Derived Differential Geometry

Lecture 5 of 14:
Differential-geometric description of d-manifolds

Dominic Joyce, Oxford University
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These slides, and references, etc., available at
`http://people.maths.ox.ac.uk/~joyce/DDG2015`

Plan of talk:

5. Differential-geometric description of d-manifolds

We have defined a strict 2-category **dSpa** of d-spaces $\mathbf{X} = (X, \mathcal{O}^\bullet_X)$, which are topological spaces X equipped with a sheaf of square zero dg C^∞-rings $\mathcal{O}^\bullet_X$. We have full (2-)subcategories $\mathbf{Man} \subset \mathbf{C^\infty Sch} \subset \mathbf{dSpa}$, so that we may regard manifolds as examples of d-spaces. All fibre products exist in **dSpa**.

A d-space $\mathbf{X}$ is called a *d-manifold* of *virtual dimension* $n \in \mathbb{Z}$ if it is locally modelled on fibre products $V \times_{0,E,s} V$ in **dSpa**, where V is a manifold, $E \to V$ a vector bundle with $\dim V - \operatorname{rank} E = n$, and $s : V \to E$ a smooth section. D-manifolds form a full 2-subcategory $\mathbf{dMan} \subset \mathbf{dSpa}$.

To actually do stuff with d-manifolds, it is very useful to be able to describe objects, 1-morphisms and 2-morphisms in **dMan** not using square zero dg C^∞-rings, but using honest differential-geometric objects: manifolds, vector bundles, sections, and smooth maps.

I will define a family of explicit 'standard model' d-manifolds $\mathbf{S}_{V,E,s}$, related to Example 4.3, depending on a manifold V, vector bundle $E \to V$ and section $s : V \to E$. We can describe 1-morphisms $\mathbf{f}, \mathbf{g} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ and 2-morphisms $\eta : \mathbf{f} \Rightarrow \mathbf{g}$ completely in terms of the differential geometry of V, E, s, W, F, t. For this we will need '$O(s)$' and '$O(s^2)$' notation, defined in §5.1.

As every d-manifold $\mathbf{X}$ is locally equivalent to standard models $\mathbf{S}_{V,E,s}$, this enables us to describe d-manifolds and their 1- and 2-morphisms locally, solely in differential-geometric language.

In fact we can use these ideas to give an alternative definition of a (weak) 2-category of derived manifolds $\mathbf{Kur_{trG}}$ involving only manifolds and differential geometry, not using (dg) C^∞-rings and C^∞-schemes at all. This is the theory of (*M*-)*Kuranishi spaces*, and will be the subject of lectures 6-8.

5.1. The $O(s)$ and $O(s^2)$ notation

Definition

Let V be a manifold, $E \to V$ a vector bundle, and $s : V \to E$ be a smooth section of E, that is, $s \in C^\infty(E)$.

- If $f : V \to \mathbb{R}$ is smooth, we write '$f = O(s)$' if $f = \alpha \cdot s$ for some $\alpha \in C^\infty(E^*)$, and '$f = O(s^2)$' if $f = \beta \cdot (s \otimes s)$ for some $\beta \in C^\infty(E^* \otimes E^*)$.
- If $F \to V$ is a another vector bundle and $t \in C^\infty(F)$, we write '$t = O(s)$' if $t = \alpha \cdot s$ for some $\alpha \in C^\infty(F \otimes E^*)$, and '$t = O(s^2)$' if $t = \beta \cdot (s \otimes s)$ for some $\beta \in C^\infty(F \otimes E^* \otimes E^*)$.

In terms of the $\mathbb{R}$-algebra (or C^∞-ring) $C^\infty(V)$, $f = O(s)$ means $f \in I_s \subseteq C^\infty(V)$, and $f = O(s^2)$ means $f \in I_s^2 \subseteq C^\infty(V)$, where $I_s = C^\infty(E^*) \cdot s$ is the ideal in $C^\infty(V)$ generated by s. Similarly $t = O(s) \Leftrightarrow t \in I_s \cdot C^\infty(F)$ and $t = O(s^2) \Leftrightarrow t \in I_s^2 \cdot C^\infty(F)$.

Definition

Let V be a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$. Let W be another manifold, and $f, g : V \to W$ be smooth maps.

- We write '$g = f + O(s)$' if for all smooth $h : W \to \mathbb{R}$ we have $h \circ g - h \circ f = O(s)$ as smooth functions $V \to \mathbb{R}$.
- Similarly, we write '$g = f + O(s^2)$' if for all smooth $h : W \to \mathbb{R}$ we have $h \circ g - h \circ f = O(s^2)$.
- Let $v \in C^\infty(f^*(TW))$ with $v = O(s)$. Then we write '$g = f + v + O(s^2)$' if $h \circ g - f^*(\mathrm{d}h) \cdot v - h \circ f = O(s^2)$ for all smooth $h : W \to \mathbb{R}$, where $f^*(\mathrm{d}h)$ lies in $C^\infty(f^*(T^*W))$.

This is more tricky: note that f, g and v do not lie in the same vector space, so '$g - f - v$' does not make sense. Nonetheless $g = f + v + O(s^2)$ makes sense.
In terms of C^∞-schemes, $g = f + O(s)$ iff $g|_{\underline{X}} = f|_{\underline{X}}$, where $\underline{X} \subseteq V$ is the C^∞-subscheme defined by $s = 0$.

Definition

Let V be a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$. Let W be another manifold, and $f, g : V \to W$ be smooth maps with $g = f + O(s)$. Let $F \to W$ be a vector bundle, and $t \in C^\infty(f^*(F))$, $u \in C^\infty(g^*(F))$.
We say that '$u = t + O(s)$' if for all $\gamma \in C^\infty(F^*)$ we have $u \cdot g^*(\gamma) - t \cdot f^*(\gamma) = O(s)$ as smooth functions $V \to \mathbb{R}$.

Note that t, u are sections of different vector bundles, so '$u - t$' does not make sense. Nonetheless '$u = t + O(s)$' makes sense. In terms of C^∞-schemes, if $\underline{X} \subseteq V$ is the C^∞-subscheme defined by $s = 0$, then $g = f + O(s)$ implies that $g|_{\underline{X}} = f|_{\underline{X}}$, so $g^*(F)|_{\underline{X}}$ and $f^*(F)|_{\underline{X}}$ are the same vector bundle. Then $u = t + O(s)$ means that $u|_{\underline{X}} = t|_{\underline{X}}$ as sections of $g^*(F)|_{\underline{X}} = f^*(F)|_{\underline{X}}$.

5.2. Standard model d-manifolds

Proposition 5.1

Let $\mathbf{X}$ be a d-manifold, with $\mathrm{vdim}\,\mathbf{X} = n$. Then the following are equivalent:

(i) $\mathbf{X} \simeq U \times_{g,W,h} V$ *in* **dSpa**, *where U, V, W are manifolds, $g : U \to W$, $h : V \to W$ are smooth, and $\dim U + \dim V - \dim W = n$.*

(ii) $\mathbf{X} \simeq U \times_{i,W,j} V$ *in* **dSpa**, *where W is a manifold, $U, V \subseteq W$ are submanifolds with inclusions $i : U \hookrightarrow W$, $j : V \hookrightarrow W$, and $\dim U + \dim V - \dim W = n$.*

(iii) $\mathbf{X} \simeq V \times_{0,E,s} V$ *in* **dSpa**, *where V is a manifold, $E \to V$ is a vector bundle, and $s \in C^\infty(E)$, with $\dim V - \mathrm{rank}\, E = n$.*

We call $\mathbf{X}$ satisfying **(i)–(iii)** *a* **principal d-manifold**.

Every d-manifold $\mathbf{X}$ can be covered by open $\mathbf{Y} \subseteq \mathbf{X}$ with $\mathbf{Y}$ principal. We prefer to use model number (iii).

Definition

Let V be a manifold, $E \to V$ a vector bundle, and $s: V \to E$ a smooth section. As in Example 4.2, define a square zero dg C^∞-ring $\mathfrak{C}^\bullet = [\mathfrak{C}^{-1} \xrightarrow{\mathrm{d}} \mathfrak{C}^0]$ by

$$\mathfrak{C}^0 = C^\infty(V)/I_s^2, \qquad \mathfrak{C}^{-1} = C^\infty(E^*)/I_s \cdot C^\infty(E^*),$$
$$\mathrm{d}(\epsilon + I_s \cdot C^\infty(E^*)) = \epsilon(s) + I_s^2,$$

where $I_s = C^\infty(E^*) \cdot s \subset C^\infty(V)$ is the ideal generated by s. Define $\mathbf{S}_{V,E,s} = \mathbf{Spec}\, \mathfrak{C}^\bullet$. We call $\mathbf{S}_{V,E,s}$ a *standard model d-manifold*. It has topological space $S_{V,E,s} = s^{-1}(0) \subseteq V$.

Then $\mathbf{S}_{V,E,s} \simeq V \times_{0,E,s} V$, as in Proposition 5.1(iii). Now writing $\mathbf{S}_{V,E,s}$ as a fibre product only characterizes it up to equivalence in the 2-category **dSpa**. But writing $\mathbf{S}_{V,E,s} = \mathbf{Spec}\, \mathfrak{C}^\bullet$ characterizes it uniquely (at least, up to canonical 1-isomorphism) in **dSpa**. This will be important.

Open d-submanifolds of standard model d-manifolds

Let V, E, s be as above, and suppose $V' \subseteq V$ is open. Write $E' = E|_{V'}$ and $s' = s|_{V'}$. Then we have standard model d-manifolds $\mathbf{S}_{V,E,s}$ and $\mathbf{S}_{V',E',s'}$, with topological spaces $S_{V,E,s} = s^{-1}(0)$ and $S_{V',E',s'} = s^{-1}(0) \cap V'$.
In fact $\mathbf{S}_{V',E',s'} \subseteq \mathbf{S}_{V,E,s}$ is an open d-submanifold.
In particular, if V' is an open neighbourhood of $s^{-1}(0)$ in V, then $\mathbf{S}_{V',E',s'} = \mathbf{S}_{V,E,s}$. This means that we can always restrict to an arbitrarily small open neighbourhood of $s^{-1}(0)$ in V without changing anything; in effect, we can take germs about $s^{-1}(0)$ in V.
We have dg C^∞-rings $\mathfrak{C}^\bullet, \mathfrak{C}'^\bullet$ from (V, E, s) and (V', E', s'), and the natural restriction morphism $\iota : \mathfrak{C}^\bullet \to \mathfrak{C}'^\bullet$ is an isomorphism.

5.3. Standard model 1- and 2-morphisms

Definition

Let V,W be manifolds, $E\to V$, $F\to W$ be vector bundles, and $s\in C^\infty(E)$, $t\in C^\infty(F)$. Let V' be an open neighbourhood of $s^{-1}(0)$ in V, and $E'=E|_{V'}$, $s'=s|_{V'}$. Write $\mathfrak{C}^\bullet,\mathfrak{C}'^\bullet,\mathfrak{D}^\bullet$ for the square zero dg C^∞-rings from $(V,E,s),(V',E',s'),(W,F,t)$, so that $\mathbf{S}_{V,E,s}=\mathbf{Spec}\,\mathfrak{C}^\bullet$, $\mathbf{S}_{W,F,t}=\mathbf{Spec}\,\mathfrak{D}^\bullet$, and $\iota:\mathfrak{C}^\bullet\xrightarrow{\cong}\mathfrak{C}'^\bullet$. Suppose $f:V'\to W$ is smooth, and $\hat f:E'\to f^*(F)$ is a morphism of vector bundles on V' with $\hat f\circ s'=f^*(t)+O(s^2)$ in $C^\infty(f^*(F))$. Define a morphism $\alpha:\mathfrak{D}^\bullet\to\mathfrak{C}'^\bullet$ of dg C^∞-rings by

$$\begin{tikzcd} \mathfrak{D}^{-1}=C^\infty(F^*)/I_t\cdot C^\infty(F^*) \arrow[r, "\mathrm{d}=t\cdot"'] \arrow[d, "\alpha^{-1}=\hat f^*\circ f^*"] & \mathfrak{D}^0=C^\infty(W)/I_t^2 \arrow[d, "\alpha^0=f^*"'] \\ \mathfrak{C}'^{-1}=C^\infty(E'^*)/I_{s'}\cdot C^\infty(E'^*) \arrow[r, "\mathrm{d}=s'\cdot"] & \mathfrak{C}'^0=C^\infty(V')/I_{s'}^2. \end{tikzcd}$$

Define $\mathbf{S}_{V',f,\hat f}=\mathbf{Spec}(\iota^{-1}\circ\alpha):\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$. We call $\mathbf{S}_{V',f,\hat f}$ a *standard model 1-morphism*.

Theorem 5.2

Let $\mathbf{S}_{V,E,s},\mathbf{S}_{W,F,t}$ be standard model d-manifolds. Then

(a) *Suppose $\mathbf{g}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ is a 1-morphism in* **dMan**. *Then $\mathbf{g}=\mathbf{S}_{V',f,\hat f}$ for some standard model 1-morphism $\mathbf{S}_{V',f,\hat f}$ defined using $s^{-1}(0)\subseteq V'\subseteq V$, $f:V'\to W$, $\hat f:E'\to f^*(F)$.*

(b) *Suppose $\mathbf{S}_{V_1',f_1,\hat f_1},\mathbf{S}_{V_2',f_2,\hat f_2}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ are standard model 1-morphisms defined for $i=1,2$ using $s^{-1}(0)\subseteq V_i'\subseteq V$, $f_i:V_i'\to W$ and $\hat f_i:E_i'\to f_i^*(F)$. Then $\mathbf{S}_{V_1',f_1,\hat f_1}=\mathbf{S}_{V_2',f_2,\hat f_2}$ iff $f_2|_{V_1'\cap V_2'}=f_1|_{V_1'\cap V_2'}+O(s^2)$ and $\hat f_2|_{V_1'\cap V_2'}=\hat f_1|_{V_1'\cap V_2'}+O(s)$.*

Sketch proof.

For (a), we show $\mathbf{g}=\mathbf{Spec}\,\alpha$ for $\alpha:\mathfrak{D}^\bullet\to\mathfrak{C}^\bullet$ a morphism of dg C^∞-rings, and then show α is induced from some $V',f,\hat f$. For (b), we show the morphisms of dg C^∞-rings $\alpha_1,\alpha_2:\mathfrak{D}^\bullet\to\mathfrak{C}^\bullet$ are equal iff $f_2=f_1+O(s^2)$ and $\hat f_2=\hat f_1+O(s)$. □

From the point of view of 2-categories, Theorem 5.2 is a perverse result: we characterize the 1-morphisms $\mathbf{g}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ completely as a set, *not* up to 2-isomorphism.

If we were to replace $\mathbf{S}_{V,E,s},\mathbf{S}_{W,F,t}$ by equivalent objects in **dMan**, then the set of 1-morphisms $\mathbf{g}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ might change (though the set of 2-isomorphism classes of 1-morphisms $\mathbf{g}$ would not), and Theorem 5.2 would be false.

The theorem depends upon using the particular model $\mathbf{S}_{V,E,s}=\mathbf{Spec}\,\mathfrak{C}^\bullet$ for the equivalence class of objects in **dSpa** representing the fibre product $V\times_{0,E,s}V$.

Next we need to understand 2-morphisms $\eta:\mathbf{S}_{V_1',f_1,\hat f_1}\Rightarrow\mathbf{S}_{V_2',f_2,\hat f_2}$ between standard model 1-morphisms.

'Standard model' 2-morphisms

Definition

Let $\mathbf{S}_{V,E,s},\mathbf{S}_{W,F,t}$ be standard model d-manifolds, and $\mathbf{S}_{V_1',f_1,\hat f_1},\mathbf{S}_{V_2',f_2,\hat f_2}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ be standard model 1-morphisms. Suppose $f_2=f_1+O(s)$ on $V'':=V_1'\cap V_2'\subseteq V$. Let $\Lambda:E\vert_{V''}\to f_1\vert_{V''}^*(TW)$ be a vector bundle morphism, with

$$f_2=f_1+\Lambda\cdot s+O(s^2)\quad\text{and}\quad \hat f_2=\hat f_1+\Lambda\cdot f^*(\mathrm{d}t)+O(s).\quad(5.1)$$

Define the 'standard model' 2-morphism $\mathbf{S}_\Lambda:\mathbf{S}_{V_1',f_1,\hat f_1}\Rightarrow\mathbf{S}_{V_2',f_2,\hat f_2}$ to be **Spec** of the composition

$$\begin{matrix}\mathfrak{D}^0= \\ C^\infty(W)/I_t^2\end{matrix}\xrightarrow{\Lambda^*\circ\mathrm{d}}\begin{matrix}\mathfrak{C}''^{-1}= \\ C^\infty(E''^*)/I_{s''}\cdot C^\infty(E''^*)\end{matrix}\xrightarrow{\iota''^{-1}}\mathfrak{C}^{-1},$$

where $\mathfrak{C}''^\bullet$ is from $(V'',E''=E\vert_{V''},s''=s\vert_{V''})$ and $\iota'':\mathfrak{C}^\bullet\to\mathfrak{C}''^\bullet$ the natural isomorphism.

Here is the analogue of Theorem 5.2, with a similar proof:

Theorem 5.3

Let $\mathbf{S}_{V,E,s}, \mathbf{S}_{W,F,t}$ be standard model d-manifolds, and $\mathbf{S}_{V_1',f_1,\hat f_1}, \mathbf{S}_{V_2',f_2,\hat f_2} : \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ be standard model 1-morphisms in **dMan**. *Then*

(a) *Suppose $\eta : \mathbf{S}_{V_1',f_1,\hat f_1} \Rightarrow \mathbf{S}_{V_2',f_2,\hat f_2}$ is a 2-morphism in* **dMan**. *Then $\eta = \mathbf{S}_\Lambda$ for some standard model 2-morphism defined using $\Lambda : E|_{V_1'\cap V_2'} \to f_1|^*_{V_1'\cap V_2'}(TW)$.*

(b) *Suppose $\mathbf{S}_{\Lambda_1}, \mathbf{S}_{\Lambda_2} : \mathbf{S}_{V_1',f_1,\hat f_1} \Rightarrow \mathbf{S}_{V_2',f_2,\hat f_2}$ are standard model 2-morphisms. Then $\mathbf{S}_{\Lambda_1} = \mathbf{S}_{\Lambda_2}$ iff $\Lambda_2 = \Lambda_1 + O(s)$.*

Conclusions

Write **SMod** for the full 2-subcategory of **dMan** with objects standard model d-manifolds $\mathbf{S}_{V,E,s}$. Then Theorems 5.2 and 5.3 allow us to describe **SMod** *completely*, up to strict isomorphism of strict 2-categories, *using only differential geometric language:*

- Objects of **SMod** correspond to triples (V, E, s), with V a manifold, $E \to V$ a vector bundle, and $s \in C^\infty(E)$.
- 1-morphisms $(V, E, s) \to (W, F, t)$ correspond to equivalence classes $[V', f, \hat f]$ of triples $(V', f, \hat f)$, where V' is an open neighbourhood of $s^{-1}(0)$ in V, and $f : V' \to W$ is smooth, and $\hat f : E' \to f^*(F)$ is a morphism of vector bundles on V' with $\hat f \circ s' = f^*(t) + O(s^2)$, where $E' = E|_{V'}$, $s' = s|_{V'}$, and two triples $(V_1', f_1, \hat f_1), (V_2', f_2, \hat f_2)$ are equivalent if $f_2|_{V_1'\cap V_2'} = f_1|_{V_1'\cap V_2'} + O(s^2)$ and $\hat f_2|_{V_1'\cap V_2'} = \hat f_1|_{V_1'\cap V_2'} + O(s)$.

Conclusions

- 2-morphisms $[V_1', f_1, \hat f_1] \Rightarrow [V_2', f_2, \hat f_2]$ exist only if $f_2 = f_1 + O(s)$, and correspond to equivalence classes $[\Lambda]$ of vector bundle morphisms $\Lambda : E|_{V_1'\cap V_2'} \to f_1|^*_{V_1'\cap V_2'}(TW)$ with $f_2 = f_1 + \Lambda\cdot s + O(s^2)$ and $\hat f_2 = \hat f_1 + \Lambda \cdot f^*(\mathrm{d}t) + O(s)$, and Λ_1, Λ_2 are equivalent if $\Lambda_2 = \Lambda_1 + O(s)$.
- We can also give differential-geometric definitions of the other structures of a strict 2-category: composition of 1-morphisms, vertical and horizontal composition of 2-morphisms, identities. For example, the composition of 1-morphisms $[V', f, \hat f] : (V, E, s) \to (W, F, t)$ and $[W', g, \hat g] : (W, F, t) \to (X, G, u)$ is
$$[W', g, \hat g] \circ [V', f, \hat f] = \bigl[f^{-1}(W'), g\circ f|_{\cdots}, f^{-1}(\hat g)\circ \hat f|_{\cdots}\bigr],$$
and $\mathrm{id}_{(V,E,s)} = [V, \mathrm{id}_V, \mathrm{id}_E]$, and $\mathrm{id}_{[V',f,\hat f]} = [0]$.

Conclusions

So, if we are happy to work only in **SMod** $\subset$ **dMan**, that is, with d-manifolds which are covered by a single Kuranishi neighbourhood (V, E, s), we can give up all the tedious mucking about with (dg) C^∞-rings, sheaves, C^∞-schemes, etc., and work only with manifolds, vector bundles, and smooth sections. We do have to get used to the $O(s), O(s^2)$ notation, though.
Later in the course we will explain the following:

Theorem 5.4

Let $\mathbf{X}$ *be a d-manifold. Then* $\mathbf{X}$ *is equivalent in* **dMan** *to a standard model d-manifold* $\mathbf{S}_{V,E,s}$ *if and only if the dimensions of 'tangent spaces'* $\dim T_x\mathbf{X}$ *are globally bounded on* $\mathbf{X}$*. For instance, this is true if* $\mathbf{X}$ *is compact.*

Because of this, almost all interesting d-manifolds can be written in the form $\mathbf{S}_{V,E,s}$, and we lose little by working in **SMod**.

5.4. Tangent spaces and obstruction spaces

Let $\mathbf{X}$ be a d-manifold, and $x \in \mathbf{X}$. Then we have the *tangent space* $T_x\mathbf{X}$ and *obstruction space* $O_x\mathbf{X}$, which are natural finite-dimensional real vector spaces with $\dim T_x\mathbf{X} - \dim O_x\mathbf{X} = \mathrm{vdim}\,\mathbf{X}$. The dual vector spaces are the *cotangent space* $T_x^*\mathbf{X}$ and *coobstruction space* $O_x^*\mathbf{X}$. If $\mathbb{L}_{\mathbf{X}} = [\mathbb{L}_{\mathbf{X}}^{-1} \xrightarrow{\mathrm{d}} \mathbb{L}_{\mathbf{X}}^0]$ is the cotangent complex of $\mathbf{X}$ as a d-space, as in §4.3, we may define these by the exact sequence

$$0 \longrightarrow O_x^*\mathbf{X} \longrightarrow \mathbb{L}_{\mathbf{X}}^{-1}|_x \xrightarrow{\mathrm{d}|_x} \mathbb{L}_{\mathbf{X}}^0|_x \longrightarrow T_x^*\mathbf{X} \longrightarrow 0. \qquad (5.2)$$

If $\mathbf{f}: \mathbf{X} \to \mathbf{Y}$ is a 1-morphism in **dMan** and $x \in \mathbf{X}$ with $\mathbf{f}(x) = y \in \mathbf{Y}$, we have natural, functorial linear maps $T_x\mathbf{f}: T_x\mathbf{X} \to T_y\mathbf{Y}$ and $O_x\mathbf{f}: O_x\mathbf{X} \to O_y\mathbf{Y}$. If $\eta: \mathbf{f} \Rightarrow \mathbf{g}$ is a 2-morphism in **dMan** then $T_x\mathbf{f} = T_x\mathbf{g}$ and $O_x\mathbf{f} = O_x\mathbf{g}$.

If $\mathbf{X}$ is a standard model d-manifold $\mathbf{S}_{V,E,s}$ then $\mathbb{L}_{\mathbf{X}} = [E^*|_{s^{-1}(0)} \xrightarrow{\mathrm{d}s} T^*V|_{s^{-1}(0)}]$. So dualizing (5.2), for each $x \in s^{-1}(0) \subseteq V$, the tangent and obstruction spaces are given by the exact sequence

$$0 \longrightarrow T_x\mathbf{S}_{V,E,s} \longrightarrow T_xV \xrightarrow{\mathrm{d}s|_x} E|_x \longrightarrow O_x\mathbf{S}_{V,E,s} \longrightarrow 0. \qquad (5.3)$$

That is, $T_x\mathbf{S}_{V,E,s}$, $O_x\mathbf{S}_{V,E,s}$ are the kernel and cokernel of $\mathrm{d}s|_x: T_xV \to E|_x$. If $\mathbf{S}_{V',f,\hat{f}}: \mathbf{S}_{V,E,s} \to \mathbf{S}_{W,F,t}$ is a standard model 1-morphism then $T_x\mathbf{S}_{V',f,\hat{f}}$, $O_x\mathbf{S}_{V',f,\hat{f}}$ are given by the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & T_x\mathbf{S}_{V,E,s} \arrow[r] \arrow[d, "T_x\mathbf{S}_{V',f,\hat{f}}"] & T_xV \arrow[r, "\mathrm{d}s|_x"'] \arrow[d, "T_xf"] & E|_x \arrow[r] \arrow[d, "\hat{f}|_x"] & O_x\mathbf{S}_{V,E,s} \arrow[r] \arrow[d, "O_x\mathbf{S}_{V',f,\hat{f}}"] & 0 \\
0 \arrow[r] & T_y\mathbf{S}_{W,F,t} \arrow[r] & T_yW \arrow[r, "\mathrm{d}t|_y"] & F|_y \arrow[r] & O_y\mathbf{S}_{W,F,t} \arrow[r] & 0.
\end{tikzcd} \qquad (5.4)$$

Étale 1-morphisms and equivalences

Definition

A 1-morphism $\mathbf{f}:\mathbf{X}\rightarrow\mathbf{Y}$ in $\mathbf{dMan}$ or $\mathbf{dSpa}$ is called *étale* if it is a local equivalence. That is, $\mathbf{f}$ is étale if for all $x\in\mathbf{X}$ there exist open d-submanifolds $x\in\mathbf{U}\subseteq\mathbf{X}$ and $f(x)\in\mathbf{V}\subseteq\mathbf{Y}$ with $\mathbf{f}(\mathbf{U})=\mathbf{V}$, such that $\mathbf{f}|_{\mathbf{U}}:\mathbf{U}\rightarrow\mathbf{V}$ is an equivalence in the 2-category $\mathbf{dMan}$.

Theorem 5.5

A 1-morphism $\mathbf{f}:\mathbf{X}\rightarrow\mathbf{Y}$ in $\mathbf{dMan}$ or $\mathbf{dSpa}$ is an equivalence if and only if it is étale and $f:X\rightarrow Y$ is a bijection of sets.

The proof involves choosing local quasi-inverses $\mathbf{g}_i:\mathbf{V}_i\rightarrow\mathbf{U}_i$ for $\mathbf{f}|_{\mathbf{U}_i}:\mathbf{U}_i\rightarrow\mathbf{V}_i$ for $\{\mathbf{U}_i:i\in I\}$, $\{\mathbf{V}_i:i\in I\}$ open covers of $\mathbf{X},\mathbf{Y}$, and then gluing the $\mathbf{g}_i$ for $i\in I$ using a partition of unity to get a global quasi-inverse for $\mathbf{f}$.

Theorem 5.6

A 1-morphism $\mathbf{f}:\mathbf{X}\rightarrow\mathbf{Y}$ in $\mathbf{dMan}$ is étale if and only if $T_x\mathbf{f}:T_x\mathbf{X}\rightarrow T_y\mathbf{Y}$ and $O_x\mathbf{f}:O_x\mathbf{X}\rightarrow O_y\mathbf{Y}$ are isomorphisms for all $x\in\mathbf{X}$ with $\mathbf{f}(x)=y\in\mathbf{Y}$.

The 'only if' part is obvious: if $\mathbf{g}:\mathbf{V}\rightarrow\mathbf{U}$ is a local quasi-inverse for $\mathbf{f}$, then $T_y\mathbf{g},O_y\mathbf{g}$ are inverses for $T_x\mathbf{f},O_x\mathbf{f}$. For the 'if' part, replacing $\mathbf{X},\mathbf{Y},\mathbf{f}$ locally by 'standard model' d-manifolds and 1-morphism, we can construct an explicit quasi-inverse at the level of dg C^∞-rings by choosing a splitting of an exact sequence of vector bundles.
The analogue is false for $\mathbf{dSpa}$.

Combining (5.4) and Theorems 5.5 and 5.6 gives a criterion for when a standard model 1-morphism is étale or an equivalence:

Theorem 5.7

Let $\mathbf{S}_{V',f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ be a standard model 1-morphism in **dMan**. *Then $\mathbf{S}_{V',f,\hat f}$ is étale if and only if for all $x\in s^{-1}(0)\subseteq V$ with $f(x)=y\in t^{-1}(0)\subseteq W$, the following sequence is exact:*

$$0 \longrightarrow T_xV \xrightarrow{\mathrm{d}s|_x\oplus T_xf} E|_x\oplus T_yW \xrightarrow{\hat f|_x\oplus -\mathrm{d}t|_y} F|_y \longrightarrow 0. \quad (5.5)$$

Also $\mathbf{S}_{V',f,\hat f}$ is an equivalence in **dMan** *if in addition $f|_{s^{-1}(0)}:s^{-1}(0)\to t^{-1}(0)$ is a bijection.*

Example 5.8

In Fukaya–Oh–Ohta–Ono Kuranishi spaces in symplectic geometry, a ‘coordinate change’ $(f,\hat f):(V,E,s)\to(W,F,t)$ of ‘Kuranishi neighbourhoods’ $(V,E,s),(W,F,t)$ is an embedding of submanifolds $f:V\hookrightarrow W$ and an embedding of vector bundles $\hat f:E\hookrightarrow f^*(F)$ with $\hat f\circ s=f^*(t)$, such that the induced morphism $(\mathrm{d}s)_*:f^*(TW)/TV\to f^*(F)/E$ is an isomorphism near $s^{-1}(0)$. Theorem 5.7 shows $\mathbf{S}_{V,f,\hat f}:\mathbf{S}_{V,E,s}\to\mathbf{S}_{W,F,t}$ is étale, or an equivalence. But FOOO coordinate changes are very special examples of equivalences; they only exist if $\dim V\leqslant\dim W$.

Derived Differential Geometry

Lecture 6 of 14: M-Kuranishi spaces

Dominic Joyce, Oxford University
Summer 2015

These slides, and references, etc., available at
http://people.maths.ox.ac.uk/∼joyce/DDG2015

Plan of talk:

6. M-Kuranishi spaces

We now explain another way to define (2-)categories of derived manifolds, using an 'atlas of charts' approach, motivated by the ideas of §5. Today we will define an ordinary category **MKur** of 'M-Kuranishi spaces'. (The 'M-' stands for 'Manifold', following Hofer's 'M-polyfolds' and 'polyfolds'.)

Recall that *orbifolds* are generalizations of manifolds locally modelled on $\mathbb{R}^n/\Gamma$, for Γ a finite group acting linearly on $\mathbb{R}^n$. Later in the course we will define a weak 2-category **Kur** of 'Kuranishi spaces', a form of derived orbifold. The full 2-subcategory $\mathbf{Kur_{trG}} \subset \mathbf{Kur}$ of Kuranishi spaces with trivial orbifold groups is a 2-category of derived manifolds. There are equivalences of categories $\mathbf{MKur} \simeq \mathrm{Ho}(\mathbf{Kur_{trG}}) \simeq \mathrm{Ho}(\mathbf{dMan})$, where $\mathrm{Ho}(\mathbf{Kur_{trG}}), \mathrm{Ho}(\mathbf{dMan})$ are the homotopy categories, and an equivalence of weak 2-categories $\mathbf{Kur_{trG}} \simeq \mathbf{dMan}$.

In fact 'Kuranishi spaces' (with a different, non-equivalent definition, which we will call 'FOOO Kuranishi spaces') have been used for many years in the work of Fukaya et al. in symplectic geometry (Fukaya and Ono 1999, Fukaya–Oh–Ohta–Ono 2009), as the geometric structure on moduli spaces of J-holomorphic curves. There are problems with their theory (e.g. there is no notion of morphism of FOOO Kuranishi space), and I claim my definition is the 'correct' definition of Kuranishi space, which should replace the FOOO definition. Any FOOO Kuranishi space $\mathbf{X}$ can be made into a Kuranishi space $\mathbf{X}'$ in my sense, uniquely up to equivalence in the 2-category **Kur**. I began working in Derived Differential Geometry to try and find the 'correct' definition of Kuranishi space, and sort out the problems in the area.

To motivate the comparison between d-manifolds and M-Kuranishi spaces, consider the following two equivalent definitions of manifold:

Definition 6.1

A *manifold* of dimension n is a Hausdorff, second countable topological space X with a sheaf $\mathcal{O}_X$ of $\mathbb{R}$-algebras (or C^∞-rings) locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$, where $\mathcal{O}_{\mathbb{R}^n}$ is the sheaf of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$.

Definition 6.2

A *manifold* of dimension n is a Hausdorff, second countable topological space X equipped with an atlas of charts $\{(V_i, \psi_i) : i \in I\}$, where $V_i \subseteq \mathbb{R}^n$ is open, and $\psi_i : V_i \to X$ is a homeomorphism with an open subset $\mathrm{Im}\,\psi_i$ of X for all $i \in I$, and $\psi_j^{-1} \circ \psi_i : \psi_i^{-1}(\mathrm{Im}\,\psi_j) \to \psi_j^{-1}(\mathrm{Im}\,\psi_i)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ for all $i, j \in I$.

If you try to define derived manifolds by generalizing Definition 6.1, you get d-manifolds (or something similar, e.g. Spivak); if you try to generalize Definition 6.2, you get (M-)Kuranishi spaces.

6.1. M-Kuranishi neighbourhoods and their morphisms

Definition 6.3

Let X be a topological space. An *M-Kuranishi neighbourhood* on X is a quadruple (V, E, s, ψ) such that:

(a) V is a smooth manifold.
(b) $\pi : E \to V$ is a vector bundle over V, the *obstruction bundle*.
(c) $s \in C^\infty(E)$ is a smooth section of E, the *Kuranishi section*.
(d) ψ is a homeomorphism from $s^{-1}(0)$ to an open subset $\mathrm{Im}\,\psi$ in X, where $\mathrm{Im}\,\psi$ is called the *footprint* of (V, E, s, ψ).

If $S \subseteq X$ is open, we call (V, E, s, ψ) an *M-Kuranishi neighbourhood over* S if $S \subseteq \mathrm{Im}\,\psi \subseteq X$.

This is the same as Fukaya–Oh–Ohta–Ono Kuranishi neighbourhoods, omitting finite groups Γ.

Definition 6.4

Let X be a topological space, (V_i, E_i, s_i, ψ_i), (V_j, E_j, s_j, ψ_j) be M-Kuranishi neighbourhoods on X, and $S \subseteq \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \subseteq X$ be an open set. Consider triples $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.

(b) $\phi_{ij} : V_{ij} \to V_j$ is smooth, with $\psi_i = \psi_j \circ \phi_{ij}$ on $s_i^{-1}(0) \cap V_{ij}$.

(c) $\hat\phi_{ij} : E_i|_{V_{ij}} \to \phi_{ij}^*(E_j)$ is a morphism of vector bundles on V_{ij}, with $\hat\phi_{ij}(s_i|_{V_{ij}}) = \phi_{ij}^*(s_j) + O(s_i^2)$.

Define an equivalence relation $\sim$ on such triples $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$ by $(V_{ij}, \phi_{ij}, \hat\phi_{ij}) \sim (V'_{ij}, \phi'_{ij}, \hat\phi'_{ij})$ if there are open $\psi_i^{-1}(S) \subseteq \dot V_{ij} \subseteq V_{ij} \cap V'_{ij}$ and a morphism $\Lambda : E_i|_{\dot V_{ij}} \to \phi_{ij}^*(TV_j)|_{\dot V_{ij}}$ of vector bundles on $\dot V_{ij}$ satisfying $\phi'_{ij} = \phi_{ij} + \Lambda\cdot s_i + O(s_i^2)$ and $\hat\phi'_{ij} = \hat\phi_{ij} + \Lambda\cdot\phi_{ij}^*(\mathrm{d}s_j) + O(s_i)$ on $\dot V_{ij}$. We write $[V_{ij}, \phi_{ij}, \hat\phi_{ij}]$ for the $\sim$-equivalence class of $(V_{ij}, \phi_{ij}, \hat\phi_{ij})$, and call $[V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ a *morphism of M-Kuranishi neighbourhoods over* S.

We can interpret all this in terms of standard model d-manifolds from §5, and their 1- and 2-morphisms:

- An M-Kuranishi neighbourhood (V, E, s, ψ) on X corresponds to a standard model d-manifold $\mathbf{S}_{V,E,s}$ together with a homeomorphism ψ from the topological space $S_{V,E,s} = s^{-1}(0)$ to an open subset $\mathrm{Im}\,\psi \subseteq X$.
- A morphism $[V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ of M-Kuranishi neighbourhoods consists of an open d-submanifold $\mathbf{S}_{V_{ij},E_i|\dots,s_i|\dots} \subseteq \mathbf{S}_{V_i,E_i,s_i}$, together with a 2-isomorphism class $[\mathbf{S}_{V_{ij},\phi_{ij},\hat\phi_{ij}}]$ of standard model 1-morphisms $\mathbf{S}_{V_{ij},\phi_{ij},\hat\phi_{ij}} : \mathbf{S}_{V_{ij},E_i|\dots,s_i|\dots} \to \mathbf{S}_{V_j,E_j,s_j}$, such that on topological spaces we have $\psi_j \circ S_{V_{ij},\phi_{ij},\hat\phi_{ij}} = \psi_i : S_{V_{ij},E_i|\dots,s_i|\dots} \to X$. The definition of $(V_{ij}, \phi_{ij}, \hat\phi_{ij}) \sim (V'_{ij}, \phi'_{ij}, \hat\phi'_{ij})$ is just the existence of a 2-isomorphism $\mathbf{S}_\Lambda : \mathbf{S}_{V_{ij},\phi_{ij},\hat\phi_{ij}} \Rightarrow \mathbf{S}_{V'_{ij},\phi'_{ij},\hat\phi'_{ij}}$.

Given morphisms $[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$, $[V_{jk},\phi_{jk},\hat\phi_{jk}]:(V_j,E_j,s_j,\psi_j)\to(V_k,E_k,s_k,\psi_k)$ of M-Kuranishi neighbourhoods over $S\subseteq X$, the *composition* is

$$[V_{jk},\phi_{jk},\hat\phi_{jk}]\circ[V_{ij},\phi_{ij},\hat\phi_{ij}]=\bigl[\phi_{ij}^{-1}(V_{jk}),\phi_{jk}\circ\phi_{ij}|_{\cdots},\phi_{ij}^{-1}(\hat\phi_{jk})\circ\hat\phi_{ij}|_{\cdots}\bigr]:$$
$$(V_i,E_i,s_i,\psi_i)\longrightarrow(V_k,E_k,s_k,\psi_k).$$

Then M-Kuranishi neighbourhoods over $S\subseteq X$ form a category $\mathrm{MKur}_S(X)$. We call $[V_{ij},\phi_{ij},\hat\phi_{ij}]$ an *M-coordinate change over S* if it is an isomorphism in $\mathrm{MKur}_S(X)$. Theorem 5.7 implies:

Theorem 6.5

A morphism $[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ is an M-coordinate change over S if and only if for all $x\in S$ with $v_i=\psi_i^{-1}(x)$ and $v_j=\psi_j^{-1}(x)$, the following sequence is exact:

$$0\longrightarrow T_{v_i}V_i\xrightarrow{\mathrm{d}s_i|_{v_i}\oplus T_{v_i}\phi_{ij}}E_i|_{v_i}\oplus T_{v_j}V_j\xrightarrow{\hat\phi_{ij}|_{v_i}\oplus -\mathrm{d}s_j|_{v_j}}E_j|_{v_j}\longrightarrow 0.$$

The sheaf property of morphisms

Theorem 6.6

Let (V_i,E_i,s_i,ψ_i), (V_j,E_j,s_j,ψ_j) be M-Kuranishi neighbourhoods on X. For each open $S\subseteq\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$, write $\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)(S)$ for the set of morphisms $\Phi_{ij}:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ over S, and for all open $T\subseteq S\subseteq\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$ define

$$\rho_{ST}:\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)(S)\longrightarrow$$
$$\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)(T)\ \text{by}\ \rho_{ST}:\Phi_{ij}\longmapsto\Phi_{ij}|_T.$$

Then $\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)$ is a sheaf of sets on $\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j$. Similarly, M-coordinate changes from (V_i,E_i,s_i,ψ_i) to (V_j,E_j,s_j,ψ_j) are a subsheaf of $\mathcal{H}om\bigl((V_i,E_i,s_i,\psi_i),(V_j,E_j,s_j,\psi_j)\bigr)$.

This is not obvious, but can be seen using the d-manifold interpretation. It means we can glue (iso)morphisms of M-Kuranishi neighbourhoods over the sets of an open cover.

We generalize Definition 6.4:

Definition 6.7

Let $f : X \to Y$ be a continuous map of topological spaces, (V_i, E_i, s_i, ψ_i), (W_j, F_j, t_j, χ_j) be M-Kuranishi neighbourhoods on X, Y, and $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j) \subseteq X$ be an open set. Consider triples $(V_{ij}, f_{ij}, \hat{f}_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.
(b) $f_{ij} : V_{ij} \to W_j$ is smooth, with $f \circ \psi_i = \chi_j \circ f_{ij}$ on $s_i^{-1}(0) \cap V_{ij}$.
(c) $\hat{f}_{ij} : E_i|_{V_{ij}} \to f_{ij}^*(F_j)$ is a morphism of vector bundles on V_{ij}, with $\hat{f}_{ij}(s_i|_{V_{ij}}) = f_{ij}^*(t_j) + O(s_i^2)$.

Define an equivalence relation $\sim$ by $(V_{ij}, f_{ij}, \hat{f}_{ij}) \sim (V'_{ij}, f'_{ij}, \hat{f}'_{ij})$ if there are open $\psi_i^{-1}(S) \subseteq \dot{V}_{ij} \subseteq V_{ij} \cap V'_{ij}$ and $\Lambda : E_i|_{\dot{V}_{ij}} \to f_{ij}^*(TW_j)|_{\dot{V}_{ij}}$ with $f'_{ij} = f_{ij} + \Lambda \cdot s_i + O(s_i^2)$ and $\hat{f}'_{ij} = \hat{f}_{ij} + \Lambda \cdot f_{ij}^*(\mathrm{d}t_j) + O(s_i)$. We write $[V_{ij}, f_{ij}, \hat{f}_{ij}]$ for the $\sim$-equivalence class of $(V_{ij}, f_{ij}, \hat{f}_{ij})$, and call $[V_{ij}, f_{ij}, \hat{f}_{ij}] : (V_i, E_i, s_i, \psi_i) \to (W_j, F_j, t_j, \chi_j)$ a *morphism over* S, f.

When $Y = X$ and $f = \mathrm{id}_X$, this recovers the notion of morphisms of M-Kuranishi neighbourhoods on X. We have the obvious notion of compositions of morphisms of M-Kuranishi neighbourhoods over $f : X \to Y$ and $g : Y \to Z$.

Here is the generalization of Theorem 6.6:

Theorem 6.8

Let (V_i, E_i, s_i, ψ_i), (W_j, F_j, t_j, χ_j) be M-Kuranishi neighbourhoods on X, Y, and $f : X \to Y$ be continuous. Then morphisms from (V_i, E_i, s_i, ψ_i) to (W_j, F_j, t_j, χ_j) over f form a sheaf $\mathcal{H}om_f\big((V_i, E_i, s_i, \psi_i), (W_j, F_j, t_j, \chi_j)\big)$ on $\mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j)$.

This will be essential for defining compositions of morphisms of M-Kuranishi spaces. The lack of such a sheaf property in the FOOO theory is why FOOO Kuranishi spaces are not a category.

6.2. M-Kuranishi spaces

Definition 6.9

Let X be a Hausdorff, second countable topological space, and $n \in \mathbb{Z}$. An *M-Kuranishi structure $\mathcal{K}$ on X of virtual dimension n* is data $\mathcal{K} = \bigl(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\ i,j\in I}\bigr)$, where:

(a) I is an indexing set.

(b) (V_i, E_i, s_i, ψ_i) is an M-Kuranishi neighbourhood on X for each $i \in I$, with $\dim V_i - \operatorname{rank} E_i = n$.

(c) $\Phi_{ij} = [V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is an M-coordinate change over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ for all $i, j \in I$.

(d) $\bigcup_{i\in I} \operatorname{Im}\psi_i = X$.

(e) $\Phi_{ii} = \mathrm{id}_{(V_i,E_i,s_i,\psi_i)}$ for all $i \in I$.

(f) $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$ for all $i, j, k \in I$ over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$.

We call $\mathbf{X} = (X, \mathcal{K})$ an *M-Kuranishi space*, of *virtual dimension* $\operatorname{vdim}\mathbf{X} = n$. When we write $x \in \mathbf{X}$, we mean that $x \in X$.

In terms of standard model d-manifolds, an M-Kuranishi structure $\mathcal{K}$ on X is the data:

- An open cover $\{\operatorname{Im}\psi_i : i \in I\}$ of X.
- Standard model d-manifolds $\mathbf{S}_{V_i,E_i,s_i}$ for $i \in I$, with homeomorphisms $\psi_i : S_{V_i,E_i,s_i} \to \operatorname{Im}\psi_i \subseteq X$.
- On each double overlap $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ for $i, j \in I$, a 2-isomorphism class $[\mathbf{S}_{V_{ij},\phi_{ij},\hat\phi_{ij}}]$ of equivalences $\mathbf{S}_{V_{ij},\phi_{ij},\hat\phi_{ij}} : \mathbf{S}_{V_{ij},E_i|\dots,s_i|\dots} \to \mathbf{S}_{V_{ji},E_j|\dots,s_j|\dots}$ in $\mathbf{dMan}$, where $\mathbf{S}_{V_{ij},E_i|\dots,s_i|\dots} \subseteq \mathbf{S}_{V_i,E_i,s_i}$ and $\mathbf{S}_{V_{ji},E_j|\dots,s_j|\dots} \subseteq \mathbf{S}_{V_j,E_j,s_j}$ are the open d-submanifolds corresponding to $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$.
- On each triple overlap $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$, there must exist a 2-isomorphism $\mathbf{S}_{V_{jk},\phi_{jk},\hat\phi_{jk}} \circ \mathbf{S}_{V_{ij},\phi_{ij},\hat\phi_{ij}} \cong \mathbf{S}_{V_{ik},\phi_{ik},\hat\phi_{ik}}$.

In the 'atlas of charts' definition of manifolds, we provide data (V_i, ψ_i) on each set $\operatorname{Im}\psi_i$ of an open cover, and verify conditions on double overlaps $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$. Here we provide data on $\operatorname{Im}\psi_i$ and $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, and verify conditions on $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$.

Definition 6.10

Let $\mathbf{X}=(X,\mathcal{K})$ with $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},\Phi_{ii',\,i,i'\in I}\bigr)$ and $\mathbf{Y}=(Y,\mathcal{L})$ with $\mathcal{L}=\bigl(J,(W_j,F_j,t_j,\chi_j)_{j\in J},\Psi_{jj',\,j,j'\in J}\bigr)$ be M-Kuranishi spaces. A *morphism* $\mathbf{f}:\mathbf{X}\to\mathbf{Y}$ is $\mathbf{f}=\bigl(f,\mathbf{f}_{ij,\,i\in I,\,j\in J}\bigr)$, where $f:X\to Y$ is a continuous map, and $\mathbf{f}_{ij}=[V_{ij},f_{ij},\hat f_{ij}]:(V_i,E_i,s_i,\psi_i)\to(W_j,F_j,t_j,\chi_j)$ is a morphism of M-Kuranishi neighbourhoods over $S=\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f for all $i\in I$, $j\in J$, satisfying the conditions:

(a) If $i,i'\in I$ and $j\in J$ then $\mathbf{f}_{i'j}\circ\Phi_{ii'}|_S=\mathbf{f}_{ij}|_S$ over $S=\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_{i'}\cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f.

(b) If $i\in I$ and $j,j'\in J$ then $\Psi_{jj'}\circ\mathbf{f}_{ij}|_S=\mathbf{f}_{ij'}|_S$ over $S=\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j\cap\mathrm{Im}\,\chi_{j'})$ and f.

If $x\in\mathbf{X}$ (i.e. $x\in X$), we will write $\mathbf{f}(x)=f(x)\in\mathbf{Y}$. When $\mathbf{Y}=\mathbf{X}$, so that $J=I$, define the identity morphism $\mathbf{id_X}:\mathbf{X}\to\mathbf{X}$ by $\mathbf{id_X}=\bigl(\mathrm{id}_X,\Phi_{ij,\,i,j\in I}\bigr)$.

Composition of morphisms

Let $\mathbf{X}=(X,\mathcal{I})$ with $\mathcal{I}=\bigl(I,(U_i,D_i,r_i,\phi_i)_{i\in I},\Phi_{ii',\,i,i'\in I}\bigr)$ and $\mathbf{Y}=(Y,\mathcal{J})$ with $\mathcal{J}=\bigl(J,(V_j,E_j,s_j,\psi_j)_{j\in J},\Psi_{jj',\,j,j'\in J}\bigr)$ and $\mathbf{Z}=(Z,\mathcal{K})$ with $\mathcal{K}=\bigl(K,(W_k,F_k,t_k,\xi_k)_{k\in K},\Xi_{kk',\,k,k'\in K}\bigr)$ be M-Kuranishi spaces, and $\mathbf{f}=(f,\mathbf{f}_{ij}):\mathbf{X}\to\mathbf{Y}$, $\mathbf{g}=(g,\mathbf{g}_{jk}):\mathbf{Y}\to\mathbf{Z}$ be morphisms. Consider the problem of how to define the composition $\mathbf{g}\circ\mathbf{f}:\mathbf{X}\to\mathbf{Y}$.

For all $i\in I$ and $k\in K$, $\mathbf{g}\circ\mathbf{f}$ must contain a morphism $(\mathbf{g}\circ\mathbf{f})_{ik}:(U_i,D_i,r_i,\phi_i)\to(W_k,F_k,t_k,\xi_k)$ defined over $S_{ik}=\mathrm{Im}\,\phi_i\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g\circ f$.

For each $j\in J$, we have a morphism $\mathbf{g}_{jk}\circ\mathbf{f}_{ij}:(U_i,D_i,r_i,\phi_i)\to(W_k,F_k,t_k,\xi_k)$, but it is defined over $S_{ijk}=\mathrm{Im}\,\phi_i\cap f^{-1}(\mathrm{Im}\,\psi_j)\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g\circ f$, not over the whole of $S_{ik}=\mathrm{Im}\,\phi_i\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$.

Composition of morphisms

The solution is to use the sheaf property of morphisms, Theorem 6.8. The sets S_{ijk} for $j \in J$ form an open cover of S_{ik}. Using Definition 6.10(a),(b) we can show that
$\mathbf{g}_{jk} \circ \mathbf{f}_{ij}|_{S_{ijk} \cap S_{ij'k}} = \mathbf{g}_{j'k} \circ \mathbf{f}_{ij'}|_{S_{ijk} \cap S_{ij'k}}$. Therefore by Theorem 6.8 there is a unique morphism of M-Kuranishi neighbourhoods $(\mathbf{g} \circ \mathbf{f})_{ik} : (U_i, D_i, r_i, \phi_i) \to (W_k, F_k, t_k, \xi_k)$ defined over S_{ik} and $g \circ f$ with $(\mathbf{g} \circ \mathbf{f})_{ik}|_{S_{ijk}} = \mathbf{g}_{jk} \circ \mathbf{f}_{ij}$ for all $j \in J$. We show that $\mathbf{g} \circ \mathbf{f} := \bigl(g \circ f, (\mathbf{g} \circ \mathbf{f})_{ik,\ i \in I,\ k \in K}\bigr)$ is a morphism $\mathbf{g} \circ \mathbf{f} : \mathbf{X} \to \mathbf{Z}$ of M-Kuranishi spaces, which we call *composition*.

Composition is associative, and makes M-Kuranishi spaces into an ordinary category **MKur**.

Using facts about standard model d-manifolds, we can prove that there is an equivalence of categories $\mathbf{MKur} \simeq \mathrm{Ho}(\mathbf{dMan})$. Thus, isomorphism classes of M-Kuranishi spaces are in 1-1 correspondence with equivalence classes of d-manifolds.

Why no higher categories?

I have been stressing throughout that to do derived geometry properly, you should work in a higher category (a 2-category, or an ∞-category) rather than an ordinary category. So why have I just defined an ordinary category **MKur** of derived manifolds?

One answer is that you can always reduce to ordinary categories by taking homotopy categories, just as $\mathbf{MKur} \simeq \mathrm{Ho}(\mathbf{dMan})$. But doing so loses important information that we want to keep, and this information is missing in **MKur**. For example, fibre products $\mathbf{X} \times_{\mathbf{g},\mathbf{Z},\mathbf{h}} \mathbf{Y}$ in **MKur**, if they exist, will generally not be the 'correct' fibre products we want for applications, because the 'correct' fibre products are characterized by a universal property involving 2-morphisms, that makes no sense in **MKur**.

I only intended **MKur** as a 'cheap' version of derived manifolds, in which we sacrifice some good behaviour for the sake of simplicity.

Why no higher categories?

However, there is more to it than this. It is surprising that our definition of **MKur** 'works' at all, in the sense that it satisfies **MKur** $\simeq \mathrm{Ho}(\mathbf{dMan}) \simeq \mathrm{Ho}(\mathbf{DerMan_{Spi}})$, so it is equivalent to the homotopy categories of some genuine higher categories of derived manifolds $\mathbf{dMan}, \mathbf{DerMan_{Spi}}$.

The reason for this is the complicated result Theorem 4.6 in §4 on gluing families of d-spaces $\mathbf{X}_i$, $i \in I$ (and hence d-manifolds) by equivalences on overlaps. Surprisingly, this theorem held in the homotopy category $\mathrm{Ho}(\mathbf{dSpa}), \mathrm{Ho}(\mathbf{dMan})$. That is, though we need the 2-category structure on **dMan** to form 'correct' fibre products, etc., we only need the ordinary category $\mathrm{Ho}(\mathbf{dMan})$ to glue by equivalences. The analogue is false for stacks, orbifolds, derived schemes, ... An M-Kuranishi space is basically a family of standard model d-manifolds $\mathbf{S}_{V_i,E_i,s_i}$ glued by equivalences on overlaps, in the homotopy category $\mathrm{Ho}(\mathbf{dMan})$.

6.3. Geometry of M-Kuranishi spaces

Example 6.11

Let X be a manifold. Then $(V, E, s, \psi) = (X, 0, 0, \mathrm{id}_X)$ is an M-Kuranishi neighbourhood on X, where $V = X$, $E = 0$ is the zero vector bundle on X, $s = 0$ is the zero section, and $\psi = \mathrm{id}_X : s^{-1}(0) = X \to X$. Define an M-Kuranishi structure $\mathcal{K} = \big(\{0\}, (X, 0, 0, \mathrm{id}_X)_0, \mathrm{id}_{(X,0,0,\mathrm{id}_X)\,00}\big)$ on X to have indexing set $I = \{0\}$, one M-Kuranishi neighbourhood $(V_0, E_0, s_0, \psi_0) = (X, 0, 0, \mathrm{id}_X)$, and one M-coordinate change $\Phi_{00} = \mathrm{id}_{(X,0,0,\mathrm{id}_X)}$. Then $\mathbf{X} = (X, \mathcal{K})$ is an M-Kuranishi space.
Similarly, any smooth map of manifolds $f : X \to Y$ induces a morphism of M-Kuranishi spaces $\mathbf{f} = (f, \mathbf{f}_{00}) : \mathbf{X} \to \mathbf{Y}$ with $\mathbf{f}_{00} = [X, f, 0]$. This defines a full and faithful functor $F_{\mathbf{Man}}^{\mathbf{MKur}} : \mathbf{Man} \to \mathbf{MKur}$ mapping $X \mapsto \mathbf{X}$, $f \mapsto \mathbf{f}$, which embeds **Man** as a full subcategory of **MKur**. We say that an M-Kuranishi space $\mathbf{X}$ *is a manifold* if $\mathbf{X} \cong F_{\mathbf{Man}}^{\mathbf{MKur}}(X')$ for some manifold X'.

As for d-manifolds, for an M-Kuranishi space $\mathbf{X}$ we can define the *tangent space* $T_x\mathbf{X}$ and *obstruction space* $O_x\mathbf{X}$ for any $x \in \mathbf{X}$, where if $\mathbf{X} = (X, \mathcal{K})$ with $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ii'},\ {}_{i,i'\in I}\big)$ and $x \in \operatorname{Im}\psi_i$ with $\psi_i^{-1}(x) = v_i \in s_i^{-1}(0) \subseteq V_i$ then as for (5.3) we have an exact sequence

$$0 \longrightarrow T_x\mathbf{X} \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}s_i|_{v_i}} E_i|_{v_i} \longrightarrow O_x\mathbf{X} \longrightarrow 0. \qquad (6.1)$$

If $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ is a morphism of M-Kuranishi spaces we get functorial linear maps $T_x\mathbf{f} : T_x\mathbf{X} \to T_y\mathbf{Y}$ and $O_x\mathbf{f} : O_x\mathbf{X} \to O_y\mathbf{Y}$.

Theorem 6.12

(a) *An M-Kuranishi space* $\mathbf{X}$ *is a manifold iff* $O_x\mathbf{X}=0$ *for all* $x\in\mathbf{X}$.
(b) *A morphism* $\mathbf{f} : \mathbf{X} \to \mathbf{Y}$ *of M-Kuranishi spaces is étale (a local isomorphism) iff* $T_x\mathbf{f} : T_x\mathbf{X} \to T_y\mathbf{Y}$ *and* $O_x\mathbf{f} : O_x\mathbf{X} \to O_y\mathbf{Y}$ *are isomorphisms for all* $x \in \mathbf{X}$ *with* $\mathbf{f}(x) = y \in \mathbf{Y}$. *And* $\mathbf{f}$ *is an isomorphism in* **MKur** *if also* $f : X \to Y$ *is a bijection.*