Chenjing 7/12/25

The AKSZ formalism.

based on Calaque – Haugseng – Scheimbauer. 2108.02473.

[He is using terminology from notes by Adeel Khan.]

Recall: derived stack:

$$X : \underbrace{dAff}^{\uparrow}{}^{op} \longrightarrow \infty\text{-gpd}$$

∞-functor

derived affine schemes

$dAff = d\,CAlg^{op} = sCRing\,[W^{-1}]^{op}$

+ étale topology.

W: invert weak equivalences of simplicial sets.

satisfying a sheaf condition.

Think of a derived stack as:

scheme = sets + geometry.

derived stack = ∞-groupoid + derived geometry.

combinatorial or topological object. Encodes (higher) stackiness.

- Every derived stack will have an ∞-groupoid of points. (field-valued points)

(co)tangent complex

If X is algebraic, locally finitely presented,

$\Rightarrow \mathbb{T}_X = \mathbb{L}_X^\vee \in \operatorname{Perf}(X) \simeq$ complex of coherent sheaves

tangent complex

cotangent complex

If X is smooth scheme, $\mathbb{T}_X$, $\mathbb{L}_X$ are vector bundles.

A n-shifted symplectic structure on X: (may need characteristic zero from here on).

$\omega \in \Gamma(X, \Lambda^2 \mathbb{L}_X(n))$ + closed structure.

or: $\omega \in \mathcal{A}^{2,cl}(X, n)$ inducing $\mathbb{T}_X \xrightarrow{\sim} \mathbb{L}_X(n)$.

Typical examples.

G reductive group (or group scheme over general base)

$\Rightarrow$ BG is 2-shifted symplectic.

Because $\mathbb{T}_{BG} = \mathfrak{g}[1]$

$\mathbb{L}_{BG} \simeq \mathfrak{g}^\vee[-1]$

Choose G-invariant nondegenerate quadratic form on $\mathfrak{g}$

$\Rightarrow$ 2-symplectic structure on BG.

A : a Calabi-Yau dg-cat / k

field of characteristic zero

+ assumptions

$\rightsquigarrow$ $\mathcal{M}_A$: moduli stack of objects in A

(Toën–Vaquié)

good situation: $\mathcal{M}_A$ locally finitely presented with $(2-n)$-shifted symplectic structure.

[Recall: $\mathbb{T}_{M_A}\big|_x \simeq \mathrm{Ext}^{1+*}(x,x)$

n-Calabi-Yau structure $\Rightarrow$ $\mathrm{Ext}^i(x,x) \simeq \mathrm{Ext}^{n-i}(x,x)^\vee$.]

Example: U : smooth, $f : U \to \mathbb{A}^1$
scheme, or higher stack.
function.

$\Rightarrow$ $\mathrm{Crit}\, f$ (-1) shifted symplectic structure.

Goal: (PTVV) Consider $\mathrm{Map}(X, Y)$:
$U \mapsto \mathrm{Hom}(U \times X, Y)$

If X : d-oriented
Y : n-symplectic
and $\mathrm{Map}(X,Y)$ algebraic + locally finitely presented
$\Rightarrow$ $\mathrm{Map}(X,Y)$ is $(n-d)$-shifted symplectic

E.g. X smooth projective CY d-fold over nice field, X is d-oriented.

$\mathrm{Bun}_G(X) = \mathrm{Map}(X, BG) \Rightarrow (2-d)$-shifted symplectic.

M: oriented, compact d-mfd.

$\Rightarrow$ as a derived stack, it is d-oriented.

(geometric points of this stack $\sim \pi_0(M)$).

$\mathrm{Loc}_G(M) = \mathrm{Map}(M, BG)$

$\Rightarrow$ $2-d$ shifted symplectic structure.

G-local systems on M.

"d-oriented": annoying clash of terminology: different meanings.

i.e. DAG, orientation has

Idea: $X \xrightarrow{\pi} S$ derived stack

d-oriented $\iff$ "volume form" on fibres of π

$\iff$ global section of $\omega_{X/S}(-d)$ satisfying conditions

dualizing complex of $X \to S$

$\iff$ global section $\in (\pi_* \mathcal{O}_X)^\vee(-d)$ satisfying conditions

cohomology of fibers of π

Poincaré / Serre duality

H^0 $\qquad\qquad$ H^d

<u>Reminder:</u> standard notion induces a Poincaré duality isomorphism on cohomology.

<u>Definition:</u> a <u>d-orientation</u> on $\pi : X \to S$ is a map $[X] : \pi_* \mathcal{O}_X \longrightarrow \mathcal{O}_S(-d)$

dual description to section of $(\pi_* \mathcal{O}_X)^\vee(-d)$

map of complexes on S

(Assume $\pi_* \mathcal{E}$ is perfect $\forall\, \mathcal{E} \in \mathrm{Perf}(X)$)

such that $\forall\, \mathcal{E} \in \mathrm{Perf}(X)$

$$\pi_*(\mathcal{E}) \xrightarrow{\sim} (\pi_* \mathcal{E}^\vee)^\vee(-d)$$

(Poincaré–Serre duality)

induced by

$$\pi_* \mathcal{E}^\vee \otimes \pi_* \mathcal{E} \longrightarrow \pi_*(\mathcal{E}^\vee \otimes \mathcal{E}) \xrightarrow{\text{unit}} \pi_* \mathcal{O}_X \xrightarrow{[X]} \mathcal{O}_S(-d)$$

and this holds after base change along any $T \to S$.

Thm (PTVV) If X is d-oriented, Y n-shifted symplectic, if $Map(X,Y)$ is algebraic + locally finitely presented, then $Map(X,Y)$ is $(n-d)$ symplectic.

$B\mathbb{G}_m = \pi^*(\mathbb{G}_m)$ is 0-oriented.

$\Rightarrow$ $Grad(Y)$ is n-shifted symplectic.

<u>How proved:</u> $$\mathbb{T}_f(Map(X,Y)) = \Gamma(\mathbb{T}_f Y/X)$$

during the proof, another

$$\pi_* \xi \xrightarrow{\sim} (\pi_! \xi^\vee)^\vee(-d).$$

— The definition of d-oriented is coined to make the proof work.

Lagrangian correspondences

Goal (AKSZ construction):

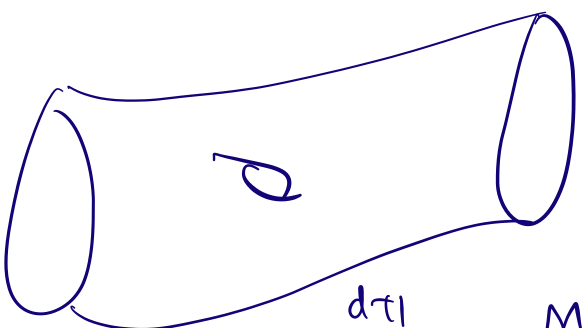
$$\underset{d\text{-oriented}}{X'} \longrightarrow X \longleftarrow \overset{d\text{-oriented}}{X''} \quad \text{"}d\text{-oriented cospan"}$$

Y n-shifted symplectic.

$$\Rightarrow \quad \underset{(n-d)\text{-symplectic}}{\mathrm{Map}(X',Y)} \longleftarrow \mathrm{Map}(X,Y) \longrightarrow \overset{(n-d)\text{-symplectic}}{\mathrm{Map}(X'',Y)}$$

$(n-d)$-Lagrangian correspondence.

E.g. cobordism of oriented manifolds



M_1^d $\quad$ N^{d+1} oriented. $\quad$ M_2^d

$$\Rightarrow \quad M_1 \longrightarrow N \longleftarrow M_2$$

d-oriented.

Example (Conjectural)

X smooth projective $(d+1)$ dim'l variety

$s \in \Gamma(X, \omega_X^{-1})$.

If $s^{-1}(0)$ is a smooth divisor, then
$\quad = X'$ $\quad X'$ is d-Calabi-Yau

$X' \longrightarrow X \circlearrowleft \phi$ is a d-oriented cospan

Conjecture: If $s^{-1}(0) = X' \cup X''$ irreducible components which intersect

$\overset{?}{\Longrightarrow}$ oriented cospan in two ways.

Eg. $* \longrightarrow \mathbb{A}^1 / \mathbb{G}_m \hookleftarrow * / \mathbb{G}_m$ is a 0-oriented cospan.

$\Rightarrow$ if Y is n-symplectic, then

$$\mathcal{Y} \longleftarrow \mathrm{Filt}(\mathcal{Y}) \longrightarrow \mathrm{Grad}(\mathcal{Y}) \leftarrow \text{n-symplectic}$$

n-symplectic is a n-Lagrangian correspondence.

e.g. If $\mathcal{Y}$ is moduli stack of objects in a dg-category, then

$\mathrm{Filt}(\mathcal{Y})$, $\mathrm{Grad}(\mathcal{Y})$ are the stacks of $\mathbb{Z}$-filtered / $\mathbb{Z}$-graded objects.

So $\mathrm{Filt}(\mathcal{Y})$ includes an open substack which is the moduli stack of short exact sequences. thick of as graded objects (E_0, E_1).

$$\mathcal{Y} \xleftarrow{\pi_2} \mathrm{Exact} \xrightarrow{\pi_1 \times \pi_3} \mathcal{Y} \times \mathcal{Y}$$

To define oriented cospans:

Definition.

$$\begin{tikzcd} X \arrow[dr, "f"] \arrow[ddr, bend right, "\pi_X"'] & & Y \arrow[dl, "g"'] \arrow[ddl, bend left, "\pi_Y"] \\ & Z \arrow[d, "\pi_Z"] & \\ & S & \end{tikzcd}$$

Cospan.

Suppose given d-orientations (X), (Y) on X, Y.

<u>Idea:</u> require $f_+(X) \sim g_+(Y)$

$[\,(X) - (Y) = [\partial Z] = d(Z)\,]$

<u>More precisely</u>

$$\begin{tikzcd} (\pi_Z)_+ \mathcal{O}_Z \arrow[r, "f^\flat"] \arrow[dr, "f_+(X)"'] & \pi_X^\flat \mathcal{O}_X \\ & \mathcal{O}_S(-d) \end{tikzcd} \qquad (X)$$

$g_+(Y)$: similar.

<u>Unoriented cobordism</u> if $f_+(X) \simeq g_+(Y)$

extra data: choose a homotopy for $f_+(X)$ to $g_+(Y)$

such that $\forall \mathcal{E} \in \mathrm{Perf}(Z)$, $(\pi_Z)_+(\mathcal{E})$

analogue of Lefschetz duality
+
Poincaré duality of manifolds with boundary.

$$
\begin{tikzcd}
 & (\pi_Z)_\flat \Sigma \arrow[dl, "f^\flat"'] \arrow[dr, "g^\flat"] & \\
(\pi_X)_\flat (f^* \Sigma) \arrow[d, "\simeq"', "(C_X)"] & & (\pi_Y)_\flat g^* \Sigma \arrow[d, "\simeq"', "(C_Y)"] \\
((\pi_X)_\flat f^* \Sigma^\vee)^\vee(-d) \arrow[rr, phantom, "\overset{\text{extra data induces}}{\underset{\text{2-morphism}}{\Longrightarrow}}"] \arrow[dr, "(f^\flat)^\vee"'] & & ((\pi_Y)_\flat g^* \Sigma^\vee)^\vee(-d) \arrow[dl, "(g^\flat)^\vee"] \\
 & ((\pi_Z)_\flat \Sigma^\vee)^\vee(-d) &
\end{tikzcd}
$$

<u>Require:</u> this square is a pull back - push forward square in stable ∞-categories $\mathrm{Perf}(S)$ (pull back $\Leftrightarrow$ push forward).

n-shifted Lagrangian correspondence: similar definition

Lagrangian correspondence.

Recall: X symplectic manifold,

$i : L \to X$ is Lagrangian if at any $x \in L$, $T_xL \subset T_xX$ is maximal isotropic

$$\Rightarrow \quad 0 \to T_xL \xrightarrow{i_x} (T_xX \simeq T_x^*X) \xrightarrow{i^*} T_x^*L \to 0$$

i.e.

$$\begin{tikzcd}
 & T_xL \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge"] & \\
0 \arrow[dr] & \text{also pullback} & T_xX \arrow[d, "\wr\simeq"] \\
 & T_x^*L & T_x^*X \arrow[l]
\end{tikzcd}$$

If X, Y are symplectic manifolds

$$X \leftarrow L \rightarrow Y$$

Lagrangian correspondence if

$$\begin{tikzcd}
 & T_xL \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge"] & \\
T_xX \arrow[d, "\wr"] & & T_xY \arrow[d, "\wr"] \\
T_x^*X \arrow[dr] & & T_x^*Y \arrow[dl] \\
 & T_x^*L &
\end{tikzcd}$$

__Def.__ For derived stacks,

$$\begin{tikzcd} X & C \arrow[l, "f"'] \arrow[r, "g"] & Y \end{tikzcd}$$

X: n-shifted symplectic

Y: n-shifted symplectic

is n-Lagrangian correspondence

if $f^*\omega_X \simeq g^*\omega_Y$

homotopy is extra data

$$\begin{tikzcd} & \mathbb{T}_C \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\wedge"] & \\ \mathbb{T}_X \cong \mathbb{L}_X(n) \arrow[dr] \arrow[rr, phantom, "\substack{\text{homotopy induces} \\ \text{2-morphism } \checkmark}"] & & \mathbb{T}_Y \cong \mathbb{L}_Y(n) \arrow[dl] \\ & \mathbb{L}_C(n) & \end{tikzcd}$$

Require this diagram to be pullback / pushout square in stable ∞-category.

Composition of Lagrangian correspondences

$$\begin{tikzcd}
 & & N \arrow[dl] \arrow[dr] \arrow[dll, bend right] \arrow[drr, bend left] & & \\
 & L \arrow[dl] \arrow[dr] & & M \arrow[dl] \arrow[dr] & \\
X & & Y & & Z
\end{tikzcd}$$

n-Lag correspondence

n-Lag cor   n-Lag cor

n-symplectic   n-symp   n-symple

Composition of oriented cobordism.

d-oriented   d-oriented   d-oriented

$$\begin{tikzcd}
T \arrow[dr] & & U \arrow[dl] \arrow[dr] & & V \arrow[dl] \\
 & P \arrow[dr] & & Q \arrow[dl] & \\
 & & R & &
\end{tikzcd}$$

oriented cobordism

does this exist?

— There is a set of conditions that makes everything work.

The $T \rightarrow R \leftarrow V$ d-oriented.

$\underline{\mathrm{Lag}}^n_S$ ∞-cat of n-Lag correspondences over S.

$\underline{Or}^d_S$ world-like categ of oriented cospans

$\underline{Or}^{d\ X\text{-good}}_S$ X fixed n-symplectic stack

X-good: T, d-oriented stk.

$\mathrm{Map}(T, X)$ stk, be finitely present + other conditions.

get well defined ∞-cat of X-oriented cospans.

<u>Thm (AKSZ)</u>: X n-symplectic

$\Rightarrow$ $\mathrm{Map}(-, X) : \underline{Or}_S^{d, X\text{ good}} \longrightarrow \underline{\mathrm{Lag}}_S^{n-d}$