Derived Differential Geometry

Lecture 11 of 14: Derived manifolds and orbifolds
with boundary and with corners

Dominic Joyce, Oxford University
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These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

11. Derived manifolds with boundary and corners

Manifolds are spaces locally modelled on $\mathbb{R}^n$. Similarly, *manifolds with boundary* are spaces locally modelled on $[0,\infty)\times\mathbb{R}^{n-1}$, and *manifolds with corners* spaces locally modelled on $[0,\infty)^k\times\mathbb{R}^{n-k}$.
We will explain how to define derived manifolds and orbifolds with boundary, and with corners. They form 2-categories $\mathbf{dMan^b}, \mathbf{dMan^c}$, $\mathbf{dOrb^b}, \mathbf{dOrb^c}, \mathbf{Kur^b}, \mathbf{Kur^c}$, with $\mathbf{dMan}\subset\mathbf{dMan^b}\subset\mathbf{dMan^c}$, etc.
Derived orbifolds (Kuranishi spaces) with corners are important in symplectic geometry, since moduli spaces of J-holomorphic curves with boundary in a Lagrangian are Kuranishi spaces with corners, and Lagrangian Floer cohomology and Fukaya categories depend on understanding such moduli spaces and their boundaries.
'Things with corners' – even basic questions, like what is a smooth map of manifolds with corners – are often more complicated than you might expect.

11.1. Manifolds with corners

Defining manifolds with corners just as objects is a straightforward generalization of the definition of manifolds.

Definition

Write $\mathbb{R}^n_k=[0,\infty)^k\times\mathbb{R}^{n-k}$ for $0\leqslant k\leqslant n$. Elements of $\mathbb{R}^n_k$ are $(x_1,\ldots,x_n)$ with $x_1,\ldots,x_k\geqslant 0$ and $x_{k+1},\ldots,x_n\in\mathbb{R}$.
A *manifold with corners* of dimension n is a Hausdorff, second countable topological space X equipped with an atlas of charts $\{(V_i,\psi_i):i\in I\}$, where $V_i\subseteq\mathbb{R}^n_k$ is open, and $\psi_i:V_i\to X$ is a homeomorphism with an open subset $\mathrm{Im}\,\psi_i$ of X for all $i\in I$, and $\psi_j^{-1}\circ\psi_i:\psi_i^{-1}(\mathrm{Im}\,\psi_j)\to\psi_j^{-1}(\mathrm{Im}\,\psi_i)$ is a diffeomorphism of open subsets of $\mathbb{R}^n_k$ for all $i,j\in I$. (That is, $\psi_j^{-1}\circ\psi_i$ must extend to a diffeomorphism of open subsets of $\mathbb{R}^n$.)

Let $U\subseteq\mathbb{R}^m_k$, $V\subseteq\mathbb{R}^n_l$ be open. There are broadly four sensible definitions of when a continuous map $g:U\to V$ is 'smooth', where $g=(g_1,\ldots,g_n)$ with $g_j=g_j(x_1,\ldots,x_m)$:

(a) We call g *weakly smooth* if there exist an open neighbourhood U' of U in $\mathbb{R}^m$, and a smooth map $g':U'\to\mathbb{R}^n$ in the usual sense, such that $g'|_U=g$. By Seeley's Extension Theorem, this holds iff all derivatives $\frac{\partial^k g}{\partial x_{i_1}\cdots\partial x_{i_k}}$ exist and are continuous, using one-sided derivatives at boundaries.

(b) (Richard Melrose) We call g *smooth* if it is weakly smooth and locally in U, for each $j=1,\ldots,l$ we have either (i) $g_j=0$, or (ii)

$$g_j(x_1,\ldots,x_n)=x_1^{a_{1,j}}\cdots x_k^{a_{k,j}}h_j(x_1,\ldots,x_n),\qquad(11.1)$$

for $a_1,\ldots,a_k\in\mathbb{N}$ and $h_j:U\to(0,\infty)$ weakly smooth.

(c) (Melrose) We call smooth g *interior* if (i) does not occur.

(d) (Joyce 2009) We call g *strongly smooth* if it is smooth, and in (11.1) we always have $a_{1,j}+\cdots+a_{k,j}=0$ or 1.

Definition

Let X,Y be manifolds with corners, and $f:X\to Y$ be continuous. We say that f is *weakly smooth*, or *smooth*, or *interior*, or *strongly smooth*, if for all charts (U_i,ϕ_i) on X with $U_i\subseteq\mathbb{R}^m_k$ and (V_j,ϕ_j) on Y with $V_j\subseteq\mathbb{R}^n_l$, the map

$$\psi_j^{-1}\circ f\circ\phi_i:(f\circ\phi_i)^{-1}(\mathrm{Im}\,\psi_j)\longrightarrow V_j$$

is weakly smooth, or smooth, or interior, or strongly smooth, respectively, as a map between open subsets of $\mathbb{R}^m_k$ and $\mathbb{R}^n_l$.

All four classes of maps are closed under composition and include identities, and so make manifolds with corners into a category. My favourite is smooth maps (which Richard Melrose calls 'b-maps'). Write $\mathbf{Man^c}$ for the category with objects manifolds with corners, and morphisms smooth maps.

Example 11.1

(i) $e : \mathbb{R} \to [0, \infty)$, $e(x) = x^2$, is weakly smooth but not smooth.
(ii) $f : [0, \infty) \to [0, \infty)$, $f(x) = x^2$, is smooth and interior, but not strongly smooth.
(iii) $g : [0, \infty)^2 \to [0, \infty)$, $g(x, y) = x + y$, is weakly smooth but not smooth.
(iv) $h : [0, \infty)^2 \to [0, \infty)$, $h(x, y) = xy$, is smooth and interior, but not strongly smooth.
(v) $i : \mathbb{R} \to [0, \infty)$, $i(x) = 0$, is (strongly) smooth but not interior.
(vi) $j : [0, \infty) \to \mathbb{R}$, $j(x) = x$, is (strongly) smooth and interior.

Derived manifolds with boundary and corners
Bordism, virtual classes, and virtual chains
Manifolds with corners
Derived manifolds and orbifolds with corners

An n-manifold with corners X has a natural *depth stratification* $X = \coprod_{l=0}^n S^l(X)$, where $S^l(X)$ is the set of $x \in X$ in a boundary stratum of codimension l. If $(x_1, \ldots, x_n) \in \mathbb{R}^n_k$ are local coordinates on X, then $(x_1, \ldots, x_n)$ lies in $S^l(X)$ (i.e. has *depth* l) iff l out of $x_1, \ldots, x_k$ are zero. Then $S^l(X)$ is a manifold without boundary of dimension $n - l$. We call $S^0(X)$ the *interior* X°.
The closure is $\overline{S^l(X)} = \coprod_{k=l}^n S^k(X)$. We call X a *manifold with boundary* if $S^l(X) = \emptyset$ for $l > 1$, and a *manifold without boundary* (i.e. an ordinary manifold) if $S^l(X) = \emptyset$ for $l > 0$.
A *local boundary component* β at $x \in X$ is a choice of local connected component of $S^1(X)$ near x. That is, β is a local boundary hypersurface containing x. If $(x_1, \ldots, x_n) \in \mathbb{R}^n_k$ are local coordinates on X, then local boundary components β at $(x_1, \ldots, x_n)$ correspond to a choice of $l = 1, \ldots, k$ such that $x_l = 0$. If $x \in S^k(X)$ then there are k distinct local boundary components $\beta_1, \ldots, \beta_k$ at x.

Boundaries and corners of manifolds with corners

Definition

Let X be a manifold with corners. The *boundary* ∂X, as a set, is $\partial X = \{(x,\beta) : x \in X,\ \beta$ is a local boundary component of X at $x\}$. Define $i_X : \partial X \to X$ by $i_X : (x,\beta) \mapsto x$. Then ∂X has the natural structure of a manifold with corners of dimension $\dim X - 1$, such that i_X is a smooth (but not interior) map. Note that $i_X^{-1}(x)$ is k points if $x \in S^k(X)$. So i_X may not be injective. We have

$$\partial^k X \cong \{(x,\beta_1,\ldots,\beta_k) : x \in X,\ \beta_1,\ldots,\beta_k \text{ are distinct local boundary components of } X \text{ at } x\}.$$

The symmetric group S_k acts on $\partial^k X$ by permuting $\beta_1,\ldots,\beta_k$. The *k-corners* $C_k(X)$ is $\partial^k X/S_k$, a manifold with corners of dimension $\dim X - k$, with $C_0(X) = X$, $C_1(X) = \partial X$. We have

$$C_k(X) \cong \{(x,\{\beta_1,\ldots,\beta_k\}) : x \in X,\ \beta_1,\ldots,\beta_k \text{ are distinct local boundary components of } X \text{ at } x\}.$$

Example 11.2

Consider $X = [0,\infty)^2$ with coordinates (x,y). The local boundary components of X at (x',y') are $\{x=0\}$ if $x'=0$ and $\{y=0\}$ if $y'=0$, so there are two local boundary components $\{x=0\}$, $\{y=0\}$ at $(0,0)$.
The boundary ∂X is $[0,\infty)\amalg[0,\infty)$, two copies of $[0,\infty)$, with the first from local boundary component $\{y=0\}$ with $i_X : x \mapsto (x,0)$, and the second from local boundary component $\{x=0\}$ with $i_X : y \mapsto (0,y)$.
Also $\partial^2 X$ is two points, both mapped to $(0,0)$ by $i_X \circ i_{\partial X}$, and $S_2 = \mathbb{Z}_2$ acts freely on $\partial^2 X$ by swapping the points. $C_2(X) = \partial^2 X/S_2$ is one point.
Note that ∂X and $\partial^2 X$ are not subsets of X, the maps $\partial X \to X$, $\partial^2 X \to X$ are 2:1 over $(0,0)$.

How do smooth maps act on boundaries and corners?

Let $f : X \to Y$ be a smooth map of manifolds with corners. In general there is no natural smooth map $\partial f : \partial X \to \partial Y$ (e.g. for $j : [0,\infty) \to \mathbb{R}$, $j(x) = x$ we have $\partial X = \{0\}$, $\partial Y = \emptyset$, so no map $\partial X \to \partial Y$). So boundaries are not a functor $\partial : \mathbf{Man^c} \to \mathbf{Man^c}$. Nonetheless boundaries do play nicely with smooth maps (though *not* with weakly smooth maps).

One way to show this is to write $\check{\mathbf{M}}\mathbf{an^c}$ for the category of *manifolds with corners of mixed dimension*, with objects $\coprod_{n\geqslant 0} X_n$ for X_n a manifold with corners of dimension n, and morphisms continuous maps $f : \coprod_{m\geqslant 0} X_m \to \coprod_{n\geqslant 0} Y_n$ smooth on each component. Then there is a natural *corner functor* $C : \mathbf{Man^c} \to \check{\mathbf{M}}\mathbf{an^c}$, with $C(X) = \coprod_{k=0}^{\dim X} C_k(X)$ on objects, and

$$C(f) : (x, \{\beta_1,\ldots,\beta_k\}) \longmapsto (y, \{\gamma_1,\ldots,\gamma_l\}),$$

where $f(x) = y$, and $\gamma_1,\ldots,\gamma_l$ are all the local boundary components of Y at y which contain $f(\beta_i)$ for all $i = 1,\ldots,k$.

We call a smooth map $f : X \to Y$ *simple* if $C(f)$ maps $C_k(X) \to C_k(Y)$ for all $k \geqslant 0$. This is a discrete condition on f. Diffeomorphisms are simple. Simple maps are closed under composition and include identities, so they give a subcategory $\mathbf{Man^c_{si}} \subset \mathbf{Man^c}$. Then boundaries and corners give functors

$$\partial : \mathbf{Man^c_{si}} \longrightarrow \mathbf{Man^c_{si}}, \quad C_k : \mathbf{Man^c_{si}} \longrightarrow \mathbf{Man^c_{si}},$$

mapping $X \mapsto \partial X$, $X \mapsto C_k(X)$ on objects, and $f \mapsto C(f)|_{C_1(X)}$, $f \mapsto C(f)|_{C_k(X)}$ on morphisms $f : X \to Y$.

Tangent bundles and b-tangent bundles

For manifolds with corners X there are *two different* notions of tangent bundle: the *tangent bundle* TX, and the *b-tangent bundle* ${}^bTX$ (due to Richard Melrose). If $(x_1,\ldots,x_n)\in\mathbb{R}^n_k$ are local coordinates on X, with $x_1,\ldots,x_k\geqslant 0$, then

$$TX=\Big\langle\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\Big\rangle,\quad {}^bTX=\Big\langle x_1\frac{\partial}{\partial x_1},\ldots,x_k\frac{\partial}{\partial x_k},\frac{\partial}{\partial x_{k+1}},\ldots,\frac{\partial}{\partial x_n}\Big\rangle.$$

Think of $C^\infty(TX)$ as the vector fields v on X, and $C^\infty({}^bTX)$ as the vector fields v on X which are tangent to each boundary stratum. Usually it is better to use ${}^bTX$ than TX.

You can only define the b-derivative ${}^bTf:{}^bTX\rightarrow f^*({}^bTY)$ if $f:X\rightarrow Y$ is an *interior* map of manifolds with corners.

There are also two cotangent bundles $T^*X=(TX)^*$ and ${}^bT^*X=({}^bTX)^*$, where in coordinates

$${}^bT^*X=\big\langle x_1^{-1}\mathrm{d}x_1,\ldots,x_k^{-1}\mathrm{d}x_k,\mathrm{d}x_{k+1},\ldots,\mathrm{d}x_n\big\rangle.$$

Transverse fibre products?

Good conditions for existence of fibre products of $g:X\rightarrow Z$, $h:Y\rightarrow Z$ in $\mathbf{Man^c}$ are complicated. Copying the usual definition of transversality using either TX,TY,TZ or ${}^bTX,{}^bTY,{}^bTZ$ is not enough: you need additional discrete conditions on how g,h act on $\partial^jX,\partial^kY,\partial^lZ$. See Joyce arXiv:0910.3518 for sufficient conditions for existence of fibre products in $\mathbf{Man^c_{ss}}$, the category of manifolds with corners and strongly smooth maps.

The nicest answer (Joyce arXiv:1501.00401) is to define *manifolds with generalized corners* (*g-corners*) $\mathbf{Man^{gc}}$, allowing more complicated local models for corners than $[0,\infty)^k\times\mathbb{R}^{n-k}$. Then b-tangent bundles ${}^bTX$ and interior maps $g:X\rightarrow Z$ make sense for X in $\mathbf{Man^{gc}}$. If $g:X\rightarrow Z$ and $h:Y\rightarrow Z$ are *b-transverse* interior maps in $\mathbf{Man^{gc}}$ (i.e. ${}^bT_xg\oplus{}^bT_yh:{}^bT_xX\oplus{}^bT_yY\rightarrow{}^bT_zZ$ is always surjective) then a fibre product $X\times_{g,Z,h}Y$ exists in $\mathbf{Man^{gc}_{in}}$, the category of manifolds with g-corners and interior maps.

11.2 Derived manifolds and orbifolds with corners

How should we define derived manifolds and orbifolds with boundary and corners?

My d-manifolds book gives definitions of strict 2-categories $\mathbf{dMan^b}, \mathbf{dMan^c}$, $\mathbf{dOrb^b}, \mathbf{dOrb^c}$ of d-manifolds and d-orbifolds with boundary and corners using C^∞-algebraic geometry. They are full 2-subcategories of strict 2-categories $\mathbf{dSpa^b}, \mathbf{dSpa^c}$, $\mathbf{dSta^b}, \mathbf{dSta^c}$ of d-spaces and d-stacks with boundary and corners, which are classes of derived C^∞-schemes and Deligne–Mumford C^∞-stacks with boundary and corners. The details are complex and messy.

However, a far less painful route is to start with my definition of (M-)Kuranishi spaces (arXiv:1409.6908), and take the V in (M-)Kuranishi neighbourhoods (V, E, s, ψ) or (V, E, Γ, s, ψ) to be manifolds with boundary, or with corners. We replace manifolds by manifolds with boundary or (g-)corners throughout, and make a few other changes (use interior/simple maps, use TV or ${}^bTV$, etc.)

Thus, we can define ordinary categories $\mathbf{MKur^b}, \mathbf{MKur^c}$ of *M-Kuranishi spaces with boundary and corners*, with $\mathbf{MKur} \subset \mathbf{MKur^b} \subset \mathbf{MKur^c}$, a kind of derived manifold with boundary, or corners.

We also have weak 2-categories $\mathbf{Kur^b}, \mathbf{Kur^c}$ of *Kuranishi spaces with boundary and corners*, and $\mathbf{Kur^{gc}}$ of *Kuranishi spaces with generalized corners* (*g-corners*), with $\mathbf{Kur} \subset \mathbf{Kur^b} \subset \mathbf{Kur^c} \subset \mathbf{Kur^{gc}}$. These are forms of derived orbifolds with boundary, or corners, or g-corners.

Write $\mathbf{Kur^b_{trG}} \subset \mathbf{Kur^b}$, $\mathbf{Kur^c_{trG}} \subset \mathbf{Kur^c}$, $\mathbf{Kur^{gc}_{trG}} \subset \mathbf{Kur^{gc}}$ for the full 2-subcategories of $\mathbf{X}$ with orbifold groups $G_x\mathbf{X} = \{1\}$ for all $x \in \mathbf{X}$.

Then $\mathbf{Kur^b_{trG}}, \mathbf{Kur^c_{trG}}, \mathbf{Kur^{gc}_{trG}}$ are 2-categories of derived manifolds with boundary, or corners, or g-corners.

Orbifold groups, (b-)tangent and (b-)obstruction spaces

As for Kuranishi spaces in §9.1, Kuranishi spaces $\mathbf{X}$ with boundary or corners have *orbifold groups* $G_x\mathbf{X}$, *tangent spaces* $T_x\mathbf{X}$, and *obstruction spaces* $O_x\mathbf{X}$ with $\dim T_x\mathbf{X} - \dim O_x\mathbf{X} = \mathrm{vdim}\,\mathbf{X}$, and for 1-morphisms $\mathbf{f}:\mathbf{X}\rightarrow\mathbf{Y}$ and $x\in\mathbf{X}$ with $\mathbf{f}(x)=y\in\mathbf{Y}$ we have morphisms $G_x\mathbf{f}:G_x\mathbf{X}\rightarrow G_y\mathbf{Y}$, $T_x\mathbf{f}:T_x\mathbf{X}\rightarrow T_y\mathbf{Y}$, $O_x\mathbf{f}:O_x\mathbf{X}\rightarrow O_y\mathbf{Y}$, and for 2-morphisms $\boldsymbol{\eta}:\mathbf{f}\Rightarrow\mathbf{g}$ we have $G_x\boldsymbol{\eta}\in G_y\mathbf{Y}$.

Just as manifolds with corners X have two tangent bundles $TX, {}^bTX$, and it is usually better to use ${}^bTX$, so Kuranishi spaces $\mathbf{X}$ with boundary or corners also have *b-tangent spaces* ${}^bT_x\mathbf{X}$, and *b-obstruction spaces* ${}^bO_x\mathbf{X}$ with $\dim {}^bT_x\mathbf{X} - \dim {}^bO_x\mathbf{X} = \mathrm{vdim}\,\mathbf{X}$.

If $\mathbf{f}:\mathbf{X}\rightarrow\mathbf{Y}$ is an *interior* 1-morphism and $x\in\mathbf{X}$ with $\mathbf{f}(x)=y\in\mathbf{Y}$ we have ${}^bT_x\mathbf{f}:{}^bT_x\mathbf{X}\rightarrow {}^bT_y\mathbf{Y}$, ${}^bO_x\mathbf{f}:{}^bO_x\mathbf{X}\rightarrow {}^bO_y\mathbf{Y}$.

It is usually better to use ${}^bT_x\mathbf{X}, {}^bO_x\mathbf{X}$ rather than $T_x\mathbf{X}, O_x\mathbf{X}$.

For Kuranishi spaces $\mathbf{X}$ with g-corners we define ${}^bT_x\mathbf{X}, {}^bO_x\mathbf{X}$ but not $T_x\mathbf{X}, O_x\mathbf{X}$.

If $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ is a Kuranishi neighbourhood in the Kuranishi structure on $\mathbf{X}$, so that V_i is a manifold with corners, and $x\in\mathrm{Im}\,\psi_i\subseteq\mathbf{X}$ with $x=\psi_i(v\Gamma_i)$ for $v\in s_i^{-1}(0)\subseteq V_i$, then $T_x\mathbf{X}, O_x\mathbf{X}, {}^bT_x\mathbf{X}, {}^bO_x\mathbf{X}$ are defined by the exact sequences

$$\begin{array}{ccccccccccc}
0 & \rightarrow & {}^bT_x\mathbf{X} & \rightarrow & {}^bT_vV_i & \xrightarrow{{}^b\mathrm{d}s_i|_v} & E_i|_v & \rightarrow & {}^bO_x\mathbf{X} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow & & \\
0 & \rightarrow & T_x\mathbf{X} & \rightarrow & T_vV_i & \xrightarrow{\mathrm{d}s_i|_v} & E_i|_v & \rightarrow & O_x\mathbf{X} & \rightarrow & 0,
\end{array}$$

where there are natural vertical morphisms making the diagram commute.

Boundaries of Kuranishi spaces with corners

If $\mathbf{X}$ is a Kuranishi space with corners, we can define a natural Kuranishi space with corners $\partial\mathbf{X}$ called the *boundary* of $\mathbf{X}$, with a morphism $\mathbf{i_X}:\partial\mathbf{X}\to\mathbf{X}$, and with $\mathrm{vdim}\,\partial\mathbf{X}=\mathrm{vdim}\,\mathbf{X}-1$. If $(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I}$ are the Kuranishi neighbourhoods in the Kuranishi structure on $\mathbf{X}$, then $(\partial V_i,E_i|_{\partial V_i},\Gamma_i,s_i|_{\partial V_i},\psi_i')_{i\in I}$ are the Kuranishi neighbourhoods in the Kuranishi structure on $\partial\mathbf{X}$.
If $\mathbf{X}$ is a Kuranishi space with boundary, then $\partial\mathbf{X}$ is a Kuranishi space without boundary. If $\mathbf{X}$ is a Kuranishi space without boundary, then $\partial\mathbf{X}=\emptyset$.
There is a strict action of the symmetric group S_k on $\partial^k\mathbf{X}$, which is free if $\mathbf{X}$ has trivial orbifold groups. Then the *k-corners* $C_k(\mathbf{X})=(\partial^k\mathbf{X})/S_k$ is a Kuranishi space with corners for $k\geqslant 0$, with $\mathrm{vdim}\,C_k(\mathbf{X})=\mathrm{vdim}\,\mathbf{X}-k$.

Write $\check{\mathbf{Kur}}^{\mathbf{c}}$ for the 2-category of *Kuranishi spaces of mixed dimension*, with objects $\coprod_{n\in\mathbb{Z}}\mathbf{X}_n$ for $\mathbf{X}_n$ a Kuranishi space with corners of virtual dimension $n\in\mathbb{Z}$, and 1- and 2-morphisms $\mathbf{f},\mathbf{g}:\coprod_{m\in\mathbb{Z}}\mathbf{X}_m\to\coprod_{n\in\mathbb{Z}}\mathbf{Y}_n$, $\boldsymbol{\eta}:\mathbf{f}\Rightarrow\mathbf{g}$ being 1- and 2-morphisms in $\mathbf{Kur^c}$ on each component.
Then there is a natural *corner 2-functor* $C:\mathbf{Kur^c}\to\check{\mathbf{Kur}}^{\mathbf{c}}$, with $C(\mathbf{X})=\coprod_{k\geqslant 0}C_k(\mathbf{X})$ on objects.
Call a 1-morphism $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in $\mathbf{Kur^c}$ *simple* if $C(\mathbf{f})$ maps $C_k(\mathbf{X})\to C_k(\mathbf{Y})$ for all $k\geqslant 0$. Simple 1-morphisms are closed under composition and include identities, so they define a 2-subcategory $\mathbf{Kur^c_{si}}\subset\mathbf{Kur^c}$. Then we have 2-functors $\partial,C_k:\mathbf{Kur^c_{si}}\to\mathbf{Kur^c_{si}}$ mapping $\mathbf{X}\mapsto\partial\mathbf{X}$, $\mathbf{X}\mapsto C_k(\mathbf{X})$ on objects, and $\mathbf{f}\mapsto C(\mathbf{f})|_{C_1(\mathbf{X})}$, $\mathbf{f}\mapsto C(\mathbf{f})|_{C_k(\mathbf{X})}$ on 1-morphisms $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$, and $\boldsymbol{\eta}\mapsto C(\boldsymbol{\eta})|_{C_1(\mathbf{X})}$, $\boldsymbol{\eta}\mapsto C(\boldsymbol{\eta})|_{C_k(\mathbf{X})}$ on 2-morphisms $\boldsymbol{\eta}:\mathbf{f}\Rightarrow\mathbf{g}$.
Any equivalence $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ in $\mathbf{Kur^c}$ is simple, and then $\partial\mathbf{f}:\partial\mathbf{X}\to\partial\mathbf{Y}$ and $C_k(\mathbf{f}):C_k(\mathbf{X})\to C_k(\mathbf{Y})$ are equivalences. So boundaries $\partial\mathbf{X}$ and corners $C_k(\mathbf{X})$ are preserved by equivalences.

Differential geometry of Kuranishi spaces with corners

All the material in lectures 9 and 10 on differential geometry of derived manifolds and orbifolds extends to the corners case, with suitable modifications: immersions, embeddings and derived submanifolds; embedding derived manifolds into manifolds; submersions; orientations; and transverse fibre products.
We give some highlights. Here is an analogue of Corollary 9.8:

Theorem 11.3

A d-manifold with corners $\mathbf{X}$ is equivalent in $\mathbf{dMan^c}$ to a standard model d-manifold with corners $\mathbf{S}_{V,E,s}$ if and only if $\dim T_x\mathbf{X} + \left|\mathbf{i}_{\mathbf{X}}^{-1}(x)\right|$ is bounded above for all $x \in \mathbf{X}$. This always holds if $\mathbf{X}$ is compact.

Here $\left|\mathbf{i}_{\mathbf{X}}^{-1}(x)\right|$ is the 'depth' of $x \in \mathbf{X}$, the codimension of the boundary stratum it lives in.

W-submersions and submersions

We can define *w-submersions* and *submersions* $f : \mathbf{X} \to \mathbf{Y}$ of derived manifolds and orbifolds with corners. The definition includes a discrete condition on how $\mathbf{f}$ acts on $\partial^k\mathbf{X}, \partial^l\mathbf{Y}$.
Here are analogues of Theorems 9.11 and 9.12.

Theorem 11.4

Suppose $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$ is a w-submersion in $\mathbf{dMan^c}$. Then for any 1-morphism $\mathbf{h} : \mathbf{Y} \to \mathbf{Z}$ in $\mathbf{dMan^c}$, the fibre product $\mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ exists in $\mathbf{dMan^c}$.

Theorem 11.5

Suppose $\mathbf{g} : \mathbf{X} \to \mathbf{Z}$ is a submersion in $\mathbf{dMan^c}$. Then for any 1-morphism $\mathbf{h} : Y \to \mathbf{Z}$ in $\mathbf{dMan}$ with Y a manifold with corners, the fibre product $\mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} Y$ exists in $\mathbf{dMan^c}$ and is a manifold with corners. In particular, the fibres $\mathbf{X}_z = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},z} *$ of $\mathbf{g}$ for $z \in \mathbf{Z}$ are manifolds with corners.

Orientations

Here is the analogue of Theorem 9.14:

Theorem 11.6

(a) *Every d-manifold with corners* $\mathbf{X}$ *has a* **canonical bundle** $K_{\mathbf{X}}$, *a* C^∞ *real line bundle over the* C^∞*-scheme* $\underline{X}$, *natural up to canonical isomorphism, with* $K_{\mathbf{X}}|_x \cong \Lambda^{\rm top}({}^bT_x^*\mathbf{X}) \otimes \Lambda^{\rm top}({}^bO_x\mathbf{X})$ *for* $x \in \mathbf{X}$.

(b) *If* $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ *is an étale* 1*-morphism (e.g. an equivalence), there is a canonical, functorial isomorphism* $K_{\mathbf{f}}: K_{\mathbf{X}} \to \underline{f}^*(K_{\mathbf{Y}})$. *If* $\mathbf{f},\mathbf{g}:\mathbf{X}\to\mathbf{Y}$ *are* 2*-isomorphic then* $K_{\mathbf{f}} = K_{\mathbf{g}}$.

(c) *If* $\mathbf{X}\simeq \mathbf{S}_{V,E,s}$, *there is a canonical isomorphism*

$$K_{\mathbf{X}} \cong \left(\Lambda^{\dim V}({}^bT^*V) \otimes \Lambda^{\operatorname{rank} E}E\right)\big|_{s^{-1}(0)}.$$

(d) *There is a canonical isomorphism* $K_{\partial\mathbf{X}} \cong \underline{i}_{\mathbf{X}}^*(K_{\mathbf{X}})$.

Analogues of **(a)–(d)** *hold for d-orbifolds and Kuranishi spaces with corners.*

We then define an *orientation* of a derived manifold or orbifold with corners $\mathbf{X}$ to be an orientation on the fibres of $K_{\mathbf{X}}$. Note that Theorem 11.6(d) implies that if $\mathbf{X}$ is oriented then so are $\partial\mathbf{X},\partial^2\mathbf{X},\partial^3\mathbf{X},\ldots$.

Recall that the k-corners $C_k(\mathbf{X})$ is $C_k(\mathbf{X}) = \partial^k\mathbf{X}/S_k$. If $k \geqslant 2$ then the action of S_k on $\partial^k\mathbf{X}$ is not orientation preserving, and $C_k(\mathbf{X})$ may not be orientable. This also happens for ordinary manifolds with corners.

Transverse fibre products

In the 2-categories $\mathbf{dMan^c}, \mathbf{dOrb^c}, \mathbf{Kur^c}$, fibre products of $\mathbf{g}: \mathbf{X} \to \mathbf{Z}$, $\mathbf{h}: \mathbf{Y} \to \mathbf{Z}$ exist provided a 'd-transversality' condition holds – $O_x\mathbf{g} \oplus O_y\mathbf{h}: O_x\mathbf{X} \oplus O_y\mathbf{Y} \to O_z\mathbf{Z}$ or ${}^bO_x\mathbf{g} \oplus {}^bO_y\mathbf{h}: {}^bO_x\mathbf{X} \oplus {}^bO_y\mathbf{Y} \to {}^bO_z\mathbf{Z}$ is surjective – *and* a discrete condition ('b-transversality' or 'c-transversality') on the action of $\mathbf{g}, \mathbf{h}$ on corners holds.

If $\mathbf{Z}$ is a manifold without boundary both of these are trivial, giving an analogue of Corollary 10.3:

Theorem 11.7

Suppose $\mathbf{g}: \mathbf{X} \to \mathbf{Z}$, $\mathbf{h}: \mathbf{Y} \to \mathbf{Z}$ *are* 1*-morphisms in* $\mathbf{dMan^c}$*, with* $\mathbf{Z}$ *a manifold without boundary. Then a fibre product* $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ *exists in* $\mathbf{dMan^c}$*, with* $\mathrm{vdim}\,\mathbf{W} = \mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Y} - \dim \mathbf{Z}$.

We can dispense with these discrete conditions on corners if we work with 'generalized corners'. Then we have:

Theorem 11.8

Let $\mathbf{g}: \mathbf{X} \to \mathbf{Z}$, $\mathbf{h}: \mathbf{Y} \to \mathbf{Z}$ *be interior* 1*-morphisms in* $\mathbf{Kur^{gc}}$*. Suppose* $\mathbf{g}, \mathbf{h}$ *are* **d-transverse***, that is,* ${}^bO_x\mathbf{g} \oplus (\gamma \cdot {}^bO_y\mathbf{h}): {}^bO_x\mathbf{X} \oplus {}^bO_y\mathbf{Y} \to {}^bO_z\mathbf{Z}$ *is surjective for all* $x \in \mathbf{X}$, $y \in \mathbf{Y}$ *with* $\mathbf{g}(x) = \mathbf{h}(y) = z \in \mathbf{Z}$ *and* $\gamma \in G_z\mathbf{Z}$*. Then a fibre product* $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ *exists in the 2-category* $\mathbf{Kur^{gc}_{in}}$ *of Kuranishi spaces with g-corners and interior* 1*-morphisms, with* $\mathrm{vdim}\,\mathbf{W} = \mathrm{vdim}\,\mathbf{X} + \mathrm{vdim}\,\mathbf{Y} - \dim \mathbf{Z}$.

For any such d-transverse fibre product $\mathbf{W} = \mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ in $\mathbf{dMan^c}, \mathbf{dOrb^c}, \mathbf{Kur^c}, \mathbf{Kur^{gc}_{in}}$, if $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ are oriented, then $\mathbf{W}$ is oriented.

Derived Differential Geometry

Lecture 12 of 14: Bordism, virtual classes, and virtual chains

Dominic Joyce, Oxford University
Summer 2015

These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

12. Bordism, virtual classes, and virtual chains

In many important areas of geometry to do with enumerative invariants (e.g. Donaldson and Seiberg–Witten invariants of 4-manifolds, Gromov–Witten invariants of symplectic manifolds, Donaldson–Thomas invariants of Calabi–Yau 3-folds, . . .), we form a moduli space $\bar{\mathcal{M}}$ with some geometric structure, and we want to 'count' $\bar{\mathcal{M}}$ to get a number in $\mathbb{Z}$ or $\mathbb{Q}$ (if $\bar{\mathcal{M}}$ has no boundary and dimension 0), or a homology class ('virtual class') $[\bar{\mathcal{M}}]_{\mathrm{virt}}$ in some homology theory (if $\bar{\mathcal{M}}$ has no boundary and dimension > 0).
For more complicated theories (Floer homology, Fukaya categories), $\bar{\mathcal{M}}$ has boundary, and then we must define a chain $[\bar{\mathcal{M}}]_{\mathrm{virt}}$ in the chain complex (C_*, ∂) of some homology theory (a 'virtual chain'), where ideally we want $\partial[\bar{\mathcal{M}}]_{\mathrm{virt}} = [\partial\bar{\mathcal{M}}]_{\mathrm{virt}}$.

In general $\bar{\mathcal{M}}$ is not a manifold (or orbifold). However, the point is to treat $\bar{\mathcal{M}}$ *as if* it were a compact, oriented manifold, so that in particular, if $\partial\bar{\mathcal{M}} = \emptyset$ then $\bar{\mathcal{M}}$ has a fundamental class $[\bar{\mathcal{M}}]$ in the homology group $H_{\dim\bar{\mathcal{M}}}(\bar{\mathcal{M}};\mathbb{Z})$.
All of these 'counting invariant' theories over $\mathbb{R}$ or $\mathbb{C}$, in both differential and algebraic geometry, can be understood using derived differential geometry. The point is that the moduli spaces $\bar{\mathcal{M}}$ should be compact, oriented derived manifolds or orbifolds (possibly with corners). Then we show that compact, oriented derived manifolds or orbifolds (with corners) have virtual classes (virtual chains), and these are used to define the invariants.
There is an easy way to define virtual classes for compact, oriented derived manifolds without boundary, using *bordism*, so we explain this first. It does not work as well in the orbifold case, though.

12.1 Bordism and derived bordism

Let Y be a manifold. Define the *bordism group* $B_k(Y)$ to have elements $\sim$-equivalence classes $[X, f]$ of pairs (X, f), where X is a compact oriented k-manifold without boundary and $f : X \to Y$ is smooth, and $(X, f) \sim (X', f')$ if there exists a $(k+1)$-manifold with boundary W and a smooth map $e : W \to Y$ with $\partial W \cong X \amalg -X'$ and $e|_{\partial W} \cong f \amalg f'$. It is an abelian group, with addition $[X, f] + [X', f'] = [X \amalg X', f \amalg f']$.

If Y is oriented of dimension n, there is a supercommutative, associative *intersection product* $\bullet : B_k(Y) \times B_l(Y) \to B_{k+l-n}(Y)$ given by $[X, f] \bullet [X', f'] = [X \times_{f,Y,f'} X', \pi_Y]$, choosing X, f, X', f' in their bordism classes with $f : X \to Y$, $f' : X' \to Y$ transverse.

Bordism is a *generalized homology theory*, i.e. it satisfies all the Eilenberg–Steenrod axioms except the Dimension Axiom.

There is a natural morphism $\Pi_{\rm bo}^{\rm hom} : B_k(Y) \to H_k(Y, \mathbb{Z})$ given by $\Pi_{\rm bo}^{\rm hom} : [X, f] \mapsto f_*([X])$, for $[X] \in H_k(X, \mathbb{Z})$ the fundamental class.

Similarly, define the *derived bordism group* $dB_k(Y)$ to have elements $\approx$-equivalence classes $[\mathbf{X}, \mathbf{f}]$ of pairs $(\mathbf{X}, \mathbf{f})$, where $\mathbf{X}$ is a compact oriented d-manifold with $\mathrm{vdim}\, \mathbf{X} = k$ and $\mathbf{f} : \mathbf{X} \to Y$ is a 1-morphism in **dMan**, and $(\mathbf{X}, \mathbf{f}) \approx (\mathbf{X}', \mathbf{f}')$ if there exists a d-manifold with boundary $\mathbf{W}$ with $\mathrm{vdim}\, \mathbf{W} = k+1$ and a 1-morphism $\mathbf{e} : \mathbf{W} \to \mathbf{Y}$ in $\mathbf{dMan^b}$ with $\partial \mathbf{W} \simeq \mathbf{X} \amalg -\mathbf{X}'$ and $\mathbf{e}|_{\partial \mathbf{W}} \cong \mathbf{f} \amalg \mathbf{f}'$. It is an abelian group, with $[\mathbf{X}, \mathbf{f}] + [\mathbf{X}', \mathbf{f}'] = [\mathbf{X} \amalg \mathbf{X}, \mathbf{f} \amalg \mathbf{f}']$.

If Y is oriented of dimension n, there is a supercommutative, associative *intersection product* $\bullet : dB_k(Y) \times dB_l(Y) \to dB_{k+l-n}(Y)$ given by $[\mathbf{X}, \mathbf{f}] \bullet [\mathbf{X}', \mathbf{f}'] = [\mathbf{X} \times_{\mathbf{f},Y,\mathbf{f}'} \mathbf{X}', \boldsymbol{\pi}_Y]$, with no transversality condition on $\mathbf{X}, \mathbf{f}, \mathbf{X}', \mathbf{f}'$.

There is a morphism $\Pi_{\rm bo}^{\rm dbo}: B_k(Y)\to dB_k(Y)$ mapping $[X,f]\mapsto[X,f]$.

Theorem 12.1 (First version due to David Spivak.)

$\Pi_{\rm bo}^{\rm dbo}: B_k(Y)\to dB_k(Y)$ is an isomorphism for all k, with $dB_k(Y)=0$ for $k<0$.

This holds as every d-manifold can be perturbed to a manifold.

Composing $(\Pi_{\rm bo}^{\rm dbo})^{-1}$ with $\Pi_{\rm bo}^{\rm hom}: B_k(Y)\to H_k(Y,\mathbb{Z})$ gives a morphism $\Pi_{\rm dbo}^{\rm hom}: dB_k(Y)\to H_k(Y,\mathbb{Z})$. We can interpret this as a *virtual class map* for compact, oriented d-manifolds. In particular, this is an easy proof that *the geometric structure on d-manifolds is strong enough to define virtual classes.*

We can also define *orbifold bordism* $B_k^{\rm orb}(Y)$ and *derived orbifold bordism* $dB_k^{\rm orb}(Y)$, replacing (derived) manifolds by (derived) orbifolds. However, the natural morphism $B_k^{\rm orb}(Y)\to dB_k^{\rm orb}(Y)$ is not an isomorphism, as derived orbifolds cannot always be perturbed to orbifolds (Lemma 9.1).

A virtual class for **X** in the homology of **X**?

In algebraic geometry, given a moduli space $\bar{\mathcal M}$, it is usual to define the virtual class in the (Chow) homology $H_{{\rm vdim}\,\bar{\mathcal M}}(\bar{\mathcal M};\mathbb{Q})$. But in differential geometry, given $\bar{\mathcal M}$, usually we find a manifold Y with a map $\bar{\mathcal M}\to Y$, and define the virtual class $[\bar{\mathcal M}]_{\rm virt}$ in the (ordinary) homology $H_{{\rm vdim}\,\bar{\mathcal M}}(Y;\mathbb{Q})$. This is because differential-geometric techniques for defining $[\bar{\mathcal M}]_{\rm virt}$ involve perturbing $\bar{\mathcal M}$, which changes it as a topological space.

Example 12.2

Define $f:\mathbb{R}\to\mathbb{R}$ by $f(x)=e^{-x^{-2}}\sin(\pi/x)$ for $x\neq 0$, and $f(0)=0$. Then f is smooth. Define $\mathbf{X}=\mathbb{R}\times_{f,\mathbb{R},0}*$. Then $\mathbf{X}$ is a compact, oriented derived manifold without boundary, with ${\rm vdim}\,\mathbf{X}=0$. As a topological space we have

$$X=\{1/n:0\neq n\in\mathbb{Z}\}\amalg\{0\}.$$

Then *no virtual class exists* for $\mathbf{X}$ in ordinary homology $H_0(X;\mathbb{Z})$.

Virtual classes in Steenrod or Čech homology

Steenrod homology $H_*^{\rm St}(X;\mathbb{Z})$ (see J. Milnor, *'On the Steenrod homology theory'*, Milnor collected works IV, 2009) is a homology theory of topological spaces. For nice topological spaces X (e.g. manifolds, or finite simplicial complexes) it equals ordinary (e.g. singular) homology $H_*(X;\mathbb{Z})$. It has a useful limiting property:

Theorem 12.3

Let X be a compact subset of a metric space Y, and suppose $W_1,W_2,\ldots$ are open neighbourhoods of X in Y with $\bigcap_{n\geqslant 1}W_n=X$ and $W_1\supseteq W_2\supseteq\cdots$. Then $H_k^{\rm St}(X;\mathbb{Z})\cong\varprojlim_{n\geqslant 1}H_k^{\rm St}(W_n;\mathbb{Z})$.

Čech homology $\check H_*(X;\mathbb{Q})$ over $\mathbb{Q}$ has the same property. Singular homology does not.

Following an idea due to Dusa McDuff, we can use this to define a virtual class $[\mathbf{X}]_{\rm virt}$ for a compact oriented d-manifold $\mathbf{X}$ in $H^{\rm St}_{\mathrm{vdim}\,\mathbf{X}}(X;\mathbb{Z})$ or $\check H_*(X;\mathbb{Q})$. We may write $\mathbf{X}\simeq\mathbf{S}_{V,E,s}$ by Corollary 9.8. This gives a homeomorphism $X\cong s^{-1}(0)$, for $s^{-1}(0)$ a compact subset of V. Choose open neighbourhoods $W_1,W_2,\ldots$ of $s^{-1}(0)$ in V with $\bigcap_{n\geqslant 1}W_n=s^{-1}(0)$ and $W_1\supseteq W_2\supseteq\cdots$. The inclusion $\mathbf{i}_n:\mathbf{X}\hookrightarrow W_n$ defines a d-bordism class $[\mathbf{X},\mathbf{i}_n]\in dB_{\mathrm{vdim}\,\mathbf{X}}(W_n)$, and hence a homology class $\Pi_{\rm dbo}^{\rm hom}([\mathbf{X},\mathbf{i}_n])$ in $H_{\mathrm{vdim}\,\mathbf{X}}(W_n;\mathbb{Z})\cong H^{\rm St}_{\mathrm{vdim}\,\mathbf{X}}(W_n;\mathbb{Z})$. These are preserved by the inclusions $W_{n+1}\hookrightarrow W_n$, and so define a class in the inverse limit $\varprojlim_{n\geqslant 1}H_k^{\rm St}(W_n;\mathbb{Z})$, and thus, by Theorem 12.3, a virtual class $[\mathbf{X}]_{\rm virt}$ in $H^{\rm St}_{\mathrm{vdim}\,\mathbf{X}}(X;\mathbb{Z})$ or $\check H_{\mathrm{vdim}\,\mathbf{X}}(X;\mathbb{Q})$.

12.2. Virtual classes for derived orbifolds

If $\mathbf{X}$ is a compact, oriented derived orbifold (d-orbifold or Kuranishi space), Y is a manifold, and $\mathbf{f}:\mathbf{X}\to Y$ is a 1-morphism, then we can define a *virtual class* $[\mathbf{X}]_{\rm virt}$ in $H_{\mathrm{vdim}\,\mathbf{X}}(Y;\mathbb{Q})$. In the orbifold case it is necessary to work in homology over $\mathbb{Q}$ rather than $\mathbb{Z}$, as points $x\in\mathbf{X}$ with orbifold group $G_x\mathbf{X}$ must be 'counted' with weight $1/|G_x\mathbf{X}|$. There is a standard method for doing this, developed by Fukaya and Ono 1999, in their definition of Gromov–Witten invariants. It is rather messy.
Alternatively, following McDuff–Wehrheim, we can forget Y and define the virtual class $[\mathbf{X}]_{\rm virt}$ in Čech homology $\check H_{\mathrm{vdim}\,\mathbf{X}}(X;\mathbb{Q})$.

The Fukaya–Ono method is to cover $\mathbf{X}$ by finitely many Kuranishi neighbourhoods $(V_i,E_i,\Gamma_i,s_i,\psi_i)$ for $i\in I$ with coordinate changes $\Phi_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\to(V_j,E_j,\Gamma_j,s_j,\psi_j)$ on overlaps between them in a particularly nice form (a 'good coordinate system'), and compatible maps $f_i:V_i\to Y$ representing $\mathbf{f}:\mathbf{X}\to Y$.
Ideally we would like to choose small perturbations $\tilde s_i$ of $s_i:V_i\to E_i$, such that $\tilde s_i$ is transverse and Γ_i-equivariant, and Φ_{ij} maps $\tilde s_i\to\tilde s_j$. Then we could glue the orbifolds $\tilde s_i^{-1}(0)/\Gamma_i$ for $i\in I$ using Φ_{ij} to get a compact oriented orbifold $\tilde{\mathfrak X}$ with morphism $\tilde{\mathfrak f}:\tilde{\mathfrak X}\to Y$, and we would set $[\mathbf{X}]_{\rm virt}=\tilde{\mathfrak f}_*([\tilde{\mathfrak X}])$, for $[\tilde{\mathfrak X}]\in H_{\dim\tilde{\mathfrak X}}(\tilde{\mathfrak X};\mathbb{Q})$ the fundamental class of $\tilde{\mathfrak X}$.
However, it is generally not possible to find perturbations $\tilde s_i$ which are both transverse to the zero section of $E_i\to V_i$, and Γ_i-equivariant.

Instead, we take the perturbations $\tilde{s}_i$ to be 'multisections', $\mathbb{Q}$-weighted mutivalued C^∞-sections of E_i, where the sum of the $\mathbb{Q}$-weights of the branches is 1. As Γ_i can permute the 'branches' of the multisections locally, we have more freedom to make $\tilde{s}_i$ both transverse and Γ_i-equivariant. Then $\tilde{\mathfrak{X}}$ is not an orbifold, but a '$\mathbb{Q}$-weighted orbifold'. Done carefully, by triangulating by simplices we can define a virtual class $[\tilde{\mathfrak{X}}]$ in singular homology $H_{\dim\tilde{\mathfrak{X}}}(\tilde{\mathfrak{X}};\mathbb{Q})$, and then $[\mathbf{X}]_{\rm virt}=\tilde{f}_*([\tilde{\mathfrak{X}}])$.

It is important that although the construction of $[\mathbf{X}]_{\rm virt}$ involves many arbitrary choices of $(V_i,E_i,\Gamma_i,s_i,\psi_i),\tilde{s}_i,\ldots$, the final result $[\mathbf{X}]_{\rm virt}$ in $H_{\mathrm{vdim}\,\mathbf{X}}(Y;\mathbb{Q})$ is independent of these choices. Furthermore, $[\mathbf{X}]_{\rm virt}$ is unchanged under bordisms of $\mathbf{f}:\mathbf{X}\to Y$, that is, it depends only on $[\mathbf{X},\mathbf{f}]\in dB^{\rm orb}_{\mathrm{vdim}\,\mathbf{X}}(Y)$. This bordism-independence makes Gromov–Witten invariants independent of the choice of almost complex structure J used to define them, so they are symplectic invariants, and so on.

12.3. Derived orbifolds with corners and virtual chains

In the Fukaya–Oh–Ono–Ohta Lagrangian Floer cohomology theory, given a symplectic manifold (M,ω) with an almost complex structure J and a Lagrangian L in M, one defines moduli spaces $\bar{\mathcal{M}}_k(\beta)$ of prestable J-holomorphic discs Σ in M with boundary in L, relative homology class $[\Sigma]=\beta\in H_2(M,L;\mathbb{Z})$, and k boundary marked points. The $\bar{\mathcal{M}}_k(\beta)$ are Kuranishi spaces with corners, with 'evaluation maps' $\mathbf{ev}_i:\bar{\mathcal{M}}_k(\beta)\to L$ for $i=1,\ldots,k$, and

$$\partial\bar{\mathcal{M}}_k(\beta)\simeq\coprod_{i+j=k}\coprod_{\beta_1+\beta_2=\beta}\bar{\mathcal{M}}_{i+1}(\beta_1)\times_{\mathbf{ev}_{i+1},L,\mathbf{ev}_{j+1}}\bar{\mathcal{M}}_{j+1}(\beta_2).\quad(12.1)$$

To define Lagrangian Floer cohomology, we have to 'count' the $\bar{\mathcal{M}}_k(\beta)$ in (co)homology, in some sense compatible with (12.1).

Roughly, we want to define a 'virtual class' for $\mathbf{ev}_1 \times \cdots \times \mathbf{ev}_k : \bar{\mathcal{M}}_k(\beta) \to L^k$ in $H_{\mathrm{vdim}\,\bar{\mathcal{M}}_k(\beta)}(L^k;\mathbb{Q})$. However, as $\partial\bar{\mathcal{M}}_k(\beta) \neq \emptyset$, we cannot define a homology class. Instead, we should write $H_*(L^k;\mathbb{Q})$ as the homology of a chain complex $\bigl(C_*(L^k;\mathbb{Q}),\partial\bigr)$, and define a *virtual chain* $[\bar{\mathcal{M}}_k(\beta)]_{\mathrm{virt}}$ in $C_{\mathrm{vdim}\,\bar{\mathcal{M}}_k(\beta)}(L^k;\mathbb{Q})$. The boundary $\partial C_{\mathrm{vdim}\,\bar{\mathcal{M}}_k(\beta)}(L^k;\mathbb{Q})$ should hopefully satisfy an equation modelled on (12.1), something like:

$$\partial[\bar{\mathcal{M}}_k(\beta)]_{\mathrm{virt}} = \sum_{i+j=k,\ \beta_1+\beta_2=\beta} [\bar{\mathcal{M}}_{i+1}(\beta_1)]_{\mathrm{virt}} \times_{\pi_{i+1},L,\pi_{j+1}} [\bar{\mathcal{M}}_{j+1}(\beta_2)]_{\mathrm{virt}}. \quad (12.2)$$

Fukaya–Oh–Ohta–Ono 2009 take their homology theory $\bigl(C_*(L^k;\mathbb{Q}),\partial\bigr)$ to be singular homology, generated by smooth maps $\sigma:\Delta_n \to L^k$. Then (12.2) does not make sense, as the fibre product (intersection product) is not defined in singular homology at the chain level. They have an alternative approach (unfinished (?), 2010) using de Rham cohomology, in which (12.2) does make sense.

There are other technical difficulties in the FOOO approach. One is that for each moduli space $\bar{\mathcal{M}}_k(\beta)$, one must choose a perturbation by multisections, and a triangulation by simplices / de Rham (co)chains, and these perturbations must be compatible at $\partial\bar{\mathcal{M}}_k(\beta)$ according to (12.1). There are infinitely many moduli spaces, and each moduli space $\bar{\mathcal{M}}_k(\beta)$ can occur in the formula for $\partial^j\bar{\mathcal{M}}_l(\beta')$ for infinitely many j,l,β', so the virtual chain for $\bar{\mathcal{M}}_k(\beta)$ should be subject to infinitely many compatibility conditions, including infinitely many smallness conditions on the size of the perturbations $s_i \rightsquigarrow \tilde{s}_i$. But we can only satisfy finitely many smallness conditions at once. And so on.

12.4. New (co)homology theories for virtual chains

A lot of the technical complexity in Fukaya–Oh–Ohta–Ono's 2009 Lagrangian Floer cohomology theory comes from the fact that the homology theory in which they do all their homological algebra – singular homology – does not play nicely with Kuranishi spaces. Their de Rham version is better, but still not ideal.

I would like to propose an alternative approach, which is to define new (co)homology theories $KH_*(Y;R)$, $KH^*(Y;R)$ of manifolds Y, in which it is easy to define virtual classes and virtual chains for compact, oriented Kuranishi spaces $\mathbf{X}$ with 1-morphisms $\mathbf{f}:\mathbf{X}\to Y$. (Joyce, work in progress 2015; prototype version using FOOO Kuranishi spaces arXiv:0707.3573, arXiv:0710.5634 – please don't read these.) I'll discuss one version of the homology theory only.

Let Y be a manifold or orbifold, and R a $\mathbb{Q}$-algebra. We define a complex of R-modules $\bigl(KC_*(Y;R),\partial\bigr)$, whose homology groups $KH_*(Y;R)$ are the *Kuranishi homology* of Y.

Similarly to the definition of d-bordism $dB_k(Y)$, chains in $KC_k(Y;R)$ for $k\in\mathbb{Z}$ are R-linear combinations of equivalence classes $[\mathbf{X},\mathbf{f},\mathbf{G}]$ with relations, where $\mathbf{X}$ is a compact, oriented Kuranishi space with corners with dimension k, $\mathbf{f}:\mathbf{X}\to Y$ is a 1-morphism in $\mathbf{Kur^c}$, and $\mathbf{G}$ is some extra 'gauge-fixing data' associated to $\mathbf{f}:\mathbf{X}\to Y$, with many possible choices.

I won't give all the relations on the $[\mathbf{X},\mathbf{f},\mathbf{G}]$. Two examples are:

$$[\mathbf{X}_1\amalg\mathbf{X}_2,\mathbf{f}_1\amalg\mathbf{f}_2,\mathbf{G}_1\amalg\mathbf{G}_2]=[\mathbf{X}_1,\mathbf{f}_1,\mathbf{G}_1]+[\mathbf{X}_2,\mathbf{f}_2,\mathbf{G}_2], \tag{12.3}$$

$$[-\mathbf{X},\mathbf{f},\mathbf{G}]=-[\mathbf{X},\mathbf{f},\mathbf{G}], \tag{12.4}$$

where $-\mathbf{X}$ is $\mathbf{X}$ with the opposite orientation.

Gauge-fixing data – first properties

Here 'gauge-fixing data' is the key to the whole story. I won't tell you what it is, but I will tell you some properties it has:

(i) For any compact Kuranishi space with corners $\mathbf{X}$ and 1-morphism $\mathbf{f} : \mathbf{X} \to Y$ we have a nonempty set $\mathrm{Gauge}(\mathbf{f} : \mathbf{X} \to Y)$ of choices of 'gauge-fixing data' $\mathbf{G}$ for $\mathbf{f}$.

(ii) If $\mathbf{g} : \mathbf{X}' \to \mathbf{X}$ is étale we have a *pullback map* $\mathbf{g}^* : \mathrm{Gauge}(\mathbf{f} : \mathbf{X} \to Y) \to \mathrm{Gauge}(\mathbf{f} \circ \mathbf{g} : \mathbf{X}' \to Y)$. If $\mathbf{g}, \mathbf{g}'$ are 2-isomorphic then $\mathbf{g}^* = \mathbf{g}'^*$. Pullbacks are functorial.

(iii) There is a *boundary map* $|_{\partial\mathbf{X}} : \mathrm{Gauge}(\mathbf{f} : \mathbf{X} \to Y) \to \mathrm{Gauge}(\mathbf{f} \circ \mathbf{i}_{\mathbf{X}} : \partial\mathbf{X} \to Y)$. We regard it as a pullback along $\mathbf{i}_{\mathbf{X}} : \partial\mathbf{X} \to \mathbf{X}$.

(iv) If $g : Y \to Z$ is a smooth map of manifolds, there is a *pushforward map* $g_* : \mathrm{Gauge}(\mathbf{f} : \mathbf{X} \to Y) \to \mathrm{Gauge}(g \circ \mathbf{f} : \mathbf{X} \to Z)$.

Boundary operators

Note that Kuranishi spaces $\mathbf{X}$ can have virtual dimension $\mathrm{vdim}\,\mathbf{X} < 0$, so $KC_k(Y;R) \neq 0$ for all $k < 0$, although $KH_k(Y;R) = 0$ for $k < 0$.
The boundary operator $\partial : KC_k(Y;R) \to KC_{k-1}(Y;R)$ maps

$$\partial : \big[\mathbf{X}, \mathbf{f}, \mathbf{G}\big] \longmapsto \big[\partial\mathbf{X}, \mathbf{f} \circ \mathbf{i}_{\mathbf{X}}, \mathbf{G}|_{\partial\mathbf{X}}\big].$$

We have a natural 1-morphism $\mathbf{i}_{\mathbf{X}} : \partial\mathbf{X} \to \mathbf{X}$ and an equivalence $\partial^2\mathbf{X} \simeq \partial\mathbf{X}_{\mathbf{i}_{\mathbf{X}},\mathbf{X},\mathbf{i}_{\mathbf{X}}}\partial\mathbf{X}$. Thus there is an orientation-reversing involution $\boldsymbol{\sigma} : \partial^2\mathbf{X} \to \partial^2\mathbf{X}$ swapping the two factors of $\partial\mathbf{X}$. This satisfies $\mathbf{i}_{\mathbf{X}} \circ \mathbf{i}_{\partial\mathbf{X}} \circ \boldsymbol{\sigma} \cong \mathbf{i}_{\mathbf{X}} \circ \mathbf{i}_{\partial\mathbf{X}}$. Hence $\mathbf{G}|_{\partial^2\mathbf{X}}$ is $\boldsymbol{\sigma}$-invariant. Using this and (12.4) we show that $\partial^2 = 0$, so $KH_*(Y;R)$ is well-defined.
Here is a property of gauge-fixing data with prescribed boundary values. 'Only if' is necessary by (i)–(iii) as above.

(v) Suppose $\mathbf{H} \in \mathrm{Gauge}(\mathbf{f} \circ \mathbf{i}_{\mathbf{X}} : \partial\mathbf{X} \to Y)$. Then there exists $\mathbf{G} \in \mathrm{Gauge}(\mathbf{f} : \mathbf{X} \to Y)$ with $\mathbf{G}|_{\partial\mathbf{X}} = \mathbf{H}$ iff $\boldsymbol{\sigma}^*(\mathbf{H}|_{\partial^2\mathbf{X}}) = \mathbf{H}|_{\partial^2\mathbf{X}}$.

Suppose $g : Y \to Z$ is a smooth map of manifolds or orbifolds. Define an R-linear *pushforward* $g_* : KC_k(Y;R) \to KC_k(Z;R)$ by $g_* : [\mathbf{X}, \mathbf{f}, \mathbf{G}] \mapsto [\mathbf{X}, g \circ \mathbf{f}, g_*(\mathbf{G})]$. Then $g_* \circ \partial = \partial \circ g_*$, so this induces $g_* : KH_k(Y;R) \to KH_k(Z;R)$. Pushforwards are functorial.

Singular homology $H_*^{\mathrm{sing}}(Y;R)$ may be defined using $(C_*^{\mathrm{sing}}(Y;R), \partial)$, where $C_k^{\mathrm{sing}}(Y;R)$ is spanned by *smooth* maps $f : \Delta_k \to Y$, for Δ_k the standard k-simplex, thought of as a manifold with corners.

We define an R-linear map $F_{\mathrm{sing}}^{\mathrm{KH}} : C_k^{\mathrm{sing}}(Y;R) \to KC_k(Y;R)$ by

$$F_{\mathrm{sing}}^{\mathrm{KH}} : f \longmapsto \left[\Delta_k, f, \mathbf{G}_{\Delta_k}\right],$$

with $\mathbf{G}_{\Delta_k}$ some standard choice of gauge-fixing data for Δ_k. Then $F_{\mathrm{sing}}^{\mathrm{KH}} \circ \partial = \partial \circ F_{\mathrm{sing}}^{\mathrm{KH}}$, so that $F_{\mathrm{sing}}^{\mathrm{KH}}$ induces morphisms $F_{\mathrm{sing}}^{\mathrm{KH}} : H_k^{\mathrm{sing}}(Y;R) \to KH_k(Y;R)$.

One of the main results of the theory will be:

Theorem 12.4

$F_{\mathrm{sing}}^{\mathrm{KH}} : H_k^{\mathrm{sing}}(Y;R) \to KH_k(Y;R)$ is an isomorphism for all $k \in \mathbb{Z}$.

- Forming virtual classes/virtual chains is easy. Suppose $\bar{\boldsymbol{\mathcal{M}}}$ is a moduli Kuranishi space, with evaluation map $\mathbf{ev} : \bar{\boldsymbol{\mathcal{M}}} \to Y$. Choose gauge-fixing data $\mathbf{G}$ for $\bar{\boldsymbol{\mathcal{M}}}$, which is possible by (i). Then $[\bar{\boldsymbol{\mathcal{M}}}, \mathbf{ev}, \mathbf{G}] \in KC_k(Y;R)$ is a virtual chain for $\bar{\boldsymbol{\mathcal{M}}}$. If $\partial \bar{\boldsymbol{\mathcal{M}}} = \emptyset$ then $[[\bar{\boldsymbol{\mathcal{M}}}, \mathbf{ev}, \mathbf{G}]] \in KH_k(Y;R)$ is a virtual class for $\bar{\boldsymbol{\mathcal{M}}}$.
- Obviously, Kuranishi homology is not a new invariant, it's just ordinary homology. The point is that it has special properties at the chain level which make it more convenient than competing homology theories (e.g. singular homology) for some tasks.
- The messy parts of defining virtual chains in [FOOO 2009] are repackaged in the proof of Theorem 12.4.