A Ringel–Hall type construction of vertex algebras

Dominic Joyce, Oxford University

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`http://people.maths.ox.ac.uk/~joyce/`.

Plan of talk:

1. Ringel–Hall algebras and vertex algebras

1.1. Ringel–Hall algebras

Let $\mathbb{K}$ be a field, and $\mathcal{A}$ a $\mathbb{K}$-linear abelian category satisfying some conditions, e.g. $\mathcal{A}$ could be the category $\text{mod-}\mathbb{K}Q$ of representations of a quiver Q, or the category $\text{coh}(X)$ of coherent sheaves on a smooth projective $\mathbb{K}$-scheme X. Write $\mathfrak{M}$ for the moduli stack of objects in $\mathcal{A}$, which should be an Artin $\mathbb{K}$-stack, locally of finite type, and $\mathfrak{M}(\mathbb{K})$ for the set of $\mathbb{K}$-points.

There are several versions of the *Ringel–Hall algebra* $\mathcal{H}$ associated to $\mathcal{A}$. In one version, $\mathcal{H}$ is a $\mathbb{Q}$-vector space of some class of functions $f : \mathfrak{M}(\mathbb{K}) \to \mathbb{Q}$ (e.g. functions with finite support, or constructible functions $\mathcal{H} = \text{CF}(\mathfrak{M})$) equipped with an associative multiplication $*$ making $\mathcal{H}$ into a $\mathbb{Q}$-algebra, with unit $\delta_0 : \mathfrak{M}(\mathbb{K}) \to \mathbb{Q}$ the function which is 1 on $0 \in \mathcal{A}$ and 0 otherwise.

Write $\mathfrak{Exact}$ for the moduli stack of exact sequences $E_\bullet = 0 \to E_1 \to E_2 \to E_3 \to 0$ in $\mathcal{A}$, with projections $\Pi_i : \mathfrak{Exact} \to \mathfrak{M}$ mapping $E_\bullet \to E_i$ for $i = 1, 2, 3$. Then $* : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is

$$f * g = (\Pi_2)_* \circ (\Pi_1, \Pi_3)^*(f \boxtimes g).$$

Here $\Pi_2 : \mathfrak{Exact} \to \mathfrak{M}$ is a representable morphism, and $(\Pi_1, \Pi_3) : \mathfrak{Exact} \to \mathfrak{M} \times \mathfrak{M}$ is a finite type morphism, and pushforwards (pullbacks) of $\mathcal{H}$-type functions should be defined for representable (finite type) morphisms of Artin $\mathbb{K}$-stacks.
Ringel–Hall algebras are studied in Geometric Representation Theory, for instance to construct Quantum Groups from $\mathcal{A} = \text{mod-}\mathbb{K}Q$ for Q an ADE quiver.
Note that $\mathcal{H}$ is also a *Lie algebra*, with Lie bracket $[f, g] = f * g - g * f$, the Jacobi identity follows from $*$ associative.

Ringel–Hall algebras are interesting for several reasons:

- If Q is a quiver and $\mathbb{F}_q$ is a finite field with q elements, one can identify a (twisted) Ringel–Hall algebra $\mathcal{H}(\text{mod-}\mathbb{F}_q Q)$ with the quantum group $U_{\sqrt{q}}(\mathfrak{n}_+)$ of the positive part $\mathfrak{n}_+$ of the Kac–Moody Lie algebra $\mathfrak{g}$ associated to the dimension vector lattice $\mathbb{Z}^{Q_0}$ with symmetrized intersection form $\chi(\boldsymbol{d},\boldsymbol{e})=\chi_Q(\boldsymbol{d},\boldsymbol{e})+\chi_Q(\boldsymbol{e},\boldsymbol{d})$. Morally one should get the full quantum group $U_{\sqrt{q}}(\mathfrak{g})$ from the derived category $\mathcal{H}(D^b\text{mod-}\mathbb{F}_q Q)$, but this doesn't work very well at the moment.
- Ringel–Hall (Lie) algebras are used to write wall-crossing formulae for enumerative invariants (e.g. D–T invariants) under change of stability condition (Joyce, 2004-8).
- Work by Grojnowski, Nakajima, ... constructs *representations* of interesting infinite-dimensional (Lie) algebras on homology of Hilbert schemes, etc., by what *looks like* a Ringel–Hall type construction; but usually *without* constructing the Ringel–Hall (Lie) algebra geometrically.

1.2. Vertex algebras

Let R be a commutative ring. A *vertex algebra* over R is an R-module V equipped with morphisms $D^{(n)}:V\to V$ for $n=0,1,2,\ldots$ with $D^{(0)}=\mathrm{id}_V$ and $v_n:V\to V$ for all $v\in V$ and $n\in\mathbb{Z}$, with v_n R-linear in v, and a distinguished element $\mathbb{1}\in V$ called the *identity* or *vacuum vector*, satisfying:

(i) For all $u,v\in V$ we have $u_n(v)=0$ for $n\gg 0$.
(ii) If $v\in V$ then $\mathbb{1}_{-1}(v)=v$ and $\mathbb{1}_n(v)=0$ for $-1\neq n\in\mathbb{Z}$.
(iii) If $v\in V$ then $v_n(\mathbb{1})=D^{(-n-1)}(v)$ for $n<0$ and $v_n(\mathbb{1})=0$ for $n\geqslant 0$.
(iv) $u_n(v)=\sum_{k\geqslant 0}(-1)^{k+n+1}D^{(k)}(v_{n+k}(u))$ for all $u,v\in V$ and $n\in\mathbb{Z}$, where the sum makes sense by (i), as it has only finitely many nonzero terms.
(v) $$(u_l(v))_m(w)=\sum_{n\geqslant 0}(-1)^n\binom{l}{n}\bigl(u_{l-n}(v_{m+n}(w))-(-1)^l v_{l+m-n}(u_n(w))\bigr)$$
for all $u,v,w\in V$ and $l,m\in\mathbb{Z}$, where the sum makes sense by (i).

We can also define *graded vertex algebras* and *vertex superalgebras*.

It is usual to encode the maps $u_n : V \to V$ for $n \in \mathbb{Z}$ in generating function form as R-linear maps for each $u \in V$

$$Y(u,z) : V \longrightarrow V[[z,z^{-1}]], \quad Y(u,z) : v \longmapsto \sum_{n\in\mathbb{Z}} u_n(v) z^{-n-1},$$

where z is a formal variable. The $Y(u,z)$ are called *fields*, and have a meaning in Physics. Parts (i)–(v) may be rewritten as properties of the $Y(u,z)$. One interesting property is this: for all $u,v,w \in V$ there exist $N \gg 0$ depending on u,v such that

$$(y-z)^N Y(u,y)Y(v,z)w = (y-z)^N Y(v,z)Y(u,y)w. \qquad (1)$$

There may be a V-valued rational function $R(y,z)$ with poles when $y=0$, $z=0$ and $y=z$, such that the l.h.s. of (1) is a formal Laurent series convergent to $R(y,z)$ when $0<|y|<|z|$, and the r.h.s. converges to $R(y,z)$ when $0<|z|<|y|$.
Think of $u *_z v = Y(u,z)v$ as a multiplication on V depending on a complex variable z, with poles at $z=0$. Very roughly, V is a commutative associative algebra under $*_z$, with identity $\mathbb{1}$, except the formal power series and poles make everything more complicated.

Any commutative algebra $(V,\mathbb{1},\cdot)$ with derivation D is a vertex algebra, with $Y(u,z)v = (e^{zD}u)\cdot v$, so no poles, where $u_n(v) = \bigl(\frac{1}{(n+1)!}D^{n+1}u\bigr)\cdot v$ for $n \geqslant -1$, and $u_n(v)=0$ for $n<-1$. We call such V a *commutative vertex algebra*. All non-commutative vertex algebras are infinite-dimensional, so even the simplest nontrivial examples are large, complicated objects, which are difficult to write down.
Let R be a field of characteristic zero. A *vertex operator algebra* (*VOA*) over R is a vertex algebra V over R, with a distinguished *conformal element* $\omega \in V$ and a *central charge* $c_V \in R$, such that writing $L_n = \omega_{n+1} : V_* \to V_*$, the L_n define an action of the *Virasoro algebra* on V_*, with central charge c_V, and $L_{-1} = D^{(1)}$.
VOAs are important in Physics. We will give a geometric construction of vertex algebras, but often they will *not* be VOAs, and even when they are, the conformal element ω does not appear naturally from the geometry, so far as I know.

If V is a (graded/super) vertex algebra then $V/\langle D^{(k)}(V),\ k \geqslant 1\rangle$ is a (graded/super) Lie algebra, with Lie bracket

$$\bigl[u+\langle D^{(k)}(V),\ k\geqslant 1\rangle, v+\langle D^{(k)}(V),\ k\geqslant 1\rangle\bigr]=u_0(v)+\langle D^{(k)}(V),\ k\geqslant 1\rangle.$$

Vertex algebras were introduced in mathematics by Borcherds, who noticed that certain infinite-dimensional Lie algebras important in Representation Theory were constructed as $V/\langle D^{(k)}(V),\ k \geqslant 1\rangle$. For example, Kac–Moody Lie algebras are (Lie subalgebras of) the Lie algebras associated to lattice vertex algebras.
Vertex algebras are used in Representation Theory, both of infinite-dimensional Lie algebras, and in Moonshine – the Monster may be characterized as the symmetry group of a certain infinite-dimensional vertex algebra.

2. Vertex algebras on homology of moduli stacks

Let $\mathcal{A}$ be a $\mathbb{K}$-linear abelian category as before, and $\mathfrak{M}$ the moduli stack of objects in $\mathcal{A}$, an Artin $\mathbb{K}$-stack, locally of finite type.
Suppose we have a homology theory $H_*(-)$ of Artin $\mathbb{K}$-stacks over a commutative ring R (e.g. $R=\mathbb{Q}$), satisfying some axioms.
Given some extra data on $\mathfrak{M}$, we will define a *vertex algebra* structure on the homology $H_*(\mathfrak{M})$. We also define a *graded Lie bracket* $[\,,\,]$ on $H_*(\mathfrak{M})$ (or rather, a modification of this), making $H_*(\mathfrak{M})$ into a *graded Lie (super)algebra* (with a nonstandard grading). This is analogous to the Ringel–Hall Lie algebra $(\mathrm{CF}(\mathfrak{M}),[\,,\,])$, but with $\mathrm{CF}(\mathfrak{M})$ replaced by $H_*(\mathfrak{M})$.
There are lots of interesting applications:

- Lie algebras in Geometric Representation Theory from quivers, etc.
- Explain Grojnowski–Nakajima on (co)homology of Hilbert schemes.
- Wall-crossing for virtual cycles in enumerative invariant problems.
- A differential-geometric version for use in gauge theory.

The extra data we need

We have $f * g = (\Pi_2)_* \circ (\Pi_1, \Pi_3)^*(f \boxtimes g)$ for Ringel–Hall algebras of constructible functions $\mathrm{CF}(\mathfrak{M})$. If we replace $\mathrm{CF}(\mathfrak{M})$ by $H_*(\mathfrak{M})$ then the pushforward $(\Pi_2)_*$ is natural, but the pullback $(\Pi_1, \Pi_3)^*$ is not. To define our substitute for $(\Pi_1, \Pi_3)^*$ we need some extra data, a perfect complex $\Theta^\bullet$ on $\mathfrak{M} \times \mathfrak{M}$ satisfying some assumptions; the formula for $[\,,\,]$ involves $\mathrm{rank}\,\Theta^\bullet$ and $c_i(\Theta^\bullet)$.
We also need signs $\epsilon_{\alpha,\beta}$ related to 'orientation data' for $\mathcal{A}$.
For graded antisymmetry of $[\,,\,]$ we need $\sigma^*(\Theta^\bullet) \cong (\Theta^\bullet)^\vee[2n]$ for some $n \in \mathbb{Z}$, where $\sigma : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M} \times \mathfrak{M}$ exchanges the factors, as then $c_i(\sigma^*(\Theta^\bullet)) = (-1)^i c_i(\Theta^\bullet)$.
In our examples there is a natural perfect complex $\mathcal{E}xt^\bullet$ on $\mathfrak{M} \times \mathfrak{M}$ with $H^i(\mathcal{E}xt^\bullet |_{([E],[F])}) \cong \mathrm{Ext}^i_{\mathcal{A}}(E, F)$ for $E, F \in \mathcal{A}$ and $i \in \mathbb{Z}$. If $\mathcal{A}$ is a $2n$-Calabi–Yau category then $\sigma^*((\mathcal{E}xt^\bullet)^\vee) \cong \mathcal{E}xt^\bullet[2n]$, and we put $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$. Otherwise we put $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee + \sigma^*(\mathcal{E}xt^\bullet)[2n]$.
Thus examples split into 'even Calabi–Yau' and 'general' vertex algebras.

More detail on the basic set-up

Let $K(\mathcal{A})$ be a quotient group of the Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ such that $\mathfrak{M} = \coprod_{\alpha \in K(\mathcal{A})} \mathfrak{M}_\alpha$, with $\mathfrak{M}_\alpha$ the moduli stack of objects $E \in \mathcal{A}$ in class α in $K(\mathcal{A})$, an open and closed substack in $\mathfrak{M}$.
We suppose we are given a biadditive map $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ called the *Euler form*, with $\chi(\alpha, \beta) = \chi(\beta, \alpha)$. The restriction $\Theta^\bullet_{\alpha,\beta} = \Theta^\bullet|_{\mathfrak{M}_\alpha \times \mathfrak{M}_\beta}$ should have $\mathrm{rank}\,\Theta^\bullet_{\alpha,\beta} = \chi(\alpha, \beta)$.
There should be an Artin stack morphism $\Phi : \mathfrak{M} \times \mathfrak{M} \to \mathfrak{M}$ mapping $\Phi(\mathbb{K}) : ([E], [F]) \mapsto [E \oplus F]$ on $\mathbb{K}$-points, from direct sum in $\mathcal{A}$. It is associative and commutative. In perfect complexes on $\mathfrak{M}_\alpha \times \mathfrak{M}_\beta \times \mathfrak{M}_\gamma$ for $\alpha, \beta, \gamma \in K(\mathcal{A})$ we should have

$$(\Phi_{\alpha,\beta} \times \mathrm{id}_{\mathfrak{M}_\gamma})^*(\Theta^\bullet_{\alpha+\beta,\gamma}) \cong \Pi^*_{\mathfrak{M}_\alpha \times \mathfrak{M}_\gamma}(\Theta^\bullet_{\alpha,\gamma}) \oplus \Pi^*_{\mathfrak{M}_\beta \times \mathfrak{M}_\gamma}(\Theta^\bullet_{\beta,\gamma}),$$

needed for the graded Jacobi identity for $[\,,\,]$, and corresponding to

$$\mathrm{Ext}^i_{\mathcal{A}}(E \oplus F, G)^* \cong \mathrm{Ext}^i_{\mathcal{A}}(E, G)^* \oplus \mathrm{Ext}^i_{\mathcal{A}}(F, G)^*.$$

The stack $[*/\mathbb{G}_m]$ and morphism Ψ

Write $\mathbb{G}_m = \mathbb{K} \setminus \{0\}$ as an algebraic $\mathbb{K}$-group under multiplication, and $[*/\mathbb{G}_m]$ for the quotient stack, where $* = \operatorname{Spec}\mathbb{K}$ is the point. If S is an Artin $\mathbb{K}$-stack and $s \in S(\mathbb{K})$ a $\mathbb{K}$-point there is an *isotropy group* $\operatorname{Iso}_S(s)$, an algebraic $\mathbb{K}$-group. We have $\operatorname{Iso}_{\mathfrak{M}}([E]) \cong \operatorname{Aut}(E)$ for $E \in \mathcal{A}$. There is a natural morphism $\mathbb{G}_m \to \operatorname{Aut}(E)$ mapping $\lambda \mapsto \lambda \cdot \mathrm{id}_E \in \operatorname{Aut}(E) \subset \operatorname{Hom}_{\mathcal{A}}(E,E)$. There should be an Artin stack morphism $\Psi : [*/\mathbb{G}_m] \times \mathfrak{M} \to \mathfrak{M}$ mapping $(*, [E]) \mapsto [E]$ on $\mathbb{K}$-points, and acting on isotropy groups by

$$\Psi_* : \operatorname{Iso}_{[*/\mathbb{G}_m]\times\mathfrak{M}}(*, [E]) \cong \mathbb{G}_m \times \operatorname{Aut}(E) \longrightarrow \operatorname{Iso}_{\mathfrak{M}}([E]) \cong \operatorname{Aut}(E),$$

$$\Psi_* : (\lambda, \mu) \longmapsto (\lambda \cdot \mathrm{id}_E) \circ \mu.$$

Here $[*/\mathbb{G}_m]$ is a *group stack*, and Ψ is an *action of* $[*/\mathbb{G}_m]$ *on* $\mathfrak{M}$, which is free except over $[0] \in \mathfrak{M}$. This Ψ encodes the natural morphisms $\mathbb{G}_m \to \operatorname{Iso}_{\mathfrak{M}}([E])$ for all $[E] \in \mathfrak{M}(\mathbb{K})$.

We require a compatibility between Ψ and $\Theta^\bullet$, roughly that

$$(\Psi \times \mathrm{id}_{\mathfrak{M}})^*(\Theta^\bullet) \cong \Pi^*_{[*/\mathbb{G}_m]}(L) \otimes \Pi^*_{\mathfrak{M}\times\mathfrak{M}}(\Theta^\bullet)$$

where L is the line bundle on $[*/\mathbb{G}_m]$ associated to the obvious representation of $\mathbb{G}_m$ on $\mathbb{K}$. This corresponds to $\lambda\,\mathrm{id}_E \in \operatorname{Aut}(E)$ acting by multiplication by $\lambda \in \mathbb{G}_m$ on $\operatorname{Ext}^i(E,F)^*$.

We should be given $\epsilon_{\alpha,\beta} = \pm 1$ for $\alpha, \beta \in K(\mathcal{A})$ satisfying

$$\epsilon_{\alpha,\beta} \cdot \epsilon_{\beta,\alpha} = (-1)^{\chi(\alpha,\beta)+\chi(\alpha,\alpha)\chi(\beta,\beta)},$$

$$\epsilon_{\alpha,\beta} \cdot \epsilon_{\alpha+\beta,\gamma} = \epsilon_{\alpha,\beta+\gamma} \cdot \epsilon_{\beta,\gamma}.$$

They are needed to correct signs in defining $[\,,\,]$. Such $\epsilon_{\alpha,\beta}$ always exist. They are related to 'orientation data' as follows: if we have chosen 'orientations' for $\mathfrak{M}_\alpha, \mathfrak{M}_\beta, \mathfrak{M}_{\alpha+\beta}$, then $\epsilon_{\alpha,\beta}$ should be the natural sign comparing the orientations at $[E] \in \mathfrak{M}_\alpha(\mathbb{K})$, $[F] \in \mathfrak{M}_\beta(\mathbb{K})$ and $[E \oplus F] = \Phi([E],[F]) \in \mathfrak{M}_{\alpha+\beta}(\mathbb{K})$. (See ongoing work by Cao–Gross–Joyce–Tanaka–Upmeier on orienting $\mathfrak{M}_\alpha$, in arXiv:1811.01096, arXiv:1811.02405, arXiv:1811.09658, ...)

The homology of $[*/\mathbb{G}_m]$, and its action on $H_*(\mathfrak{M})$

Let $H_*(-)$ be a homology theory of Artin $\mathbb{K}$-stacks over a commutative ring R, satisfying some natural axioms. Then

$$H_i([*/\mathbb{G}_m]) \cong \begin{cases} R, & i = 0, 2, 4, 6, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

(This holds as the 'classifying space' of $[*/\mathbb{G}_m]$ is $\mathbb{KP}^\infty$.) So we may write $H_*([*/\mathbb{G}_m]) \cong R[t]$, for t a formal variable of degree 2, such that t^n is a basis element for $H_{2n}([*/\mathbb{G}_m])$.
Let $\Omega : [*/\mathbb{G}_m] \times [*/\mathbb{G}_m] \to [*/\mathbb{G}_m]$ be the stack morphism induced by the group morphism $\omega : \mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ mapping $\omega : (\lambda, \mu) \mapsto \lambda\mu$. Define $\star : H_*([*/\mathbb{G}_m]) \times H_*([*/\mathbb{G}_m]) \to H_*([*/\mathbb{G}_m])$ by $\zeta \star \eta = H_*(\Omega)(\zeta \boxtimes \eta)$. Then $\star$ makes $H_*([*/\mathbb{G}_m]) \cong R[t]$ into a commutative R-algebra, with $t^m \star t^n = \binom{m+n}{m} t^{m+n}$.
Define $\diamond : H_*([*/\mathbb{G}_m]) \times H_*(\mathfrak{M}) \to H_*(\mathfrak{M})$ by $\zeta \diamond \theta = H_*(\Psi)(\zeta \boxtimes \theta)$. Then $\diamond$ makes $H_*(\mathfrak{M})$ into a module over $H_*([*/\mathbb{G}_m]) \cong R[t]$.

Bilinear operations $u_n(v)$ on $H_*(\mathfrak{M})$ and vertex algebras

Let $\alpha, \beta \in K(\mathcal{A})$ and $a, b \geqslant 0$, $n \in \mathbb{Z}$. Define an R-bilinear operation

$$H_a(\mathfrak{M}_\alpha) \times H_b(\mathfrak{M}_\beta) \longrightarrow H_{a+b-2n-2\chi(\alpha,\beta)-2}(\mathfrak{M}_{\alpha+\beta})$$

by, for all $u \in H_a(\mathfrak{M}_\alpha)$ and $v \in H_b(\mathfrak{M}_\beta)$,

$$u_n(v) = \sum_{\substack{i \geqslant 0:\ 2i \leqslant a+b, \\ i \geqslant n+\chi(\alpha,\beta)+1}} \begin{aligned}[t] &\epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} \cdot \\ &H_{a+b-2n-2\chi(\alpha,\beta)-2}(\Phi_{\alpha,\beta} \circ (\Psi_\alpha \times \mathrm{id}_{\mathfrak{M}_\beta})) \\ &\bigl(t^{i-n-\chi(\alpha,\beta)-1} \boxtimes \bigl[(u \boxtimes v) \cap c_i([\Theta^\bullet_{\alpha,\beta}])\bigr]\bigr), \end{aligned} \qquad (2)$$

where $t^k \in H_{2k}([*/\mathbb{G}_m])$ as above. Define $D^{(k)}(u) = t^k \diamond u$, and let the vacuum vector $\mathbb{1}$ be $1 \in H_0(\mathfrak{M}_0)$.

Theorem

All this makes $H_*(\mathfrak{M}) = \bigoplus_{\alpha \in K(\mathcal{A})} H_*(\mathfrak{M}_\alpha)$ into a **graded vertex superalgebra** *over R, with the shifted grading*

$$\tilde{H}_i(\mathfrak{M}_\alpha) = H_{i+2-\chi(\alpha,\alpha)}(\mathfrak{M}_\alpha).$$

The proof uses properties of Chern classes, and combinatorial identities.
When R is a $\mathbb{Q}$-algebra, for $u \in H_a(\mathfrak{M}_\alpha)$ and $v \in H_b(\mathfrak{M}_\beta)$, we can rewrite (2) using Chern characters $\mathrm{ch}_j(-)$ in the suggestive form

$$\begin{aligned} &Y(u,z)v = \textstyle\sum_{n\in\mathbb{Z}} u_n(v) z^{-n-1} \\ &= \epsilon_{\alpha,\beta}(-1)^{a\chi(\beta,\beta)} z^{\chi(\alpha,\beta)} \cdot H_*\bigl(\Phi_{\alpha,\beta}\circ(\Psi_\alpha\times \mathrm{id}_{\mathfrak{M}_\beta})\bigr) \\ &\Bigl\{\bigl(\sum_{i\geqslant 0} z^i t^i\bigr)\boxtimes\bigl[(u\boxtimes v)\cap \exp\bigl(\sum_{j\geqslant 1}(-1)^{j-1}(j-1)! z^{-j}\,\mathrm{ch}_j([\Theta^\bullet_{\alpha,\beta}])\bigr)\bigr]\Bigr\}. \end{aligned}$$

Question

What is the interpretation of these vertex algebras in Physics?

3. Lie algebras on homology of moduli stacks

There are many different versions of our Lie algebra construction. Here is one of the simplest, which is well known in the theory of vertex algebras. Write $I_t = \langle t, t^2, t^3, \ldots\rangle_R$ for the ideal in $H_*([*/\mathbb{G}_m]) = R[t]$ spanned over R by all positive powers of t. For each $\alpha \in K(\mathcal{A})$, define

$$H_*(\mathfrak{M}_\alpha)^{t=0} = H_*(\mathfrak{M}_\alpha)/(I_t \diamond H_*(\mathfrak{M}_\alpha)),$$

using the representation $\diamond$ of $(R[t],\star)$ on $H_*(\mathfrak{M}_\alpha)$. Now define

$$[\,,\,]^{t=0} : H_a(\mathfrak{M}_\alpha)^{t=0} \times H_b(\mathfrak{M}_\beta)^{t=0} \longrightarrow H_{a+b-2\chi(\alpha,\beta)-2}(\mathfrak{M}_{\alpha+\beta})^{t=0}$$

by $\bigl[u+(I_t\diamond H_*(\mathfrak{M}_\alpha)), v+(I_t\diamond H_*(\mathfrak{M}_\beta))\bigr]^{t=0} = u_0(v)+(I_t\diamond H_*(\mathfrak{M}_{\alpha+\beta}))$.

Define an alternative grading on $H_*(\mathfrak{M}_\alpha)^{t=0}$ by

$$\tilde{H}_i(\mathfrak{M}_\alpha)^{t=0} = H_{i+2-\chi(\alpha,\alpha)}(\mathfrak{M}_\alpha)^{t=0}.$$

Then using $\chi(\alpha,\beta)=\chi(\beta,\alpha)$ we find that $[\,,\,]^{t=0}$ maps

$$[\,,\,]^{t=0}:\tilde{H}_{\tilde{a}}(\mathfrak{M}_\alpha)^{t=0}\times\tilde{H}_{\tilde{b}}(\mathfrak{M}_\beta)^{t=0}\longrightarrow\tilde{H}_{\tilde{a}+\tilde{b}}(\mathfrak{M}_{\alpha+\beta})^{t=0}.$$

Using identities on the $u_n(v)$, we find that if $u\in\tilde{H}_{\tilde{a}}(\mathfrak{M}_\alpha)^{t=0}$, $v\in\tilde{H}_{\tilde{b}}(\mathfrak{M}_\beta)^{t=0}$ and $w\in\tilde{H}_{\tilde{c}}(\mathfrak{M}_\gamma)^{t=0}$ then

$$\begin{aligned}[v,u]^{t=0}&=(-1)^{\tilde{a}\tilde{b}+1}[u,v]^{t=0},\\ (-1)^{\tilde{c}\tilde{a}}[[u,v]^{t=0},w]^{t=0}&+(-1)^{\tilde{a}\tilde{b}}[[v,w]^{t=0},u]^{t=0}\\ &+(-1)^{\tilde{b}\tilde{c}}[[w,u]^{t=0},v]^{t=0}=0.\end{aligned}$$

That is, $[\,,\,]^{t=0}$ is a *graded Lie bracket* on $\tilde{H}_*(\mathfrak{M})^{t=0}=\bigoplus_{\alpha\in K(\mathcal{A})}\tilde{H}_*(\mathfrak{M}_\alpha)^{t=0}$, as we want.

The 'projective linear' Lie algebra

A disadvantage of the '$t=0$' version is that $H_*(\mathfrak{M})^{t=0}$ is not presented as the homology of a nice space. The 'projective linear' version corrects this. Recall that $[*/\mathbb{G}_m]$ is a group stack, and $\Psi:[*/\mathbb{G}_m]\times\mathfrak{M}\to\mathfrak{M}$ is an action of $[*/\mathbb{G}_m]$ on $\mathfrak{M}$, which is free on $\mathfrak{M}'=\mathfrak{M}\setminus\{[0]\}$. We can form a quotient $\mathfrak{M}^{\rm pl}=\mathfrak{M}'/[*/\mathbb{G}_m]$ called the 'projective linear moduli stack', with a morphism $\Pi^{\rm pl}:\mathfrak{M}'\to\mathfrak{M}^{\rm pl}$ which is a principal $[*/\mathbb{G}_m]$-bundle.
Then $\mathbb{K}$-points of $\mathfrak{M}^{\rm pl}$ are isomorphism classes $[E]$ of nonzero $E\in\mathcal{A}$, and isotropy groups are

$$\mathrm{Iso}_{\mathfrak{M}^{\rm pl}}([E])\cong\mathrm{Aut}(E)/(\mathbb{G}_m\cdot\mathrm{id}_E).$$

That is, we make $\mathfrak{M}^{\rm pl}$ from $\mathfrak{M}'$ by quotienting out $\mathbb{G}_m$ from each isotropy group, a process called 'rigidification'. For moduli of stable coherent sheaves, the stable moduli scheme is the rigidification of the stable moduli stack.

Under some assumptions (including R a $\mathbb{Q}$-algebra) we can show that $H_*(\Pi^{\mathrm{pl}}) : H_*(\mathfrak{M}') \to H_*(\mathfrak{M}^{\mathrm{pl}})$ induces an isomorphism $H_*(\mathfrak{M}')^{t=0} \cong H_*(\mathfrak{M}^{\mathrm{pl}})$. Thus, the Lie bracket $[\,,\,]^{t=0}$ on $H_*(\mathfrak{M}')^{t=0}$ induces a Lie bracket $[\,,\,]^{\mathrm{pl}}$ on $H_*(\mathfrak{M}^{\mathrm{pl}})$. Actually, even without an isomorphism $H_*(\mathfrak{M}')^{t=0} \cong H_*(\mathfrak{M}^{\mathrm{pl}})$ we can define a graded Lie bracket $[\,,\,]^{\mathrm{pl}}$ on $H_*(\mathfrak{M}^{\mathrm{pl}})$ in a different way.
Here $[\,,\,]^{\mathrm{pl}}$ is graded for the alternative grading

$$\tilde{H}_i(\mathfrak{M}^{\mathrm{pl}}_\alpha) = H_{i+2-\chi(\alpha,\alpha)}(\mathfrak{M}^{\mathrm{pl}}_\alpha).$$

We should interpret $2-\chi(\alpha,\alpha)$ as the (homological) *virtual dimension* of $\mathfrak{M}^{\mathrm{pl}}_\alpha$, where the 2 is the (real) dimension of $\mathbb{G}_m$, which we quotiented from the isotropy groups to make $\mathfrak{M}^{\mathrm{pl}}$.
There is also a triangulated category version of the construction, using higher stacks, which we can apply to moduli of objects in categories such as $D^b\,\mathrm{coh}(X)$ for X a smooth projective $\mathbb{K}$-scheme.

4. Examples from quivers and coherent sheaves

Let $Q=(Q_0,Q_1,h,t)$ be a quiver and $\mathbb{K}=\mathbb{C}$, and apply our constructions to the abelian category $\mathcal{A}=\mathrm{mod}\text{-}\mathbb{C}Q$ and the triangulated category $\mathcal{T}=D^b\,\mathrm{mod}\text{-}\mathbb{C}Q$. Write $K(\mathcal{A})=K(\mathcal{T})=\mathbb{Z}^{Q_0}$ for the lattice of dimension vectors. Define $\chi : K(\mathcal{A})\times K(\mathcal{A})\to\mathbb{Z}$ by $\chi(\boldsymbol{d},\boldsymbol{e})=\sum_{v,w\in Q_0} a_{vw}\boldsymbol{d}(v)\boldsymbol{e}(w)$, where $a_{vw}=2\delta_{vw}-n_{vw}-n_{wv}$, for n_{vw} the number of edges $\overset{v}{\bullet}\to\overset{w}{\bullet}$ in Q, so that $A=(a_{vw})_{v,w\in Q_0}$ is the generalized Cartan matrix of Q. Write $\mathfrak{M}$ and $\bar{\mathfrak{M}}$ for the (higher) moduli stacks of objects in $\mathcal{A}$ and $\mathcal{T}$. Then we can work everything out very explicitly. We find:

- The vertex algebra $H_*(\bar{\mathfrak{M}})$ is the lattice vertex algebra of $(\mathbb{Z}^{Q_0},\chi)$.
- The full Lie algebra $\tilde{H}_*(\bar{\mathfrak{M}}^{\mathrm{pl}})$ is rather large, but (for Q with no vertex loops) $\tilde{H}_0(\bar{\mathfrak{M}}^{\mathrm{pl}})$ contains the derived Kac–Moody algebra $\mathfrak{g}'(A)$ with Cartan matrix A, with $\tilde{H}_0(\bar{\mathfrak{M}}^{\mathrm{pl}})=\mathfrak{g}'(A)$ if A is positive definite. Similarly, $\tilde{H}_0(\mathfrak{M}^{\mathrm{pl}})$ contains/equals the positive part $\mathfrak{n}_+$ of $\mathfrak{g}'(A)$.
- If $Q=\bullet$ has one vertex and no edges then $\tilde{H}_0(\bar{\mathfrak{M}}^{\mathrm{pl}})\cong\mathfrak{sl}(2,\mathbb{C})$.

Let X be a smooth projective $\mathbb{C}$-scheme, and apply our theory to the abelian category $\mathcal{A} = \mathrm{coh}(X)$, with moduli stack $\mathfrak{M}$, and the triangulated category $\mathcal{T} = D^b \mathrm{coh}(X)$, with moduli stack $\bar{\mathfrak{M}}$. We either take X to be $2n$-Calabi–Yau and set $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee$, or we set $\Theta^\bullet = (\mathcal{E}xt^\bullet)^\vee + \sigma^*(\mathcal{E}xt^\bullet)$ for any X. Note that if X is $(2n+1)$-Calabi–Yau this gives $c_i(\Theta^\bullet) = 0$, so our vertex algebras and Lie algebras are abelian, and boring.

I haven't worked out the details yet, but here are some highlights:

- For some nice classes of X (e.g. curves, some surfaces) we can compute $H_*(\bar{\mathfrak{M}})$ fairly explicitly as a vertex algebra. It is the tensor product of a lattice-type vertex algebra defined using $K^0(X)$ or $H^{\mathrm{even}}(X)$, and a fermion vertex algebra defined using $K^1(X)$ or $H^{\mathrm{odd}}(X)$. For general X we can produce vertex algebra morphisms from $H_*(\mathfrak{M}), H_*(\bar{\mathfrak{M}})$ to an explicit vertex algebra of this type.
- The Heisenberg algebra acting on homology of Hilbert schemes in Grojnowski–Nakajima should appear as a Lie subalgebra of $\tilde{H}_*(\bar{\mathfrak{M}}^{\mathrm{pl}}_{\dim 0})$ for dimension 0 sheaves and complexes on X.

- When X is a Calabi–Yau 4-fold, one can define Donaldson–Thomas type invariants 'counting' moduli spaces $\mathfrak{M}^{\mathrm{ss}}(\alpha)$ of (semi)stable coherent sheaves on X (Borisov–Joyce, Cao–Leung). We can think of these (i.e. the virtual classes of the moduli spaces) as taking values in $H_*(\mathfrak{M}^{\mathrm{pl}})$.

I have a conjecture that the wall-crossing formula for these DT4 invariants under change of stability condition may be written using the Lie bracket $[\,,\,]^{\mathrm{pl}}$ on $H_*(\mathfrak{M}^{\mathrm{pl}})$, using the same universal wall-crossing formula in a Lie algebra that appears in my previous work on motivic and DT3 invariants.

The way I discovered the vertex algebra structure on $H_*(\mathfrak{M})$ was while trying to write down this Lie bracket $[\,,\,]^{\mathrm{pl}}$, I accidentally reinvented the Borcherds definition of vertex algebra, without knowing what these were at the time.

I expect the same wall-crossing formula will also work for Mochizuki invariants counting semistable coherent sheaves on surfaces, and other classes of enumerative invariants with wall-crossing.