Constructing compact 7-manifolds with holonomy G_2

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Based on J. Diff. Geom. 43 (1996), 291–328 and 329–375; and 'Compact Manifolds with Special Holonomy', OUP, 2000.

These slides available at
http://people.maths.ox.ac.uk/∼joyce/.

Plan of talk:

Apology

This talk contains no new work since 2000.

1. The holonomy group G_2

Let (X, g) be a Riemannian manifold, and $x \in X$. The *holonomy group* $\mathrm{Hol}(g)$ is the group of isometries of T_xX given by parallel transport using the Levi-Civita connection ∇ around loops in X based at x. They were classified by Berger:

Theorem (Berger, 1955)

Suppose X is simply-connected of dimension n and g is irreducible and nonsymmetric. Then either: (i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$ *[generic];*

(ii) $n = 2m \geqslant 4$ *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$, *[Kähler manifolds];*

(iii) $n = 2m \geqslant 4$ *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$, *[Calabi–Yau m-folds];*

(iv) $n = 4m \geqslant 8$ *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$, *[hyperkähler];*

(v) $n=4m\geqslant 8$ *and* $\mathrm{Hol}(g)=\mathrm{Sp}(m)\,\mathrm{Sp}(1)$, *[quaternionic Kähler];*

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$, *[exceptional holonomy] or*

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *[exceptional holonomy].*

The action of G_2 on $\mathbb{R}^7$ preserves the Euclidean metric $g_0 = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_7^2$, the orientation, and the 3- and 4-forms

$$\varphi_0 = \mathrm{d}\mathbf{x}_{123} + \mathrm{d}\mathbf{x}_{145} + \mathrm{d}\mathbf{x}_{167} + \mathrm{d}\mathbf{x}_{246} - \mathrm{d}\mathbf{x}_{257} - \mathrm{d}\mathbf{x}_{347} - \mathrm{d}\mathbf{x}_{356},$$
$$*\varphi_0 = \mathrm{d}\mathbf{x}_{4567} + \mathrm{d}\mathbf{x}_{2367} + \mathrm{d}\mathbf{x}_{2345} + \mathrm{d}\mathbf{x}_{1357} - \mathrm{d}\mathbf{x}_{1346} - \mathrm{d}\mathbf{x}_{1256} - \mathrm{d}\mathbf{x}_{1247},$$

where $\mathrm{d}\mathbf{x}_{ijk} = \mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k$, etc. If (X, g) is a Riemannian 7-manifold with holonomy G_2 then X has a natural orientation, 3-form φ, and Hodge dual 4-form $*\varphi$ with $\nabla\varphi = \nabla(*\varphi) = 0$, such that for each $x \in X$ there is an oriented isomorphism $T_xX \cong \mathbb{R}^7$ identifying $g|_x, \varphi|_x, *\varphi|_x$ with $g_0, \varphi_0, *\varphi_0$. Also g is Ricci-flat.
We call a pair (φ, g) a *G_2-structure* on X if at each $x \in X$ there is an isomorphism $T_xX \cong \mathbb{R}^7$ identifying $(\varphi|_x, g|_x)$ with (φ_0, g_0). We call (φ, g) *torsion-free* if $\nabla\varphi = 0$ for ∇ the Levi-Civita connection of g, or equivalently if $\mathrm{d}\varphi = \mathrm{d}(*\varphi) = 0$ (though this is apparently weaker). Then $\mathrm{Hol}(g) \subseteq G_2$.

The subgroup of $\mathrm{GL}(7,\mathbb{R})$ preserving φ_0 is G_2, a compact exceptional Lie group of dimension 14. Hence the orbit of φ_0 in $\Lambda^3(\mathbb{R}^7)^*$ is $\mathrm{GL}(7,\mathbb{R})/G_2$, with dimension $49-14=35$. But the $\dim\Lambda^3(\mathbb{R}^7)^*=\binom{7}{3}=35$, so $\mathrm{GL}(7,\mathbb{R})\cdot\varphi_0$ is open in $\Lambda^3(\mathbb{R}^7)^*$. That is, G_2-forms are generic.
Let X be an oriented 7-manifold. Write $\mathcal{P}^3X$ and $\mathcal{P}^4X$ for the subbundles of 3- and 4-forms in Λ^3T^*X and Λ^4T^*X such that $x\in X$ there is an oriented isomorphism $T_xX\cong\mathbb{R}^7$ identifying elements of $\mathcal{P}^3_xX$ and $\mathcal{P}^4_xX$ with $\varphi_0,*\varphi_0$. Then $\mathcal{P}^3X,\mathcal{P}^4X$ are open subsets of $\Lambda^3T^*X,\Lambda^4T^*X$ which are bundles over X with fibre $\mathrm{GL}_+(7,\mathbb{R})/G_2$. Note that they are not vector bundles.
There is a natural, nonlinear smooth map $\Theta:\mathcal{P}^3X\to\mathcal{P}^4X$, an isomorphism of bundles, such that an isomorphism $T_xX\cong\mathbb{R}^7$ identifying $\alpha\in\mathcal{P}^3_xX$ with $\varphi_0\in\Lambda^3(\mathbb{R}^7)^*$ also identifies $\Theta(\alpha)\in\mathcal{P}^4_xX$ with $*\varphi_0\in\Lambda^4(R^7)^*$. Note that Θ depends only on X as an oriented 7-manifold, it is independent of any φ,g.

Smooth sections $\varphi\in\Gamma^\infty(\mathcal{P}^3X)$ are called *positive* 3-forms. Every positive 3-form extends to a unique G_2-structure (φ,g), and then $\Theta(\varphi)=*_g\varphi$, where $*_g$ is the Hodge star of g.
Let (X,φ,g) be a G_2-manifold. Then we have natural decompositions of exterior forms

$$\begin{aligned}
&\Lambda^1T^*X=\Lambda^1_7, \quad \Lambda^2T^*X=\Lambda^2_7\oplus\Lambda^2_{14}, \quad \Lambda^3T^*X=\Lambda^3_1\oplus\Lambda^3_7\oplus\Lambda^3_{27},\\
&\Lambda^6T^*X=\Lambda^6_7, \quad \Lambda^5T^*X=\Lambda^5_7\oplus\Lambda^5_{14}, \quad \Lambda^4T^*X=\Lambda^4_1\oplus\Lambda^4_7\oplus\Lambda^4_{27},
\end{aligned}$$

where Λ^k_l is a vector bundle of rank l. Write $\pi_l:\Gamma^\infty(\Lambda^kT^*X)\to\Gamma^\infty(\Lambda^k_l)$ for the projection.
Let (φ,g) be a G_2-structure. Then for C^0-small $\chi\in\Gamma^\infty(\Lambda^3T^*X)$ we have $\varphi+\chi\in\Gamma^\infty(\mathcal{P}^3X)$ and

$$\Theta(\varphi+\chi)=*\varphi+\tfrac{4}{3}\pi_1(\chi)+\pi_7(\chi)-\pi_{27}(\chi)+O(|\chi|^2).$$

This computes the derivative $D\Theta$ at φ.

We have inclusions of holonomy groups $\mathrm{SU}(2) \subset \mathrm{SU}(3) \subset G_2$. Thus if Y is a Calabi–Yau 2-fold (hyperkähler 4-manifold) then $Y \times \mathbb{R}^3$ and $Y \times T^3$ have torsion-free G_2-structures, and if Z is a Calabi–Yau 3-fold then $Z \times \mathbb{R}$ and $Z \times \mathcal{S}^1$ have torsion-free G_2-structures.
If we construct a torsion-free G_2-manifold (X, φ, g), we would like to check that $\mathrm{Hol}(g)$ is G_2 and not some proper subgroup. Here is a topological test that allows us to do this.

Theorem 1

Suppose (X, φ, g) is a compact torsion-free G_2-manifold. Then $\mathrm{Hol}(g) = G_2$ if and only if $\pi_1(X)$ is finite.

This is an easy consequence of Berger's theorem: if $\mathrm{Hol}(g) \neq G_2$ then the universal cover $\tilde{X}$ of X must be $\mathbb{R}^7$ or $(\mathrm{SU}(2)\text{-manifold} \times \mathbb{R}^3)$ or $(\mathrm{SU}(3)\text{-manifold} \times \mathbb{R})$, so $\tilde{X}$ is noncompact, and $\pi_1(X)$ is infinite.

The moduli space of torsion-free G_2-structures

Theorem 2 (Bryant–Harvey; Joyce)

Let X be a compact 7-manifold. Then the moduli space $\mathcal{M}$ of torsion-free G_2-structures (φ, g) on X, modulo diffeomorphisms isotopic to the identity, is a smooth manifold of dimension $\dim \mathcal{M} = b^3(X)$. The map $\iota : \mathcal{M} \to H^3(X; \mathbb{R})$ taking $\iota : [(\varphi, g)] \mapsto [\varphi]$ is a local diffeomorphism.

We can also consider the map $\jmath : \mathcal{M} \to H^3(X; \mathbb{R}) \times H^4(X; \mathbb{R})$ mapping $\jmath : [(\varphi, g)] \mapsto ([\varphi], [*\varphi])$. The image of $\jmath$ is an immersed Lagrangian in $H^3(X; \mathbb{R}) \times H^4(X; \mathbb{R})$ with the obvious symplectic form from $H^4(X; \mathbb{R}) \cong H^3(X; \mathbb{R})^*$. It seems to be difficult to compute the image of $\jmath$ in examples.

In 1987, Robert Bryant proved the local existence of many metrics with holonomy G_2, using EDS. In 1989, Robert Bryant and Simon Salamon produced explicit, complete examples of holonomy G_2 manifolds. Examples of compact 7-manifolds with holonomy G_2 were constructed by me (1996, 2000) by resolving torus orbifolds T^7/Γ — the subject of this talk – and by Kovalev (2003) and Corti–Haskins–Nordström–Pacini (2015) using the 'twisted connect sum' construction – see later talks.

Compact 7-manifolds with holonomy G_2 are important in M-theory, as ingredients you need to bake your universe. Really M-theorists want compact G_2-manifolds with conical singularities of a certain type, to make physically realistic models – see Bobby Acharya's talk. Constructing examples of such singular G_2-manifolds is an important open problem.

If (X,φ,g) is a G_2-manifold then $\varphi,*\varphi$ are calibrations on (X,g), in the sense of Harvey–Lawson, so we have natural classes of calibrated submanifolds, called *associative* 3*-folds*, and *coassociative* 4*-folds*. They are minimal submanifolds in (X,g). The deformation theory of compact associatives and coassociatives was studied by McLean (1998). Both are controlled by an elliptic equation, and so are well behaved. Moduli spaces of associative 3-folds N may be obstructed, and have virtual dimension 0. If $\mathrm{d}\varphi=0$, moduli of coassociative 4-folds C are smooth, of dimension $b^2_+(C)$.

Let (X,φ,g) be a G_2-manifold, $P\rightarrow X$ be a principal bundle, and A a connection on P with curvature F_A. We call (P,A) a G_2*-instanton* if $\pi_7(F_A)=0$, where π_7 is the projection to $\mathrm{ad}(P)\otimes\Lambda^2_7\subset\mathrm{ad}(P)\otimes\Lambda^2T^*X$. For general G_2-structures this is overdetermined equation, but if $\mathrm{d}(*\varphi)=0$ the Bianchi identity for A gives a relation which makes the equation elliptic modulo gauge, so G_2-instantons form well-behaved moduli spaces. They are the subject of the Donaldson–Segal programme.

2. Constructing compact 7-manifolds with holonomy G_2

Compact G_2-manifolds (X, φ, g) with $\mathrm{Hol}(g) = G_2$ do exist, but we cannot write down φ, g explicitly, and probably we never will – I expect them to be transcendental objects satisfying no nice algebraic equations. For comparison, the Ricci-flat metrics on compact Calabi–Yau m-folds exist by the Calabi conjecture, but we cannot write them down in any example (except flat T^{2m}).
To construct examples, the key fact is that G_2-manifolds (X, φ, g) occur in moduli spaces $\mathcal{M}$ of dimension $b^3(X) > 0$. These moduli spaces may admit partial compactifications $\bar{\mathcal{M}}$, whose boundary $\partial\bar{\mathcal{M}} = \bar{\mathcal{M}} \setminus \mathcal{M}$ consists of singular, limiting G_2-manifolds $(\bar{X}, \bar{\varphi}, \bar{g})$. These $(\bar{X}, \bar{\varphi}, \bar{g})$ may be built of simpler pieces $(\bar{X}_i, \bar{\varphi}_i, \bar{g}_i)$ which are flat, or have holonomy $\mathrm{SU}(2)$ or $\mathrm{SU}(3)$, and can be written down explicitly, or constructed using Calabi conjecture analysis.
We first construct these singular limits $(\bar{X}, \bar{\varphi}, \bar{g})$, and then we deform to (X, φ, g) in $\mathcal{M}$ close to $(\bar{X}, \bar{\varphi}, \bar{g})$ in $\partial\bar{\mathcal{M}}$ by 'gluing'.

There are at least three ways of doing this:

(a) (Joyce 1996, 2000.) Start with $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ with a flat G_2-structure, and Γ a finite group of isomorphisms. Arrange that T^7/Γ has only orbifold singularities resolvable using (Quasi-)ALE manifolds with holonomy $\mathrm{SU}(2)$ or $\mathrm{SU}(3)$.

(b) (Kovalev, 2003; Corti–Haskins–Nordström–Pacini 2015.) Construct noncompact, Asymptotically Cylindrical Calabi–Yau 3-folds X_1, X_2 with asymptotic ends $Y_i \times \mathcal{S}^1 \times (0, \infty)$ for Y_1, Y_2 hyperkähler 4-folds, which are isomorphic under a 'hyperkähler twist'. Then $X_1 \times \mathcal{S}^1$, $X_2 \times \mathcal{S}^1$ are torsion-free ACyl G_2-manifolds which can be glued at their infinite ends to give a compact G_2-manifold, a 'twisted connect sum'.

(c) (Joyce–Karigiannis, this conference.) Let X be a Calabi–Yau 3-fold and $\sigma : X \to X$ an antiholomorphic involution with fixed points L. Suppose α is a nonvanishing harmonic 1-form on L. Then $(X \times \mathcal{S}^1)/\langle(\sigma, -1)\rangle$ is a G_2-orbifold, with singular set $L \times \{0, \frac{1}{2}\}$, locally $\mathbb{R}^3 \times \mathbb{R}^4/\{\pm 1\}$. We resolve singularities using a family of Eguchi–Hanson spaces depending on α.

The general method is the same for all the constructions:

(i) Construct the ingredients $(\bar{X}_i, \bar{\varphi}_i, \bar{g}_i)$ of the limit $(\bar{X}, \bar{\varphi}, \bar{g})$, which are flat or have holonomy $\mathrm{SU}(2)$ or $\mathrm{SU}(3)$. These may be explicit, or involve Calabi Conjecture analysis. Ensure the ingredients satisfy matching conditions, so they can be glued.

(ii) Glue the pieces $\bar{X}_i$ together to get a compact 7-manifold X. Glue the G_2-structures $(\bar{\varphi}_i, \bar{g}_i)$ on the pieces by a partition of unity to get a family (φ_t, g_t), $t \in (0, \epsilon]$ of G_2-structures on X, such that $(X, \varphi_t, g_t) \to (\bar{X}, \bar{\varphi}, \bar{g})$ as $t \to 0$, and the torsion of (φ_t, g_t) tends to zero as $t \to 0$, in appropriate Banach norms.

(iii) Apply an analytic theorem (later), that a G_2-manifold with small torsion can be deformed to a torsion-free G_2-manifold. So (φ_t, g_t) deforms to torsion-free $(\hat{\varphi}_t, \hat{g}_t)$ for small t.

(iv) Check that $\pi_1(X)$ is finite. Then Theorem 1 says that $\mathrm{Hol}(\hat{g}_t) = G_2$, so $(X, \hat{\varphi}_t, \hat{g}_t)$ is what we want.

The Kummer construction for the $K3$ surface

The T^7/Γ construction for G_2-manifolds is based on the Kummer construction for hyperkähler metrics on the $K3$ surface. Let $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ have the flat $\mathrm{SU}(2)$-structure, and let $\mathbb{Z}_2 = \{1, \sigma\}$ act on T^4 by, for $x_1, \ldots, x_4 \in \mathbb{R}/\mathbb{Z}$,

$$(x_1, x_2, x_3, x_4) \longmapsto (-x_1, -x_2, -x_3, -x_4).$$

Then σ has 16 fixed points $x_i \in \{0 + \mathbb{Z}, \frac{1}{2} + \mathbb{Z}\}$, and $T^4/\langle\sigma\rangle$ has 16 orbifold points locally modelled on $\mathbb{R}^4/\{\pm 1\}$.
The *Eguchi–Hanson space* (Y, h) is an explicit Asymptotically Locally Euclidean manifold with holonomy $\mathrm{SU}(2)$ asymptotic to $\mathbb{R}^4/\{\pm 1\}$, where $Y \cong T^*\mathbb{CP}^1$. There is an isomorphism $\iota : Y \setminus \{\text{compact}\} \to (\mathbb{R}^4 \setminus \{\text{ball}\})/\{\pm 1\}$ such that $\iota_*(h) = g_0 + O(r^{-4})$. Scaling the metric by $t > 0$, $(Y, t^2 h)$ is also ALE, with an isomorphism $\iota^t : Y \setminus \{\text{compact}\} \to (\mathbb{R}^4 \setminus \{\text{ball}\})/\{\pm 1\}$ such that $\iota^t_*(t^2 h) = g_0 + O(t^4 r^{-4})$.

We make a $K3$ surface X by gluing in 16 copies of the Eguchi–Hanson space Y at the 16 orbifold points of $T^4/\langle\sigma\rangle$. Then we make an $\mathrm{SU}(2)$-structure $(\omega_1^t,\omega_2^t,\omega_3^t,g^t)$ on X for small $t>0$ by gluing the $\mathrm{SU}(2)$-structures on $T^4/\langle\sigma\rangle$ and on (Y,t^2h) by a partition of unity, in an annulus of radii $r\in[\epsilon,2\epsilon]$ about each orbifold point. As $\iota_*^t(t^2h)=g_0+O(t^4r^{-4})$, the error (torsion of the $\mathrm{SU}(2)$-structure) is $O(t^4\epsilon^{-4})$, which is small when t is small. We then prove that for small t we can deform $(\omega_1^t,\omega_2^t,\omega_3^t,g^t)$ to a torsion-free $\mathrm{SU}(2)$-structure $(\hat\omega_1^t,\hat\omega_2^t,\hat\omega_3^t,\hat g^t)$ on X, and $\mathrm{Hol}(\hat g^t)=\mathrm{SU}(2)$. The proof is a balancing act: for small t, the torsion is $O(t^4)$, so the metric is close to hyperkähler (good), but the injectivity radius is $O(t)$ and curvature $O(t^{-2})$, so the metric is close to singular (bad). The good wins.
This is called the *Kummer construction*, and was the motivation for my G_2-manifold construction.

An example of an orbifold T^7/Γ

Let $T^7=\mathbb{R}^7/\mathbb{Z}^7$ with standard G_2-structure (φ_0,g_0). Let $\Gamma=\langle\alpha,\beta,\gamma\rangle\cong\mathbb{Z}_2^3$, where α,β,γ are involutions acting by

$$\alpha(x_1,\dots,x_7)=(-x_1,-x_2,-x_3,-x_4,x_5,x_6,x_7),$$
$$\beta(x_1,\dots,x_7)=(-x_1,\tfrac12-x_2,x_3,x_4,-x_5,-x_6,x_7),$$
$$\gamma(x_1,\dots,x_7)=(\tfrac12-x_1,x_2,\tfrac12-x_3,x_4,-x_5,x_6,-x_7).$$

These preserve the G_2-structure (φ_0,g_0). The only elements of Γ with fixed points are $1,\alpha,\beta,\gamma$, where α,β,γ each fix 16 T^3. The quotient T^7/Γ is simply-connected, with singular set:

(a) 4 copies of T^3 from the fixed points of α.
(b) 4 copies of T^3 from the fixed points of β.
(c) 4 copies of T^3 from the fixed points of γ.

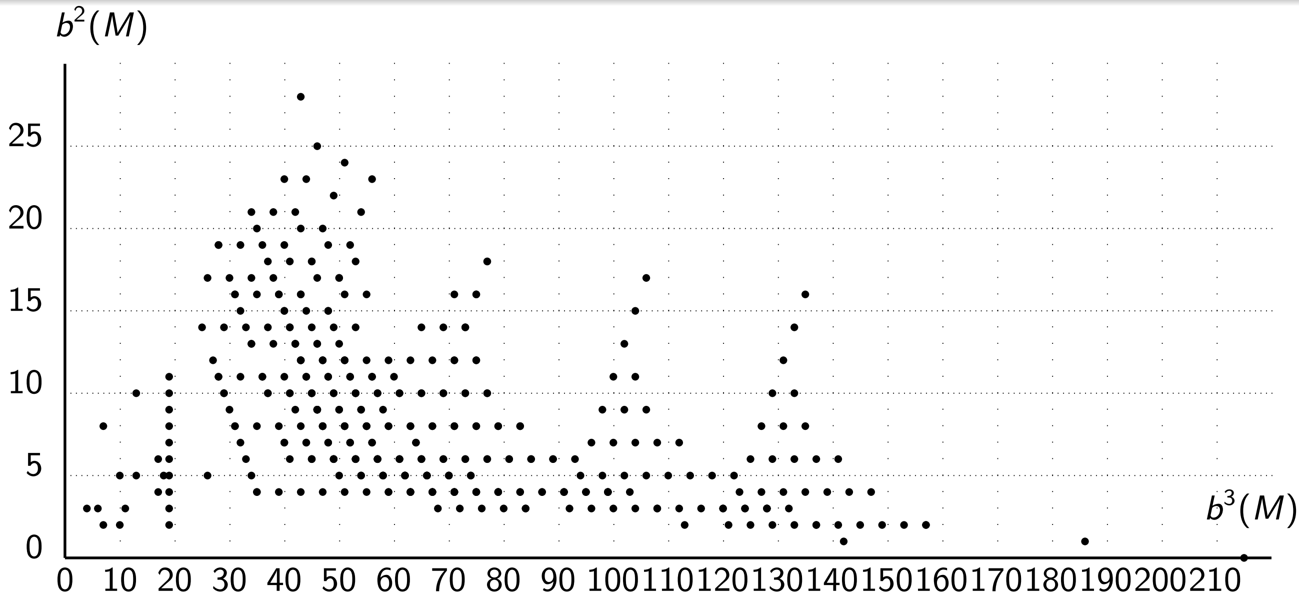
These 12 T^3 are disjoint. Near each T^3, T^7/Γ is modelled on $T^3\times(\mathbb{R}^4/\{\pm1\})$.

As for the Kummer construction, we can resolve T^7/Γ to get a compact 7-manifold X by gluing in 12 copies of $T^3 \times Y$, for Y the Eguchi–Hanson space, to resolve the 12 singular T^3 in T^7/Γ. This X is simply-connected with $b^2 = 12$ and $b^3 = 43$. We can construct a 43-dimensional family of metrics with holonomy G_2 on X.

Here are the important considerations in choosing Γ:

- For X to have holonomy G_2 we need $\pi_1(T^7/\Gamma)$ finite. If Γ acts freely then $\pi_1(T^7/\Gamma) = \Gamma \ltimes \mathbb{Z}^7$; fixed points of elements of Γ make $\pi_1(T^7/\Gamma)$ smaller, so Γ must be large enough (in particular, Γ cannot be conjugate to a subgroup of $\mathrm{SU}(2) \subset G_2$ or $\mathrm{SU}(3) \subset G_2$), and enough elements of Γ must have fixed points.
- For every $x \in T^7$, the stabilizer group $\mathrm{Stab}_\Gamma(x)$ must be conjugate to a subgroup of $\mathrm{SU}(2) \subset G_2$ or $\mathrm{SU}(3) \subset G_2$, so that we can resolve within holonomy $\mathrm{SU}(2)$ or $\mathrm{SU}(3)$. So, $\mathrm{Stab}_\Gamma(x)$ cannot be too large for all $x \in T^7$, in particular, $\mathrm{Stab}_\Gamma(x) \neq \Gamma$.

In our example, we put in the shifts $\frac{1}{2} - x_1$, etc., to prevent fixed point loci of elements of Γ intersecting, and keep $\mathrm{Stab}_\Gamma(x)$ small.



Here is a graph of Betti numbers (b^2, b^3) of compact, simply-connected 7-manifolds with holonomy G_2 found in my book by resolving T^7/Γ. There are 252 sets, with $0 \leqslant b^2 \leqslant 28$ and $4 \leqslant b^3 \leqslant 215$. The Betti numbers from twisted connect sums tend to have a different profile, with b^2 small and b^3 large.

3. Deforming small torsion G_2-structures to zero torsion

Theorem 3

Let $\alpha,\lambda,\mu,\nu>0$. Then there exist $\kappa,K>0$ depending only on α,λ,μ,ν such that whenever $0<t\leqslant\kappa$, the following holds. Suppose (X,φ_t,g_t) is a compact G_2-manifold with $\mathrm{d}\varphi_t=0$, and ψ_t is a closed 4-form on X such that:

(i) $\|\Theta(\varphi_t)-\psi_t\|_{C^0}\leqslant\lambda t^\alpha$, $\|\Theta(\varphi_t)-\psi_t\|_{L^2}\leqslant\lambda t^{\frac{7}{2}+\alpha}$, *and* $\|\mathrm{d}(\Theta(\varphi_t)-\psi_t)\|_{L^{14}}\leqslant\lambda t^{-\frac{1}{2}+\alpha}$,

(ii) *the injectivity radius* $\delta(g_t)$ *satisfies* $\delta(g_t)\geqslant\mu t$, *and*

(iii) *the Riemann curvature* $R(g_t)$ *satisfies* $\|R(g_t)\|_{C^0}\leqslant\nu t^{-2}$.

Then there exists a torsion-free G_2-structure $(\hat\varphi_t,\hat g_t)$ on X with $\|\hat\varphi_t-\varphi_t\|_{C^0}\leqslant Kt^\alpha$.

The theorem as stated is intended for the T^7/Γ construction, in which one shrinks an ALE manifold (Y,h) by small $t>0$ to get (Y,t^2h) before gluing in, giving a G_2-manifold with injectivity radius $O(t)$ and curvature $O(t^{-2})$, as for t^2h.

A rescaled version of the same theorem

If we rescale by t^{-1}, so $g_t\mapsto t^{-2}g_t$, $\varphi_t\mapsto t^{-3}\varphi_t$ etc., we get:

Theorem 4

Let $\alpha,\lambda,\mu,\nu>0$. Then there exist $\kappa,K>0$ depending only on α,λ,μ,ν such that whenever $0<t\leqslant\kappa$, the following holds. Suppose (X,φ_t,g_t) is a compact G_2-manifold with $\mathrm{d}\varphi_t=0$, and ψ_t is a closed 4-form on X such that:

(i) $\|\Theta(\varphi_t)-\psi_t\|_{C^0}\leqslant\lambda t^\alpha$, $\|\Theta(\varphi_t)-\psi_t\|_{L^2}\leqslant\lambda t^{\alpha}$, *and* $\|\mathrm{d}(\Theta(\varphi_t)-\psi_t)\|_{L^{14}}\leqslant\lambda t^{\alpha}$,

(ii) *the injectivity radius* $\delta(g_t)$ *satisfies* $\delta(g_t)\geqslant\mu$, *and*

(iii) *the Riemann curvature* $R(g_t)$ *satisfies* $\|R(g_t)\|_{C^0}\leqslant\nu$.

Then there exists a torsion-free G_2-structure $(\hat\varphi_t,\hat g_t)$ on X with $\|\hat\varphi_t-\varphi_t\|_{C^0}\leqslant Kt^\alpha$.

This is in a form suitable for the twisted connect sum construction. We can simplify further: omit t, and in (i) just require the norms of $\Theta(\varphi_t)-\psi_t$ to be sufficiently small in terms of μ,ν.

Remarks on Theorems 3,4

- We assume that $\mathrm{d}\varphi_t = 0$ and $\mathrm{d}\psi_t = 0$. If $\Theta(\varphi_t) - \psi_t = 0$ then $\mathrm{d}\varphi_t = \mathrm{d}\Theta(\varphi_t) = 0$, so (φ_t, g_t) is torsion-free. Thus we can regard $\Theta(\varphi_t) - \psi_t$ as a measure of the torsion of (φ_t, g_t), and part (i) of Theorems 3,4 as saying that (φ_t, g_t) has small torsion.
- Thus, the theorem roughly says that if the torsion of (φ_t, g_t) is small enough (in C^0, L^2 and L^{14}_1) compared to the injectivity radius and curvature, then we can deform to zero torsion.
- We do not require a volume or diameter bound. So in principle the same proof should work for noncompact G_2-manifolds, provided we have suitable Fredholm-type results for the elliptic operators in the proof.
- To apply the theorems, we glue the closed 3-forms $\bar{\varphi}_i$ on the pieces with a partition of unity to get a closed 3-form φ_t, and we glue the closed 4-forms $*\bar{\varphi}_i$ to get a closed 4-form ψ_t. These do not satisfy $\Theta(\varphi_t) = \psi_t$ because of errors introduced by the gluing, but $\Theta(\varphi_t) - \psi_t$ is small provided the forms we glue are close.

Outline of the proof of Theorems 3, 4

The idea is to find a small 2-form η_t on X satisfying

$$\mathrm{d}\Theta(\varphi_t + \mathrm{d}\eta_t) = 0 \quad \text{and} \quad \mathrm{d}^*\eta_t = 0. \tag{1}$$

Then $\hat{\varphi}_t = \varphi_t + \mathrm{d}\eta_t$ has $\mathrm{d}\hat{\varphi}_t = \mathrm{d}\Theta(\hat{\varphi}_t) = 0$, and so is a torsion-free G_2-structure as we want.

In fact we can show that (1) is equivalent to

$$\begin{aligned} &\mathrm{d}\Theta(\varphi_t + \mathrm{d}\eta_t) = \tfrac{7}{3}\mathrm{d}(*\pi_1(\mathrm{d}\eta_t)) + 2\mathrm{d}(*\pi_7(\mathrm{d}\eta_t)) - \epsilon_t \mathrm{d}\Theta(\varphi_t), \\ &\epsilon_t = \frac{1}{3\,\mathrm{vol}(X)} \int_X \mathrm{d}\eta_t \wedge (\varphi_t - *\psi_t), \quad \text{and} \quad \mathrm{d}^*\eta_t = 0, \end{aligned} \tag{2}$$

as both sides of the first equation vanish, proved using a magic fact (Bryant) that for a G_2-structure (φ_t, g_t), $\mathrm{d}\varphi_t$ and $\mathrm{d}(*\varphi_t)$ are not independent, but have a common component.

Then (2) is equivalent to

$$\begin{aligned} &\mathrm{d}\mathrm{d}^*\eta_t + *\mathrm{d}\Theta(\varphi_t + \mathrm{d}\eta_t) - *\tfrac{7}{3}\mathrm{d}(*\pi_1(\mathrm{d}\eta_t)) \\ &-2 * \mathrm{d}(*\pi_7(\mathrm{d}\eta_t)) + \epsilon_t * \mathrm{d}\Theta(\varphi_t) = 0, \end{aligned} \qquad (3)$$

since (exact form)+(coexact form)=0 implies that (exact form)= (coexact form)=0, and $\mathrm{d}\mathrm{d}^*\eta_t = 0$ implies that $\mathrm{d}^*\eta_t = 0$. Now

$$\Theta(\varphi + \chi) = *\varphi + \tfrac{4}{3}\pi_1(\chi) + \pi_7(\chi) - \pi_{27}(\chi) + F(\chi),$$

where $F(\chi)$ is a nonlinear function with $F(\chi) = O(|\chi|^2)$. Hence as $\mathrm{d}^* = - * \mathrm{d} *$ on 3-forms, (3) is equivalent to

$$(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\eta_t = -(1 + \epsilon_t) * \mathrm{d}(\Theta(\varphi_t) - \psi_t) + *\mathrm{d}F(\mathrm{d}\eta_t). \qquad (4)$$

This is of the form $\Delta_{\mathrm{d}}\eta_t =$ (small error)+(nonlinear term).

We solve (4) by a sequence method: we define a series $(\eta_t^i)_{i=0}^{\infty}$ by induction, with $\eta_t^0 = 0$ and for $i = 0, 1, \ldots$

$$(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\eta_t^{i+1} = -(1 + \epsilon_t^i) * \mathrm{d}(\Theta(\varphi_t) - \psi_t) + *\mathrm{d}F(\mathrm{d}\eta_t^i). \qquad (5)$$

Here as the r.h.s. is coexact, for given η_t^i there is a unique η_t^{i+1} satisfying (5) and L^2-orthogonal to the harmonic 2-forms $\mathcal{H}^2$ on X. We want to prove the sequence converges as $i \to \infty$ to a limit η_t. For the first term we have

$$(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\eta_t^1 = - * \mathrm{d}(\Theta(\varphi_t) - \psi_t),$$

so we can bound η_t^1 using norms of $\Theta(\varphi_t) - \psi_t$, as in Theorem 3(i).

Subtracting (5) for $i, i-1$ gives

$$(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})(\eta_t^{i+1} - \eta_t^i) = -(\epsilon_t^i - \epsilon_t^{i-1}) * \mathrm{d}(\Theta(\varphi_t) - \psi_t) + * \,\mathrm{d}(F(\mathrm{d}\eta_t^i) - F(\mathrm{d}\eta_t^{i-1})).$$

Here $\epsilon_t^i - \epsilon_t^{i-1}$ is bounded in terms of $\|\eta_t^i - \eta_t^{i-1}\|_{L^1}$, and $*\mathrm{d}(\Theta(\varphi_t) - \psi_t)$ is small by Theorem 3(i), and

$$\left|F(\mathrm{d}\eta_t^i) - F(\mathrm{d}\eta_t^{i-1})\right| \leqslant C|\mathrm{d}\eta_t^i - \mathrm{d}\eta_t^{i-1}|\big(|\mathrm{d}\eta_t^i| + |\mathrm{d}\eta_t^{i-1}|\big).$$

If η_t^1 is sufficiently small (which happens if $\Theta(\varphi_t) - \psi_t$ is sufficiently small in suitable norms) then we can use this to show by induction that $\|\eta_t^{i+1} - \eta_t^i\| \leqslant \frac{1}{2}\|\eta_t^i - \eta_t^{i-1}\|$ in suitable norms, so the sequence $(\eta_t^i)_{i=0}^\infty$ is Cauchy, and converges. We use elliptic regularity to show that the limit η_t is smooth.