A new theory of Kuranishi spaces

Dominic Joyce, Oxford University

May 2016

based on arXiv:1409.6908, arXiv:1509.05672, arXiv:1510.07444, and work in progress.
Also see website
`people.maths.ox.ac.uk/~joyce/dmanifolds.html`.
These slides available at
`people.maths.ox.ac.uk/~joyce/talks.html`.

1. Introduction

Several important areas in Symplectic Geometry are concerned with *moduli spaces of stable J-holomorphic curves*. Here (S,ω) is a symplectic manifold, J is a choice of almost complex structure on S compatible with ω. We have to consider *moduli spaces* $\bar{\mathcal{M}}(S,J,\beta)$ of J-holomorphic curves $u:\Sigma\rightarrow S$ in S, where Σ is a Riemann surface, and β is some fixed topological data for Σ,u. By including 'stable' curves with nodal singularities, we can usually make $\bar{\mathcal{M}}(S,J,\beta)$ compact. The idea is to 'count' the moduli space $\bar{\mathcal{M}}(S,J,\beta)$ to get a 'number' of J-holomorphic curves with topological data β. This 'number' could be in $\mathbb{Z}$, or in $\mathbb{Q}$, or a chain or homology class in some homology theory. To do this 'counting' we need an extra geometric structure on $\bar{\mathcal{M}}(S,J,\beta)$. The goal is to make the 'number' of J-holomorphic curves independent of choice of J (possibly up to some kind of homotopy in a homology theory), so that it is an *invariant* of (S,ω).

Any such J-holomorphic curve programme must solve four problems:
(a) Define the kind of geometric structure $\mathcal{G}$ you want to put on moduli spaces $\bar{\mathcal{M}}(S, J, \beta)$.
(b) Prove that moduli spaces $\bar{\mathcal{M}}(S, J, \beta)$ really do have geometric structure $\mathcal{G}$. Prove that relationships between curve moduli spaces (e.g. boundary formulae) lift to relations between the structures $\mathcal{G}$.
(c) Prove that given a compact topological space X with structure $\mathcal{G}$ you can define a 'number of points' in X, in $\mathbb{Z}$ or $\mathbb{Q}$ or some homology theory. (This is a *virtual cycle* or *virtual chain* construction.)
(d) Derive some interesting consequences in symplectic geometry – define Gromov–Witten invariants, Lagrangian Floer cohomology, Fukaya categories, Symplectic Field Theory, etc.

Of these, (b) is the most difficult. My focus today is on (a),(c).

There are broadly three current approaches in the literature:
(i) (The Fukaya school.) Geometric structure $\mathcal{G}$ is called a *Kuranishi space*. Fukaya–Ono 1999, Fukaya–Oh–Ohta–Ono 2009–.
(ii) (The Hofer school.) Geometric structure $\mathcal{G}$ is called a *polyfold*. (Actually, a 'Fredholm section of a polyfold bundle over a polyfold'.) Hofer, Wysocki and Zehnder 2005–2025 (?).
(iii) (The rest of the world.) Make strong assumptions on geometry, e.g. (S, ω) exact, J generic. Then ensure that moduli spaces $\bar{\mathcal{M}}(S, J, \beta)$ are manifolds (or at least 'pseudomanifolds').

Kuranishi spaces and polyfolds are philosophically opposed: Kuranishi spaces remember only minimal information about the moduli problem, but polyfolds remember essentially everything. There has been a lot of debate about the Fukaya–Oh–Ohta–Ono (FOOO) theory. The early versions certainly had serious problems. Their more recent work looks correct to me, as far as I can tell.

Derived Differential Geometry and Kuranishi spaces

The Derived Algebraic Geometry of Jacob Lurie and Toën-Vezzosi is a major subject. It studies 'derived schemes' and 'derived stacks', enhanced versions of classical schemes and stacks with a richer geometric structure. In moduli problems, this geometric structure remembers about obstructions as well as deformations.

There is a much less well-known subject of Derived Differential Geometry, the study of 'derived' smooth manifolds and orbifolds. It began in 2008 with the thesis of Lurie's student David Spivak on derived smooth manifolds, which was simplified by Borisov–Noël (2011, 2012) and myself (2010-).

If you know FOOO Kuranishi spaces well, and you read Spivak's thesis on derived manifolds, you immediately have a revelation:

- FOOO Kuranishi spaces ought to be defined to be derived orbifolds (possibly with corners).
- Some of the problems in the FOOO theory are due to a lack of 'derived' ideas, especially the need for higher categories.

The first definitions of derived manifolds (Spivak; Borisov–Noël; my 2-category of 'd-manifolds' **dMan**) were special examples of 'derived C^∞-schemes', derived schemes over C^∞-rings, using heavy machinery from algebraic geometry. However, I now have a new definition (arXiv:1409.6908) based on FOOO Kuranishi spaces, which I will explain today. The two definitions are equivalent in the sense that my 'm-Kuranishi spaces' form a 2-category **mKur** with an equivalence **dMan** $\simeq$ **mKur**.

To see why there should be two definitions, note that an n-manifold structure on a Hausdorff, second countable topological space X is:

(a) A sheaf $\mathcal{O}_X$ of $\mathbb{R}$-algebras locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$, for $\mathcal{O}_{\mathbb{R}^n}$ the sheaf of smooth $f : \mathbb{R}^n \to \mathbb{R}$. (Compare d-manifolds.)

(b) An atlas of charts $\{(V_i, \psi_i) : i \in I\}$, where $V_i \subseteq \mathbb{R}^n$ is open, and $\psi_i : V_i \to X$ is a homeomorphism with open $\operatorname{Im}\psi_i \subseteq X$ for $i \in I$, and $\psi_j^{-1} \circ \psi_i : \psi_i^{-1}(\operatorname{Im}\psi_j) \to \psi_j^{-1}(\operatorname{Im}\psi_i)$ is a diffeomorphism for $i, j \in I$. (Compare Kuranishi spaces.)

2. The definition of μ-Kuranishi spaces

I define 2-categories **mKur** of *m-Kuranishi spaces*, the 'manifold' version, and **Kur** of *Kuranishi spaces*, the 'orbifold' version, with **mKur** $\subset$ **Kur**. I also define a simplified ordinary category $\boldsymbol{\mu}$**Kur** of 'μ-Kuranishi spaces', with $\boldsymbol{\mu}$**Kur** $\simeq \mathrm{Ho}($**mKur**$)$. For simplicity, today I will only explain μ-Kuranishi spaces.

Definition 2.1

Let X be a topological space. A *μ-Kuranishi neighbourhood* on X is a quadruple (V, E, s, ψ) such that:

(a) V is a smooth manifold.

(b) $\pi : E \to V$ is a vector bundle over V, the *obstruction bundle*.

(c) $s \in C^\infty(E)$ is a smooth section of E, the *Kuranishi section*.

(d) ψ is a homeomorphism from $s^{-1}(0)$ to an open subset $\mathrm{Im}\,\psi$ in X, where $\mathrm{Im}\,\psi$ is called the *footprint* of (V, E, s, ψ).

If $S \subseteq X$ is open, we call (V, E, s, ψ) a *μ-Kuranishi neighbourhood over S* if $S \subseteq \mathrm{Im}\,\psi \subseteq X$.

Definition 2.2

Let X be a topological space, (V_i, E_i, s_i, ψ_i), (V_j, E_j, s_j, ψ_j) be μ-Kuranishi neighbourhoods on X, and $S \subseteq \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \subseteq X$ be an open set. Consider triples $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.

(b) $\phi_{ij} : V_{ij} \to V_j$ is smooth, with $\psi_i = \psi_j \circ \phi_{ij}$ on $s_i^{-1}(0) \cap V_{ij}$.

(c) $\hat\phi_{ij} : E_i|_{V_{ij}} \to \phi_{ij}^*(E_j)$ is a morphism of vector bundles on V_{ij}, with $\hat\phi_{ij}(s_i|_{V_{ij}}) = \phi_{ij}^*(s_j) + O(s_i^2)$.

Define an equivalence relation $\sim$ on such triples $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$ by $(V_{ij}, \phi_{ij}, \hat\phi_{ij}) \sim (V'_{ij}, \phi'_{ij}, \hat\phi'_{ij})$ if there are open $\psi_i^{-1}(S) \subseteq \dot V_{ij} \subseteq V_{ij} \cap V'_{ij}$ and a morphism $\Lambda : E_i|_{\dot V_{ij}} \to \phi_{ij}^*(TV_j)|_{\dot V_{ij}}$ of vector bundles on $\dot V_{ij}$ satisfying $\phi'_{ij} = \phi_{ij} + \Lambda \cdot s_i + O(s_i^2)$ and $\hat\phi'_{ij} = \hat\phi_{ij} + \Lambda \cdot \phi_{ij}^*(\mathrm{d}s_j) + O(s_i)$ on $\dot V_{ij}$. We write $[V_{ij}, \phi_{ij}, \hat\phi_{ij}]$ for the $\sim$-equivalence class of $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$, and call $[V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ a *morphism of μ-Kuranishi neighbourhoods over S*.

Given morphisms $[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$, $[V_{jk},\phi_{jk},\hat\phi_{jk}]:(V_j,E_j,s_j,\psi_j)\rightarrow(V_k,E_k,s_k,\psi_k)$ of μ-Kuranishi neighbourhoods over $S\subseteq X$, the *composition* is

$$[V_{jk},\phi_{jk},\hat\phi_{jk}]\circ[V_{ij},\phi_{ij},\hat\phi_{ij}]=\bigl[\phi_{ij}^{-1}(V_{jk}),\phi_{jk}\circ\phi_{ij}|_{\cdots},\phi_{ij}^{-1}(\hat\phi_{jk})\circ\hat\phi_{ij}|_{\cdots}\bigr]:\\ (V_i,E_i,s_i,\psi_i)\longrightarrow(V_k,E_k,s_k,\psi_k).$$

Then μ-Kuranishi neighbourhoods over $S\subseteq X$ form a category $\mathrm{mKur}_S(X)$. We call $[V_{ij},\phi_{ij},\hat\phi_{ij}]$ a *μ-coordinate change over S* if it is an isomorphism in $\mathrm{mKur}_S(X)$. We have:

Theorem 2.3

A morphism $[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ is a μ-coordinate change over S if and only if for all $x\in S$ with $v_i=\psi_i^{-1}(x)$ and $v_j=\psi_j^{-1}(x)$, the following sequence is exact:

$$0\longrightarrow T_{v_i}V_i\xrightarrow{\mathrm{d}s_i|_{v_i}\oplus T_{v_i}\phi_{ij}}E_i|_{v_i}\oplus T_{v_j}V_j\xrightarrow{\hat\phi_{ij}|_{v_i}\oplus-\mathrm{d}s_j|_{v_j}}E_j|_{v_j}\longrightarrow 0.$$

The sheaf property of morphisms

Theorem 2.4

Let (V_i,E_i,s_i,ψ_i), (V_j,E_j,s_j,ψ_j) be μ-Kuranishi neighbourhoods on X. For each open $S\subseteq\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$, write $\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)(S)$ for the set of morphisms $\Phi_{ij}:(V_i,E_i,s_i,\psi_i)\rightarrow(V_j,E_j,s_j,\psi_j)$ over S, and for all open $T\subseteq S\subseteq\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$ define

$$\rho_{ST}:\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)(S)\longrightarrow\\ \mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)(T)\ \text{ by }\ \rho_{ST}:\Phi_{ij}\longmapsto\Phi_{ij}|_T.$$

Then $\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)$ is a sheaf of sets on $\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$. Similarly, μ-coordinate changes from (V_i,E_i,s_i,ψ_i) to (V_j,E_j,s_j,ψ_j) are a subsheaf of $\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)$.

This is not obvious. It means we can glue (iso)morphisms of μ-Kuranishi neighbourhoods over the sets of an open cover.

Definition 2.5

Let X be a Hausdorff, second countable topological space, and $n \in \mathbb{Z}$. A *μ-Kuranishi structure $\mathcal{K}$ on X of virtual dimension n* is data $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\ i,j\in I}\big)$, where:

(a) I is an indexing set.

(b) (V_i, E_i, s_i, ψ_i) is a μ-Kuranishi neighbourhood on X for each $i \in I$, with $\dim V_i - \operatorname{rank} E_i = n$.

(c) $\Phi_{ij} = [V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a μ-coordinate change over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ for all $i, j \in I$.

(d) $\bigcup_{i\in I} \operatorname{Im}\psi_i = X$.

(e) $\Phi_{ii} = \mathrm{id}_{(V_i,E_i,s_i,\psi_i)}$ for all $i \in I$.

(f) $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$ for all $i, j, k \in I$ over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$.

We call $\boldsymbol{X} = (X, \mathcal{K})$ a *μ-Kuranishi space*, of *virtual dimension* $\operatorname{vdim}\boldsymbol{X} = n$.

Definition 2.6

Let $f : X \to Y$ be a continuous map of topological spaces, (V_i, E_i, s_i, ψ_i), (W_j, F_j, t_j, χ_j) be μ-Kuranishi neighbourhoods on X, Y, and $S \subseteq \operatorname{Im}\psi_i \cap f^{-1}(\operatorname{Im}\chi_j) \subseteq X$ be an open set. Consider triples $(V_{ij}, f_{ij}, \hat f_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.

(b) $f_{ij} : V_{ij} \to W_j$ is smooth, with $f \circ \psi_i = \chi_j \circ f_{ij}$ on $s_i^{-1}(0) \cap V_{ij}$.

(c) $\hat f_{ij} : E_i|_{V_{ij}} \to f_{ij}^*(F_j)$ is a morphism of vector bundles on V_{ij}, with $\hat f_{ij}(s_i|_{V_{ij}}) = f_{ij}^*(t_j) + O(s_i^2)$.

Define an equivalence relation $\sim$ by $(V_{ij}, f_{ij}, \hat f_{ij}) \sim (V'_{ij}, f'_{ij}, \hat f'_{ij})$ if there are open $\psi_i^{-1}(S) \subseteq \dot V_{ij} \subseteq V_{ij} \cap V'_{ij}$ and $\Lambda : E_i|_{\dot V_{ij}} \to f_{ij}^*(TW_j)|_{\dot V_{ij}}$ with $f'_{ij} = f_{ij} + \Lambda\cdot s_i + O(s_i^2)$ and $\hat f'_{ij} = \hat f_{ij} + \Lambda\cdot f_{ij}^*(\mathrm{d}t_j) + O(s_i)$. We write $[V_{ij}, f_{ij}, \hat f_{ij}]$ for the $\sim$-equivalence class of $(V_{ij}, f_{ij}, \hat f_{ij})$, and call $[V_{ij}, f_{ij}, \hat f_{ij}] : (V_i, E_i, s_i, \psi_i) \to (W_j, F_j, t_j, \chi_j)$ a *morphism over S, f*.

When $Y = X$ and $f = \mathrm{id}_X$, this recovers the notion of morphisms of μ-Kuranishi neighbourhoods on X. We have the obvious notion of compositions of morphisms of μ-Kuranishi neighbourhoods over $f : X \to Y$ and $g : Y \to Z$.
Here is the generalization of Theorem 2.4:

Theorem 2.7

Let (V_i, E_i, s_i, ψ_i), (W_j, F_j, t_j, χ_j) be μ-Kuranishi neighbourhoods on X, Y, and $f : X \to Y$ be continuous. Then morphisms from (V_i, E_i, s_i, ψ_i) to (W_j, F_j, t_j, χ_j) over f form a sheaf $\mathcal{H}om_f\big((V_i, E_i, s_i, \psi_i), (W_j, F_j, t_j, \chi_j)\big)$ on $\mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j)$.

This will be essential for defining compositions of morphisms of μ-Kuranishi spaces. The lack of such a sheaf property in the FOOO theory is why FOOO Kuranishi spaces are not a category.

Definition 2.8

Let $\boldsymbol{X} = (X, \mathcal{K})$ with $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i \in I}, \Phi_{ii',\, i,i' \in I}\big)$ and $\boldsymbol{Y} = (Y, \mathcal{L})$ with $\mathcal{L} = \big(J, (W_j, F_j, t_j, \chi_j)_{j \in J}, \Psi_{jj',\, j,j' \in J}\big)$ be μ-Kuranishi spaces. A *morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is $\boldsymbol{f} = \big(f, \boldsymbol{f}_{ij,\, i \in I,\, j \in J}\big)$, where $f : X \to Y$ is a continuous map, and $\boldsymbol{f}_{ij} = [V_{ij}, f_{ij}, \hat{f}_{ij}] : (V_i, E_i, s_i, \psi_i) \to (W_j, F_j, t_j, \chi_j)$ is a morphism of μ-Kuranishi neighbourhoods over $S = \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f for all $i \in I$, $j \in J$, satisfying the conditions:

(a) If $i, i' \in I$ and $j \in J$ then $\boldsymbol{f}_{i'j} \circ \Phi_{ii'}|_S = \boldsymbol{f}_{ij}|_S$ over $S = \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_{i'} \cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f.

(b) If $i \in I$ and $j, j' \in J$ then $\Psi_{jj'} \circ \boldsymbol{f}_{ij}|_S = \boldsymbol{f}_{ij'}|_S$ over $S = \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j \cap \mathrm{Im}\,\chi_{j'})$ and f.

When $\boldsymbol{Y} = \boldsymbol{X}$, so that $J = I$, define the identity morphism $\mathbf{id}_{\boldsymbol{X}} : \boldsymbol{X} \to \boldsymbol{X}$ by $\mathbf{id}_{\boldsymbol{X}} = \big(\mathrm{id}_X, \Phi_{ij,\, i,j \in I}\big)$.

Composition of morphisms in $\boldsymbol{\mu}$Kur

Let $\boldsymbol{X} = (X, \mathcal{I})$ with $\mathcal{I} = \bigl(I, (U_i, D_i, r_i, \phi_i)_{i\in I}, \Phi_{ii',\, i,i'\in I}\bigr)$ and $\boldsymbol{Y} = (Y, \mathcal{J})$ with $\mathcal{J} = \bigl(J, (V_j, E_j, s_j, \psi_j)_{j\in J}, \Psi_{jj',\, j,j'\in J}\bigr)$ and $\boldsymbol{Z} = (Z, \mathcal{K})$ with $\mathcal{K} = \bigl(K, (W_k, F_k, t_k, \xi_k)_{k\in K}, \Xi_{kk',\, k,k'\in K}\bigr)$ be μ-Kuranishi spaces, and $\boldsymbol{f} = (f, \boldsymbol{f}_{ij}) : \boldsymbol{X} \to \boldsymbol{Y}$, $\boldsymbol{g} = (g, \boldsymbol{g}_{jk}) : \boldsymbol{Y} \to \boldsymbol{Z}$ be morphisms. Consider the problem of how to define the composition $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$.

For all $i \in I$ and $k \in K$, $\boldsymbol{g} \circ \boldsymbol{f}$ must contain a morphism $(\boldsymbol{g} \circ \boldsymbol{f})_{ik} : (U_i, D_i, r_i, \phi_i) \to (W_k, F_k, t_k, \xi_k)$ defined over $S_{ik} = \mathrm{Im}\,\phi_i \cap (g \circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g \circ f$.

For each $j \in J$, we have a morphism $\boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij} : (U_i, D_i, r_i, \phi_i) \to (W_k, F_k, t_k, \xi_k)$, but it is defined over $S_{ijk} = \mathrm{Im}\,\phi_i \cap f^{-1}(\mathrm{Im}\,\psi_j) \cap (g \circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g \circ f$, not over the whole of $S_{ik} = \mathrm{Im}\,\phi_i \cap (g \circ f)^{-1}(\mathrm{Im}\,\xi_k)$.

Composition of morphisms in $\boldsymbol{\mu}$Kur

The solution is to use the sheaf property of morphisms, Theorem 2.7. The sets S_{ijk} for $j \in J$ form an open cover of S_{ik}. Using Definition 2.8(a),(b) we can show that $\boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij}|_{S_{ijk}\cap S_{ij'k}} = \boldsymbol{g}_{j'k} \circ \boldsymbol{f}_{ij'}|_{S_{ijk}\cap S_{ij'k}}$. Therefore by Theorem 2.7 there is a unique morphism of μ-Kuranishi neighbourhoods $(\boldsymbol{g} \circ \boldsymbol{f})_{ik} : (U_i, D_i, r_i, \phi_i) \to (W_k, F_k, t_k, \xi_k)$ defined over S_{ik} and $g \circ f$ with $(\boldsymbol{g} \circ \boldsymbol{f})_{ik}|_{S_{ijk}} = \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij}$ for all $j \in J$. We show that $\boldsymbol{g} \circ \boldsymbol{f} := \bigl(g \circ f, (\boldsymbol{g} \circ \boldsymbol{f})_{ik,\, i\in I,\, k\in K}\bigr)$ is a morphism $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ of μ-Kuranishi spaces, which we call *composition*.

Composition is associative, and makes μ-Kuranishi spaces into a category $\boldsymbol{\mu}$**Kur**.

Comparison with m-Kuranishi spaces and Kuranishi spaces

To define **mKur** and **Kur** instead of $\boldsymbol{\mu}$**Kur**, we must work with 2-categories throughout. So we define (m-)Kuranishi neighbourhoods (V_i, E_i, s_i, ψ_i) or $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ on X, and 1-morphisms Φ_{ij} between them, and 2-morphisms $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ between 1-morphisms. A Kuranishi structure $\mathcal{K}$ on X assigns an open cover of X by (m-)Kuranishi neighbourhoods, with coordinate changes Φ_{ij} on double overlaps $\mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$, and 2-morphisms $\Lambda_{ijk} : \Phi_{jk} \circ \Phi_{ij} \Rightarrow \Phi_{ik}$ on triple overlaps $\mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \cap \mathrm{Im}\,\psi_k$. This leads to an awful lot of notation.

We need 2-categories to do orbifolds/Kuranishi spaces properly (the analogues of Theorems 2.4 and 2.7 are false in the orbifold case if we try to work with ordinary categories). Also, some important constructions such as fibre products of (m-)Kuranishi spaces need the 2-category structure, and don't work in $\boldsymbol{\mu}$**Kur**, or Ho(**mKur**), or Ho(**Kur**).

Differential geometry of (m-)Kuranishi spaces

Manifolds and orbifolds include into m-Kuranishi spaces and Kuranishi spaces, in a diagram of 2-categories

$$\begin{array}{ccc} \mathbf{Man} & \xrightarrow{\subset} & \mathbf{mKur} \\ \downarrow \subset & & \subset \downarrow \\ \mathbf{Orb} & \xrightarrow{\subset} & \mathbf{Kur}. \end{array}$$

Much of the differential geometry of ordinary manifolds extends nicely to Kuranishi spaces. There are good notions of dimension, orientation, submersions, immersions, embeddings, transversality and fibre products, gluing by equivalences on open covers. There are also good notions of (m-)Kuranishi space with boundary and corners, forming 2-categories $\mathbf{mKur} \subset \mathbf{mKur^b} \subset \mathbf{mKur^c}$ and $\mathbf{Kur} \subset \mathbf{Kur^b} \subset \mathbf{Kur^c}$. Some results are stronger than the classical case. For example, if $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are 1-morphisms in **Kur** with $\boldsymbol{Z}$ a manifold or orbifold then a fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in **Kur**, without further transversality conditions.

3. M-(co)homology and virtual cycles

Virtual cycles/virtual chains in symplectic geometry are usually set up as follows, at least in the FOOO theory. Suppose $\boldsymbol{X}$ is a compact, oriented Kuranishi space (possibly with corners) with dimension $\mathrm{vdim}\,\boldsymbol{X}=n$, Y an (oriented) manifold or orbifold, and $\boldsymbol{f}:\boldsymbol{X}\to Y$ a 1-morphism (smooth map). If $\partial\boldsymbol{X}=\emptyset$ we would like to define a *virtual class* $[[\boldsymbol{X}]_{\rm virt}]$ in the homology group $H_n(Y;\mathbb{Q})$ (or the Poincaré dual cohomology group $H^{\dim Y-n}_{\rm cs}(Y;\mathbb{Q})$).
For example, if $\boldsymbol{X}=\bar{\boldsymbol{\mathcal{M}}}_{g,k}(S,J,\beta)$ is a Gromov–Witten moduli space with evaluation maps $\mathbf{ev}_i:\bar{\boldsymbol{\mathcal{M}}}_{g,k}(S,J,\beta)\to S$, $i=1,\ldots,k$, $\boldsymbol{\pi}:\bar{\boldsymbol{\mathcal{M}}}_{g,k}(S,J,\beta)\to\bar{\mathcal{M}}_{g,k}$ then $[[\bar{\boldsymbol{\mathcal{M}}}_{g,k}(S,J,\beta)]_{\rm virt}]$ in $H_n(S^k\times\bar{\mathcal{M}}_{g,k};\mathbb{Q})$ are basically the Gromov–Witten invariants of S.
If $\partial\boldsymbol{X}\neq\emptyset$ we must work at the (co)chain level, so we have to choose a chain complex $\bigl(C_*(Y;\mathbb{Q}),\partial\bigr)$ computing $H_*(Y;\mathbb{Q})$, and define a *virtual (co)chain* $[\boldsymbol{X}]_{\rm virt}\in C_n(Y;\mathbb{Q})$. This should have nice properties including $\partial[\boldsymbol{X}]_{\rm virt}=[\partial\boldsymbol{X}]_{\rm virt}$, so that if $\partial\boldsymbol{X}=\emptyset$ then again we have a virtual class $[[\boldsymbol{X}]_{\rm virt}]\in H_n(Y;\mathbb{Q})$.

In the Fukaya–Oh–Ono–Ohta Lagrangian Floer cohomology theory, one studies moduli spaces $\bar{\boldsymbol{\mathcal{M}}}_k(\beta)$ of prestable J-holomorphic discs Σ in S with boundary in a Lagrangian L, and k boundary marked points. The $\bar{\boldsymbol{\mathcal{M}}}_k(\beta)$ are Kuranishi spaces with corners, with 'evaluation maps' $\mathbf{ev}_i:\bar{\boldsymbol{\mathcal{M}}}_k(\beta)\to L$ for $i=1,\ldots,k$, and

$$\partial\bar{\boldsymbol{\mathcal{M}}}_k(\beta)\simeq\coprod_{i+j=k}\coprod_{\beta_1+\beta_2=\beta}\bar{\boldsymbol{\mathcal{M}}}_{i+1}(\beta_1)\times_{\mathbf{ev}_{i+1},L,\mathbf{ev}_{j+1}}\bar{\boldsymbol{\mathcal{M}}}_{j+1}(\beta_2).\qquad(3.1)$$

To make the homological algebra in the theory work nicely, we would like to be able to choose our (co)chain theory $\bigl(C_*(-;\mathbb{Q}),\partial\bigr)$ and virtual chains $[-]_{\rm virt}$ such that (3.1) translates to a (co)chain-level equation

$$\partial[\bar{\boldsymbol{\mathcal{M}}}_k(\beta)]_{\rm virt}=\sum_{i+j=k,\;\beta_1+\beta_2=\beta}[\bar{\boldsymbol{\mathcal{M}}}_{i+1}(\beta_1)]_{\rm virt}\cup_{\pi_{i+1},L,\pi_{j+1}}[\bar{\boldsymbol{\mathcal{M}}}_{j+1}(\beta_2)]_{\rm virt},\qquad(3.2)$$

where $\cup$ is a (co)chain-level cup product/intersection product.

Which (co)homology theory of manifolds/orbifolds is best?

FOOO 2009 used singular homology for the chain complex $(C_*(Y;\mathbb{Q}),\partial)$. Then (3.2) makes no sense, as singular homology of manifolds has no chain-level intersection product. This made the homological algebra much more complicated than it need be. Their more recent work uses de Rham cohomology, where (3.2) makes sense with $\cup$ the wedge product of exterior forms. They are forced to work with (co)homology over $\mathbb{R}$, not $\mathbb{Q}$ or $\mathbb{Z}$. Defining virtual (co)chains is messy, involving perturbing Kuranishi spaces.
In arXiv:1509.05672 I define new (co)homology theories $MH_*(Y;R), MH^*(Y;R)$ of manifolds and orbifolds Y, called M-(co)homology, which have very good behaviour at the (co)chain level, and are specially designed for constructing virtual (co)chains for Kuranishi spaces in. They satisfy the Eilenberg–Steenrod axioms, and so have canonical isomorphisms $MH_*(Y;R)\cong H_*(Y;R)$, $MH^*(Y;R)\cong H^*(Y;R)$ with ordinary (co)homology.

M-chains and M-cochains

Let Y be a manifold (or effective orbifold), R a commutative ring, and $k\in\mathbb{Z}$. The *M-chain group* $MC_k(Y;R)$ is the R-module spanned by isomorphism classes $[V,n,s,t]$ of quadruples (V,n,s,t), where V is an oriented manifold with corners, and $n\in\mathbb{N}$ with $\dim V=n+k$, and $s:V\to\mathbb{R}^n$ is a smooth map in $\mathbf{Man^c}$ which is proper near $0\in\mathbb{R}^n$ (so that $s^{-1}(0)$ is compact), and $t:V\to Y$ is a smooth map. These generators $[V,n,s,t]$ in $MC_k(Y;R)$ are subject to some complicated relations we will not give. The boundary operator $\partial: MC_k(Y;R)\to MC_{k-1}(Y;R)$ is $\partial:[V,n,s,t]\mapsto[\partial V,n,s,t]$.
The story for M-cohomology and M-cochains $MC^k(Y;R)$ is similar, except that $t:V\to Y$ must be a cooriented submersion rather than V oriented, and $\dim V+k=\dim Y+n$.
There is a second version called *rational M-(co)homology* defined for R a $\mathbb{Q}$-algebra, with different relations. The cup product is supercommutative at the cochain level in rational M-cohomology.

Why M-(co)homology is good for virtual (co)chains

Let $\boldsymbol{X}$ be a compact, oriented m-Kuranishi space with corners and $\boldsymbol{t} : \boldsymbol{X} \to Y$ a 1-morphism. Then $\boldsymbol{X}$ is covered by m-Kuranishi neighbourhoods (V_i, E_i, s_i, ψ_i), where V_i is a manifold with corners, and $\boldsymbol{t}$ is represented on each by a smooth map $t_i : V_i \to Y$. Suppose that $E_i \to V_i$ is a trivial vector bundle $V_i \times \mathbb{R}^{n_i} \to V_i$. Then we can regard $s_i : V_i \to E_i$ as a smooth map $s_i : V_i \to \mathbb{R}^{n_i}$. Neglecting properness issues, $[V_i, n_i, s_i, t_i]$ is an M-chain in $MC_*(Y; R)$. Thus, locally on $\boldsymbol{X}$, to define a virtual chain for $\boldsymbol{t} : \boldsymbol{X} \to Y$ we just have to trivialize the obstruction bundles E_i. The relations in $MC_*(Y; R)$ are compatible with coordinate changes. **(The rest is work in progress.)** I intend to define notions of *strong (m-)Kuranishi spaces*, which are special examples of (m-)Kuranishi spaces with trivial obstruction bundles $E_i \to V_i$. I will prove that strong (m-)Kuranishi spaces have *canonical virtual chains* in M-(co)homology. I will show that any (m-)Kuranishi space is equivalent in **mKur**/**Kur** to a strong (m-)Kuranishi space.

Postscript: integrality properties of G–W invariants

Here is a future project that I believe will be possible in my Kuranishi space theory. Given $\boldsymbol{f} : \boldsymbol{X} \to Y$, if $\boldsymbol{X}$ is a general Kuranishi space (the orbifold version), we define virtual cycles $[\boldsymbol{X}]_{\rm virt}$ in rational homology $H_*(Y; \mathbb{Q})$, but if $\boldsymbol{X}$ is an m-Kuranishi space (the manifold version), we can do it in integral homology $H_*(Y; \mathbb{Q})$. I believe that if $\boldsymbol{X}$ is a Kuranishi space with a little extra structure — an 'almost complex structure' on the 'virtual normal bundle' of each orbifold stratum — which is natural for G–W moduli spaces, then there is a method to define a blow-up $\boldsymbol{\pi} : \tilde{\boldsymbol{X}} \to \boldsymbol{X}$ of $\boldsymbol{X}$ at its orbifold strata, with $\tilde{\boldsymbol{X}}$ an m-Kuranishi space (actually, an m-Kuranishi space in a category of singular manifolds). Then $\boldsymbol{f} \circ \boldsymbol{\pi} : \tilde{\boldsymbol{X}} \to Y$ has a virtual class in *integral* homology $H_*(Y; \mathbb{Z})$. This provides a way to systematically define Gromov–Witten type invariants in integral homology, and hopefully to study integrality properties of ordinary G–W invariants. I would be interested in discussing this with people.