Continuous-Time Mean–Risk Portfolio Selection

Jin Hanqing

Based on joint work with

Prof. Harry Markowitz, Prof. Xun Yu Zhou and Prof. Yan Jia An

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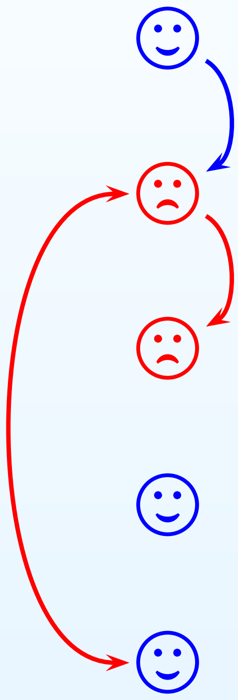
  Equivalent condition
- How about the single period mean-semivariance?
  Optimal solution existing



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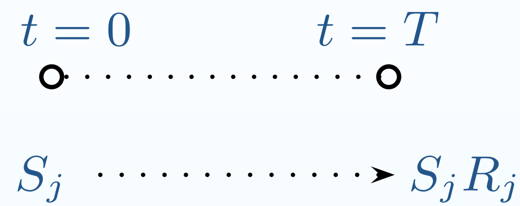

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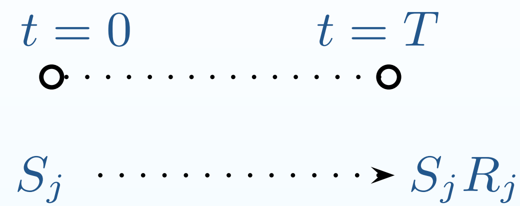

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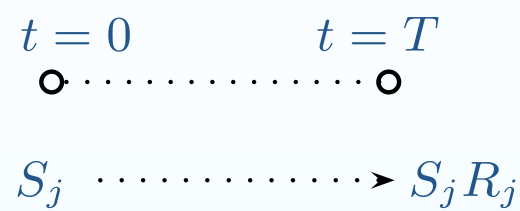

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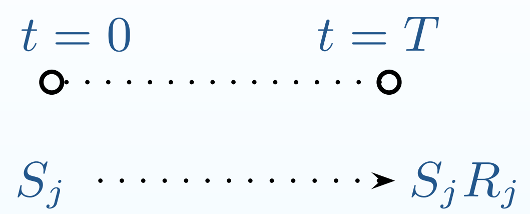

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$$\text{Mimimize} \quad \text{Var}(\textstyle\sum_j \pi_j R_j) = \sum_{i,j=1}^m \pi_i \sigma_{ij} \pi_j \qquad \text{(risk)}$$

$$\text{subject to} \quad \begin{cases} \sum_i \pi_i = x_0 & \text{(budget constraints)} \\ E(\sum_j \pi_j R_j) = \sum_i r_i \pi_i = z & \text{(targeted payoff)} \\ [\pi_i \geq 0 & \text{(no shorting)}] \end{cases}$$

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- Given expectation return level, minimizing the risk

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- *Semivariance* proposed where only the return below its mean or a target level counted as risk (Markowitz 1959: *"semivariance seems more plausible than variance as a measure of risk"*)
- Generalization of semivariance: *Downside risk* (Fishburn 1977, Sortino and van der Meer 1991)

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- A *bond (or bank account)* whose price process $S_0(t)$ satisfies

$$dS_0(t) = r(t)S_0(t)dt, \; t \in [0,T]; \quad S_0(0) = s_0 > 0,$$

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- m *stocks* whose price processes $S_1(t), \cdots S_m(t)$ satisfy stochastic differential equation (SDE)

$$\begin{cases} dS_i(t) = S_i(t)\big\{\mu_i(t)dt + \sum_{j=1}^m \sigma_{ij}(t)dW^j(t)\big\}, \; t \in [0,T]; \\ S_i(0) = s_i > 0, \end{cases}$$

where $\mu_i(t)$: *appreciate rate*; $\sigma_{ij}(t)$: *volatility* (*dispersion*) *rate*

Wealth and Portfolio

- Define *covariance matrix* $\sigma(t)$ and *excess rate of return process* $B(t)$ by

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- $\pi(\cdot)$ is called *admissible* if it is $\mathcal{F}_t$-adapted and $E\int_0^T |\pi(t)|^2 dt < +\infty$.
- An investor's *wealth process* $x(t)$ follows *wealth equation*

$$\begin{cases} dx(t) &= [r(t)x(t) + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t) \\ x(0) &= x_0 \end{cases} \qquad (1)$$

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- Problem (2) is a dynamic optimization problem

A Static Problem

- Define $\theta(t) = \sigma(t)^{-1}B(t)$ and $\rho(t) := e^{-\int_0^t [r(s)+|\theta(s)|^2/2]ds - \int_0^t \theta(s)dW(s)}$

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- The market is complete. *i.e.*, $\forall\, \xi \in L^2(\mathcal{F}_T, \mathrm{R})$, there exists an admissible wealth-portfolio pair $(x(\cdot), \pi(\cdot))$ such that $x(T) = \xi$ (replication).

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Consider the following *static* optimization problem:

$$\begin{array}{ll} \text{Minimize} & Ef(X - EX), \\ \text{subject to} & \left\{ \; EX = z; \quad E[\rho(T)X] = x_0; \quad X \in L^2(\mathcal{F}_T, \mathbf{R}) \right. \end{array} \tag{3}$$

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Problem: Solve the static problem (3)

Weighted Mean-Variance Model

Let $f(x) = \alpha x_+^2 + \beta x_-^2$, where $\alpha > 0, \beta > 0$.

- $Ef(X(T) - EX(T))$ the weighted variance of $X(T)$

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Changing variable $Y := X - z$, the static problem (3) specializes to

$$\begin{array}{ll} \text{Minimize} & E[\alpha Y_+^2 + \beta Y_-^2] \\ \text{subject to} & \begin{cases} EY = 0 \\ E[Y\rho] = y_0 \\ Y \in L^2(\mathcal{F}_T, \mathbf{R}) \end{cases} \end{array}$$

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where $\rho := \rho(T)$ and $y_0 := x_0 - zE\rho$.

Introducing two *Lagrange multipliers* (λ, μ) for the two constraints, one needs only to solve

$$\min_{Y \in L^2(\mathcal{F}_T, \mathbf{R})} E[\alpha Y_+^2 + \beta Y_-^2 - 2(\lambda - \mu\rho)Y] \tag{4}$$

And then determine (λ, μ) by the two constraints $EY = 0, E[Y\rho] = y_0$.

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Theorem 2: *The unique optimal portfolio for the weighted MV problem corresponding to (x_0, z) is the one to replicate the contingent claim*

$$X^* = \frac{(\lambda - \mu\rho)_+}{\alpha} - \frac{(\lambda - \mu\rho)_-}{\beta} + z$$

with (λ, μ) being the unique solution to the system of algebraic equations

$$\frac{E(\lambda - \mu\rho)_+}{\alpha} - \frac{E(\lambda - \mu\rho)_-}{\beta} = 0, \qquad \frac{E[\rho(\lambda - \mu\rho)_+]}{\alpha} - \frac{(\lambda - \mu\rho)_-}{\beta} = x_0 - zE\rho.$$

Moreover, the minimum risk value is given by $J^*(x_0, z) = -\mu(x_0 - zE\rho)$.

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Remark 1: When the market coefficients are deterministic, the optimal portfolio can be obtained more explicitly via some *Black-Scholes* type equation.

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- Define

$$\rho_0 := \inf\{\eta \in \mathbf{R} : P(\rho < \eta) > 0\} \qquad \rho_1 := \sup\{\eta \in \mathbf{R} : P(\rho > \eta) > 0\}$$

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- View the mean-semivariance problem as the limiting problem of the weighted MV problem with $\beta = 1 - \alpha$ and $\alpha \to 0$
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Remark 3: Although the mean-semivariance problem in general does not admit optimal solutions, the infimum of the problem has been obtained explicitly, which is $\frac{y_0^2}{E(\rho-\rho_0)^2}$ if $y_0 < 0$ and $\frac{y_0^2}{E(\rho_1-\rho)^2}$ if $y_0 > 0$. Moreover, asymptotically optimal portfolios can be obtained by replicating $Y(\alpha) + z$ as $\alpha \to 0$.

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Idea of proof: Find out the infimum $\inf Ef(Y)$, and show it is not attainable.

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Question: When the optimal solutions exist for a general mean-risk problem?

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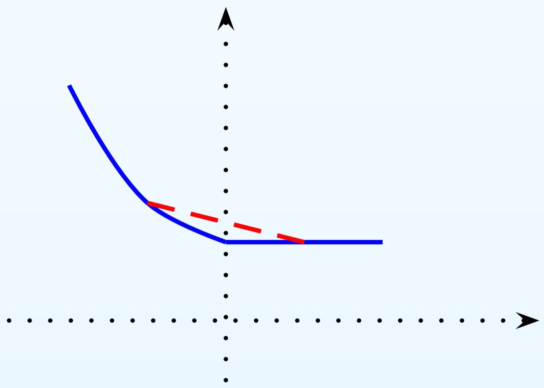
i.e. $kf(x) + (1-k)f(y) > f(0)$ for any $k \in (0,1), kx + (1-k)y = 0$)



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$$\begin{cases} g(y) := \text{argmin}_{x \in \mathbf{R} : y \in \partial f(x)} |x| \\ \Lambda := \{\lambda \in \mathbf{R} : \exists \mu = \mu(\lambda) \in \mathbf{R} \text{ so that } g(\lambda - \mu(\lambda)\rho) \in L^2(\mathcal{F}_T, \mathbf{R}), \\ \qquad\qquad Eg(\lambda - \mu(\lambda)\rho) = 0, \rho g(\lambda - \mu(\lambda)\rho) \in L^1(\mathcal{F}_T, \mathbf{R})\ \} \\ \tilde{g}(\lambda) = E[\rho g(\lambda - \mu(\lambda)\rho)],\ \forall \lambda \in \Lambda \\ \bar{\lambda} = \sup_{\lambda \in \Lambda} \lambda, \quad \underline{\lambda} = \inf_{\lambda \in \Lambda} \lambda \\ \underline{y} := \lim_{\lambda \uparrow \bar{\lambda}} \tilde{g}(\lambda), \quad \bar{y} := \lim_{\lambda \downarrow \underline{\lambda}} \tilde{g}(\lambda) \end{cases}$$

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Remark 5: All these definition can be calculated offline

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Solution of General MR problem (Cont'd)

(iii) *Assume that either* $\cup_{x\in \boldsymbol{R}} \partial f(x) = (-\infty, \bar{k}]$ *or* $\cup_{x\in \boldsymbol{R}} \partial f(x) = (-\infty, \bar{k})$ *for some* $\bar{k} \in \boldsymbol{R}$. *Then* $\underline{\lambda} = 0 \in \Lambda$. *If* $\bar{\lambda} \notin \Lambda$, *then the problem admits an optimal solution* ***if and only if*** $x_0 - zE\rho \in (\underline{y}, 0]$. *If* $\bar{\lambda} \in \Lambda$, *then the problem admits an optimal solution* ***if and only if*** $x_0 - zE\rho \in \{\tilde{g}(\bar{\lambda})\} \cup (\underline{y}, 0]$. *If in addition* $\bar{\lambda} < \bar{k}$, *then* $\tilde{g}(\bar{\lambda}) = \underline{y}$

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f is strictly convex, $\cup_{x \in \mathbf{R}} \partial f(x) = (-\infty, 0)$.(Case (iii)). The MR problem admits an optimal solution iff $x_0 - zE\rho \in [(E\rho)(E\ln\rho) - E(\rho\ln\rho), 0]$ or, equivalently, $z \in [\frac{x_0}{E\rho}, \frac{x_0 - (E\rho)(E\ln\rho) + E(\rho\ln\rho)}{E\rho}]$. When the problem does admit an optimal solution, the optimal portfolio is the one replicating the claim $z - \ln(-\lambda + \mu\rho)$, where (λ, μ) is the unique solution pair to the following algebraic equation (which must admit a solution):

$$\begin{cases} E\ln(-\lambda + \mu\rho) = 0 \\ E[\rho\ln(-\lambda + \mu\rho)] = zE\rho - x_0 \end{cases}$$

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  It can be shown:

  - The mean-risk portfolio selection problem with risk function f_α must admit an optimal solution
  - The corresponding optimal portfolio π_α is asymptotically optimal for the original problem when $\alpha \downarrow 0$

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- Concentrate on numerical solution (as analytical solution impossible) and comparison with mean-variance model

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- Concentrate on numerical solution (as analytical solution impossible) and comparison with mean-variance model
- Existence of efficient portfolios/frontier not addressed

Back to Single Period

- Continuous-time mean-semivariance problem admits no optimal solution. How about its single period counterpart?
- $m(\geq 2)$ securities, each with return rate R_j, $ER_j = r_j$, $\mathrm{Cov}(R_i, R_j) = \sigma_{ij}$
- The single period Mean-semivariance problem

$$\begin{array}{ll} \text{Mimimize} & E[(\sum_j \pi_j R_j - E\sum_j \pi_j R_j)_-]^2 \\ \text{subject to} & \begin{cases} \sum_i \pi_i = x_0 \\ E(\sum_j \pi_j R_j) = \sum_i r_i \pi_i = z \end{cases} \end{array} \tag{6}$$

- Vast literature on single-period mean-semivariance models
- Concentrate on numerical solution (as analytical solution impossible) and comparison with mean-variance model
- Existence of efficient portfolios/frontier not addressed
- Technically non-trivial as the feasible region generally unbounded, and the objective not coercive

  ($f : \mathbf{R}^d \mapsto \mathbf{R}$ called *coercive* if $\lim_{|x|\to+\infty} f(x) = +\infty$)

A Lemma

Consider

$$\min_{x \in \mathbf{R}^d} E[(A + B'x)_-]^2, \tag{7}$$

where $B \equiv (B_1, \cdots, B_d)'$, and A, B_i are random variables with $EA^2 < +\infty$, $EB_i^2 < +\infty, \;\; i = 1, \cdots, d$.

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Counter example: Let $A = -1, B = (e^{W_1}, \cdots, e^{W_d})'$, where $(W_1, \cdots, W_d)$ follow $N(0, I_d)$. For any $0 \neq x \in \mathbf{R}_+^d$, $\lim_{\alpha\to+\infty} E[(A+B'(\alpha x))_-]^2 = 0$. This implies the optimal value of (7) is 0. However, this value cannot be achieved since $E[(A+B'(\alpha x))_-]^2 > 0$ for any $x \in \mathbf{R}^d$.

Existence for Single-Period M-S

Theorem 6: For any $x_0 \in \mathbf{R}$ and $z \in \mathbf{R}$, Problem (7) admits optimal solutions if and only if it admits feasible solutions.

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Idea of Proof. Let $\xi_i = R_i - r_i$. After eliminating π_1 and π_2 from the constraints, one gets the following equivalent problem

$$\min_{(\pi_3,\cdots,\pi_m)\in\mathbf{R}^{m-2}} E[(A + \sum_{i=3}^{m} \pi_i B_i)_-]^2,$$

where

$$A = x_0\xi_1 + \frac{z - x_0 r_1}{r_2 - r_1}(\xi_2 - \xi_1), \quad B_i = \xi_i - \xi_1 - (r_i - r_1)\frac{\xi_2 - \xi_1}{r_2 - r_1}.$$

Then Lemma 3 applies.

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Then Lemma 3 applies.

Sharply contrast Continuous-time mean-semivariance Vs Single-period mean-semivariance

Further Research

- *Incomplete* market (the replicating problem becomes significant)

  - Some work has been done on the mean-variance problem. ...

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- Other risk measures: safety first, Var, minimax, ...

END

Comments and questions are appreciated ... ...

Thank you very much!