From algebraic integers to global fields

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A few intersections.

1. (January 1986, Oberwolfach.) ACF definable groups are algebraic.

2. From pseudo-finite fields to ACFA and fields with automorphisms.

3. (courses 1981 Yale, 2011 UCLA, Goldbring) Hilbert 5. A connected locally compact group is pro-Lie (Gleason, Yamabe, Montgomery-Zippin; discrete version by Breuillard, Green, Tao.)

4. (1988) The theory of the algebraic integers.

Theorem 1 (van den Dries)**.** *The theory of the algebraic integers is decidable.*

Theorem 2 (van den Dries-Macintyre)**.** *So is the analogue for global fields of positive characteristic.*

Elements of proof.

$K = \mathbb{Q}^a$ or $K = F_p(t)^a$.
$\mathcal{O}$ = integral closure of $\mathbb{Z}$ or $F_p[t]$.
$= \{a \in K : (\forall v)(v(a) \geq 0)$.

1. The space X of maximal ideals of $\mathcal{O}$, a locally compact space; compact above each rational prime.

2. X is accessible to the logic via the clopen subsets = finitely generated nonzero radical ideals = 2-generated radical ideals.

3. These form a definable family B_{fin}; it is a maximal ideal in a Boolean algebra B.

4. For $p \in X$, $\mathcal{O}_p$ is the valuation ring of $K_p \models ACVF$, the theory of nontrivially valued algebraically closed fields.

5. For any formula $\phi(x_1, \ldots, x_n)$ of the language of valued fields, and any $a_1, \ldots, a_n \in K$, a 'boolean truth value'

$$[\phi(a_1, \ldots, a_n)] \in B$$

defined as:

$$\{p : K_p \models \phi(a_1, \ldots, a_n)\}$$

6. Theory is axiomatized by: $[ACVF] = 1$ or by: ACF + $[VAL] = 1$, and characteristic. ($(0, p)$ with nonzero truth value for each p in arithmetic case, (p, p) in geometric case.)

It can thus be described as the Booleanization of ACVF relative to the theory of fields [1].

7. The validity of these axioms in the specific structure $\widetilde{\mathbb{Z}}$ uses a deep theory of Rumely, that we will return to later. Actually here a qualitative version of Moret-Bailly suffices.

8. A similar theory is available if any completion of the global field is omitted, but at least one must be. (E.g. totally real algebraic numbers, such that $2^n x$ is integral for some n.) In this case, when the archimedean primes are included, adjustments must be made allowing absolute values, i.e. allow ACVF' whose models include $((\mathbb{C}, +, \cdot, v))$, $v(x) = -\log|x|$. I will pass over these adjustments in silence when they occur.

[1] with $\mathrm{ACVF}_{0,0}$'s at a distinguished point of X.

Some questions:

1. What about geometric case, $\mathbb{C}[t]$?

2. What about *all* primes?

   A lot of tame mathematics is not visible at the level of a single place or even 'all but one'. E.g.

   (a) Finite dimensionality of spaces of holomorphic maps, or sections, on compact Riemann surfaces;

   (b) Irreducible lattices in $SL_n(\mathbb{Q}_2) \times SL_n(\mathbb{R})$(Discreteness of $G(\mathbb{Q})$ in $G(\mathbb{A})$.)

   (c) quadratic reciprocity.

The case of $\mathbb{C}[t]$ and the entire algebraic functions.

Let $\mathcal{O}$ be the ring of entire algebraic functions. So $\mathcal{O}$ is the integral closure of $\mathbb{C}[x]$ in $\mathbb{C}(x)^{alg}$.

Theorem 3 ([2])**.** *$\mathcal{O}$ interprets $(\mathbb{Q}, +, \cdot)$.*

Consider family of smooth curves $C = (C, j)$ along with finite morphisms $j : C \to \mathbb{C}$.)

$$X = \varprojlim C$$

As before the algebra of clopen subsets of X is interpretable.

[2] w. Dupuis.

Let $D(C)$ be the group of maps with finite support $C \to \mathbb{Z}$. (Divisors on C.)

For $f \in \mathbb{C}(C)$ we have $(f) \in D(C)$, the principal divisor $(f)(p) = val_p(f)$.

$$J(C) := D(C)/\{(f) : f \in \mathbb{C}(C)$$

$=$ Jacobian of completion $\bar{C}$ of C, modulo the differences $a - b$ of two elements mapping to ∞.

Natural map $D(C) \to J(C)$

Let D, J be the direct limits[3] over all curves C. These are partially ordered $\mathbb{Q}$-vector spaces.

Interpretation of D, J in $\mathcal{O}$. Any positive function is the minimum of two principal positive elements.

For $b \in B_{fin}$, we have $D(b)$, the elements supported on b; these are sections of a sheaf and the model theory is well-understood.

[3]for appropriate maps $D(C) \to D(C')$, taking ramification into account.

Let $J(b)$ be the image in J. Let C be a curve of genus g.

Lemma. *Let $a, b \in B_{fin}(C) \smallsetminus (0)$, $|a + b + C_\infty| \geq 2g$. Then*

$$J(a) \cap J(b) \neq (0)$$

This uses curves C' whose definition depends on a, b; it would not be true for a direct limit over a static index set.

On the other hand, if we consider only elements $J_{C'/C}(a)$ of $J(a)$ arising from a given curve C' (images in J of elements of $J(C')$ mapping to $0 \in J(C_1)$) we can see the opposite phenomenon:

Lemma. *Assume $p : C' \to C_1$ is defined over k_0. Let $a \in C_1(k_0)$, $b \in C_1(k) \smallsetminus C_1(k_0)^{alg}$, $c = a - b$. Then*

$$J_{C'/C}(\{a\}) \cap (J(\{b, c\}) = (0)$$

Lemma. *Let $c \in C_1$. If $c' \in J$ and c' is supported on b whenever c is, then $c' \in \mathbb{Q}c$.*

This leads to an intepretation of the field $\mathbb{Q}$.

What about *all* primes?

1. Julia Robinson. The totally real algebraic integers are undecidable. Via finite subsets of roots of 1: $(\forall v)(v(x) = 0)$ + additional condition.

2. In function field case, given access to all valuations, the same formula interprets the *field of constants* k. But here we have an exercise: *Let* R *be an integral domain,* k *an infinite subfield,* $t \in R$ *such that* $t - \alpha$ *is not invertible for* $\alpha \in k$. *Then* $(R, k, +, \cdot)$ *is undecidable.*

But the 'tame', purely global geometries mentioned earlier suggest that (1,2) should not be the last word.

In the rest of this talk I will discuss an attempt at the *algebraically closed case*. We take essentially the minimal reasonable language capable of expressing the product formula. Thus it is a global version of Robinson's ACVF, and not yet of Ax-Kochen's Henselian fields. Also, no *discriminants*. Work in progress with Ben Yaacov.

New universal law, but *infinitary* in nature: the product formula, generalizing Eucilid's theorem on unique decomposition into primes: $n = \pm\Pi_p p^{v_p(n)}$, or

$$-\log|n| = \sum_p v_p(n)$$

To formulate it, we move (at least implicitly) to $L_1(X, \mathbb{R})$ in place of $C(X, 2)$ as our algebraic model of X.

Continuous logic will be used, but lightly: we will have a field K, functions $K^n \to \mathbb{R}$, and the use of continuous logic amounts to demanding that $\mathbb{R}$, viewed as a sort, remain the standard model. The main, field sort will be discrete.

1 The language

The terms in the field sort are polynomials over $\mathbb{Z}$; equality is a $\{0,1\}$-valued relation as usual. We also consider tropical terms on $\mathbb{R}$, namely terms in the language $+, \min, 0, \alpha \cdot x$ of divisible ordered Abelian groups.

Basic relations R_t: A symbol R_t for each tropical term $t = t(x_1, \ldots, x_n)$ to be interpreted as functions $(F^*)^n \to \mathbb{R}$.

Local interpretation of R_t Let (K, v) be a valued field, or a subfield of $\mathbb{C}$ with $v(x) = -\alpha \log |x|$. For x with $x_i \neq 0$, interpret ${R_t}^v(x)$ as $t(vx_1, \ldots, vx_n)$.

Global intended intepretation: We think of $R_t(x)$ [4] as an *expected value* of $t(v(x))$ with respect to an implied measure on valuations.

Among them, the Weil height: for $x = (x_0 : \cdots : x_n) \in \mathbb{P}^n$,

$$ht(x) = -\int \min_i v(x_i) dv$$

[4] or rather of ratios between them

Height has a structural role in the definition of quantifiers and limits of GVF's, but I will not go into this here.

Connectives $\min, \max, 0, +, \alpha \cdot x$.

Quantifiers The analogue of quantifiers in real-valued logic is inf and sup operators. Let $\psi_{n,\epsilon}(t)$ be 1 on $[-n, n]$, 0 on $|t| > n + \epsilon$, and a linear interpolation on $[n, n + \epsilon]$. Let $\phi(x, y)$ be a formula. Then so is $\sup_x \psi_{n,\epsilon}(ht(x))\phi(x, y)$.

We view this as a quantifier over x of height up to about n.

All formulas are preserved by ultrapowers.

2 GVF axioms

1. $(F, +, \cdot)$ is an integral domain.

2. The R_t are compatible with permutations of variables and dummy variables.

3. (Linearity:) $R_{t_1+t_2} = R_{t_1} + R_{t_2}$. $R_{\alpha t} = \alpha R_t$.

4. (Local-global positivity) For $\phi(x_1, \ldots, x_n) \in L_{rings}$ [5] and t a term in $+, -, \min$ such that

$$VF \models (\forall x)(\phi(x) \implies t(v(x_1), \ldots, v(x_n)) \geq 0$$

an axiom:

$$\phi(a_1, \ldots, a_n) \implies R_t(a_1, \ldots, a_n) \geq 0$$

5. (Product formula) $R_x = 0$

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[5] implying $x_1, \ldots, x_n \neq 0$

Axioms 1-4 are similar to the Boolean local/global axioms; but Boolean truth values are replaced by 'expectations'. The local-global axiom now states that everywhere locally positive implies positive expectation.

From now on we will write $\int t(vx_1, \ldots, vx_n)dv$ in place of $R_t(x_1, \ldots, x_n)$.

(This is in fact justified at the level of models.)

(Product formula) $\int v(x)dv = 0$

Implicitly, the algebraic avatar of X has shifted from $C(X, 2)$ to $(L^1(X), +, \min)$. Henson et al have shown this, in itself, to be tame.

3 Classical structures

$\mathbb{Q}, k(t)$ and their algebraic extensions have natural GVF structures. There is also one arising asymptotically from Nevanlinna theory. In the case of $k(t)$, valuations over k = points of $\mathbb{P}^1(k)$; take the measure giving each point mass 1, and interpret the formulas naturally. The product formula then states that rational functions have as many poles as zeroes.

Let C be a curve over a field $k = k^{alg}$, and $K = k(C)$; let $\alpha > 0$. Define:

$$R_t(x_1, \ldots, x_n) = \alpha \sum_{p \in C(k)} t(v_p(x_1), \ldots, v_p(x_n))$$

This makes $K = k(C)$ into a GVF, with k the field of constants (and there is no other way.)

Similarly for number fields K, with appropriate weights.

At the limit, K^a has a unique GVF structure over k, up to a scalar renomalization.

Incidentally, this is not true for all GVF's K; but it is true for K such that K^a is e.c. There is always a unique Galois-invariant extension to K^a. .

Conjecture 3.1. *Let* $K = \mathbb{Q}$ *or* $K = \mathbb{F}_p(t)$. *Then* K^a *is existentially closed in the GVF language.*

Further conjectures, not discussed today:

1. The theory of GVF's admits a model companion; in other words, the class of existentially closed GVF's is axiomatizable.
2. The class of GVF's admits amalgamation over structures A that are qf-algebraically closed within some existentially closed extension. NB: GVF is not algebraically bounded.
3. The theory of GVF's is stable, at least at the qf level.

Definition. M *is an* existentially closed *model of* T *if for any structure* $N \geq M$, *with* $N \models T$, *basic formulas* $\phi_i(X,Y) \in L$, $(i = 1,\ldots,l)$, $\epsilon > 0$, *and any* b *from* M *and* a *from* N, *there exists* a' *from* M *with* $|\phi_i(a,b) - \phi_i(a',b)| < \epsilon$, $i = 1,\ldots,l$.

Theorem 4. *For any field* k, $\mathrm{k}(t)^a$ *is existentially closed as a GVF.*

Three corollaries of existential closedness.

Corollary. *Automatic effectiveness, Lefschetz principle.*

We exemplify this by means of Cinkir's theorem; it is one of a number of known cases of the Bogomolov conjecture for function fields. (Gubler, Yamaki; Ullmo and Zhang for number fields.

Let $f : C \to U$ be a non-isotrivial, generically smooth family of projective curves of genus ≥ 2 over $k = \mathbb{Q}^a$, $K = k(C)$, and embed C_K canonically in its Jacobian A_K (via some rational point.)

Theorem 5 (Cinkir). *For some $b \in \mathbb{N}, h > 0$, $C(K^a)$ has $\leq b$ points of canonical height $\leq h$ on A.*

Equivalently: for any h, for some n, $C(K^a)$ has $\leq b$ points $a_1, \ldots, a_b$ such that na_i has *Weil height* $\leq h$ on A.

Automatic effectivity

An algorithm guaranteed to produce h, b and (the degree of) these points p_i. ('search for $a_1, \ldots, a_b$ *and* a proof from GVF that no further solutions exist.')

Lefschetz principle char. 0 $\leftrightarrow$ large char. p

By Theorem 4, the same is true for $\mathbb{F}_p(U)$, for almost all primes p.

Lefschetz principle, towards number fields

There exists $b \in \mathbb{N}, \epsilon > 0$ such that if $t \in U(\mathbb{Q}^a)$ has large enough height h, $C = C_t$, then $C(\mathbb{Q}^a)$ has at most b points a with

$$ht_{can}(a) \leq \epsilon h$$

This follows from Theorem 4 and Cinkir's theorem, though neither mentions number fields at all. To prove it one uses the family of GVFs $\mathbb{Q}^a[r]$; they are just renormalizations of $\mathbb{Q}^a$:

$$R_t(f_1, \ldots, f_n)^{\mathbb{Q}^a[r]} := \frac{1}{r} R_t(f_1, \ldots, f_n)^{\mathbb{Q}^a}$$

In $\mathbb{Q}^a[r]$, the height of 2 is $\log(2)/r$; the ultraproducts are purely non-archimedean GVFs.

To prove the corollary, we suppose it is false; then there exists a sequence of curves $C_i = C_{t_i}$, $t_i \in U(\mathbb{Q}^a)$ of height r_i, and at least i distinct points $a_{i,j} \in C_i(\mathbb{Q}^a)$ with $ht_{can}(a_{i,j}) \leq \epsilon r_i$. Then in $\mathbb{Q}^a[r]$, a_i has height 1, and $ht_{can}(a_{i,j}) \leq \epsilon$. It follows that in the non-archimedean GVF in $K = \mathbb{Q}(x)^a$ there exists a curve $C = C_t$ with $t \in U(K)$ of height 1, so C_t is not isotrivial, and with as any points of $C(K)$ as desired, of arbitrarily small canonical height; this contradicts Cinkir's theorem. (In fact one can even find a sequence $a_j \in C(K)$, such that $ht_{can}(a_j) \to 0$, using transitivity of $Aut(K)$ on height-1 elements of $\mathbb{P}^1$)

An arithmetic-geometric curve selection theorem

Let X be a smooth projective variety over $\mathbb{Q}$; let Y, $Y_1, \ldots, Y_m$ be subschemes.

For $x \in X(\mathbb{Q}^a)$ there is a standard definition of an adelic distance $\delta(x, Y)$ from Y (Schmidt, Faltings-Wusztholz, Vojta...). Product of local distances smaller than one.

On $\mathbb{A}^1$, the distance from 0 is essentially $e^{-ht(x)}$.

Note that $-\log \delta(x, Y)$ is the value of a certain quantifier-free formula $\phi_Y(x)$ in the language of GVF's.

Proposition. *Assume $a_n \in X(\mathbb{Q}^a)$ form a Zariski dense sequence of unbounded height, with $\lim_{n\to\infty} \delta_{Y_k}(a_n)/ht(a_n) = e_k$; let $\epsilon > 0$. Then there exists a curve C on X (not lying on Y_i) such that for any sequence $a'_n \in C(\mathbb{Q}^a)$, $ht(a'_n) \to \infty$, we have $\lim_{n\to\infty} |\delta_{Y_k}(a'_n)/ht(a'_n) - e_k| < \epsilon$.*

The proof in fact concerns the intersection of C with each Y_k, normalized by the degree of C. In particular, if algebraic points of exponential height H have minimal adelic distance $\sim r/H$ fro Y, then there exist curves of degree d with $\sim rd$ intersection points with Y.

Proof. Choose $r_i = ht(2)/ht(a_i)$ so that $\mathbb{Q}^a[r_i]$ gives a_i height 1. Consider any non-principal ultrafilter u on the index set $\mathbb{N}$, and let (L, a) be the GVF ultraproduct of $(\mathbb{Q}^a[r_i], a_i)$. Then (L, a) is a purely non-archimedean GVF, and $\delta_{Y_k}(a) = \phi(a)^L = e$. There exists $a' \in K = k(t)^a$ with $e' = \phi(a')^K$ satisfying $|e' - e| < \epsilon$. In fact $a' \in k(C)$ for some curve C, so a' corresponds to a morphism $g : C \to X$. We may choose a' so that $g(C)$ avoids any given proper subvariety of X. By computing the meaning of ϕ in $k(t)^a$ we see that $\bar{i}(C, Y_k) = e'$.

Conversely, if C is a curve on X defined over $\mathbb{Q}^a$, then for any sequence of distinct $a_i \in C(\mathbb{Q}^a)$ of bounded degree over $\mathbb{Q}$, $\delta_Y(a) \to i_Y(C)$. This follows upon taking normalized ultraproducts as above, from the uniqueness of the GVF structure on $k(C)$. $\square$

In particular, there exists such a sequence a'_i of bounded degree over $\mathbb{Q}$.

Distributional Fekete-Szegő

Fekete-Szegő (1953) asked: *When does a compact subset C of $\mathbb{C}$ contain infinitely many Galois orbits of algebraic integers?* (Polya, Schur 1918 for intervals: iff length ≥ 4.)

This, they did not succeed in answering, but they gave a beautiful answer to a toplogical relaxation of the question: *There exists a sequence of Galois orbits, whose Hausdorff limit is an infinite subset of C*, if and only if C has *capacity* ≥ 1.

The *capacity* can be defined in several ways, including the Chebyshev number and the transfinite diameter.

Modern formulation (Cantor, Rumely, Chambert-Loir (Berkovich spaces)): [6]

[6] The Berkovich space is just the quantifier-free type space of the theory of $\mathbb{R}$-valued fields.

Let $A = C \times \Pi_{p \neq \infty} \widehat{\mathcal{O}_p}$, a compact affine subset of the adelic Berkovich space of $\mathbb{P}^1$ over $\mathbb{Q}$. Does there exist a sequence of Galois orbits approaching A?

In general for compact affine subset of the adelic Berkovich space, the *Chebyshev constant* $\mathrm{Ch}(A)$ is defined by:

$$-\log \mathrm{Ch}(A) = \lim_{d \to \infty} \frac{1}{d} \sup_{\deg(f) \leq d} \Sigma_p \inf_{v \in A_p} v(f)$$

This describes the asymptotic volume of the smallest adelic enveloping cube, with sides described by degree d polynomial inequalities.

This formulation immediately extends to global fields K, and to varieties X other than $\mathbb{P}^1$. Rumely fully generalized the theory to *curves.*[7]. Several of the definitions of capacity have been generalized to higher dimensions (Chinburg 1991, ..., Chinburg-Moret-Bailly-

[7] In the $\mathbb{C}_p$-formulation

Pappas-Taylor 2012), with some implications for Galois orbits, but no sharp characterization so far.

Here we will look at a measure-theoretic formulation: we do not ask whether *all* points of the Galois orbit are in a neighborhood of C, but *almost all* (a fraction approaching 1). For this we obtain a sharp characterization in all dimensions.

Given a finite set S, the characteristic measure is the probability measure giving equal mass to each point of S.

Let K be a function field; so that K^a is existentially closed as a GVF. A an adelic set.

Theorem 6. *Let X be a projective variety, A a compact affine subset of X as above. Then there exists a Zariski dense sequence of Galois orbits approaching A* distributionally *iff* $\mathrm{Ch}(A) \geq 1$.

Proof: the Chebyshev number condition allows a soft construction of a GVF extension $L \cong K(X)$, $a \in X(L)$, with $\int t(v(a))dv = 0$ for t supported away from A. By Theorem 4, there exist approximations a_i to a within K. Now integrating t over all valuations above a place p of K = fixing one v_p above p and summing over the Galois orbit.

Remark on the transfinite capacity.

Proof of Theorem 4

1. Reduction to equations *over the constant subfield.* This is a standard model-theoretic lemma, using the large automorphism group of K^*. *approximate* automorphisms of $K = k(x)^a$, e.g. with $x \mapsto x^{1+1/100}$, become automorphisms at the limit. This works also over $\bar{Q}[1]$, but is not known to me for $\bar{Q}[r]$ for nonconstant r.)

2. Geometric description of formulas.

X a normal projective variety over $k = k^a$. $K = k(X)$.

Given a very ample Cartier divisor H on X, consider the associated projective embedding; recall that the Weil height with respect to this embedding is given by a formula:

$$ht_H = \int -\min(v(s_0), \ldots, v(s_m))dv$$

where $s_0, \ldots, s_m \in K$ are a basis of the linear system of H $((s_i)+H \geq 0.)$

This gives a pairing:
(GVF structures on K/k) $\times$ (very ample divisors on X) $\to \mathbb{R}$.

For a fixed GVF structure p on K, $H \mapsto ht^p_H$, is a linear map on $Pic_{\mathbb{R}}(X)$, positive on the effective cone. From this it follows that it must vanish on the bounded subgroup $Pic^0(X)$.

Let $N_1(X)$ be the $\mathbb{R}$-space generated by the curves on X, up to numerical equivalence.

$$N_1^+(X) = \{c \in N_1(X) : (\forall \text{effective Cartier } H)(c, H) \geq 0\}$$

We have just described a map from GVF structures on K/k to $N_1^+(X)$.

3. Dually, for any H, $p \mapsto c(p) \cdot H$ is described by a formula, and if H is allowed to range over Cartier divisors of blowups, such formulas generate all. (Given the fixed field structure.)

$N^1(X) \rightsquigarrow \{formulas\}, \quad D \mapsto \phi_D$.

4. On the other hand, points of X in K are given by irreducible curves on X. To approximate a given structure, these curves need to avoid a hypersurface, and approach the class of the structural 1-cycle $c \in N_1^+(X)$. We are thus led to the following problem:

5. (*) Multiples of irreducible curves on Zariski open sets, are dense in the nef 1-cycles.

From this point on, the problem is purely geometric.

Example: smooth projective surfaces. (*) follows from: nef divisors are approximated by ample divisors. Here Nakai-Moishezon + Bertini irreducibility suffice. In the case of arithmetic surfaces, does an ample line bundle have a pluri-section whose poles form a single Galois orbit?

Higher dimensional case: nef 1-cycles approximated by A^{n-1}, A ample on a blowup.

A theorem of Boucksom-Demailly-Pau-Peternell 2004 (in char. 0) asserts that the *convex hull* of such divisors is dense. A proof using positive intersection products is given in BFJ. We use the same methods, along with Legendre duality, to obtain:

Theorem 7. *Let X be a normal variety over an algebraically closed field. Let c be in the open cone of curves dual to the pseudo-effective cone of X. Then c is the increasing limit of cycles A^{n-1}, A ample on a blowup of X.*

In fact $c = B^{<n-1>}$, B a big $\mathbb{R}$-Cartier divisor on X, $< n-1 >$ the 'positive intersection product'.

Along with Bertini, this gives (*).

Big divisors

X a normal projective variety over k, of dimension n.

We consider $\mathbb{R}$-Cartier divisors; $D = \sum \alpha_i D_i$, $\alpha_i \in \mathbb{R}$. If each $\alpha_i \geq 0$, write $D \geq 0$, D effective.

An rational function $f \in K = k(X)$ is a *section* of D if $(f) + D \geq 0$. $L(mD)$ is the space of sections of mD.

An effective divisor D is *big* if some $L(mD)$ has algebraically independent sections.

Positive intersection product

We define $< D >^k$ with $k = n$ or $k = n - 1$.

Let $O \subset X$ be Zariski open. Let $m \in \mathbb{N}$.

Let $s_1, \ldots, s_k$ be generic sections of $L(mD)$. Let Z be the Zariski closure of their common zero locus in O. (A curve if $k = n - 1$, a 0-dimensional scheme if $k = n$.) The class $[Z]$ in $N_{n-k}(X)$ does not depend on the generic choice, and decreases with O but stabilizes for small enough O. Define $< X >_m^{<k>} = \frac{[Z]}{m}$.

$$< D >^k := \lim_{m \to \infty} < X >_m^{<k>}$$

Remark: taking $k = n$, Demailly (1993) gave this definition of *volume*, i.e. the leading coefficient of the section growth function.

BFJ showed vol is differentiable, and

$$\mathrm{vol}'(D) \cdot H = \lim_{t \to 0} \frac{\mathrm{vol}(D + tH) - \mathrm{vol}(D)}{t} = n < D >^{n-1} \cdot H$$

Also, $\mathrm{vol}^{1/n}$ is concave on the big cone.

This was initially in char. 0, but using Okounkov's methods, it is easy to obtain the same in all characteristics.

Theorem 2 follows using a version of Legendre duality, concerning the derivative of a concave function.

Non-archimedean Yau-type theorems

On a smooth Kähler variety, the Monge-Ampère operator takes metrized line bundles to volume forms.

A non-archimedean analogue was developed by Kontsevich-Tschinkel and Chambert-Loir.It maps metrized line bundles to measures on Berkovich space.

The general definition uses a limit procedure; here I will discuss only the purely geometric level, which is easily defined.

Let k be an algebraically closed field, U a smooth projective curve over k, $\pi : X \to U$ a normal projective variety over U, $\dim(X) = n+1$. The divisors c lying above divisors of U are the *vertical* divisors of X/U, and a measure will just be a positive real-valued function on them. Each such c has a multiplicity m_c in its fiber.

Let L be an ample line bundle on X.

$$\mu_L(c) := m_c c \cdot L^n$$

We extend this to big divisors B using the positive intersection product:

$$B \mapsto \hat{\mu}_B$$

$$\hat{\mu}_B(c) = m_c c \cdot < B >^n$$

In the 1950's, Calabi proved injectivity of the Ampère-Monge operator on smooth Kähler metrizations of a given line bundle, (up to a scalar), and conjectured surjectivity to (appropriately normalized) volume forms; this was proved by Yau in 1977.

A non-archimedean version in characteristic zero appears in a recent theorem of Boucksom-Favre-Jonsson, with antecedents in Kontsevich-Tschinkel (2001, unpublished text). They obtain uniform convergence to general semi-positive metrized line bundles, but ask for additional information when beginning with a model measure.

With a little adjustment, Theorem 7 implies (but does not seem to follow from) a relative version:

Theorem 8. *Let L be an ample line bundle on X. Let μ be a nowhere vanishing positive measure on the vertical divisors, such that the total mass of each fiber X_t is $\mu(I_t) = \deg(L)$. Then there exists a big $\mathbb{R}$-Cartier divisor B on X with generic part L, $\hat{\mu}_B = \mu$.*

Remarks:

Assume X is smooth, and let v be a valuation of $K = k(U)$ over k. Then μ determines a measure on the Berkovich space of X over (K, v); the theorem implies that there exists a metrized line bundle, positive increasing limit of ones arising from nef models, whose Monge-Ampere measure is μ. The same follows for any measure on the Berkovich space with total mass prescribed as above, and with a certain non-vanishing condition.

B can be taken to be *quasi-free*, i.e. determined as a supremum of sections of mB.

One can add a multiple of a divisor arising from U (analogous to scalar multiples in Calabi.) I do not know other sources of non-uniqueness (for quasi-free B.)

Remark. The *topological* Fekete-Szegő theorem amounts to finding an irreducible curve, close to a given ray of nef 1-cycles, and at the same time orthogonal to a certain finite set of irreducible Weil divisors. In the case of curves over a function field, the set of divisors in question can be contracted in a morphism to an Artin algebraic space, probably giving another proof of Rumely's function-field Fekete-Szegő, and showing that the distributional and topological conditions coincide.

Value distribution theory

Let $\mathcal{M}$ be the field of meromorphic functions. (Or a countably generated algebraically closed subfield.) Fix a function $\eta(r)$ (say $\log(r)$ or r^d), and also an ultrafilter u on $\mathbb{R}^{>0}$, avoiding finite measure sets.

Let μ_r be the measure space on $\{a : 0 < |a| \leq r\}$ giving mass $\log(r/a)/\eta(r)$ to each point $0 < |a| < r$, and the uniform measure of mass $1/\eta(r)$ to the circle $|t| = r$. Define

$$v_a(f) = ord_a f \text{ for } |a| < r, \quad v_t(f) = -\log|f(t)|$$

$$ht_{\eta,u}(f) = \lim_{r \to u} \max(v_a f, 0) d\mu_r a$$

$$\mathcal{M}[\eta, u] = \{f \in \mathcal{M} : ht_{\eta,u}(f) < \infty\}$$

$$R_t(f_1, \ldots, f_n) := \lim_{r \to u} \int t(v_a f_1, \ldots, v_a f_n) d\mu_r a$$

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The product axiom is Jensen's formula:

$$\sum_{0<|a|<r} \log \frac{r}{a} ord_a(f) + \frac{1}{2\pi} \int_0^{2\pi} - \log |f(re^{i\theta})| \, d\theta = O(1)$$

The $O(1)$ error term, divided by $\eta(r)$, goes to 0 so that we have asymptotically purely non-archimedean GVF.

In GVF language, by Theorem 4, $\mathcal{M}[\eta]$, has the same *universal* theory as the ultraproduct of the $\mathbb{Q}^a[r]$, and also as $\mathbb{C}(t)^a[1]$. This formalizes a (small!) part of Vojta's dictionary between number theory and value distribution theory, and sets a goal of formalizing more.