What is Derived Geometry?

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These slides available at
`http://people.maths.ox.ac.uk/~joyce/`.

References: for Derived Algebraic Geometry,
see Toën arXiv:1401.1044.
For Derived Differential Geometry, see
`http://people.maths.ox.ac.uk/~joyce/dmanifolds.html` and
`http://people.maths.ox.ac.uk/~joyce/Kuranishi.html`

Different kinds of spaces in Algebraic Geometry

Two important kinds of spaces in Algebraic Geometry over $\mathbb{C}$ are *varieties* and *schemes*. A variety is a space X locally modelled on the set of solutions of polynomial equations in $\mathbb{C}^n$, i.e. on

$$\{(z_1,\ldots,z_n)\in\mathbb{C}^n : p_1(z_1,\ldots,z_n)=\cdots=p_k(z_1,\ldots,z_n)=0\},$$

for $p_1,\ldots,p_k\in\mathbb{C}[z_1,\ldots,z_n]$. A scheme is similar, but also remembers the quotient algebra $\mathbb{C}[z_1,\ldots,z_n]/(p_1,\ldots,p_n)$ of functions on the solution set. Therefore the scheme $\{z=0\}\subset\mathbb{C}$ is different to the scheme $\{z^2=0\}\subset\mathbb{C}$, though they are the same as varieties, as the algebras $\langle 1\rangle_{\mathbb{C}}$ and $\langle 1,z\rangle_{\mathbb{C}}$ are different. Think of $\{z^2=0\}$ as a 'double point', two points on top of each other, so if we 'count' the points in $\{z^2=0\}$ correctly, we get 2.

Derived geometry is an enhancement of classical geometry.

A *derived scheme* (difficult to define) also remembers degeneracies between the p_i. So for example, $\{z_1^2=z_2^2=0\}$ and $\{z_1^2=z_2^2=z_1^2+z_2^2=0\}$ are the same as schemes, but different as derived schemes: the derived schemes have a well defined 'virtual dimension', which is 0 and -1 in these examples.

Smooth spaces, singular spaces, and quasi-smoothness

A variety or scheme X is called *smooth* if it is locally modelled on $\mathbb{C}^n$. A point $x \in X$ is *singular* if X is not smooth near x. For example, the scheme $\{z_1 z_2 = 0\} \subset \mathbb{C}^2$ is singular at $(0, 0)$.

Smooth varieties and smooth schemes are the same. The extra data in a scheme is only nontrivial near singularities.

Many interesting spaces in Algebraic Geometry (e.g. moduli spaces) are singular, so Algebraic Geometers are really good at understanding singularities. (By comparison, Differential Geometers are pants at singularities.)

There is a notion of *quasi-smooth* derived scheme. It is significantly weaker than smoothness: the underlying classical scheme can be singular. Many derived moduli spaces are quasi-smooth, although the classical moduli space is singular.

Part of the magic of Derived Geometry is that quasi-smooth spaces behave in some ways like smooth spaces (Kontsevich's 'hidden smoothness' philosophy). For example, a compact quasi-smooth derived scheme $\boldsymbol{X}$ of dimension 0 has a 'number of points' $\#\boldsymbol{X}$.

Differential Geometry, manifolds, and derived manifolds

Differential Geometers mostly study *manifolds*, locally modelled on $\mathbb{R}^n$, but the basic functions on $\mathbb{R}^n$ are not polynomials but smooth functions. There is a (very little used) notion of scheme in Differential Geometry, *C^∞-schemes*, locally modelled on

$$\{(x_1,\ldots,x_n)\in\mathbb{R}^n : f_1(x_1,\ldots,x_n)=\cdots=f_k(x_1,\ldots,x_n)=0\},$$

for $f_1,\ldots,f_k:\mathbb{R}^n\to\mathbb{R}$ smooth (C^∞). There are also (even less used, and really difficult to define) notions of *derived C^∞-scheme*, and *quasi-smooth derived C^∞-scheme*.

One definition of a *derived manifold* is of a quasi-smooth derived C^∞-scheme. Derived manifolds have their own (rather beautiful) differential geometry. They behave in some ways like manifolds, although the underlying topological spaces can be very singular (e.g. the Mandelbrot set can be made into a 1-dimensional derived manifold). Many Differential-Geometric moduli spaces are naturally derived manifolds (e.g. the moduli space of solutions of any nonlinear elliptic equation on a compact manifold).

Moduli spaces

A *moduli space* $\mathfrak{M}$ is a geometric space whose points $[E]$ parametrize isomorphism classes of geometric objects E you care about (e.g. holomorphic vector bundles $E \to X$ over a fixed complex manifold X). But this defines $\mathfrak{M}$ only as a set. The important thing is to put a geometric structure on $\mathfrak{M}$ encoding the behaviour of *families* of objects E. For example, if you have a notion of continuous family $(E_t)_{t\in T}$ over a base topological space T, you would make $\mathfrak{M}$ into a topological space, such that $(E_t)_{t\in T}$ induces a continuous map $T \to \mathfrak{M}$.

In Algebraic Geometry, we get a notion of (smooth) family $(E_t)_{t\in T}$ over a (smooth) base scheme T. In Differential Geometry, we get a notion of smooth family $(E_t)_{t\in T}$ over a manifold T. So we might hope to make $\mathfrak{M}$ into a (smooth?) scheme, or manifold.

Unfortunately, moduli spaces are rarely smooth. *Murphy's Law* says that some classes of moduli spaces (e.g. vector bundles on a complex surface) can have arbitrarily horrible singularities (i.e. all local singularities of schemes over $\mathbb{Z}$ occur).

Enumerative geometry

Geometers often want to 'count' points in moduli spaces, an area known as enumerative geometry.

For example, let $X = \{[z_0, z_1, z_2] \in \mathbb{CP}^3 : P(z_0, z_1, z_2) = 0\}$ be a smooth cubic surface in $\mathbb{CP}^3$, where $P(z_0, z_1, z_2)$ is a *generic* choice of homogeneous cubic polynomial. It is well known that X contains 27 lines $\mathbb{CP}^1$ in $\mathbb{CP}^3$. That is, we can form the moduli space $\mathfrak{M}_X$ of lines $\mathbb{CP}^1$ in X, and for P generic $\mathfrak{M}_X$ is 27 points.

If P is not generic, then $\mathfrak{M}_X$ could be fewer than 27 points, or it could be infinite. But we would like a way to define a 'virtual count' $\#\mathfrak{M}_X$, depending on the geometric structure on $\mathfrak{M}_X$, which always gives $\#\mathfrak{M}_X = 27$ in this example.

The geometric structure that gives a good notion of 'virtual count' is a *proper quasi-smooth derived scheme* $\boldsymbol{Z}$ in Algebraic Geometry, or a *compact oriented derived manifold* $\boldsymbol{Z}$ in Differential Geometry. These behave like compact oriented manifolds, in that they have a *fundamental class* $[\boldsymbol{Z}]_{\mathrm{virt}}$ (*virtual class*) in the homology group $H_{\mathrm{vdim}\,\boldsymbol{Z}}(\boldsymbol{Z}, \mathbb{Z})$. If $\boldsymbol{Z}$ has dimension 0 then $\#\boldsymbol{Z} = \int_{[\boldsymbol{Z}]_{\mathrm{virt}}} 1 \in \mathbb{Z}$.

Virtual classes are *deformation-invariant*, so $\#\mathfrak{M}_X$ is independent of P.

Example: Bézout's Theorem

Let C and D be algebraic curves in $\mathbb{CP}^2$, defined by polynomials of degrees m, n. Bézout's Theorem says that if C, D intersect transversely, then $C \cap D$ is mn points. If C, D intersect non-transversely, but in finitely many points, then the classical scheme $C \cap D$ has 'length' mn. So in this case, the scheme (but not the variety $C \cap D$) knows about the virtual count mn.

If $C \cap D$ is infinite (e.g. if $C = D$) then we cannot recover the number mn from the classical scheme $C \cap D$. However, if we define $\boldsymbol{C} \cap \boldsymbol{D}$ as a derived scheme, it is quasi-smooth, and its virtual class is $[\boldsymbol{C} \cap \boldsymbol{D}]_{\mathrm{virt}} = mn \in H_0(\mathbb{CP}^2, \mathbb{Z}) = \mathbb{Z}$.

Principle

Many things that work in classical geometry under a transversality assumption, work in derived geometry without transversality.

Categories, 2-categories, and ∞-categories

Many things in mathematics form a *category*: you have objects and morphisms, e.g. topological spaces X, Y and continuous maps $f : X \to Y$. Sometimes you want to consider a 2-*category*, in which you have objects, and 1-morphisms between objects, and 2-morphisms between 1-morphisms. For example, there is a 2-category whose objects are topological spaces X, Y, 1-morphisms are continuous maps $f, g : X \to Y$, and 2-morphisms are homotopies $\eta : f \Rightarrow g$. Or we could take objects to be categories, 1-morphisms to be functors, and 2-morphisms to be natural transformations. In fact there are also 3-categories, 4-categories, $\ldots$, ∞-categories. For technical reasons, if a 2-category is not sufficient you usually go all the way to ∞-categories.

Principle

To define 'derived' spaces, you have to use higher categories: always ∞-categories in Algebraic Geometry, and in Differential geometry, 2-categories are usually enough.

Why higher categories?

A higher category can always be truncated to an ordinary category (the *homotopy category*) by taking morphisms to be 2-isomorphism classes of 1-morphisms. But in doing so you lose information. For example, a *fibre product* in a 2-category is defined by a universal property involving 2-morphisms. If you truncate to a category, there is no universal property. Also, one way to define a derived scheme $\boldsymbol{X}$ would be as a topological space X with an ∞-sheaf (homotopy sheaf) of ∞-algebras (simplicial algebras, or cdgas). But truncating an ∞-sheaf to ordinary categories loses the sheaf property (locality).

Suppose you have a category $\mathcal{C}$ with a class $\mathcal{W}$ of morphisms (weak equivalences) that you want to invert, giving a new category $\mathcal{C}[\mathcal{W}^{-1}]$. For example, in homotopy theory one wants to invert homotopy equivalences of topological spaces. Nearly always, it is better to make $\mathcal{C}[\mathcal{W}^{-1}]$ into an ∞-*category*. That is, $\mathcal{C}[\mathcal{W}^{-1}]$ has good properties at the ∞-category level which are invisible in ordinary categories. This applies to *derived categories* $D^b\,\mathrm{coh}(X)$.

Commutative differential graded algebras

The next bit only works over a field $\mathbb{K}$ of characteristic zero; in general, one should use simplicial algebras.

A $\mathbb{K}$-scheme $(X, \mathcal{O}_X)$ is a topological space X with a sheaf $\mathcal{O}_X$ of commutative $\mathbb{K}$-algebras $\mathcal{O}_X(U)$ for open $U \subset X$, where we think of $\mathcal{O}_X(U)$ as the algebraic functions $U \to \mathbb{K}$. To define derived $\mathbb{K}$-schemes, we replace commutative $\mathbb{K}$-algebras by 'derived $\mathbb{K}$-algebras'. One model for these is *commutative differential graded algebras* (*cdgas*) $A^\bullet = (A^*, \mathrm{d})$ in degrees $\leqslant 0$. That is, $A^* = \bigoplus_{k \leqslant 0} A^k$ for A^k a $\mathbb{K}$-vector space, with a supercommutative graded multiplication $\cdot : A^k \times A^l \to A^{k+l}$, an identity $1 \in A^0$, and a differential $\mathrm{d} : A^k \to A^{k+1}$ satisfyig the Leibnitz rule $\mathrm{d}(ab) = (\mathrm{d}a)b + (-1)^k a\, \mathrm{d}b$ for $a \in A^k$, $b \in A^l$. (This is like exterior forms on a manifold, but in negative degrees.)

Then $H^*(A^\bullet)$ is a graded algebra, and $H^0(A^\bullet)$ is a commutative $\mathbb{K}$-algebra, considered as the classical truncation of $A^\bullet$.

Quasi-isomorphisms

There is an obvious notion of morphism $\phi : A^\bullet \to B^\bullet$ preserving all the structure, giving a category $\mathbf{cdga}^{\mathbb{K}}_{\leqslant 0}$. We call ϕ a *quasi-isomorphism* if $H^k(\phi) : H^k(A^\bullet) \to H^k(B^\bullet)$ is an isomorphism for all $k \leqslant 0$. Quasi-isomorphic cdgas are thought of as 'the same'. We would like to invert quasi-isomorphisms to get a new category $\mathbf{cdga}^{\mathbb{K}}_{\leqslant 0}[\mathcal{Q}^{-1}]$. But this must be regarded as an ∞-category. Then a derived scheme $\boldsymbol{X}$ could be defined as a topological space X with an ∞-sheaf $\mathcal{O}_X$ of objects in $\mathbf{cdga}^{\mathbb{K}}_{\leqslant 0}[\mathcal{Q}^{-1}]$, satisfying conditions. Inverting quasi-isomorphisms makes the morphisms (or n-morphisms for $n \geqslant 1$) in $\mathbf{cdga}^{\mathbb{K}}_{\leqslant 0}[\mathcal{Q}^{-1}]$ difficult to understand.

Principle

In Derived Geometry, the local models for things (e.g. cdgas) are often nice objects, and easy to describe. However, the ways in which these local models are glued together, and the conditions under which two local models are 'the same', can be mysterious and difficult to work with explicitly.

Examples of quasi-smooth cdgas

Return to the example of

$$\{(z_1,\ldots,z_n)\in\mathbb{C}^n : p_1(z_1,\ldots,z_n)=\cdots=p_k(z_1,\ldots,z_n)=0\}.$$

We encode this as a cdga $A^\bullet$ by taking $A^*=\mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_k]$ to be the graded polynomial algebra generated by degree 0 even variables $x_1,\ldots,x_n$ and degree -1 odd variables $y_1,\ldots,y_k$. The differential $\mathrm{d}:A^*\to A^{*+1}$ is generated by $\mathrm{d}x_i=0$ and $\mathrm{d}y_j=p_j(x_1,\ldots,x_n)$. Then $H^0(A^\bullet)=\mathbb{C}[x_1,\ldots,x_n]/(y_1,\ldots,y_k)$ is the corresponding classical algebra. This cdga is *quasi-smooth* (it is quasi-isomorphic to a free superpolynomial algebra with generators in degrees $0,-1$ only). It has *virtual dimension* $n-k$ (the expected dimension of the solutions of k equations on n variables; can be negative).
For example, the cdgas of schemes $\{z_1^2=z_2^2=0\}$ and $\{z_1^2=z_2^2=z_1^2+z_2^2=0\}$ in $\mathbb{C}^2$ are different, as one has generators z_1,z_2,y_1,y_2 and virtual dimension 0, and the second generators z_1,z_2,y_1,y_2,y_3 and virtual dimension -1.

Derived categories, (co)tangent complexes

Complexes $E^\bullet = \bigl(\cdots \xrightarrow{\mathrm{d}} E^k \xrightarrow{\mathrm{d}} E^{k+1} \xrightarrow{\mathrm{d}} \cdots\bigr)$, in which each E^k is some linear object (e.g. vector spaces, or vector bundles over some space), and $k \in \mathbb{Z}$ (or maybe $k \leqslant 0$ or $k \geqslant 0$), and $\mathrm{d}^2 = 0$, are very important in derived geometry. For example, cdgas are a modification of $\mathbb{K}$-algebras in which the $\mathbb{K}$-vector space A is replaced by a complex. Usually one cares about complexes up to quasi-isomorphism (morphisms inducing isomorphisms on cohomology). Inverting quasi-isomorphisms gives an ∞-category.

Principle

To pass from classical to derived geometry, replace vector spaces or vector bundles by complexes of vector spaces or vector bundles up to quasi-isomorphism.

E.g. a smooth scheme or manifold X has a *tangent bundle* $TX \to X$ and *cotangent bundle* $T^*X \to X$. Derived schemes or manifolds $\boldsymbol{X}$ have a *tangent complex* $\mathbb{T}_{\boldsymbol{X}} \to \boldsymbol{X}$ and *cotangent complex* $\mathbb{L}_{\boldsymbol{X}} \to \boldsymbol{X}$. Then $\boldsymbol{X}$ is *quasi-smooth* if $\mathbb{L}_{\boldsymbol{X}}$ is concentrated in degrees $-1, 0$.

Stacks and derived stacks

Actually I have been lying a little bit. Derived schemes are not usually defined as topological spaces with an ∞-sheaf of functions. In classical Algebraic Geometry there is a more general class of spaces than schemes, called *stacks*. Stacks are needed as many moduli spaces cannot be defined as schemes, but do exist as stacks. Given a $\mathbb{K}$-scheme X, one can can define a functor $\mathbf{Alg}_{\mathbb{K}} \to \mathbf{Sets}$ acting on objects by $A \mapsto \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{K}}}(\mathrm{Spec}\, A, X)$. By the Yoneda Lemma, this embeds the category of $\mathbb{K}$-schemes $\mathbf{Sch}_{\mathbb{K}}$ as a full subcategory of the functor category $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Sets})$. Stacks are defined as a full subcategory of the functor category $\mathrm{Fun}(\mathbf{Alg}_{\mathbb{K}}, \mathbf{Groupoids})$, where *groupoids* are categories in which all morphisms are isomorphisms. (A set is a groupoid with only identity morphisms.) Groupoids encode automorphisms of objects. Then *derived stacks* are defined as a full ∞-subcategory of the ∞-functor category $\mathrm{Fun}_{\infty}(\mathbf{Alg}_{\mathbb{K}}^{\infty}, \mathbf{Groupoids}^{\infty})$. We make everything ∞-categorical: ∞-algebras (cdgas) and ∞-groupoids (the ∞-category analogue of sets). Derived schemes are defined as derived stacks whose classical stack is a scheme.

Derived Differential Geometry, a very brief history

In Jacob Lurie's epic series DAG I – DAG ∞, as a throwaway comment in the final paragraph of DAG V, he explained how to define an ∞-category of derived C^∞-schemes, and a subcategory of derived smooth manifolds. Lurie's student David Spivak worked out the details (technically beautiful, but unreadable by humans).

Some years before DAG, Fukaya–Oh–Ohta–Ono developed a theory of Symplectic Geometry (also unreadable by humans) involving *Kuranishi spaces*, the geometric structure they put on moduli spaces of J-holomorphic curves. The main purpose of a Kuranishi space was to define a virtual class/chain in homology. In the beginning there were problems with the FOOO theory (now mostly fixed), which made some people quite grumpy.

I may have been the first person to care about and read both of these unreadable theories. When I did, I realized that **Kuranishi spaces are actually derived smooth orbifolds**. This explained many of the problems in the FOOO theory: they were lacking ideas from Derived Geometry (especially, higher categories), so they were banging nails in with a screwdriver. I then wrote a lot of stuff on Derived Differential Geometry (also unreadable by humans, sorry).

The C^∞-scheme approach to Derived Differential Geometry

I was unable to get much geometric intuition for what the higher morphisms in Spivak's ∞-category of derived manifolds (as derived C^∞-schemes) really meant. So I defined a 2-category truncation of Spivak's versions, in which 1- and 2-morphisms were much more explicit. I called the 2-category version of derived C^∞ schemes *d-spaces*, and the 2-category derived manifolds *d-manifolds*. The 2-category truncation preserved all the properties of derived manifolds I cared about (e.g. existence of fibre products, gluing). This 2-category truncation would not work in DAG, where you really do need ∞-categories. The reason it works in DDG is that *partitions of unity* exist in smooth functions on manifolds. These make it easier to glue local models together up to quasi-isomorphism; you do not need the extra freedom of the ∞-category to glue together local models.

My 2-category truncation is roughly the universal truncation of the Lurie/Spivak model (which is itself universal) in which for $f : X \to Y$, 2-morphisms $\eta : f \Rightarrow f$ are a vector space and $C^\infty(X)$-module.

The Kuranishi space approach to DDG

In my d-manifold theory, the objects, 1-morphisms, and 2-morphisms all have explicit local descriptions (up to quasi-isomorphism) in terms of manifolds, vector bundles, and sections. In 2014 I realized that I could use this to define a 2-category of Kuranishi spaces, in the style of Fukaya–Oh–Ohta–Ono, which would be equivalent to the orbifold version (d-orbifolds) of my 2-category of d-manifolds, thus fixing the problems with the FOOO definition.

This Kuranishi space approach to DDG has the advantage that you can replace manifolds by many other categories of 'manifolds' satisfying a short list of assumptions, such as manifolds with corners. This is useful for applications in Symplectic Geometry, which involve various categories of Kuranishi spaces with corners. To do this in the C^∞-scheme approach you have to rewrite the foundations from the bottom up, with C^∞-rings with corners (see Francis-Staite–Joyce 2019).

The category of μ-Kuranishi spaces

Kuranishi spaces are still pretty complicated. I will explain how to define an ordinary category of 'μ-Kuranishi spaces', a simplified version using ordinary category rather than 2-category methods. It is equivalent to the homotopy category of the 2-category of derived manifolds ('m-Kuranishi spaces').

Definition

Let X be a topological space. A *μ-Kuranishi neighbourhood* on X is a quadruple (V, E, s, ψ) such that:

(a) V is a smooth manifold.

(b) $E \to V$ is a vector bundle over V, the *obstruction bundle*.

(c) $s \in \Gamma^\infty(E)$ is a smooth section of E, the *Kuranishi section*.

(d) ψ is a homeomorphism from $s^{-1}(0)$ to an open subset $\mathrm{Im}\,\psi$ in X, where $\mathrm{Im}\,\psi$ is called the *footprint* of (V, E, s, ψ).

These are the charts in our 'atlas of charts' approach.

Morphisms of μ-Kuranishi neighbourhoods

Definition

Let $f : X \to Y$ be a continuous map of topological spaces, (V_i, E_i, s_i, ψ_i), (W_j, F_j, t_j, χ_j) be μ-Kuranishi neighbourhoods on X, Y, and $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j) \subseteq X$ be an open set. Consider triples $(V_{ij}, f_{ij}, \hat f_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.

(b) $f_{ij} : V_{ij} \to W_j$ is smooth, with $f \circ \psi_i = \chi_j \circ f_{ij}$ on $s_i^{-1}(0) \cap V_{ij}$.

(c) $\hat f_{ij} : E_i|_{V_{ij}} \to f_{ij}^*(F_j)$ is a morphism of vector bundles on V_{ij}, with $\hat f_{ij}(s_i|_{V_{ij}}) = f_{ij}^*(t_j) + O(s_i^2)$.

Define an equivalence relation $\sim$ by $(V_{ij}, f_{ij}, \hat f_{ij}) \sim (V'_{ij}, f'_{ij}, \hat f'_{ij})$ if there are open $\psi_i^{-1}(S) \subseteq \dot V_{ij} \subseteq V_{ij} \cap V'_{ij}$ and $\Lambda : E_i|_{\dot V_{ij}} \to f_{ij}^*(TW_j)|_{\dot V_{ij}}$ with $f'_{ij} = f_{ij} + \Lambda \cdot s_i + O(s_i^2)$ and $\hat f'_{ij} = \hat f_{ij} + \Lambda \cdot f_{ij}^*(\mathrm{d}t_j) + O(s_i)$. We write $[V_{ij}, f_{ij}, \hat f_{ij}]$ for the $\sim$-equivalence class of $(V_{ij}, f_{ij}, \hat f_{ij})$, and call $[V_{ij}, f_{ij}, \hat f_{ij}] : (V_i, E_i, s_i, \psi_i) \to (W_j, F_j, t_j, \chi_j)$ a *morphism over* S, f.

Here the equivalence relation $\sim$ is weird, but crucial for later. Given continuous maps $f : X \to Y$ and $g : Y \to Z$, open $S \subseteq X$, $T \subseteq Y$, morphisms $[U_{ij}, \phi_{ij}, \hat{\phi}_{ij}] : (U_i, D_i, r_i, \phi_i) \to (V_j, E_j, s_j, \psi_j)$ over S, f and $[V_{jk}, \psi_{jk}, \hat{\psi}_{jk}] : (V_j, E_j, s_j, \psi_j) \to (W_k, F_k, t_k, \chi_k)$ over T, g, the *composition* over $S \cap f^{-1}(T)$, $g \circ f$ is

$$[V_{jk}, \psi_{jk}, \hat{\psi}_{jk}] \circ [U_{ij}, \phi_{ij}, \hat{\phi}_{ij}] = \left[\phi_{ij}^{-1}(V_{jk}), \psi_{jk} \circ \phi_{ij}|_{\cdots}, \phi_{ij}^{-1}(\hat{\psi}_{jk}) \circ \hat{\phi}_{ij}|_{\cdots}\right] : (U_i, D_i, r_i, \phi_i) \longrightarrow (W_k, F_k, t_k, \chi_k).$$

Theorem (Sheaf property of μ-Kuranishi morphisms.)

Let (V_i, E_i, s_i, ψ_i), (W_j, F_j, t_j, χ_j) be μ-Kuranishi neighbourhoods on X, Y, and $f : X \to Y$ be continuous. Then morphisms from (V_i, E_i, s_i, ψ_i) to (W_j, F_j, t_j, χ_j) over f form a sheaf $\mathcal{H}om_f\big((V_i, E_i, s_i, \psi_i), (W_j, F_j, t_j, \chi_j)\big)$ on $\mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j)$.

This will be essential for defining compositions of morphisms of μ-Kuranishi spaces. The lack of such a sheaf property in the FOOO theory is why FOOO Kuranishi spaces are not a category.

Coordinate changes of μ-Kuranishi neighbourhoods

Take $Y = X$ and $f = \mathrm{id}_X$. A morphism $\Phi_{ij} = [V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ over id_X is called a *coordinate change* if there exists $\Phi_{ji} = [V_{ji}, \phi_{ji}, \hat\phi_{ji}] : (V_j, E_j, s_j, \psi_j) \to (V_i, E_i, s_i, \psi_i)$ such that $\Phi_{ji} \circ \Phi_{ij} = [V_i, \mathrm{id}_{V_i}, \mathrm{id}_{E_i}]$ and $\Phi_{ij} \circ \Phi_{ji} = [V_j, \mathrm{id}_{V_j}, \mathrm{id}_{E_j}]$.
This does not require $\phi_{ji} \circ \phi_{ij} = \mathrm{id}_{V_i}$, $\hat\phi_{ji} \circ \hat\phi_{ij} = \mathrm{id}_{E_i}$, but only that $\phi_{ji} \circ \phi_{ij} = \mathrm{id}_{V_i} + \Lambda \cdot s_i + O(s_i^2)$ and $\hat\phi_{ji} \circ \hat\phi_{ij} = \mathrm{id}_{E_i} + \Lambda \cdot f_{ij}^*(\mathrm{d}t_j) + O(s_i)$.
Coordinate changes exist even if $\dim V_i \neq \dim V_j$.

Theorem

A morphism $[V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a coordinate change over S if and only if for all $x \in S$ with $v_i = \psi_i^{-1}(x)$ and $v_j = \psi_j^{-1}(x)$, the following sequence is exact:

$$0 \longrightarrow T_{v_i} V_i \xrightarrow{\mathrm{d}s_i|_{v_i} \oplus \, T_{v_i}\phi_{ij}} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat\phi_{ij}|_{v_i} \oplus -\mathrm{d}s_j|_{v_j}} E_j|_{v_j} \longrightarrow 0.$$

The definition of μ-Kuranishi space

Definition

Let X be a Hausdorff, second countable topological space, and $n \in \mathbb{Z}$. A *μ-Kuranishi structure $\mathcal{K}$ on X of virtual dimension n* is data $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i \in I}, \Phi_{ij,\ i,j \in I}\big)$, where:

(a) I is an indexing set.

(b) (V_i, E_i, s_i, ψ_i) is a μ-Kuranishi neighbourhood on X for each $i \in I$, with $\dim V_i - \operatorname{rank} E_i = n$.

(c) $\Phi_{ij} = [V_{ij}, \phi_{ij}, \hat{\phi}_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a coordinate change over $S = \operatorname{Im} \psi_i \cap \operatorname{Im} \psi_j$ for all $i, j \in I$.

(d) $\bigcup_{i \in I} \operatorname{Im} \psi_i = X$.

(e) $\Phi_{ii} = \operatorname{id}_{(V_i, E_i, s_i, \psi_i)}$ for all $i \in I$.

(f) $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$ for all $i, j, k \in I$ over $S = \operatorname{Im} \psi_i \cap \operatorname{Im} \psi_j \cap \operatorname{Im} \psi_k$.

We call $\boldsymbol{X} = (X, \mathcal{K})$ a *μ-Kuranishi space*, of *virtual dimension* $\operatorname{vdim} \boldsymbol{X} = n$.

Definition

Let $\boldsymbol{X} = (X, \mathcal{K})$ with $\mathcal{K} = \bigl(I, (V_i, E_i, s_i, \psi_i)_{i \in I}, \Phi_{ii',\ i,i' \in I}\bigr)$ and $\boldsymbol{Y} = (Y, \mathcal{L})$ with $\mathcal{L} = \bigl(J, (W_j, F_j, t_j, \chi_j)_{j \in J}, \Psi_{jj',\ j,j' \in J}\bigr)$ be μ-Kuranishi spaces. A *morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is $\boldsymbol{f} = \bigl(f, \boldsymbol{f}_{ij,\ i \in I,\ j \in J}\bigr)$, where $f : X \to Y$ is a continuous map, and $\boldsymbol{f}_{ij} = [V_{ij}, f_{ij}, \hat{f}_{ij}] : (V_i, E_i, s_i, \psi_i) \to (W_j, F_j, t_j, \chi_j)$ is a morphism of μ-Kuranishi neighbourhoods over $S = \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f for all $i \in I$, $j \in J$, satisfying the conditions:

(a) If $i, i' \in I$ and $j \in J$ then $\boldsymbol{f}_{i'j} \circ \Phi_{ii'}|_S = \boldsymbol{f}_{ij}|_S$ over $S = \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_{i'} \cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f.

(b) If $i \in I$ and $j, j' \in J$ then $\Psi_{jj'} \circ \boldsymbol{f}_{ij}|_S = \boldsymbol{f}_{ij'}|_S$ over $S = \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j \cap \mathrm{Im}\,\chi_{j'})$ and f.

When $\boldsymbol{Y} = \boldsymbol{X}$, so that $J = I$, define the identity morphism $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ by $\mathbf{id}_{\boldsymbol{X}} = \bigl(\mathrm{id}_X, \Phi_{ij,\ i,j \in I}\bigr)$.

Let $\boldsymbol{X}=(X,\mathcal{I})$ with $\mathcal{I}=\bigl(I,(U_i,D_i,r_i,\phi_i)_{i\in I},\Phi_{ii',\,i,i'\in I}\bigr)$ and $\boldsymbol{Y}=(Y,\mathcal{J})$ with $\mathcal{J}=\bigl(J,(V_j,E_j,s_j,\psi_j)_{j\in J},\Psi_{jj',\,j,j'\in J}\bigr)$ and $\boldsymbol{Z}=(Z,\mathcal{K})$ with $\mathcal{K}=\bigl(K,(W_k,F_k,t_k,\xi_k)_{k\in K},\Xi_{kk',\,k,k'\in K}\bigr)$ be μ-Kuranishi spaces, and $\boldsymbol{f}=(f,\boldsymbol{f}_{ij}):\boldsymbol{X}\to\boldsymbol{Y}$, $\boldsymbol{g}=(g,\boldsymbol{g}_{jk}):\boldsymbol{Y}\to\boldsymbol{Z}$ be morphisms. Consider the problem of how to define the composition $\boldsymbol{g}\circ\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Z}$.

For all $i\in I$ and $k\in K$, $\boldsymbol{g}\circ\boldsymbol{f}$ must contain a morphism $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}:(U_i,D_i,r_i,\phi_i)\to(W_k,F_k,t_k,\xi_k)$ defined over $S_{ik}=\mathrm{Im}\,\phi_i\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g\circ f$.

For each $j\in J$, we have a morphism $\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij}:(U_i,D_i,r_i,\phi_i)\to(W_k,F_k,t_k,\xi_k)$, but it is defined over $S_{ijk}=\mathrm{Im}\,\phi_i\cap f^{-1}(\mathrm{Im}\,\psi_j)\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g\circ f$, not over the whole of $S_{ik}=\mathrm{Im}\,\phi_i\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$.

Using the sheaf property of morphisms, we can glue these uniquely over all $j\in J$ to $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}$. So μ-Kuranishi spaces form a well behaved category.