tao, Croot-Sisask, Sanders, BGT

(Stability) proof of stabilizer theorem

- xS_ny iff $\mu\{z : |\mu(R(x)\triangle R(z)) - \mu(R(y)\triangle R(z))| \geq 2^{-n}\} \leq 2^{-n}\}$
- At limit, $\cap_n S_n$: for almost all z, $\mu(R(x)\triangle R(z)) = \mu(R(y)\triangle R(z))$.
- $S_n \circ S_{n+1} \subset S_n$.
- $S_n \subset R^{\circ 4}$, for large n.
- S_n is definable in terms of R using a *probability logic* (Keisler.) This definability will be essential, showing that (approximate) symmetries of the graph, are (approximate) symmetries of the associated refining metric.

A locally compact limit

- Define a finer metric d, with values in $2^{-m}\mathbb{N}$: $d(x, y) = 2^{-m}$ if $S_m(x, y)$ but not $S_{m+1}(x, y)$.
- At the limit we obtain a locally compact metric space, with a locally finite measure.
- Formal construction: Take ultraproduct, factor out equivalence relation: $d(x, y)$ infinitesimal.
- Exercise: (Ω, d) a metric space, $(\Omega, 2^m d)$ commensurable with (Ω, d) at scale 1, then (Ω, d) is totally bounded so the completion is locally compact.

Summary so far

We managed to raise the resolution of a metric given at scale 1; but we lost sight of the doubling property. We next aim to show that assuming approximate symmetry, we can maintain doubling at all scales.

This requires calling upon the Gleason, Yamabe connection between locally compact groups and Lie groups: essentially, locally compact = compact - by -finite-dimensional - by-totally disconnected.

Approximate symmetry of a graph (Ω, R)

Let N be the set of graphs on $m+1$ vertices.
Given $a \in \Omega$, and $\gamma \in N$, let $C(\gamma, a)$ be the set of graph embeddings $\gamma \to \Omega$ with $0 \mapsto a$.
Define the local statistics function $LS_m : \Omega \to [0,1]^N$

$$LS_m(a)(\nu) = |C(\gamma, a)|/|\Omega|^m$$

Definition
(Ω, R) is *m, ϵ-homogeneous* if the range of LS_m is concentrated in an ϵ-ball (for sup metric on $\mathbb{R}^N$.)

- From the point of view of CS complexity, e.g. graph isomorphism problem, LS_m is computable in time polynomial in $|\Omega|$, and in computation models with random bits, even in $log(|\Omega|)$. (cf. Nati's talk?) See also Benjamini-Schramm convergence, and Lovasz-Szegedy graphon convergence, and their closely related results. (thanks to E.Breuillard and Nati Linial for these references).
- The definition of LS makes sense for infinite metric spaces, if they come with a measure (Gromov's mm spaces.) Gromov showed that two measured metric spaces (locally finite measures) with same local data, are isomorphic.
- Similarly, given two points with same local data on one mm space, an automorphism takes one to the other.
- Local homogeneity, at limit, yields group-theoretic homogeneity.
- For graphs without a doubling condition, the natural model-theoretic function measuring distance to homogenity involves games and is stronger. Lindstrom. Keisler.

Keisler, Gromov, Vershik

Proposition

Let (Ω, μ, R) be an approximate equivalence relation, with respect to a measure μ. Then up to measure 0, the completion with respect to d is determined by the local statistics of Ω.

Proof.

Suppose (Ω', μ', R') has the same local statistics.
Let (a_n) be a random sequence in Ω, and (b_n) a random sequence in Ω', with $R(a_i, a_j) \iff R'(b_i, b_j)$.
Then the map $a_n \to b_n$ is an isomorphism preserving not only R, but also any probability-logic definable relation; in particular S_n.
Hence it is an isometry.
Extend to completion.
The same holds in the pointed case; in particular if $a, b \in \Omega$ and $LS(a) = LS(b)$ then there exists an isometry with $a \mapsto b$.

□

Fix a degree of approximateness K, also a fast growing function Ψ (say 2^{2^n}).

Theorem

For some $c, e \in \mathbb{N}$, for any K- approximate equivalence relation (X, R), the fibers of $LS : X \to \mathbb{R}^N$ admit a fine structure of dimension $\leq c$, , distortion $\leq e$, and scale $\Psi(c + e)$.

- No groups in hypothesis or conclusion. But proof uses group theory. (Locally compact groups - "Hilbert 5".)
- Stronger statements about fibers: curvature
- In fact the fibers approach Riemannian homogeneous spaces G/K.
- ****But this statement only concerns cases of precise symmetry. ***work out for approximate fibers or state under approx homogeneity assumption.*****

Approximately homogeneous spaces

Fix a degree of approximateness K, also a fast growing function Ψ

Theorem

For some $c \in \mathbb{N}$, for any $(c, 1/c)$-homogeneous K- approximate equivalence relation (X, R) admits a fine structure of dimension $\leq c$,distortion $\leq c$, and scale $\Psi(c)$. In fact, any sequence of increasingly homogeneous K- approximate equivalence relation has a subsequence converging, in the sense of LS, to a Riemannian homogeneous space. At least up to compacts, it is uniquely determined by the sequence.

Proof

- Ultraproduct. Obtain two equivalence relations: $\widetilde{E}$ = finite distance. Γ = infinitesimal distance.
- Let Ω be a class of $\widetilde{E}$; then Ω/Γ is locally compact.
- $G := Aut(\Omega/\Gamma)$ acts transitively on Ω, by isometries of the fine metric. Keisler,Gromov-Vershik,
- A locally compact structure on G (compact-open topology.) The stabilizer of a point is compact.
- By Gleason-Yamabe, an open subgroup H, a small normal compact subgroup N, with H/N a Lie group.
- From Ω to an H-orbit: locally bounded distortion. (R induces a graph of bounded degree on Ω/H.)

Proof (contd)

- Factor out N. Obtain a coarser equivalence relation than the original distance-zero, but still contained in $d_R \leq 4$.
- Now the Lie group H/N acts transitively on Ω/Γ, compact point stabilizer. Find an invariant Riemannian metric. This metric is doubling up to distance 1, and the "distance -1" relation is commensurable with d_R.
- Return information to finite factors, up to scale $\Psi(c)$.

Theorem (Benjamini- Finucane-Tessera 2012)

1. *Let* (X_n) *be an unbounded sequence of finite, connected, vertex transitive graphs with bounded degree such that* $|X_n| = o(diam(X_n)^q)$ *for some* $q > 0$. *After rescaling by the diameter, some subsequence converges in the Gromov Hausdorff distance to a torus of dimension* $< q$, *with an invariant metric.*
2. *If* q *is close to* 1, *then the scaling limit of* (X_n) *is* S^1, even if X_n is only roughly transitive

Approximate ...

- *Sub*groups (Tao): $X \subset G$, $1 \in X = X^{-1}$, $XX \subset \cup_{i=1}^{k} a_i X$.
- Equivalence relations $\subset \Omega^2$: $Id_\Omega \subset R = R^t$, $R \circ R(c) \subset \cup_{i=1}^{k} R(a_i)$
- Sub*categories*
- Groups ?
- In this talk, *approximate homogeneous spaces*.

Problem (Gromov, *Ergo*?)

Define, and describe the structure of, approximate categories.