Sam Moore 28/12/25

Generators in formal deformations of categories

(Blanc - Katzarkov - Pandit '17)

— Fix field k. Look at deforming a k-linear ∞-category of augmented k-algebra (everything is derived). (presentable, stable, k-linear)

Informally: deformation of $\mathcal{C}$ over A is $\tilde{\mathcal{C}}$ (A-linear) with $\tilde{\mathcal{C}} \otimes_A k \xrightarrow{\sim} \mathcal{C}$.

e.g. $Qcoh(X)$ deformations, and how they relate to X.

— Sits in general theory of "formal moduli problems"

Issue: $CatDef_{\mathcal{C}} \neq FMP$ (big problem!)

— There is an allowed FMP

$$\widehat{\mathrm{Cat\,Def}_C} \quad \text{— completion}$$

— introduces new "curved" deformations
— not actually deformations, but things that appear when you try and make it behave nicely.

<u>Main result:</u> $\mathcal{C}$ tamely compactly generated k-linear category, with compact generator E.

(e.g. X qcqs scheme, $Q(\mathrm{Coh}\,X)$)

Then the comparison map

$$\mathrm{Cat\,Def}_C^{\text{compactly gen}}(K[[t]]) \xrightarrow{\sim} \widehat{\mathrm{Cat\,Def}_C}(K[[t]])$$

$$X(K[[t]]) = \varprojlim_n X(K[t]/t^n).$$

§2 Formal Moduli Problems

Review: $\mathrm{Alg}_k^{(n)} = \{\mathbb{E}_n\text{-algs}\}$

$$\mathbb{E}_1 \leftarrow \mathbb{E}_2 \quad \cdots \quad \leftarrow \mathbb{E}_\infty$$

$\uparrow$ associative $\uparrow$ (A_∞-algebras)

$\uparrow$ coherently commutative $\uparrow$ (cdgas in ch 0)

$\mathbb{E}_2$ k-algebras $A \rightsquigarrow \mathrm{LMod}_A$ has a monoidal $(\mathbb{E}_1)$-structure.

Definition $A \in \mathrm{Alg}_k^{(n)}$ is artinian if

$\uparrow$ means: $\mathbb{E}_n$ k-algebras

Algebra objects within k-module spectra. So, $\pi_{<0}$ makes sense.

(0) local ($\pi_0 A$ local ring)

(1) connective $\pi_{<0} A \simeq 0$

(2) $\dim_k \left(\bigoplus_n \pi_n A \right) < \infty$

Definition An $\mathbb{E}_n$-FMP is a functor

$$X : \mathrm{Alg}_k^{(n),\,\mathrm{art},\,\mathrm{aug}} \longrightarrow \mathcal{S}$$

artinian augmented

$\mathcal{S}$ = spaces

s.t. (1) $X(k) \simeq *$

(2) X preserves all these

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

s.t. $\pi_0 C \twoheadrightarrow \pi_0 D \twoheadleftarrow \pi_0 B$

Intuition: (1) pt is $\mathcal{O}$, get $\mathcal{O}$.

(2)

$$\begin{tikzcd} \mathrm{Spec}\, D \arrow[r, hook] \arrow[d, "f"] & \mathrm{Spec}\, B \arrow[d] \\ \mathrm{Spec}\, C \arrow[r] \arrow[ur, phantom, "\square"] & \mathrm{Spec}\, A \end{tikzcd}$$

Unfortunately, we need to look at things which aren't $\mathbb{E}_n$-FMPs: approximate FMPs.

Definition A functor $X : \mathrm{Alg}_k^{(n),\mathrm{art},\mathrm{aug}} \to S$

is a *n-proximate* FMP $(n \geq 0)$ if

it satisfies ① and

weakened ②': for each pullback square

$$X(A) \to X(B) \times_{X(D)} X(C)$$

need not be an equivalence, but it is

$(n-2)$-truncated, *ie*

$$\pi_{>n-2}(F, x) = 0 \quad \forall x \in F$$

F: fibre of $X(A) \to X(B) \times_{X(D)} X(C)$

-2 truncated $=$ contractible. $\quad$ 0-prox FMP $=$ FMP

-1 truncated: fibre $*$ or $\emptyset$.

Remark Classical def functors are often "π_0 of a FMP."
But lots of information in higher π_n.

Proposition $\forall m, n$, $\{\mathbb{E}_n^k\text{-FMP}\} \hookrightarrow \{m\text{-proximate FMPs on } \mathbb{E}_n\text{-algs}_k\}$

$\qquad\qquad\qquad\qquad \mathrm{Moduli}_k^{(n)}$

(1) $\exists$ a left adjoint $(\ )^\wedge$

(2) $\forall$ n-prox X, $X \to X^\wedge$ is $(m-2)$ truncated.

(3) $X^\wedge(k \oplus k[\ell]) \simeq$

$\qquad \Omega^m X^\wedge(k \oplus k(\ell+m))$

$\qquad \simeq \Omega^m X(k \oplus k(\ell+m))$

So $X^\wedge$ often v. computable from X on square zero extensions

— Think of $X \to X^\wedge$ as a kind of specification.

③ Some deformation problems in DAG.

Category deformations (objects, algebras, etc.)

Let Pr^L = (∞-category of presentable ∞-categories)

every object is built out of a small set of objects.

e.g. S, Sp, Mod_A, ...

— This has a 'Lurie tensor product' $\otimes$

$$Pr^L_k = Mod_{Mod_k}(Pr^L)$$

— modules over $Mod_k^{\otimes}$ in Pr^L.

call them (presentable) 'k-linear categories' stable ∞-categories dg categories$/k$, or A_∞-categories$/k$ with some adjectives, should be equivalent to Pr_k^L.

If A is an $\mathbb{E}_2$ k-algebra, then

$LMod_A$ is monoidal

(Need $\mathbb{E}_2$ to be able to tensor modules of set of k-modules).

$$\mathrm{LinCat}_A = \mathrm{RMod}_{LMod_A^{\otimes}}(\mathrm{Pr}^L_k)$$

i.e. right modules over A in Pr^L_k

Informally: if $C \in \mathrm{Pr}^L_k$, deformation over $\mathbb{E}_2^{\text{aug}}$-$k$-alg A is

$\overline{C} \in \mathrm{LinCat}\,A$, equipped with

$\overline{C} \otimes_A k \xrightarrow{\sim} C$

<u>Note:</u> this is really a deformation over Spec A. But for good notion of derived schemes, want a $\mathbb{E}_\infty$-algebra A, not just $\mathbb{E}_2$ algebra.

So, this is considerably more general deformation than in derived schemes, etc.

This gives a deformation functor

$$\mathrm{CatDef}_C : \mathrm{Alg}^{2,\mathrm{art}}_k \longrightarrow \widehat{S} \quad (\text{large})$$

<u>Lurie:</u> CatDef_C is a 2-proximate FMP

$\exists$ examples where not FMP (Lowen–Van den Bergh).

(<u>Compare:</u> perfect of bounded functions, Only one way to sheafify, but not a sheaf.)

There are ways to 'improve' this functor.

① $\widehat{\mathrm{Cat\,Def}}_C$ (associated fmap)

$$\mathrm{Cat\,Def}_C \longrightarrow \widehat{\mathrm{Cat\,Def}}_C$$ 'cured' definition,

② $\mathrm{Cat\,Def}_C^{\text{compactly gen}} \subset \mathrm{Cat\,Def}_C$

<u>Defn</u> Call C <u>compactly generated</u> if of form $\simeq \underbrace{\mathrm{Ind}(C_0)}_{\text{subcategory of } \mathrm{Fun}(C_0^{op}, S) \text{ generated by } C_0 \text{ under filtered colimits.}}$ small C_0 with finite colimits,

C 'generated by' C_0, $C_0 \hookrightarrow C$ <u>compact objects</u>

($\mathrm{Map}(C, -)$ preserve filtered colimits).

e.g. Mod_k, $\mathrm{mod}_k^{\text{perf}}$ finitely generated.

Definition $\mathcal{C}$ a k-linear category.

Call $\mathcal{C}$ <u>tamely compactly generated</u> if it's compactly generated as above, and $\forall c, d \in \underset{\text{compact objs}}{\mathcal{C}^c}$

$$\mathrm{Ext}^n(c,d) = 0 \quad \text{for } n \gg 0.$$

— like: perfect complexes.

Expect for $QCoh\ X$, X smooth quasi-compact.

Thm: $\mathrm{CatDef}_C^{c \leftarrow \text{compactly gen}} \hookrightarrow \mathrm{CatDef}_C \to \widehat{\mathrm{CatDef}_C}$

Theorem (Lurie DAG X) the composite θ is <u>(-1) truncated</u> if $\mathcal{C}$ is tamely compactly generated.

(fibers $\emptyset$ or $*$)

Equivalently: for all A, $\mathrm{CatDef}_C^c(A)$ $\overset{\text{union of}}{\underset{\text{component of}}{\longrightarrow}}$ $\overbrace{\mathrm{CatDef}_C(A)}^{\text{connected}}$

Remark CatDef_C is 1-prox fmp.

Other deformation problems:

(1) Objects $x \in C$ a k-linear category.

infs: object $\bar{X} \in C \otimes_k A$,

equipped with iso $\bar{X} \otimes_A k \xrightarrow{\sim} X$

(category varies in a trivial way, but objects allowed to vary.)

(2) Simultaneous deformations (BKP).

Given $(C, x \in C)$, a simultaneous

deformation over A is

$\bar{C}$, $\bar{x} \in \bar{C}$ (along with compatible isomorphisms).

$\uparrow$

A-linear

$\to \mathrm{SimDef}(C, x)$.

$$\mathrm{Sim\,Ref}(C, X) \xrightarrow{(\text{forget } X)} \mathrm{Cat\,Ref}_C$$

$$\mathrm{Obj\,Ref}^U_X \longrightarrow ?x$$

this one

is a fibre sequence, as

cat space is a trivial ones for

objects)

<u>Remark</u> This remains a fibre sequence after

complete + from $(\ )^\wedge$, also

when restrict to compactly generated things,

<u>Fact:</u> that you can do algebra in

Pr^L + binary tensor product is one of the

most powerful things in higher algebra.

— Lots of stuff is derived algebra works in

Pr^L Nice clean definitions

(3) Algebra deformations.

B is an $\mathbb{E}_1$-k-algebra.

$$\mathrm{AlgDef}_B : \mathrm{Alg}_k^{(2),\mathrm{art}} \to \mathcal{S}$$

$$A \mapsto (B' \text{ an } \mathbb{E}_1\text{-alg}/A,\ B' \otimes_A k \xrightarrow{\sim} B)$$

(1-proximate FMP in general.)

from now on: [BKP].

§4. Formal deformations

<u>Definition</u> Given $X : \mathrm{Alg}_k^{(2),\mathrm{art}} \to \mathcal{S}$,

write $X(k[[t]]) := \varprojlim_n X(k[t]/t^n)$

<u>Main result:</u>

<u>Theorem</u> (BKP) Let $\mathcal{C}$ be a finitely

compactly generated k-linear category.

Suppose there exists a compact generator E.

(ie $\underline{\mathrm{Map}}(E, f) = 0 \Rightarrow f = 0$.)

eg R in R-mod

(derived) Ext sense as well as Hom

(e.g. qcqs scheme, then (Toën) $\mathrm{QCoh}\, X$ satisfies this.)

Thm $\theta : \mathrm{CatDef}_E^{cg} \longrightarrow \mathrm{CatDef}_C$

induces an equivalence

$$\theta_\infty : \mathrm{CatDef}^{cg}(K[[t]]) \xrightarrow{\sim} \widehat{\mathrm{CatDef}_C}(K[[t]])$$

"there are no curved formal deformations".

<u>Note:</u> θ_∞ is defined by hand, θ_∞ is surjective on π_0.

so suffices to prove θ_∞ is

Ideas behind the proof:

Proposition. Let C, E be as before.

$B = \underline{\mathrm{End}}_C(E)$ (endomorphism spectrum).

There are maps $\mathrm{Alg}\,\mathrm{Def}^{(A)}_B \xrightarrow{M_B} \mathrm{Cat}\,\mathrm{Def}^{(A)}_{\widehat{C}}$

$B' \longmapsto C\mathrm{Mod}_{B'}$. ($A$-linear)

$(\mathrm{or}\ A)$

$\mathrm{Sh}\,\mathrm{Def}^{(A)}(C, E) \xrightarrow{\mathcal{E}} \mathrm{Alg}\,\mathrm{Def}_{\underline{\mathrm{End}}_C(E)}^{(A)}$

$(C', E') \longmapsto \underline{\mathrm{End}}_{C'}(E')$.

Proposition Let (C, E) be as above.

$\alpha \in \widehat{\mathrm{Cat}\,\mathrm{Def}}_C(K(\mathbb{C}[\varepsilon]))$

Then $\exists$ another compact gen E^u,

and $\exists$ formal def B^u of $B = \underline{\mathrm{End}}_C(E^u)$

s.t.

$$\begin{tikzcd}
\mathrm{Alg}\,\mathrm{Def}_B(K(C(E))) \arrow[r] & \mathrm{CatDef}^c_C(K(C(E))) \arrow[d, "\theta_c"] \\
 & \widehat{\mathrm{CatDef}}_C(K(C(E)))
\end{tikzcd}$$

Sol) π_0 vanish of B^n + this of α.

Idea:

$$\begin{tikzcd}
\mathrm{Sinkf}^\wedge(C,E) \arrow[r, "\text{Koszul}", "\sim"'] \arrow[d] & \{K(n)\text{-str on }(C,E)\} \arrow[l, bend left, "\text{pullback}"] \arrow[dd, "\pi"] \\
\widehat{\mathrm{Alg}}\,\mathrm{End}(E) \arrow[d] & \\
\mathrm{CatDef}_C & = \{K(n)\text{-str on }C\} \arrow[uu, bend left, "\text{lift}"]
\end{tikzcd}$$

up to π_2

cares for actual deformation

Alg def

For α : $\Phi \curvearrowright E$ ..

kill Φ by open

$E^u = \mathrm{Fib}(\Phi)$

Then $K(n)$ acts trivially on E^u. lift E^u along π.