Continuous-Time Mean–Risk Portfolio Selection

Jin Hanqing

Based on joint work with
Prof. Harry Markowitz, Prof. Xun Yu Zhou and Prof. Yan Jia An
Outline

• We will study the continuous-time optimal portfolio selection problem in a mean-risk framework
Outline

• We will study the continuous-time optimal portfolio selection problem in a mean-risk framework

• Weighted-mean-variance problem. Explicit solution
Outline

• We will study the continuous-time optimal portfolio selection problem in a mean-risk framework

• Weighted-mean-variance problem.
  Explicit solution

• Mean-semivariance problem.
  No optimal solution
Outline

- We will study the continuous-time optimal portfolio selection problem in a mean-risk framework

- Weighted-mean-variance problem.
  Explicit solution

- Mean-semivariance problem.
  No optimal solution

- Mean-downside-risk problem.
  No optimal solution
Outline

- We will study the continuous-time optimal portfolio selection problem in a mean-risk framework
- Weighted-mean-variance problem.
  Explicit solution
- Mean-semivariance problem.
  No optimal solution
- Mean-downside-risk problem.
  No optimal solution
- When a general mean-risk problem admits optimal solutions?
  Equivalent condition
Outline

• We will study the continuous-time optimal portfolio selection problem in a mean-risk framework

• Weighted-mean-variance problem.  
  Explicit solution

• Mean-semivariance problem.  
  No optimal solution

• Mean-downside-risk problem.  
  No optimal solution

• When a general mean-risk problem admits optimal solutions?  
  Equivalent condition

• How about the single period mean-semivariance?  
  Optimal solution existing
Markowitz’s (Original) Model

- Single period

$t = 0 \quad t = T$
Markowitz’s (Original) Model

- Single period
  
- \( m(\geq 2) \) securities, each with return rate \( R_j \)
Markowitz’s (Original) Model

- Single period
- \( m(\geq 2) \) securities, each with return rate \( R_j \)
- \( ER_j = r_j, \text{Cov}(R_i, R_j) = \sigma_{ij} \)
Markowitz’s (Original) Model

- Single period
- \( m(\geq 2) \) securities, each with return rate \( R_j \)
- \( ER_j = r_j, \text{Cov}(R_i, R_j) = \sigma_{ij} \)
- An agent with fund \( x_0 \), and a targeted expected payoff \( z \) at the end of the investment period
Markowitz’s (Original) Model

- Single period
  - \( t = 0 \) to \( t = T \)

- \( m( \geq 2) \) securities, each with return rate \( R_j \)
  - \( S_j \) to \( S_j R_j \)

- \( ER_j = r_j, \text{ Cov}(R_i, R_j) = \sigma_{ij} \)

- An agent with fund \( x_0 \), and a targeted expected payoff \( z \) at the end of the investment period

- To find a portfolio \( \pi = (\pi_1, \cdots, \pi_m) \) so as to

\[
\begin{align*}
\text{Minimize} & \quad \text{Var}(\sum_j \pi_j R_j) = \sum_{i,j=1}^m \pi_i \sigma_{ij} \pi_j \quad \text{(risk)} \\
\text{subject to} & \quad \sum_i \pi_i = x_0 \quad \text{(budget constraints)} \\
& \quad E(\sum_j \pi_j R_j) = \sum_i r_i \pi_i = z \quad \text{(targeted payoff)} \\
& \quad [\pi_i \geq 0] \quad \text{(no shorting)}
\end{align*}
\]
Markowitz’s (Original) Model

• Single period

\[ t = 0 \quad t = T \]

• \( m(\geq 2) \) securities, each with return rate \( R_j \)

\[ S_j \quad \cdots \cdots \quad \rightarrow S_j R_j \]

\[ E(R_j) = r_j, \text{Cov}(R_i, R_j) = \sigma_{ij} \]

• An agent with fund \( x_0 \), and a targeted expected payoff \( z \) at the end of the investment period

• To find a portfolio \( \pi = (\pi_1, \cdots, \pi_m) \) so as to

\[
\text{Minimize} \quad \text{Var}(\sum_j \pi_j R_j) = \sum_{i,j=1}^{m} \pi_i \sigma_{ij} \pi_j \quad \text{(risk)}
\]

\[
\begin{cases}
\sum_i \pi_i = x_0 \quad \text{(budget constraints)} \\
E(\sum_j \pi_j R_j) = \sum_i r_i \pi_i = z \quad \text{(targeted payoff)} \\
[\pi_i \geq 0 \quad \text{(no shorting)}]
\end{cases}
\]

• Given expectation return level, minimizing the risk
Risk Measures

• *What is risk?* Chance of bad consequences (Oxford Dictionary)
Risk Measures

- *What is risk?* Chance of bad consequences (Oxford Dictionary)
- A subjective notion as opposed to return
Risk Measures

- *What is risk?* Chance of bad consequences (Oxford Dictionary)
- A subjective notion as opposed to return
- *Variance / covariance* used to measure risk by Markowitz (1952)
Risk Measures

- **What is risk?** Chance of bad consequences (Oxford Dictionary)
- A subjective notion as opposed to return
- *Variance / covariance* used to measure risk by Markowitz (1952)
- Criticisms on using variance include
  - penalty on upside return
  - weight on upside and downside equal whereas asset return distribution generally asymmetric
Risk Measures

- *What is risk?* Chance of bad consequences (Oxford Dictionary)
- A subjective notion as opposed to return
- *Variance / covariance* used to measure risk by Markowitz (1952)
- Criticisms on using variance include
  - penalty on *upside* return
  - weight on upside and downside *equal* whereas asset return distribution generally asymmetric
- *Semivariance* proposed where only the return below its mean or a target level counted as risk (Markowitz 1959: “*semivariance seems more plausible than variance as a measure of risk*”)
Risk Measures

- **What is risk?** Chance of bad consequences (Oxford Dictionary)
- A subjective notion as opposed to return
- *Variance / covariance* used to measure risk by Markowitz (1952)
- Criticisms on using variance include
  - penalty on upside return
  - weight on upside and downside equal whereas asset return distribution generally asymmetric
- *Semivariance* proposed where only the return below its mean or a target level counted as risk (Markowitz 1959: “*semivariance seems more plausible than variance as a measure of risk*”)
- Generalization of semivariance: *Downside risk* (Fishburn 1977, Sortino and van der Meer 1991)
A Continuous-Time Market

- A market in which $m + 1$ securities (assets) traded continuously
A Continuous-Time Market

- A market in which \( m + 1 \) securities (assets) traded continuously

- Market randomness described by a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) along with an \( m \)-dimensional, \( \mathcal{F}_t \)-adapted standard Brownian motion \( W(t) = (W^1(t), \ldots, W^m(t))' \) with \( \{\mathcal{F}_t\}_{t \geq 0} \) generated by \( W(\cdot) \)
A Continuous-Time Market

- A market in which \( m + 1 \) securities (assets) traded continuously

- Market randomness described by a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) along with an \( m \)-dimensional, \( \mathcal{F}_t \)-adapted standard Brownian motion \( W(t) = (W^1(t), \cdots, W^m(t))' \) with \( \{\mathcal{F}_t\}_{t \geq 0} \) generated by \( W(\cdot) \)

- A bond (or bank account) whose price process \( S_0(t) \) satisfies

\[
dS_0(t) = r(t)S_0(t)dt, \quad t \in [0, T]; \quad S_0(0) = s_0 > 0,
\]

where \( r(\cdot) \): interest rate
A Continuous-Time Market

• A market in which \( m + 1 \) securities (assets) traded continuously

• Market randomness described by a complete filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) along with an \( m \)-dimensional, \( \mathcal{F}_t \)-adapted standard Brownian motion \( W(t) = (W^1(t), \cdots, W^m(t))' \) with \( \{\mathcal{F}_t\}_{t \geq 0} \) generated by \( W(\cdot) \)

• A bond (or bank account) whose price process \( S_0(t) \) satisfies
  \[
  dS_0(t) = r(t)S_0(t)dt, \quad t \in [0, T]; \quad S_0(0) = s_0 > 0,
  \]
  where \( r(\cdot) : \text{interest rate} \)

• \( m \) stocks whose price processes \( S_1(t), \cdots S_m(t) \) satisfy stochastic differential equation (SDE)
  \[
  \begin{cases} 
  dS_i(t) = S_i(t)\{\mu_i(t)dt + \sum_{j=1}^{m} \sigma_{ij}(t)dW^j(t)\}, \quad t \in [0, T]; \\
  S_i(0) = s_i > 0,
  \end{cases}
  \]
  where \( \mu_i(t) : \text{appreciate rate}; \quad \sigma_{ij}(t) : \text{volatility (dispersion) rate} \)
Wealth and Portfolio

- Define **covariance matrix** \( \sigma(t) \) and **excess rate of return process** \( B(t) \) by

\[
\sigma(t) := (\sigma_{ij})_{m \times m}, \quad B(t) := (\mu_1(t) - r(t), \ldots, \mu_m(t) - r(t))'
\]
Wealth and Portfolio

• Define **covariance matrix** $\sigma(t)$ and **excess rate of return process** $B(t)$ by

$$
\sigma(t) := (\sigma_{ij})_{m \times m}, \quad B(t) := (\mu_1(t) - r(t), \ldots, \mu_m(t) - r(t))'
$$

• **Basic Assumption:**
  ◦ $r(\cdot), B_i(\cdot), \sigma_{ij}(\cdot)$ are all **uniformly bounded**, $\mathcal{F}_t$-adapted stochastic process
  ◦ $\sigma(t)\sigma(t)' \geq \delta I$ for some $\delta > 0$
Wealth and Portfolio

- Define covariance matrix $\sigma(t)$ and excess rate of return process $B(t)$ by
  \[ \sigma(t) := (\sigma_{ij})_{m \times m}, \quad B(t) := (\mu_1(t) - r(t), \ldots, \mu_m(t) - r(t))' \]

- Basic Assumption:
  - $r(\cdot), B_i(\cdot), \sigma_{ij}(\cdot)$ are all uniformly bounded, $\mathcal{F}_t$-adapted stochastic process
  - $\sigma(t)\sigma(t)' \geq \delta I$ for some $\delta > 0$

- $\pi(t) = (\pi_1(t), \ldots, \pi_m(t))'$, where $\pi_i(t)$ is the capital amount invested in stock $i$, is called (monetary) portfolio
Wealth and Portfolio

- Define *covariance matrix* $\sigma(t)$ and *excess rate of return process* $B(t)$ by
  $$\sigma(t) := (\sigma_{ij})_{m \times m}, \quad B(t) := (\mu_1(t) - r(t), \cdots, \mu_m(t) - r(t))^\prime$$

- Basic Assumption:
  - $r(\cdot), B_i(\cdot), \sigma_{ij}(\cdot)$ are all uniformly bounded, $\mathcal{F}_t$-adapted stochastic process
  - $\sigma(t)\sigma(t)^\prime \geq \delta I$ for some $\delta > 0$

- $\pi(t) = (\pi_1(t), \cdots, \pi_m(t))^\prime$, where $\pi_i(t)$ is the capital amount invested in stock $i$, is called (monetary) *portfolio*

- $\pi(\cdot)$ is called *admissible* if it is $\mathcal{F}_t$-adapted and $E \int_0^T |\pi(t)|^2 dt < +\infty$. 
Wealth and Portfolio

- Define *covariance matrix* $\sigma(t)$ and *excess rate of return process* $B(t)$ by
  
  $$
  \sigma(t) := (\sigma_{ij})_{m \times m}, \quad B(t) := (\mu_1(t) - r(t), \ldots, \mu_m(t) - r(t))'
  $$

- **Basic Assumption:**
  - $r(\cdot), B_i(\cdot), \sigma_{ij}(\cdot)$ are all *uniformly bounded*, $\mathcal{F}_t$-adapted stochastic process
  - $\sigma(t)\sigma(t)' \geq \delta I$ for some $\delta > 0$

- $\pi(t) = (\pi_1(t), \ldots, \pi_m(t))'$, where $\pi_i(t)$ is the *capital amount* invested in stock $i$, is called (monetary) *portfolio*

- $\pi(\cdot)$ is called *admissible* if it is $\mathcal{F}_t$-adapted and $E \int_0^T |\pi(t)|^2 dt < +\infty$.

- An investor’s *wealth process* $x(t)$ follows *wealth equation*

  $$
  \left\{ \begin{array}{l}
  dx(t) = [r(t)x(t) + B(t)\pi(t)]dt + \pi(t)'\sigma(t)dW(t) \\
  x(0) = x_0
  \end{array} \right.
  $$

  (1)
Continuous-time Mean-risk Problem

Continuous-time mean–Risk portfolio selection is formulated as the following optimization problem parameterized by a pair of scalar \((x_0, z)\):
Continuous-time Mean-risk Problem

Continuous-time mean–Risk portfolio selection is formulated as the following optimization problem parameterized by a pair of scalar \((x_0, z)\):

Minimize 

\[
J_{x_0,z}(\pi(\cdot)) := Ef(x(T) - Ex(T))
\]

subject to

\[
\begin{align*}
Ex(T) &= z, \\
(x(\cdot), \pi(\cdot)) &\text{ admissible pair,} \\
x(0) &= x_0
\end{align*}
\]  

(2)
Continuous-time Mean-risk Problem

Continuous-time mean–Risk portfolio selection is formulated as the following optimization problem parameterized by a pair of scalar \((x_0, z)\):

\[
\begin{align*}
\text{Minimize} & \quad J_{x_0,z}(\pi(\cdot)) := Ef(x(T) - Ex(T)) \\
\text{subject to} & \quad \begin{cases} \\
Ex(T) = z, \\
(x(\cdot), \pi(\cdot)) \text{ admissible pair,} \\
x(0) = x_0
\end{cases}
\end{align*}
\]

where \(f : \mathbb{R} \mapsto \mathbb{R}\) is a given risk function.
Continuous-time Mean-risk Problem

Continuous-time mean–Risk portfolio selection is formulated as the following optimization problem parameterized by a pair of scalar \((x_0, z)\):

Minimize \( J_{x_0,z}(\pi(\cdot)) := Ef(x(T) - Ex(T)) \)

subject to

\[
\begin{align*}
Ex(T) &= z, \\
(x(\cdot), \pi(\cdot)) &\text{ admissible pair,} \\
x(0) &= x_0
\end{align*}
\]

where \( f : \mathbb{R} \mapsto \mathbb{R} \) is a given risk function.

- The continuous-time Markowitz model: \( f(x) = x^2 \)
  
  studied extensively recently (Zhou and Li 2000, Lim and Zhou; Lim 2004; Heunis and Labbe 2004; Bielecki, Jin, Pliska and Zhou 2005 ......)
Continuous-time Mean-risk Problem

Continuous-time mean–Risk portfolio selection is formulated as the following optimization problem parameterized by a pair of scalar \((x_0, z)\):

\[
\begin{align*}
\text{Minimize} & \quad J_{x_0, z}(\pi(\cdot)) := Ef(x(T) - Ex(T)) \\
\text{subject to} & \quad \begin{cases}
Ex(T) = z, \\
(x(\cdot), \pi(\cdot)) \text{ admissible pair,} \\
x(0) = x_0
\end{cases}
\end{align*}
\]

where \(f : \mathbb{R} \mapsto \mathbb{R}\) is a given risk function.

- The continuous-time Markowitz model: \(f(x) = x^2\)
  - studied extensively recently (Zhou and Li 2000, Lim and Zhou; Lim 2004; Heunis and Labbe 2004; Bielecki, Jin, Pliska and Zhou 2005 ......)
- Problem (2) is a dynamic optimization problem
A Static Problem

- Define $\theta(t) = \sigma(t)^{-1} B(t)$ and $\rho(t) := e^{-\int_0^t [r(s) + |\theta(s)|^2 / 2] ds - \int_0^t \theta(s) dW(s)}$
A Static Problem

- Define $\theta(t) = \sigma(t)^{-1} B(t)$ and $\rho(t) := e^{-\int_0^t [r(s) + |\theta(s)|^2/2] ds - \int_0^t \theta(s) dW(s)}$
- Wealth equation is equivalent to $x(t) = \rho(t)^{-1} E[x(T)\rho(T)|\mathcal{F}_t]$
- The market is complete. i.e., $\forall \xi \in L^2(\mathcal{F}_T, \mathbb{R})$, there exists an admissible wealth-portfolio pair $(x(\cdot), \pi(\cdot))$ such that $x(T) = \xi$ (replication).
A Static Problem

- Define $\theta(t) = \sigma(t)^{-1} B(t)$ and $\rho(t) := e^{-\int_0^t [r(s) + |\theta(s)|^2 / 2] ds - \int_0^t \theta(s) dW(s)}$

- Wealth equation is equivalent to $x(t) = \rho(t)^{-1} E[x(T)\rho(T)|F_t]$

- The market is complete.

- Budget constraint becomes $x_0 = E[x(T)\rho(T)]$
A Static Problem

- Define $\theta(t) = \sigma(t)^{-1} B(t)$ and $\rho(t) := e^{-\int_0^t [r(s) + |\theta(s)|^2 / 2] ds - \int_0^t \theta(s) dW(s)}$
- Wealth equation is equivalent to $x(t) = \rho(t)^{-1} E[x(T) \rho(T) | \mathcal{F}_t]$
- The market is complete.
- Budget constraint becomes $x_0 = E[x(T) \rho(T)]$

Consider the following static optimization problem:

Minimize $Ef(X - EX)$,
subject to $\left\{ EX = z; \quad E[\rho(T)X] = x_0; \quad X \in L^2(\mathcal{F}_T, \mathbb{R}) \right\}$  \hspace{1cm} (3)
A Static Problem

- Define $\theta(t) = \sigma(t)^{-1} B(t)$ and $\rho(t) := e^{-\int_0^t [r(s) + \theta(s)^2 / 2] ds - \int_0^t \theta(s) dW(s)}$
- Wealth equation is equivalent to $x(t) = \rho(t)^{-1} E[x(T)\rho(T)|\mathcal{F}_t]$
- The market is complete.
- Budget constraint becomes $x_0 = E[x(T)\rho(T)]$

Consider the following static optimization problem:

Minimize $Ef(X - EX)$,
subject to $\left\{ \begin{array}{l} EX = z; \quad E[\rho(T)X] = x_0; \quad X \in L^2(\mathcal{F}_T, \mathbb{R}) \end{array} \right.$ \hspace{1cm} (3)

**Theorem 1:** If $X^*$ is the optimal solution for the static optimization problem (3), then the optimal portfolio for the mean-risk problem (2) is the one to replicate $X^*$. On the other hand, if $(x^*(\cdot), \pi^*(\cdot))$ is the optimal wealth-portfolio pair, then $X^* = x^*(T)$. 
A Static Problem

- Define $\theta(t) = \sigma(t)^{-1} B(t)$ and $\rho(t) := e^{-\int_0^t [r(s) + |\theta(s)|^2/2] ds - \int_0^t \theta(s) dW(s)}$
- Wealth equation is equivalent to $x(t) = \rho(t)^{-1} E[x(T)\rho(T)|\mathcal{F}_t]$
- The market is complete.
- Budget constraint becomes $x_0 = E[x(T)\rho(T)]$

Consider the following static optimization problem:

\[
\begin{align*}
\text{Minimize} & \quad Ef(X - EX), \\
\text{subject to} & \quad \begin{cases} 
EX = z; \\
E[\rho(T)X] = x_0; \\
X \in L^2(\mathcal{F}_T, \mathbb{R})
\end{cases}
\end{align*}
\]  

(3)

**Theorem 1:** If $X^*$ is the optimal solution for the static optimization problem (3), then the optimal portfolio for the mean-risk problem (2) is the one to replicate $X^*$. On the other hand, if $\left(x^*(\cdot), \pi^*(\cdot)\right)$ is the optimal wealth-portfolio pair, then $X^* = x^*(T)$.

**Problem:** Solve the static problem (3)
Let $f(x) = \alpha x_+^2 + \beta x_-^2$, where $\alpha > 0, \beta > 0$.

- $Ef(X(T) - EX(T))$ the weighted variance of $X(T)$
Let \( f(x) = \alpha x_+^2 + \beta x_-^2 \), where \( \alpha > 0, \beta > 0 \).

\[ Ef(X(T) - EX(T)) \] the weighted variance of \( X(T) \)

Changing variable \( Y := X - z \), the static problem (3) specializes to

Minimize \[ E[\alpha Y_+^2 + \beta Y_-^2] \]

subject to

\[ \begin{cases} 
  EY = 0 \\
  E[Y \rho] = y_0 \\
  Y \in L^2(F_T, \mathbb{R})
\end{cases} \]

where \( \rho := \rho(T) \) and \( y_0 := x_0 - zE\rho \).
Weighted Mean-Variance Model

Let \( f(x) = \alpha x_+^2 + \beta x_-^2 \), where \( \alpha > 0, \beta > 0 \).

- \( Ef(X(T) - EX(T)) \) the weighted variance of \( X(T) \)

Changing variable \( Y := X - z \), the static problem (3) specializes to

Minimize \( \mathbb{E}[\alpha Y_+^2 + \beta Y_-^2] \)

subject to

\[
\begin{align*}
\mathbb{E}Y &= 0 \\
\mathbb{E}[Y \rho] &= y_0 \\
Y &\in L^2(\mathcal{F}_T, \mathbb{R})
\end{align*}
\]

where \( \rho := \rho(T) \) and \( y_0 := x_0 - z \mathbb{E}\rho \).

Introducing two Lagrange multipliers \( (\lambda, \mu) \) for the two constraints, one needs only to solve

\[
\min_{Y \in L^2(\mathcal{F}_T, \mathbb{R})} \mathbb{E}[\alpha Y_+^2 + \beta Y_-^2 - 2(\lambda - \mu \rho)Y]
\]

(4)

And then determine \( (\lambda, \mu) \) by the two constraints \( \mathbb{E}Y = 0, \mathbb{E}[Y \rho] = y_0 \).
**Solution to Weighted MV Problem**

**Lemma 1:** The optimal solution of Problem (4) is \( Y^* = \frac{(\lambda - \mu \rho)_+}{\alpha} - \frac{(\lambda - \mu \rho)_-}{\beta} \).
Solution to Weighted MV Problem

**Lemma 1:** The optimal solution of Problem (4) is 
\[ Y^* = \frac{(\lambda - \mu \rho)_+}{\alpha} - \frac{(\lambda - \mu \rho)_-}{\beta}. \]

**Lemma 2:** For any \( y_0 \), there exists a unique pair \( (\lambda, \mu) \) such that the optimal solution \( Y^* \) in Lemma 1 satisfies \( EY^* = 0 \), \( E[Y^* \rho] = y_0 \). Moreover, 
\( \lambda < 0, \mu < 0 \) if \( y_0 > 0 \); \( \lambda > 0, \mu > 0 \) if \( y_0 < 0 \); \( \lambda = \mu = 0 \) if \( y_0 = 0 \).
Solution to Weighted MV Problem

Lemma 1: The optimal solution of Problem (4) is 
\[ Y^* = \frac{(\lambda - \mu \rho)_+}{\alpha} - \frac{(\lambda - \mu \rho)_-}{\beta}. \]

Lemma 2: For any \( y_0 \), there exists a unique pair \((\lambda, \mu)\) such that the optimal solution \( Y^* \) in Lemma 1 satisfies \( EY^* = 0, E[Y^* \rho] = y_0 \). Moreover, \( \lambda < 0, \mu < 0 \) if \( y_0 > 0 \); \( \lambda > 0, \mu > 0 \) if \( y_0 < 0 \); \( \lambda = \mu = 0 \) if \( y_0 = 0 \).

Theorem 2: The unique optimal portfolio for the weighted MV problem corresponding to \((x_0, z)\) is the one to replicate the contingent claim
\[ X^* = \frac{(\lambda - \mu \rho)_+}{\alpha} - \frac{(\lambda - \mu \rho)_-}{\beta} + z \]
with \((\lambda, \mu)\) being the unique solution to the system of algebraic equations
\[ \frac{E(\lambda - \mu \rho)_+}{\alpha} - \frac{E(\lambda - \mu \rho)_-}{\beta} = 0, \quad \frac{E[\rho(\lambda - \mu \rho)_+]}{\alpha} - \frac{(\lambda - \mu \rho)_-}{\beta} = x_0 - zE\rho. \]

Moreover, the minimum risk value is given by 
\[ J^*(x_0, z) = -\mu(x_0 - zE\rho). \]
Solution to Weighted MV Problem

Lemma 1: The optimal solution of Problem (4) is
\[ Y^* = \frac{\lambda - \mu \rho}{\alpha} + \frac{\lambda - \mu \rho}{\beta}. \]

Lemma 2: For any \( y_0 \), there exists a unique pair \( (\lambda, \mu) \) such that the optimal solution \( Y^* \) in Lemma 1 satisfies \( EY^* = 0, E[Y^* \rho] = y_0 \). Moreover, \( \lambda < 0, \mu < 0 \) if \( y_0 > 0 \); \( \lambda > 0, \mu > 0 \) if \( y_0 < 0 \); \( \lambda = \mu = 0 \) if \( y_0 = 0 \).

Theorem 2: The unique optimal portfolio for the weighted MV problem corresponding to \( (x_0, z) \) is the one to replicate the contingent claim
\[ X^* = \frac{\lambda - \mu \rho}{\alpha} - \frac{\lambda - \mu \rho}{\beta} + z \]
with \( (\lambda, \mu) \) being the unique solution to the system of algebraic equations
\[ \frac{E(\lambda - \mu \rho)_+}{\alpha} - \frac{E(\lambda - \mu \rho)_-}{\beta} = 0, \quad \frac{E[\rho(\lambda - \mu \rho)_+]}{\alpha} - \frac{(\lambda - \mu \rho)_-}{\beta} = x_0 - zE\rho. \]
Moreover, the minimum risk value is given by \( J^*(x_0, z) = -\mu(x_0 - zE\rho) \).

Remark 1: When the market coefficients are deterministic, the optimal portfolio can be obtained more explicitly via some Black-Scholes type equation.
Mean-Semivariance Model

Let \( f(x) = x^2 \). Mean-risk problem specialized to mean-semivariance problem

\[
\begin{align*}
\text{Minimize} & \quad EY^2 \\
\text{subject to} & \quad \begin{cases}
EY = 0 \\
E[Y \rho] = y_0 := x_0 - zE \rho \\
Y \in L^2(\mathcal{F}_T, \mathbb{R})
\end{cases}
\end{align*}
\]

(5)
Mean-Semivariance Model

- Let $f(x) = x^2$. Mean-risk problem specialized to mean-semivariance problem

  Minimize \[ EY^2 \]
  subject to \[ \begin{aligned}
  EY &= 0 \\
  E[Y \rho] &= y_0 := x_0 - zE\rho \\
  Y &\in L^2(F_T, \mathbb{R})
  \end{aligned} \] \hspace{1cm} (5)

- Define

  \[
  \rho_0 := \inf\{\eta \in \mathbb{R} : P(\rho < \eta) > 0\} \quad \rho_1 := \sup\{\eta \in \mathbb{R} : P(\rho > \eta) > 0\}
  \]
Mean-Semivariance Model

- Let $f(x) = x^2$. Mean-risk problem specialized to mean-semivariance problem

\[
\begin{align*}
\text{Minimize} & \quad EY^2 \\
\text{subject to} & \quad EY = 0 \\
& \quad E[Y \rho] = y_0 := x_0 - zE\rho \\
& \quad Y \in L^2(\mathcal{F}_T, \mathbb{R}) \\
\end{align*}
\]

- Define

\[
\rho_0 := \inf\{\eta \in \mathbb{R} : P(\rho < \eta) > 0\} \quad \rho_1 := \sup\{\eta \in \mathbb{R} : P(\rho > \eta) > 0\}
\]

- If $r(\cdot)$ and $\theta(\cdot)$ are deterministic and $\int_0^T |\theta(t)|^2 dt > 0$, then $\rho_0 = 0$ and $\rho_1 = +\infty$. 
Mean-Semivariance Model

• Let \( f(x) = x^2 \). Mean-risk problem specialized to mean-semivariance problem

\[
\begin{align*}
\text{Minimize} & \quad EY^2 \\
\text{subject to} & \quad EY = 0 \\
& \quad E[Y \rho] = y_0 := x_0 - zE\rho \\
& \quad Y \in L^2(\mathcal{F}_T, \mathbb{R})
\end{align*}
\] (5)

• Define

\[
\rho_0 := \inf \{ \eta \in \mathbb{R} : P(\rho < \eta) > 0 \} \quad \rho_1 := \sup \{ \eta \in \mathbb{R} : P(\rho > \eta) > 0 \}
\]

- If \( r(\cdot) \) and \( \theta(\cdot) \) are deterministic and \( \int_0^T |\theta(t)|^2 dt > 0 \), then \( \rho_0 = 0 \) and \( \rho_1 = +\infty \).

**Theorem 3:** The mean-semivariance problem does not admit any optimal solution so long as \( z \neq \frac{x_0}{E\rho} \).
Mean-Semivariance Model

- Let $f(x) = x^2$. Mean-risk problem specialized to mean-semivariance problem

  \[
  \begin{align*}
  \text{Minimize} & \quad EY^2 \\
  \text{subject to} & \quad \begin{cases}
  EY = 0 \\
  E[Y \rho] = y_0 := x_0 - zE\rho \\
  Y \in L^2(\mathcal{F}_T, \mathbb{R})
  \end{cases}
  \end{align*}
  \tag{5}
  \]

- Define

  \[
  \rho_0 := \inf\{\eta \in \mathbb{R} : P(\rho < \eta) > 0\} \quad \rho_1 := \sup\{\eta \in \mathbb{R} : P(\rho > \eta) > 0\}
  \]

  - If $r(\cdot)$ and $\theta(\cdot)$ are deterministic and \(\int_0^T |\theta(t)|^2 dt > 0\), then $\rho_0 = 0$ and $\rho_1 = +\infty$.

**Theorem 3:** The mean-semivariance problem does not admit any optimal solution so long as $z \neq \frac{x_0}{E\rho}$.

**Remark 2:** $z = \frac{x_0}{E\rho}$ is a trivial case, where the optimal portfolio is the risk-free one.
Idea of Proof

• View the mean-semivariance problem as the limiting problem of the weighted MV problem with $\beta = 1 - \alpha$ and $\alpha \to 0$

• Let $Y(\alpha)$ be the optimal solution to the weighted MV problem with $\beta = 1 - \alpha$ and $\alpha > 0$
Idea of Proof

- View the mean-semivariance problem as the limiting problem of the weighted MV problem with $\beta = 1 - \alpha$ and $\alpha \to 0$.

- Let $Y(\alpha)$ be the optimal solution to the weighted MV problem with $\beta = 1 - \alpha$ and $\alpha > 0$.

- If $y_0 < 0$, one can show that $E[Y(\alpha)_-]^2 \to \frac{y_0^2}{E(\rho_1 - \rho_0)^2}$ as $\alpha \to 0$. However, for any feasible solution $Y$ of (6), one can show via Cauchy-Schwartz’s inequality that $EY_2 > \frac{y_0^2}{E(\rho_1 - \rho_0)^2}$. Hence there is no optimal solution.

- If $y_0 > 0$, one can show that $E[Y(\alpha)_-]^2 \to \frac{y_0^2}{E(\rho_1 - \rho)^2}$ as $\alpha \to 0$ (where $\frac{y_0^2}{E(\rho_1 - \rho)^2}$ is defined to be 0 when $\rho_1 = +\infty$). whereas $E[Y(\alpha)_-]^2 > \frac{y_0^2}{E(\rho_1 - \rho)^2}$ for any feasible solution $Y$. Again there is no optimal solution.
Idea of Proof

• View the mean-semivariance problem as the limiting problem of the weighted MV problem with $\beta = 1 - \alpha$ and $\alpha \to 0$

• Let $Y(\alpha)$ be the optimal solution to the weighted MV problem with $\beta = 1 - \alpha$ and $\alpha > 0$

• If $y_0 < 0$, one can show that $E[Y(\alpha)_-]^2 \to \frac{y_0^2}{E(\rho - \rho_0)^2}$ as $\alpha \to 0$. However, for any feasible solution $Y$ of (6), one can show via Cauchy-Schwartz’s inequality that $EY^2 > \frac{y_0^2}{E(\rho - \rho_0)^2}$. Hence there is no optimal solution.

• If $y_0 > 0$, one can show that $E[Y(\alpha)_-]^2 \to \frac{y_0^2}{E(\rho_1 - \rho)^2}$ as $\alpha \to 0$ ($\frac{y_0^2}{E(\rho_1 - \rho)^2}$ is defined to be 0 when $\rho_1 = +\infty$). whereas $E[Y(\alpha)_-]^2 > \frac{y_0^2}{E(\rho_1 - \rho)^2}$ for any feasible solution $Y$. Again there is no optimal solution.

Remark 3: Although the mean-semivariance problem in general does not admit optimal solutions, the infimum of the problem has been obtained explicitly, which is $\frac{y_0^2}{E(\rho - \rho_0)^2}$ if $y_0 < 0$ and $\frac{y_0^2}{E(\rho_1 - \rho)^2}$ if $y_0 > 0$. Moreover, asymptotically optimal portfolios can be obtained by replicating $Y(\alpha) + z$ as $\alpha \to 0$. 
Mean-Downside-Risk Model

- Let \( f \geq 0 \), left continuous at 0, strictly decreasing on \( \mathbb{R}^- \), and for \( \forall x \in \mathbb{R}^+ \), \( f(x) = 0 \) (an example: \( f(x) = (x_-)^p \) for some \( p \geq 0 \)).
Mean-Downside-Risk Model

- Let $f \geq 0$, left continuous at 0, strictly decreasing on $\mathbb{R}^-$, and for $\forall x \in \mathbb{R}^+$, $f(x) = 0$ (an example: $f(x) = (x_-)^p$ for some $p \geq 0$).
- The corresponding risk $E f(X(T) - E X(T))$ only punish the downside part of the deviation of $X(T)$. (downside-risk)
Mean-Downside-Risk Model

• Let $f \geq 0$, left continuous at 0, strictly decreasing on $\mathbb{R}^-$, and for $\forall x \in \mathbb{R}^+, f(x) = 0$ (an example: $f(x) = (x_-)^p$ for some $p \geq 0$).

• The corresponding risk $Ef(X(T) - EX(T))$ only punish the downside part of the deviation of $X(T)$. (downside-risk)

**Assumption (B)** For any $0 \leq M_1 < M_2 \leq +\infty$, $P\{\rho(T) \in (M_1, M_2)\} > 0$ and $P\{\rho(T) = M_1\} = P\{\rho(T) = M_2\} = 0$. 
Mean-Downside-Risk Model

- Let $f \geq 0$, left continuous at 0, strictly decreasing on $\mathbb{R}^-$, and for $\forall x \in \mathbb{R}^+, f(x) = 0$ (an example: $f(x) = (x_-)^p$ for some $p \geq 0$).

- The corresponding risk $E f(X(T) - E X(T))$ only punish the downside part of the deviation of $X(T)$. *(downside-risk)*

**Assumption (B)** For any $0 \leq M_1 < M_2 \leq +\infty$, $P\{\rho(T) \in (M_1, M_2)\} > 0$ and $P\{\rho(T) = M_1\} = \{\rho(T) = M_2\} = 0$.

**Remark 4:** This assumption is satisfied when, say, $r(\cdot)$ and $\theta(\cdot)$ are deterministic and $\int_0^T |\theta(s)|^2 ds > 0$. 
Mean-Downside-Risk Model

- Let $f \geq 0$, left continuous at 0, strictly decreasing on $\mathbb{R}^-$, and for $\forall x \in \mathbb{R}^+, f(x) = 0$ (an example: $f(x) = (x^-)^p$ for some $p \geq 0$).

- The corresponding risk $Ef(X(T) - EX(T))$ only punish the downside part of the deviation of $X(T)$. (downside-risk)

**Assumption (B)** For any $0 \leq M_1 < M_2 \leq +\infty$, $P\{\rho(T) \in (M_1, M_2)\} > 0$ and $P\{\rho(T) = M_1\} = \{\rho(T) = M_2\} = 0$.

**Remark 4:** This assumption is satisfied when, say, $r(\cdot)$ and $\theta(\cdot)$ are deterministic and $\int_0^T |\theta(s)|^2 ds > 0$.

**Theorem 4:** When Assumption (B) hold, the mean-downside-risk model with $f(\cdot)$ as the risk measure admits no optimal solution for any $z \neq \frac{x_0}{E\rho}$. On the other hand, if $z = \frac{x_0}{E\rho}$, then the model has an optimal portfolio which is the risk-free portfolio.
Mean-Downside-Risk Model

• Let \( f \geq 0 \), left continuous at 0, strictly decreasing on \( \mathbb{R}^- \), and for \( \forall x \in \mathbb{R}^+ \), \( f(x) = 0 \) (an example: \( f(x) = (x^-)^p \) for some \( p \geq 0 \)).

• The corresponding risk \( Ef(X(T)) - EX(T) \) only punish the downside part of the deviation of \( X(T) \). (downside-risk)

**Assumption (B)** For any \( 0 \leq M_1 < M_2 \leq +\infty \), \( P\{\rho(T) \in (M_1, M_2)\} > 0 \) and \( P\{\rho(T) = M_1\} = \{\rho(T) = M_2\} = 0 \).

**Remark 4:** This assumption is satisfied when, say, \( r(\cdot) \) and \( \theta(\cdot) \) are deterministic and \( \int_0^T |\theta(s)|^2 ds > 0 \).

**Theorem 4:** When Assumption (B) hold, the mean-downside-risk model with \( f(\cdot) \) as the risk measure admits no optimal solution for any \( z \neq \frac{x_0}{E\rho} \). On the other hand, if \( z = \frac{x_0}{E\rho} \), then the model has an optimal portfolio which is the risk-free portfolio.

**Idea of proof:** Find out the infimum \( \inf Ef(Y) \), and show it is not attainable.
Mean-Downside-Risk Model

- Let $f \geq 0$, left continuous at 0, strictly decreasing on $\mathbb{R}^-$, and for $\forall x \in \mathbb{R}^+$, $f(x) = 0$ (an example: $f(x) = (x_-)^p$ for some $p \geq 0$).
- The corresponding risk $E f(X(T) - EX(T))$ only punish the downside part of the deviation of $X(T)$. (*downside-risk*)

**Assumption (B)** For any $0 \leq M_1 < M_2 \leq +\infty$, $P\{\rho(T) \in (M_1, M_2)\} > 0$ and $P\{\rho(T) = M_1\} = \{\rho(T) = M_2\} = 0$.

**Remark 4:** This assumption is satisfied when, say, $r(\cdot)$ and $\theta(\cdot)$ are deterministic and $\int_0^T |\theta(s)|^2 ds > 0$.

**Theorem 4:** When Assumption (B) hold, the mean-downside-risk model with $f(\cdot)$ as the risk measure admits no optimal solution for any $z \neq \frac{x_0}{E\rho}$. On the other hand, if $z = \frac{x_0}{E\rho}$, then the model has an optimal portfolio which is the risk-free portfolio.

**Question:** When the optimal solutions exist for a general mean-risk problem?
General Mean-Risk Model

- Let $f(\cdot)$ be convex, and strictly convex at 0.
General Mean-Risk Model

- Let $f(\cdot)$ be convex, and strictly convex at 0.

i.e. $kf(x) + (1 - k)f(y) > f(0)$ for any $k \in (0, 1), kx + (1 - k)y = 0$
General Mean-Risk Model

- Let $f(\cdot)$ be convex, and strictly convex at 0.
- Define the subdifferential $\partial f(x)$ in the sense of convex analysis

$$\partial f(x) := \{x^* \in \mathbb{R} : f(y) - f(x) \geq x^*(y - x), \forall y \in \mathbb{R}\}$$
General Mean-Risk Model

- Let $f(\cdot)$ be convex, and strictly convex at 0.
- Define the subdifferential $\partial f(x)$ in the sense of convex analysis

$$\partial f(x) := \{ x^* \in \mathbb{R} : f(y) - f(x) \geq x^*(y - x), \forall y \in \mathbb{R} \}$$

We maintain the Assumption (B). And define

$$g(y) := \arg\min_{x \in \mathbb{R} : y \in \partial f(x)} |x|$$
$$\Lambda := \{ \lambda \in \mathbb{R} : \exists \mu = \mu(\lambda) \in \mathbb{R} \text{ so that } g(\lambda - \mu(\lambda)\rho) \in L^2(\mathcal{F}_T, \mathbb{R}), \quad E g(\lambda - \mu(\lambda)\rho) = 0, \rho g(\lambda - \mu(\lambda)\rho) \in L^1(\mathcal{F}_T, \mathbb{R}) \}$$

$$\tilde{g}(\lambda) = E[\rho g(\lambda - \mu(\lambda)\rho)], \forall \lambda \in \Lambda$$
$$\bar{\lambda} = \sup_{\lambda \in \Lambda} \lambda, \quad \underline{\lambda} = \inf_{\lambda \in \Lambda} \lambda$$
$$\bar{y} := \lim_{\lambda \uparrow \bar{\lambda}} \tilde{g}(\lambda), \quad \underline{y} := \lim_{\lambda \downarrow \underline{\lambda}} \tilde{g}(\lambda)$$
General Mean-Risk Model

- Let $f(\cdot)$ be convex, and strictly convex at $0$.
- Define the subdifferential $\partial f(x)$ in the sense of convex analysis

$$\partial f(x) := \{x^* \in \mathbb{R} : f(y) - f(x) \geq x^*(y - x), \forall y \in \mathbb{R}\}$$

We maintain the Assumption (B). And define

$$g(y) := \arg\min_{x \in \mathbb{R} : y \in \partial f(x)} |x|$$

$$\Lambda := \{\lambda \in \mathbb{R} : \exists \mu = \mu(\lambda) \in \mathbb{R} \text{ so that } g(\lambda - \mu(\lambda)\rho) \in L^2(\mathcal{F}_T, \mathbb{R}), \quad Eg(\lambda - \mu(\lambda)\rho) = 0, \rho g(\lambda - \mu(\lambda)\rho) \in L^1(\mathcal{F}_T, \mathbb{R}) \}$$

$$\tilde{g}(\lambda) = E[\rho g(\lambda - \mu(\lambda)\rho)], \forall \lambda \in \Lambda$$

$$\bar{\lambda} = \sup_{\lambda \in \Lambda} \lambda, \quad \underline{\lambda} = \inf_{\lambda \in \Lambda} \lambda$$

$$\bar{y} := \lim_{\lambda \uparrow \bar{\lambda}} \tilde{g}(\lambda), \quad \underline{y} := \lim_{\lambda \downarrow \underline{\lambda}} \tilde{g}(\lambda)$$

Remark 5: All these definition can be calculated offline
Theorem 5: One has the following conclusions regarding the solution to the mean-risk portfolio selection problem with the general $f$: 
Solution of the General MR problem

**Theorem 5:** One has the following conclusions regarding the solution to the mean-risk portfolio selection problem with the general $f$:

(i) Assume that $\bigcup_{x \in \mathbb{R}} \partial f(x) = \mathbb{R}$. Then the problem admits an optimal solution if and only if $x_0 - zE\rho \in A \cup B$, where

$$A = \begin{cases} [y, 0], & \text{if } \bar{\lambda} \in \Lambda \\ (y, 0), & \text{if } \bar{\lambda} \notin \Lambda \end{cases}$$

$$B = \begin{cases} [0, \bar{y}], & \text{if } \lambda \in \Lambda \\ (0, \bar{y}), & \text{if } \lambda \notin \Lambda \end{cases}$$
Solution of the General MR problem

**Theorem 5:** One has the following conclusions regarding the solution to the mean-risk portfolio selection problem with the general $f$:

(i) Assume that $\bigcup_{x \in \mathbb{R}} \partial f(x) = \mathbb{R}$. Then the problem admits an optimal solution if and only if $x_0 - zE\rho \in A \cup B$, where

$$A = \begin{cases} [y, 0], & \text{if } \bar{\lambda} \in \Lambda \\ (y, 0], & \text{if } \bar{\lambda} \notin \Lambda \end{cases}, \quad B = \begin{cases} [0, \bar{y}], & \text{if } \lambda \in \Lambda \\ (0, \bar{y}), & \text{if } \lambda \notin \Lambda \end{cases}$$

(ii) Assume that there exists $M_1, M_2 \in \mathbb{R}$ such that $\bigcup_{x \in \mathbb{R}} \partial f(x) \subset [M_1, M_2]$. Then the problem admits an optimal solution if and only if $z = x_0 / E\rho$. 
(iii) Assume that either $\bigcup_{x \in \mathbb{R}} \partial f(x) = (-\infty, k]$ or $\bigcup_{x \in \mathbb{R}} \partial f(x) = (-\infty, k)$ for some $k \in \mathbb{R}$. Then $\lambda = 0 \in \Lambda$. If $\lambda \not\in \Lambda$, then the problem admits an optimal solution if and only if $x_0 - zE\rho \in (y, 0]$. If $\lambda \in \Lambda$, then the problem admits an optimal solution if and only if $x_0 - zE\rho \in \{\tilde{g}(\lambda)\} \cup (y, 0]$. If in addition $\lambda < k$, then $\tilde{g}(\lambda) = y$.
Solution of General MR problem (Cont’d)

(iii) Assume that either \( \bigcup_{x \in \mathbb{R}} \partial f(x) = (-\infty, \bar{k}] \) or \( \bigcup_{x \in \mathbb{R}} \partial f(x) = (-\infty, \bar{k}) \) for some \( \bar{k} \in \mathbb{R} \). Then \( \bar{\lambda} = 0 \in \Lambda \). If \( \bar{\lambda} \notin \Lambda \), then the problem admits an optimal solution if and only if \( x_0 - zE\rho \in (y, 0] \). If \( \bar{\lambda} \in \Lambda \), then the problem admits an optimal solution if and only if \( x_0 - zE\rho \in \{\tilde{g}(\bar{\lambda})\} \cup (y, 0] \). If in addition \( \bar{\lambda} < \bar{k} \), then \( \tilde{g}(\bar{\lambda}) = y \).

(iv) Assume that either \( \bigcup_{x \in \mathbb{R}} \partial f(x) = [\bar{k}, \infty) \) or \( \bigcup_{x \in \mathbb{R}} \partial f(x) = (\bar{k}, \infty) \) for some \( \bar{k} \in \mathbb{R} \), then \( \bar{\lambda} = 0 \in \Lambda \). If \( \bar{\lambda} \notin \Lambda \), then the problem admits an optimal solution if and only if \( x_0 - zE\rho \in [0, \bar{y}) \). If \( \bar{\lambda} \in \Lambda \), then the problem admits an optimal solution if and only if \( x_0 - zE\rho \in \{\tilde{g}(\bar{\lambda})\} \cup [0, \bar{y}) \). If in addition \( \bar{\lambda} > \bar{k} \), then \( \tilde{g}(\bar{\lambda}) = \bar{y} \).
Examples

Example 1: \( f(x) = |x| \) (mean-absolute-deviation model).
Examples

**Example 1:** \( f(x) = |x| \) (mean-absolute-deviation model).

\( f(\cdot) \) is strictly convex at 0, and \( \bigcup_{x \in \mathbb{R}} \partial f(x) = [-1, 1] \). (Case (ii)).
Examples

Example 1: \( f(x) = |x| \) (mean-absolute-deviation model).

\( f(\cdot) \) is strictly convex at 0, and \( \bigcup_{x \in \mathbb{R}} \partial f(x) = [-1, 1] \). (Case (ii)). Thus the continuous-time mean-absolute-deviation model admits an optimal solution if and only if \( z = x_0 / E\rho \), in which case the optimal portfolio is simply the risk-free one.
Examples

**Example 1:** \( f(x) = |x| \) (*mean-absolute-deviation model*).

\( f(\cdot) \) is strictly convex at 0, and \( \bigcup_{x \in \mathbb{R}} \partial f(x) = [-1, 1] \). (Case (ii)). Thus the continuous-time mean-absolute-deviation model admits an optimal solution if and only if \( z = x_0 / E \rho \), in which case the optimal portfolio is simply the risk-free one.

**Example 2:** \( f(x) = e^{-x} \) (*more sensitive to large loss*).
Examples

Example 1: \( f(x) = |x| \) (mean-absolute-deviation model).

\( f(\cdot) \) is strictly convex at 0, and \( \cup_{x \in \mathbb{R}} \partial f(x) = [-1, 1]. \) (Case (ii)). Thus the continuous-time mean-absolute-deviation model admits an optimal solution if and only if \( z = x_0 / E\rho \), in which case the optimal portfolio is simply the risk-free one.

Example 2: \( f(x) = e^{-x} \) (more sensitive to large loss).

\( f \) is strictly convex, \( \cup_{x \in \mathbb{R}} \partial f(x) = (-\infty, 0). \) (Case (iii)).
Examples

Example 1: \( f(x) = |x| \) (mean-absolute-deviation model).

\( f(\cdot) \) is strictly convex at 0, and \( \cup_{x \in \mathbb{R}} \partial f(x) = [-1, 1] \). (Case (ii)). Thus the continuous-time mean-absolute-deviation model admits an optimal solution if and only if \( z = x_0 / E\rho \), in which case the optimal portfolio is simply the risk-free one.

Example 2: \( f(x) = e^{-x} \) (more sensitive to large loss).

\( f \) is strictly convex, \( \cup_{x \in \mathbb{R}} \partial f(x) = (-\infty, 0) \). (Case (iii)). The MR problem admits an optimal solution iff \( x_0 - zE\rho \in [(E\rho)(E\ln\rho) - E(\rho \ln \rho), 0] \) or, equivalently, \( z \in \left[ \frac{x_0}{E\rho}, \frac{x_0 - (E\rho)(E\ln\rho) + E(\rho \ln \rho)}{E\rho} \right] \). When the problem does admit an optimal solution, the optimal portfolio is the one replicating the claim \( z - \ln(-\lambda + \mu\rho) \), where \((\lambda, \mu)\) is the unique solution pair to the following algebraic equation (which must admit a solution):

\[
\begin{align*}
E \ln(-\lambda + \mu\rho) &= 0 \\
E[\rho \ln(-\lambda + \mu\rho)] &= zE\rho - x_0
\end{align*}
\]
Examples (Cont’d)

Example 3: \[ f(x) = (x - 1)^2 \] (shift of the mean-semivariance model).
Example 3: $f(x) = (x - 1)^2$ (shift of the mean-semivariance model).

$f$ is not strictly convex everywhere; but it is indeed strictly convex at 0. $igcup_{x \in \mathbb{R}} \partial f(x) = (-\infty, 0]$. (Case (iii)).
Example 3: \( f(x) = (x - 1)^2 \) (shift of the mean-semivariance model).

\( f \) is not strictly convex everywhere; but it is indeed strictly convex at 0. \( \cup_{x \in \mathbb{R}} \partial f(x) = (-\infty, 0] \). (Case (iii)). The original portfolio selection problem admits an optimal solution if and only if \( x_0 - zE\rho \in [E\rho - E\rho^2/E\rho, 0] \) or, equivalently, \( z \in \left[ \frac{x_0}{E\rho}, \frac{x_0}{E\rho} + \frac{E\rho^2}{(E\rho)^2} - 1 \right] \). When the problem does admit an optimal solution, the optimal portfolio is the one replicating the claim \( z + 1 + \frac{\lambda - \mu \rho}{2} \) where \((\lambda, \mu)\) is the unique solution pair to the following linear algebraic equation:

\[
\begin{align*}
\lambda - \mu E\rho &= -2 \\
\lambda E\rho - \mu E\rho^2 &= 2x_0 - 2(1 + z)E\rho
\end{align*}
\]
Asymptotic Optimal Portfolios

- In all the cases, no matter the optimal portfolios exist or not, the infimum of the risk is finite \( Ef(X - EX) \geq f(E[X - EX]) = f(0) \).
Asymptotic Optimal Portfolios

- In all the cases, no matter the optimal portfolios exist or not, the infimum of the risk is finite \( Ef(X - EX) \geq f(E[X - EX]) = f(0) \).

- When optimal portfolios do not exist, consider a perturbed risk function

\[
f_\alpha(x) = f(x) + \alpha x^2, \quad \alpha > 0
\]
Asymptotic Optimal Portfolios

• In all the cases, no matter the optimal portfolios exist or not, the infimum of the risk is finite \( Ef(X - EX) \geq f(E[X - EX]) = f(0) \).

• When optimal portfolios do not exist, consider a perturbed risk function

\[
f_\alpha(x) = f(x) + \alpha x^2, \quad \alpha > 0
\]

It can be shown:

◦ The mean-risk portfolio selection problem with risk function \( f_\alpha \) must admit an optimal solution

◦ The corresponding optimal portfolio \( \pi_\alpha \) is asymptotically optimal for the original problem when \( \alpha \downarrow 0 \)
Back to Single Period

- Continuous-time mean-semivariance problem admits no optimal solution. How about its single period counterpart?
Continuous-time mean-semivariance problem admits no optimal solution. How about its single period counterpart?

- $m (\geq 2)$ securities, each with return rate $R_j$, $E R_j = r_j$, $Cov(R_i, R_j) = \sigma_{i,j}$

- The single period Mean-semivariance problem

\[
\begin{align*}
\text{Minimize} & \quad E \left[ (\sum_j \pi_j R_j - E \sum_j \pi_j R_j)^- \right]^2 \\
\text{subject to} & \quad \sum_i \pi_i = x_0 \\
& \quad E(\sum_j \pi_j R_j) = \sum_i r_i \pi_i = z
\end{align*}
\]
Back to Single Period

- Continuous-time mean-semivariance problem admits no optimal solution. How about its single period counterpart?
- \( m(\geq 2) \) securities, each with return rate \( R_j, \ E R_j = r_j, \ \text{Cov}(R_i, R_j) = \sigma_{ij} \)
- The single period Mean-semivariance problem

\[
\text{Minimize} \quad E[(\sum_j \pi_j R_j - E(\sum_j \pi_j R_j))^2]
\]
\[
\text{subject to} \quad \begin{cases} \sum_i \pi_i = x_0 \\ E(\sum_j \pi_j R_j) = \sum_i r_i \pi_i = z \end{cases}
\]  

- Vast literature on single-period mean-semivariance models
Back to Single Period

- Continuous-time mean-semivariance problem admits no optimal solution. How about its single period counterpart?
- \( m(\geq 2) \) securities, each with return rate \( R_j, ER_j = r_j, \text{Cov}(R_i, R_j) = \sigma_{ij} \)
- The single period Mean-semivariance problem

\[
\begin{align*}
\text{Minimize} & \quad E \left[ (\sum_j \pi_j R_j - E \sum_j \pi_j R_j) - \right]^2 \\
\text{subject to} & \quad \sum_i \pi_i = x_0 \\
& \quad E(\sum_j \pi_j R_j) = \sum_i r_i \pi_i = z
\end{align*}
\]

- Vast literature on single-period mean-semivariance models
- Concentrate on numerical solution (as analytical solution impossible) and comparison with mean-variance model
Continuous-time mean-semivariance problem admits no optimal solution. How about its single period counterpart?

- \( m(\geq 2) \) securities, each with return rate \( R_j, \ ER_j = r_j, \ \text{Cov}(R_i, R_j) = \sigma_{ij} \)

- The single period Mean-semivariance problem

\[
\begin{align*}
\text{Minimize} & \quad E\left[ (\sum_j \pi_j R_j - E\sum_j \pi_j R_j)^- \right]^2 \\
\text{subject to} & \quad \sum_i \pi_i = x_0 \\
& \quad E(\sum_j \pi_j R_j) = \sum_i r_i \pi_i = z
\end{align*}
\]

- Vast literature on single-period mean-semivariance models
- Concentrate on numerical solution (as analytical solution impossible) and comparison with mean-variance model
- Existence of efficient portfolios/frontier not addressed
Continuous-time mean-semivariance problem admits no optimal solution. How about its single period counterpart?

- $m(\geq 2)$ securities, each with return rate $R_j, \ E R_j = r_j, \ \text{Cov}(R_i, R_j) = \sigma_{ij}$

- The single period Mean-semivariance problem

\[
\text{Minimize} \quad E[(\sum_j \pi_j R_j - E \sum_j \pi_j R_j)^-]^2
\]
\[
\text{subject to} \quad \begin{cases} 
\sum_i \pi_i = x_0 \\
E(\sum_j \pi_j R_j) = \sum_i r_i \pi_i = z
\end{cases}
\]

- Vast literature on single-period mean-semivariance models
- Concentrate on numerical solution (as analytical solution impossible) and comparison with mean-variance model
- Existence of efficient portfolios/frontier not addressed
- Technically non-trivial as the feasible region generally unbounded, and the objective not coercive

($f : \mathbb{R}^d \mapsto \mathbb{R}$ called coercive if $\lim_{|x| \to +\infty} f(x) = +\infty$)
A Lemma

Consider

$$\min_{x \in \mathbb{R}^d} E[(A + B'x)^2],$$

(7)

where $B \equiv (B_1, \cdots, B_d)'$, and $A, B_i$ are random variables with $EA^2 < +\infty$, $EB_i^2 < +\infty$, $i = 1, \cdots, d$. 
A Lemma

Consider

$$\min_{x \in \mathbb{R}^d} E[(A + B'x)^+]^2,$$

(7)

where $B \equiv (B_1, \cdots, B_d)'$, and $A, B_i$ are random variables with $EA^2 < +\infty$, $EB_i^2 < +\infty$, $i = 1, \cdots, d$.

**Lemma 3:** If $EB_i = 0$, $i = 1, \cdots, d$, then problem (7) admits optimal solutions.
A Lemma

Consider

\[
\min_{x \in \mathbb{R}^d} E[(A + B'x)^2],
\]

where \( B \equiv (B_1, \cdots, B_d)' \), and \( A, B_i \) are random variables with \( EA^2 < +\infty, \)
\( EB_i^2 < +\infty, \ i = 1, \cdots, d. \)

**Lemma 3:** If \( EB_i = 0, \ i = 1, \cdots, d, \) then problem (7) admits optimal solutions.

**Remark 6:** Assumption that \( EB_i = 0, i = 1, \cdots, d \) is crucial.
A Lemma

Consider

$$\min_{x \in \mathbb{R}^d} E[(A + B'x)^2],$$

(7)

where $B \equiv (B_1, \cdots, B_d)'$, and $A, B_i$ are random variables with $EA^2 < +\infty$, $EB_i^2 < +\infty$, $i = 1, \cdots, d$.

Lemma 3: If $EB_i = 0$, $i = 1, \cdots, d$, then problem (7) admits optimal solutions.

Remark 6: Assumption that $EB_i = 0$ $i = 1, \cdots, d$ is crucial.

Counter example: Let $A = -1$, $B = (e^{W_1}, \cdots, e^{W_d})'$, where $(W_1, \cdots, W_d)$ follow $N(0, I_d)$. For any $0 \neq x \in \mathbb{R}^d$, $\lim_{\alpha \to +\infty} E[(A + B'(\alpha x))^2] = 0$. This implies the optimal value of (7) is 0. However, this value cannot be achieved since $E[(A + B'(\alpha x))^2] > 0$ for any $x \in \mathbb{R}^d$. 
Existence for Single-Period M-S

**Theorem 6:** For any $x_0 \in \mathbb{R}$ and $z \in \mathbb{R}$, Problem (7) admits optimal solutions if and only if it admits feasible solutions.
Theorem 6: For any $x_0 \in \mathbb{R}$ and $z \in \mathbb{R}$, Problem (7) admits optimal solutions if and only if it admits feasible solutions.

Idea of Proof. Let $\xi_i = R_i - r_i$. After eliminating $\pi_1$ and $\pi_2$ from the constraints, one gets the following equivalent problem

$$
\min_{(\pi_3, \ldots, \pi_m) \in \mathbb{R}^{m-2}} E[(A + \sum_{i=3}^{m} \pi_i B_i)^2],
$$

where

$$
A = x_0 \xi_1 + \frac{z - x_0 r_1}{r_2 - r_1} (\xi_2 - \xi_1), \quad B_i = \xi_i - \xi_1 - (r_i - r_1) \frac{\xi_2 - \xi_1}{r_2 - r_1}.
$$

Then Lemma 3 applies.
Existence for Single-Period M-S

**Theorem 6:** For any \( x_0 \in \mathbb{R} \) and \( z \in \mathbb{R} \), Problem (7) admits optimal solutions if and only if it admits feasible solutions.

**Idea of Proof.** Let \( \xi_i = R_i - r_i \). After eliminating \( \pi_1 \) and \( \pi_2 \) from the constraints, one gets the following equivalent problem

\[
\min_{(\pi_3, \ldots, \pi_m) \in \mathbb{R}^{m-2}} E[(A + \sum_{i=3}^{m} \pi_i B_i)^2],
\]

where

\[
A = x_0 \xi_1 + \frac{z - x_0 r_1}{r_2 - r_1} (\xi_2 - \xi_1), \quad B_i = \xi_i - \xi_1 - (r_i - r_1) \frac{\xi_2 - \xi_1}{r_2 - r_1}.
\]

Then Lemma 3 applies.

**Sharply contrast** Continuous-time mean-semivariance Vs Single-period mean-semivariance
Further Research

- *Incomplete* market (the replicating problem becomes significant)

  - Some work has been done on the mean-variance problem. ...
Further Research

- *Incomplete* market (the replicating problem becomes significant)

  - Some work has been done on the mean-variance problem. ...

- Other risk measures: safety first, Var, minimax, ......
Comments and questions are appreciated ... ...

Thank you very much!