On orientations for gauge-theoretic moduli spaces

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Abstract

Let X be a compact manifold, $D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ a real elliptic operator on X, G a Lie group, $P \to X$ a principal G-bundle, and $\mathcal{M}_P$ the infinite-dimensional moduli space of all connections ∇_P on P, as a topological stack. For each $[\nabla_P] \in \mathcal{M}_P$, we can consider the twisted elliptic operator $D^{\nabla_{\mathrm{Ad}(P)}} : \Gamma^\infty(\mathrm{Ad}(P) \otimes E_0) \to \Gamma^\infty(\mathrm{Ad}(P) \otimes E_1)$ on X. This is a continuous family of elliptic operators over the base $\mathcal{M}_P$, and so has an orientation bundle $O_P^{E\bullet} \to \mathcal{M}_P$, a principal $\mathbb{Z}_2$-bundle parametrizing orientations of $\operatorname{Ker} D^{\nabla_{\mathrm{Ad}(P)}} \oplus \operatorname{Coker} D^{\nabla_{\mathrm{Ad}(P)}}$ at each $[\nabla_P]$. An *orientation* on $\mathcal{M}_P$ is a trivialization $O_P^{E\bullet} \cong \mathcal{M}_P \times \mathbb{Z}_2$.

In gauge theory one studies moduli spaces $\mathcal{M}_P^{\mathrm{ga}}$ of connections ∇_P on P satisfying some curvature condition, such as anti-self-dual instantons on Riemannian 4-manifolds (X, g). Under good conditions $\mathcal{M}_P^{\mathrm{ga}}$ is a smooth manifold, and orientations on $\mathcal{M}_P$ pull back to orientations on $\mathcal{M}_P^{\mathrm{ga}}$ in the usual sense of differential geometry under the inclusion $\mathcal{M}_P^{\mathrm{ga}} \hookrightarrow \mathcal{M}_P$. This is important in areas such as Donaldson theory, where one needs an orientation on $\mathcal{M}_P^{\mathrm{ga}}$ to define enumerative invariants.

We explain a package of techniques, some known and some new, for proving orientability and constructing canonical orientations on $\mathcal{M}_P$, after fixing some algebro-topological information on X. We use these to construct canonical orientations on gauge theory moduli spaces, including new results for moduli spaces of flat connections on 2- and 3-manifolds, instantons, the Kapustin–Witten equations, and the Vafa–Witten equations on 4-manifolds, and the Haydys–Witten equations on 5-manifolds.

Contents

1 Introduction

We first set up the problem we wish to discuss.

Definition 1.1. Suppose we are given the following data:

(a) A compact, connected manifold X, of dimension $n > 0$.

(b) A Lie group G, with $\dim G > 0$, and centre $Z(G) \subseteq G$, and Lie algebra $\mathfrak{g}$.

(c) A principal G-bundle $\pi : P \to X$. We write $\mathrm{Ad}(P) \to X$ for the vector bundle with fibre $\mathfrak{g}$ defined by $\mathrm{Ad}(P) = (P \times \mathfrak{g})/G$, where G acts on P by the principal bundle action, and on $\mathfrak{g}$ by the adjoint action.

Write $\mathcal{A}_P$ for the set of connections ∇_P on the principal bundle $P \to X$. This is a real affine space modelled on the infinite-dimensional vector space $\Gamma^\infty(\mathrm{Ad}(P))$, and we make $\mathcal{A}_P$ into a topological space using the C^∞ topology on $\Gamma^\infty(\mathrm{Ad}(P))$. Here if $E \to X$ is a vector bundle then $\Gamma^\infty(E)$ denotes the vector space of smooth sections of E. Note that $\mathcal{A}_P$ is contractible.

We write $\mathcal{G} = \mathrm{Map}_{C^\infty}(X, G)$ for the infinite-dimensional Lie group of smooth maps $\gamma : X \to G$. Then $\mathcal{G}$ acts on P, and hence on $\mathcal{A}_P$ by gauge transformations, and the action is continuous for the topology on $\mathcal{A}_P$.

There is a natural inclusion $Z(G) \hookrightarrow \mathcal{G}$ mapping $z \in Z(G)$ to the constant map $\gamma : X \to G$ with value z. As X is connected, this identifies $Z(G)$ with the centre $Z(\mathcal{G})$ of $\mathcal{G}$, so we may take the quotient group $\mathcal{G}/Z(G)$. The action of $Z(G) \subset \mathcal{G}$ on $\mathcal{A}_P$ is trivial, so the $\mathcal{G}$-action on $\mathcal{A}_P$ descends to a $\mathcal{G}/Z(G)$-action.

Each $\nabla_P \in \mathcal{A}_P$ has a (finite-dimensional) *stabilizer group* $\mathrm{Stab}_{\mathcal{G}}(\nabla_P) \subset \mathcal{G}$ under the $\mathcal{G}$-action on $\mathcal{A}_P$, with $Z(G) \subseteq \mathrm{Stab}_{\mathcal{G}}(\nabla_P)$. As X is connected, $\mathrm{Stab}_{\mathcal{G}}(\nabla_P)$ is isomorphic to a closed Lie subgroup H of G with $Z(G) \subseteq H$. As in [18, p. 133] we call ∇_P *irreducible* if $\mathrm{Stab}_{\mathcal{G}}(\nabla_P) = Z(G)$, and *reducible* otherwise. Write $\mathcal{A}_P^{\mathrm{irr}}, \mathcal{A}_P^{\mathrm{red}}$ for the subsets of irreducible and reducible connections in $\mathcal{A}_P$. Then $\mathcal{A}_P^{\mathrm{irr}}$ is open and dense in $\mathcal{A}_P$, and $\mathcal{A}_P^{\mathrm{red}}$ is closed and of infinite codimension in the infinite-dimensional affine space $\mathcal{A}_P$. Hence the inclusion $\mathcal{A}_P^{\mathrm{irr}} \hookrightarrow \mathcal{A}_P$ is a weak homotopy equivalence, and $\mathcal{A}_P^{\mathrm{irr}}$ is weakly contractible.

We write $\mathcal{M}_P = \mathcal{A}_P/(\mathcal{G}/Z(G))$ for the moduli space of gauge equivalence classes of connections on P, and $\mathcal{M}_P^{\mathrm{irr}} = \mathcal{A}_P^{\mathrm{irr}}/(\mathcal{G}/Z(G))$ for the subspace $\mathcal{M}_P^{\mathrm{irr}} \subseteq \mathcal{M}_P$ of irreducible connections. We take $\mathcal{M}_P^{\mathrm{irr}}$ to be a topological space, with the

quotient topology. However, as we explain in Remark 2.1, sometimes we need to regard $\mathcal{M}_P$ as a *topological stack* rather than just as a topological space.

We define orientation bundles $O_P^{E_\bullet}$ on moduli spaces $\mathcal{M}_P$:

Definition 1.2. Work in the situation of Definition 1.1, with the same notation. Suppose we are given real vector bundles $E_0, E_1 \to X$, of the same rank r, and a linear elliptic partial differential operator $D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$, of degree d. As a shorthand we write $E_\bullet = (E_0, E_1, D)$. With respect to connections ∇_{E_0} on $E_0 \otimes \bigotimes^i T^*X$ for $0 \leqslant i < d$, when $e \in \Gamma^\infty(E_0)$ we may write

$$D(e) = \sum_{i=0}^{d} a_i \cdot \nabla_{E_0}^i e, \tag{1.1}$$

where $a_i \in \Gamma^\infty(E_0^* \otimes E_1 \otimes S^i TX)$ for $i = 0, \ldots, d$. The condition that D is *elliptic* is that $a_d|_x \cdot \otimes^d \xi : E_0|_x \to E_1|_x$ is an isomorphism for all $x \in X$ and $0 \neq \xi \in T_x^*X$, and the *symbol* $\sigma(D)$ of D is defined using a_d.

Let $\nabla_P \in \mathcal{A}_P$. Then ∇_P induces a connection $\nabla_{\mathrm{Ad}(P)}$ on the vector bundle $\mathrm{Ad}(P) \to X$. Thus we may form the twisted elliptic operator

$$\begin{aligned} &D^{\nabla_{\mathrm{Ad}(P)}} : \Gamma^\infty(\mathrm{Ad}(P) \otimes E_0) \longrightarrow \Gamma^\infty(\mathrm{Ad}(P) \otimes E_1), \\ &D^{\nabla_{\mathrm{Ad}(P)}} : e \longmapsto \sum_{i=0}^{d} (\mathrm{id}_{\mathrm{Ad}(P)} \otimes a_i) \cdot \nabla_{\mathrm{Ad}(P)\otimes E_0}^i e, \end{aligned} \tag{1.2}$$

where $\nabla_{\mathrm{Ad}(P)\otimes E_0}$ are the connections on $\mathrm{Ad}(P) \otimes E_0 \otimes \bigotimes^i T^*X$ for $0 \leqslant i < d$ induced by $\nabla_{\mathrm{Ad}(P)}$ and ∇_{E_0}.

Since $D^{\nabla_{\mathrm{Ad}(P)}}$ is a linear elliptic operator on a compact manifold X, it has finite-dimensional kernel $\mathrm{Ker}(D^{\nabla_{\mathrm{Ad}(P)}})$ and cokernel $\mathrm{Coker}(D^{\nabla_{\mathrm{Ad}(P)}})$, where the *index* of $D^{\nabla_{\mathrm{Ad}(P)}}$ is $\mathrm{ind}(D^{\nabla_{\mathrm{Ad}(P)}}) = \dim \mathrm{Ker}(D^{\nabla_{\mathrm{Ad}(P)}}) - \dim \mathrm{Coker}(D^{\nabla_{\mathrm{Ad}(P)}})$. This index is independent of $\nabla_P \in \mathcal{M}_P$, so we write $\mathrm{ind}_P^{E_\bullet} := \mathrm{ind}(D^{\nabla_{\mathrm{Ad}(P)}})$. The *determinant* $\det(D^{\nabla_{\mathrm{Ad}(P)}})$ is the 1-dimensional real vector space

$$\det(D^{\nabla_{\mathrm{Ad}(P)}}) = \det \mathrm{Ker}(D^{\nabla_{\mathrm{Ad}(P)}}) \otimes \big(\det \mathrm{Coker}(D^{\nabla_{\mathrm{Ad}(P)}})\big)^*, \tag{1.3}$$

where if V is a finite-dimensional real vector space then $\det V = \Lambda^{\dim V} V$.

These operators $D^{\nabla_{\mathrm{Ad}(P)}}$ vary continuously with $\nabla_P \in \mathcal{A}_P$, so they form a family of elliptic operators over the base topological space $\mathcal{A}_P$. Thus as in Atiyah and Singer [7], there is a natural real line bundle $L_P'^{E_\bullet} \to \mathcal{A}_P$ with fibre $L_P'^{E_\bullet}|_{\Delta_P} = \det(D^{\nabla_{\mathrm{Ad}(P)}})$ at each $\Delta_P \in \mathcal{A}_P$. It is naturally equivariant under the action of $\mathcal{G}/Z(G)$ on $\mathcal{A}_P$, and so pushes down to a real line bundle $L_P^{E_\bullet} \to \mathcal{M}_P$ on the topological stack $\mathcal{M}_P = \mathcal{A}_P/(\mathcal{G}/Z(G))$. We call $L_P^{E_\bullet}$ the *determinant line bundle* of $\mathcal{M}_P$. The restriction $L_P^{E_\bullet}|_{\mathcal{M}_P^{\mathrm{irr}}}$ is a topological real line bundle in the usual sense on the topological space $\mathcal{M}_P^{\mathrm{irr}}$.

Define the *orientation bundle* $O_P^{E_\bullet}$ of $\mathcal{M}_P$ by $O_P^{E_\bullet} = (L_P^{E_\bullet} \setminus 0(\mathcal{M}_P))/(0, \infty)$. That is, we take the complement $L_P^{E_\bullet} \setminus 0(\mathcal{M}_P)$ of the zero section $0(\mathcal{M}_P)$ in $L_P^{E_\bullet}$,

and quotient by the action of $(0,\infty)$ on the fibres of $L_P^{E\bullet}\setminus 0(\mathcal{M}_P)\rightarrow\mathcal{M}_P$ by multiplication. The projection $L_P^{E\bullet}\rightarrow\mathcal{M}_P$ descends to $\pi:O_P^{E\bullet}\rightarrow\mathcal{M}_P$, which is a fibre bundle with fibre $(\mathbb{R}\setminus\{0\})/(0,\infty)\cong\{1,-1\}=\mathbb{Z}_2$, since $L_P^{E\bullet}\rightarrow\mathcal{M}_P$ is a fibration with fibre $\mathbb{R}$. That is, $\pi:O_P^{E\bullet}\rightarrow\mathcal{M}_P$ is a *principal $\mathbb{Z}_2$-bundle*, in the sense of topological stacks. The fibres of $O_P^{E\bullet}\rightarrow\mathcal{M}_P$ are orientations on the real line fibres of $L_P^{E\bullet}\rightarrow\mathcal{M}_P$. The restriction $O_P^{E\bullet}|_{\mathcal{M}_P^{\rm irr}}$ is a principal $\mathbb{Z}_2$-bundle on the topological space $\mathcal{M}_P^{\rm irr}$, in the usual sense.

We say that $\mathcal{M}_P$ is *orientable* if $O_P^{E\bullet}$ is isomorphic to the trivial principal $\mathbb{Z}_2$-bundle $\mathcal{M}_P\times\mathbb{Z}_2\rightarrow\mathcal{M}_P$. An *orientation* ω on $\mathcal{M}_P$ is an isomorphism $\omega:O_P^{E\bullet}\xrightarrow{\cong}\mathcal{M}_P\times\mathbb{Z}_2$ of principal $\mathbb{Z}_2$-bundles. If ω is an orientation, we write $-\omega$ for the opposite orientation.

By characteristic class theory, $\mathcal{M}_P$ is orientable if and only if the first Stiefel–Whitney class $w_1(L_P^{E\bullet})$ is zero in $H^1(\mathcal{M}_P,\mathbb{Z}_2)$, which may be identified with the equivariant cohomology group $H^1_{\mathcal{G}/Z(G)}(\mathcal{A}_P,\mathbb{Z}_2)$. As $\mathcal{M}_P$ is connected, if $\mathcal{M}_P$ is orientable it has exactly two orientations.

We also define the *normalized orientation bundle*, or *n-orientation bundle*, a principal $\mathbb{Z}_2$-bundle $\check{O}_P^{E\bullet}\rightarrow\mathcal{M}_P$, by

$$\check{O}_P^{E\bullet}=O_P^{E\bullet}\otimes_{\mathbb{Z}_2}O_{X\times G}^{E\bullet}|_{[\nabla^0]}. \tag{1.4}$$

That is, we tensor the $O_P^{E\bullet}$ with the orientation torsor $O_{X\times G}^{E\bullet}|_{[\nabla^0]}$ of the trivial principal G-bundle $X\times G\rightarrow X$ at the trivial connection ∇^0. A *normalized orientation*, or *n-orientation*, of $\mathcal{M}_P$ is an isomorphism $\check{\omega}:\check{O}_P^{E\bullet}\xrightarrow{\cong}\mathcal{M}_P\times\mathbb{Z}_2$. There is a natural n-orientation of $\mathcal{M}_{X\times G}$ at $[\nabla^0]$.

Since we have natural isomorphisms

$$\operatorname{Ker}(D^{\nabla^0_{\mathrm{Ad}(P)}})\cong\mathfrak{g}\otimes\operatorname{Ker}D,\qquad \operatorname{Coker}(D^{\nabla^0_{\mathrm{Ad}(P)}})\cong\mathfrak{g}\otimes\operatorname{Coker}D, \tag{1.5}$$

we see that (using an orientation convention) there is a natural isomorphism

$$L_P^{E\bullet}|_{[\nabla^0]}\cong(\det D)^{\otimes^{\dim\mathfrak{g}}}\otimes(\Lambda^{\dim\mathfrak{g}}\mathfrak{g})^{\otimes^{\operatorname{ind}D}}, \tag{1.6}$$

which yields

$$O_{X\times G}^{E\bullet}|_{[\nabla^0]}\cong\operatorname{Or}(\det D)^{\otimes^{\dim\mathfrak{g}}}\otimes_{\mathbb{Z}_2}\operatorname{Or}(\mathfrak{g})^{\otimes^{\operatorname{ind}D}}, \tag{1.7}$$

where $\operatorname{Or}(\det D),\operatorname{Or}(\mathfrak{g})$ are the $\mathbb{Z}_2$-torsors of orientations on $\det D$ and $\mathfrak{g}$. Thus, choosing orientations for $\det D$ and $\mathfrak{g}$ gives an isomorphism $\check{O}_P^{E\bullet}\cong O_P^{E\bullet}$. (But see Remark 2.3 for an important technical point about this.)

N-orientation bundles are convenient because they behave nicely under the Excision Theorem, Theorem 3.1 below. Note that $O_P^{E\bullet}$ is trivializable if and only if $\check{O}_P^{E\bullet}$ is, so for questions of orientability there is no difference.

We can now state the central problem we consider in this paper:

Problem 1.3. *In the situation of Definition* 1.2, *we can ask:*

(a) **(Orientability.)** *Under what conditions on $X, G, P, E_\bullet$ is $\mathcal{M}_P$ orientable?*

(b) **(Canonical orientations.)** *If $\mathcal{M}_P$ is orientable, then possibly after choosing a small amount of extra data on X, can we construct a natural orientation (or n-orientation) ω_P on $\mathcal{M}_P$?*

(c) **(Relations between canonical orientations.)** *Suppose X and $E_\bullet$ are fixed, but we consider a family of pairs (G_i, P_i) for $i \in I$. Then there may be natural relations between moduli spaces $\mathcal{M}_{P_i}$ and their orientation bundles $O^{E_\bullet}_{P_i}$, which allow us to compare orientations on different $\mathcal{M}_{P_i}$. Can we construct natural orientations (or n-orientations) ω_{P_i} on $\mathcal{M}_{P_i}$ for $i \in I$ as in* **(b)**, *such that under each relation between moduli spaces $\mathcal{M}_{P_i}$, the ω_{P_i} are related by a sign ± 1 given by an explicit formula?*

Here is an example of what we have in mind in **(c)**. *Consider the family of all principal $\mathrm{U}(m)$-bundles $P \to X$ for all $m \geqslant 1$. If P_1, P_2 are $\mathrm{U}(m_1)$- and $\mathrm{U}(m_2)$-bundles we can form the direct sum $P_1 \oplus P_2$, a principal $\mathrm{U}(m_1 + m_2)$-bundle. There is a natural morphism $\Phi_{P_1,P_2} : \mathcal{M}_{P_1} \times \mathcal{M}_{P_2} \to \mathcal{M}_{P_1 \oplus P_2}$ taking direct sums of connections, and we can construct a natural isomorphism*

$$\phi^{E_\bullet}_{P_1,P_2} : O^{E_\bullet}_{P_1} \boxtimes_{\mathbb{Z}_2} O^{E_\bullet}_{P_2} \longrightarrow \Phi^*_{P_1,P_2}\big(O^{E_\bullet}_{P_1 \oplus P_2}\big)$$

of principal $\mathbb{Z}_2$-bundles on $\mathcal{M}_{P_1} \times \mathcal{M}_{P_2}$. Thus, if $\omega_{P_1}, \omega_{P_2}, \omega_{P_1 \oplus P_2}$ are orientations on $\mathcal{M}_{P_1}, \mathcal{M}_{P_2}, \mathcal{M}_{P_1 \oplus P_2}$, for some unique $\epsilon_{P_1,P_2} = \pm 1$ we have

$$(\phi^{E_\bullet}_{P_1,P_2})_*(\omega_{P_1} \boxtimes \omega_{P_2}) = \epsilon_{P_1,P_2} \cdot \Phi^*_{P_1,P_2}(\omega_{P_1 \oplus P_2}).$$

The aim is to construct orientations ω_P for all P such that ϵ_{P_1,P_2} is given by an explicit formula, perhaps involving the Chern classes $c_i(P_1), c_j(P_2)$. In good cases we might just arrange that $\epsilon_{P_1,P_2} = 1$ for all P_1, P_2.

Remark 1.4. (Orientations on gauge theory moduli spaces.) We will explain the following in detail in §4. In gauge theory one studies moduli spaces $\mathcal{M}^{\mathrm{ga}}_P$ of (irreducible) connections ∇_P on a principal bundle $P \to X$ satisfying some curvature condition, such as moduli spaces of instantons on oriented Riemannian 4-manifolds in Donaldson theory [18]. Under suitable genericity conditions, these moduli spaces $\mathcal{M}^{\mathrm{ga}}_P$ will be smooth manifolds.

Problem 1.3 is important for constructing orientations on such moduli spaces $\mathcal{M}^{\mathrm{ga}}_P$. There is a natural inclusion $\iota : \mathcal{M}^{\mathrm{ga}}_P \hookrightarrow \mathcal{M}_P$ such that $\iota^*(L^{E_\bullet}_P) \cong \det T^*\mathcal{M}^{\mathrm{ga}}_P$, for an elliptic complex $E_\bullet$ on X related to the curvature condition. Hence an orientation on $\mathcal{M}_P$ pulls back under ι to an orientation on $\mathcal{M}^{\mathrm{ga}}_P$. We can also use similar ideas to construct orientations on moduli spaces of connections ∇_P plus extra data, such as a Higgs field.

Thus, constructing orientations as in Problem 1.3 is an essential part of any programme to define enumerative invariants by 'counting' gauge theory moduli spaces, such as Casson invariants of 3-manifolds [9,53], Donaldson and Seiberg–Witten invariants of 4-manifolds [18,40,42], and proposed invariants counting G_2-instantons on 7-manifolds with holonomy G_2 [19], or Spin(7)-instantons on 8-manifolds with holonomy Spin(7) or SU(4) [11,13,19].

There are already various results on Problem 1.3 in the literature, aimed at orienting gauge theory moduli spaces. The general method was pioneered by Donaldson [15, Lem. 10], [16, §3], [18, §5.4 & §7.1.6], for moduli of instantons on 4-manifolds. We also mention Taubes [53, §2] for 3-manifolds, Walpuski [56, §6.1] for 7-manifolds, Cao and Leung [13, §10.4] and Muñoz and Shahbazi [41] for 8-manifolds, and Cao and Leung [14, Th. 2.1] for $8k$-manifolds.

We will mostly be interested in solving Problem 1.3 not just for a single choice of $X, G, P, E_\bullet$, but for whole classes at once. We make this precise:

Definition 1.5. A *Gauge Orientation Problem* (*GOP*) is a problem of the following kind. We consider compact n-manifolds X for fixed n, equipped with some particular kind of geometric structure $\mathcal{T}$, such that using $\mathcal{T}$ we can define a real elliptic operator $E_\bullet$ on X as in Definition 1.2. We also choose a family $\mathcal{G}$ of Lie groups G, such as $\mathcal{G} = \{\mathrm{SU}(m) : m = 1, 2, \ldots\}$. Then we seek to solve Problem 1.3 for all $X, G, P, E_\bullet$ arising from geometric structures $(X, \mathcal{T})$ of the chosen kind, and Lie groups $G \in \mathcal{G}$.

Often we aim to construct canonical (n-)orientations on all such $\mathcal{M}_P$, satisfying compatibility conditions comparing the (n-)orientations for different manifolds X^+, X^- using the Excision Theorem (see Theorem 3.1 and Problem 3.2).

We give some examples of Gauge Orientation Problems:

Example 1.6. Here are some possibilities for n, the geometric structure $\mathcal{T}$, and elliptic operator $E_\bullet$:

(a) Consider compact Riemannian n-manifolds (X, g) for any fixed n, and let $E_\bullet$ be the elliptic operator

$$\mathrm{d} + \mathrm{d}^* : \Gamma^\infty\bigl(\textstyle\bigoplus_{i=0}^{[n/2]} \Lambda^{2i} T^*X\bigr) \longrightarrow \Gamma^\infty\bigl(\textstyle\bigoplus_{i=0}^{[(n-1)/2]} \Lambda^{2i+1} T^*X\bigr).$$

(b) Consider compact, oriented Riemannian n-manifolds (X, g) for $n = 4k$, and let $E_\bullet$ be the elliptic operator

$$\mathrm{d} + \mathrm{d}^* : \Gamma^\infty\bigl(\textstyle\bigoplus_{i=0}^{k-1} \Lambda^{2i} T^*X \oplus \Lambda^{2k}_+ T^*X\bigr) \longrightarrow \Gamma^\infty\bigl(\textstyle\bigoplus_{i=0}^{k-1} \Lambda^{2i+1} T^*X\bigr),$$

where $\Lambda^{2k}_+ T^*X \subset \Lambda^{2k} T^*X$ is the subbundle of $2k$-forms self-dual under the Hodge star.

(c) Consider compact Riemannian n-manifolds (X, g) for any fixed n with a spin structure with spin bundle $S \to X$, and let $E_\bullet$ be the Dirac operator $D : \Gamma^\infty(S) \to \Gamma^\infty(S)$.

(d) Consider compact, oriented Riemannian n-manifolds (X, g) for $n = 4k$ with a spin structure with spin bundle $S = S_+ \oplus S_- \to X$, and let $E_\bullet$ be the positive Dirac operator $D_+ : \Gamma^\infty(S_+) \to \Gamma^\infty(S_-)$.

(e) Consider triples (X, J, g) of a compact n-manifold X for $n = 2k$, an almost complex structure J on X, and a Hermitian metric g on (X, J). Let $E_\bullet$ be the elliptic operator

$$\bar\partial + \bar\partial^* : \Gamma^\infty\bigl(\textstyle\bigoplus_{i=0}^{k} \Lambda^{0,2i} T^*X\bigr) \longrightarrow \Gamma^\infty\bigl(\textstyle\bigoplus_{i=0}^{k-1} \Lambda^{0,2i+1} T^*X\bigr).$$

For example, solving GOP (b) with $n = 4$ would give orientations for moduli spaces of anti-self-dual instantons on 4-manifolds [18]. Solving GOP (c) with $n = 7$ would give orientations for moduli spaces of G_2-instantons, as in §4.2.9. Solving GOP (d) with $n = 8$ would give orientations for moduli spaces of Spin(7)-instantons, as in §4.2.10.

In this paper we first collect together in §2 some results and methods for solving Problem 1.3. Some of these are new, and some have been used in the literature in particular cases, but we state them in general. Section 3 discusses techniques for solving Gauge Orientation Problems. Finally, §4 applies the results of §2–§3 to prove new results on orientability and canonical orientations for interesting families of gauge theory moduli spaces, and reviews the main results of the sequels [12, 32].

An important motivation for this paper was the first author's new theory [31] defining vertex algebra structures on the homology $H_*(\mathcal{M})$ of certain moduli stacks $\mathcal{M}$ in Algebraic Geometry and Differential Geometry. For fixed $X, E_\bullet$, an ingredient required in one version of this theory is canonical orientations on moduli stacks $\mathcal{M}_P$ for all principal $\mathrm{U}(m)$-bundles $P \to X$ and all $m \geqslant 1$ as in Problem 1.3(b), satisfying relations under direct sum as in Problem 1.3(c), and the theory dictates the structure of these relations.

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2 Results and methods for solving Problem 1.3

2.1 Remarks on the definitions in §1

Here are some remarks on Definitions 1.1 and 1.2, omitted from §1 for brevity.

Remark 2.1. (i) There is a theory of topological stacks, due to Metzler [39] and Noohi [43, 44]. Topological stacks form a 2-category **TopSta**, with homotopy category Ho(**TopSta**). The category of topological spaces **Top** has a full and faithful embedding $I : \mathbf{Top} \hookrightarrow \mathrm{Ho}(\mathbf{TopSta})$, so we can consider topological spaces to be examples of topological stacks. There is also a functor $\Pi : \mathrm{Ho}(\mathbf{TopSta}) \to \mathbf{Top}$ mapping a topological stack S to its *coarse moduli space* S^{coa} [43, §4.3], with $\Pi \circ I \cong \mathrm{Id}_{\mathbf{Top}}$. Thus, we can regard a topological stack S as a topological space S^{coa} with extra structure.

The most important extra structure is *isotropy groups*. If S is a topological stack, and s is a point of S (i.e. a point of S^{coa}) we have an isotropy group $\mathrm{Iso}_S(s)$, a topological group, with $\mathrm{Iso}_S(s) = \{1\}$ if S is a topological space.

If T is a topological space and H a topological group acting continuously on T we can form a *quotient stack* $[T/H]$ in **TopSta**, with $[T/H]^{\mathrm{coa}}$ the quotient topological space T/H. Points of $[T/H]$ correspond to H-orbits tH in T, and the isotropy groups are $\mathrm{Iso}_{[T/H]}(tH) \cong \mathrm{Stab}_H(t)$.

For the quotient topological stack $\mathcal{M}_P = [\mathcal{A}_P/(\mathcal{G}/Z(G))]$, the points are $\mathcal{G}/Z(G)$-orbits $[\nabla_P]$ in $\mathcal{A}$, and the isotropy groups are

$$\mathrm{Iso}_{\mathcal{M}_P}([\nabla_P]) \cong \mathrm{Stab}_{\mathcal{G}/Z(G)}(\nabla_P) \cong \mathrm{Stab}_{\mathcal{G}}(\nabla_P)/Z(G).$$

Since $\mathcal{M}^{\mathrm{irr}}$ has trivial isotropy groups as a topological stack, it is actually a topological space, and we do not need topological stacks to study $\mathcal{M}^{\mathrm{irr}}$.

(ii) Set $\widetilde{\mathcal{M}}_P = \mathcal{A}_P/\mathcal{G}$, which is a simpler and more obvious definition than $\mathcal{M}_P = \mathcal{A}_P/(\mathcal{G}/Z(G))$. Then $\mathcal{M}_P$ and $\widetilde{\mathcal{M}}_P$ coincide as sets or topological spaces. But as a topological stack, $\widetilde{\mathcal{M}}_P$ has isotropy groups $\mathrm{Iso}_{\widetilde{\mathcal{M}}_P}([\nabla_P]) \cong \mathrm{Stab}_{\mathcal{G}}(\nabla_P)$. Thus if $Z(G) \neq \{1\}$ then $\mathcal{M}_P$ and $\widetilde{\mathcal{M}}_P$ are different as topological stacks, and $\widetilde{\mathcal{M}}_P^{\mathrm{irr}}$ is not a topological space.

(iii) As in Definition 1.1 the inclusion $\mathcal{A}_P^{\mathrm{irr}} \hookrightarrow \mathcal{A}_P$ is a weak homotopy equivalence, so the inclusion $\mathcal{M}^{\mathrm{irr}} \hookrightarrow \mathcal{M}$ is a weak homotopy equivalence of topological stacks in the sense of Noohi [44]. Therefore, for the algebraic topological questions that concern us, working on $\mathcal{M}^{\mathrm{irr}}$ and on $\mathcal{M}$ are essentially equivalent, so we could just restrict our attention to the topological space $\mathcal{M}^{\mathrm{irr}}$, and not worry about topological stacks at all, following most other authors in the area.

The main reason we do not do this is that to relate orientations on different moduli spaces we consider direct sums of connections, which are generally reducible, so restricting to irreducible connections would cause problems.

(iv) Here is why we sometimes need $\mathcal{M}_P$ to be a topological stack rather than a topological space. We will be studying certain *real line bundles* $L \to \mathcal{M}_P$. A line bundle $L \to \mathcal{M}_P$ is equivalent to a $\mathcal{G}/Z(G)$-equivariant line bundle $L' \to \mathcal{A}_P$. At each point $[\nabla_P]$ in $\mathcal{M}_P$ the fibre L_{∇_P} is a 1-dimensional real vector space, and the isotropy group $\mathrm{Iso}_{\mathcal{M}_P}([\nabla_P])$ has a natural representation on L_{∇_P}.

Under some circumstances, this representation of $\mathrm{Iso}_{\mathcal{M}_P}([\nabla_P])$ on L_{∇_P} may not be trivial. Then L does not descend to the coarse moduli space $\mathcal{M}_P^{\mathrm{coa}}$. That is, if we consider $\mathcal{M}_P = \mathcal{A}_P/(\mathcal{G}/Z(G))$ as a topological space rather than a topological stack, then the orientation line bundles we are interested in *may not exist on the topological space* $\mathcal{M}_P$, though they are defined on $\mathcal{M}_P^{\mathrm{irr}} \subset \mathcal{M}_P$.

Remark 2.2. (i) Up to continuous isotopy (and hence up to isomorphism), $L_P^{E\bullet}, O_P^{E\bullet}$ in Definition 1.2 depend on the elliptic operator $D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ up to continuous deformation amongst elliptic operators, and hence only on the *symbol* $\sigma(D)$ of D (essentially, the highest order coefficients a_d in (1.1), up to deformation).

This can mean that superficially different geometric problems have isomorphic orientation bundles, or that orientations depend on less data than you think. For example, as in §4.2.9–§4.2.10 orientations for moduli spaces of G_2-instantons on a G_2-manifold (X, φ, g), or of Spin(7)-instantons on a Spin(7)-

manifold (X,Ω,g), depend only on the underlying compact spin 7- or 8-manifold X, not on the G_2-structure (φ,g) or Spin(7)-structure (Ω,g).

(ii) For orienting moduli spaces of 'instantons' in gauge theory, as in §4, we usually start not with an elliptic operator on X, but with an *elliptic complex*

$$0 \longrightarrow \Gamma^\infty(E_0) \xrightarrow{D_0} \Gamma^\infty(E_1) \xrightarrow{D_1} \cdots \xrightarrow{D_{k-1}} \Gamma^\infty(E_k) \longrightarrow 0. \tag{2.1}$$

If $k>1$ and ∇_P is an arbitrary connection on a principal G-bundle $P\to X$ then twisting (2.1) by $(\mathrm{Ad}(P),\nabla_{\mathrm{Ad}(P)})$ as in (1.2) may not yield a complex (that is, we may have $D_{i+1}^{\nabla_{\mathrm{Ad}(P)}}\circ D_i^{\nabla_{\mathrm{Ad}(P)}}\neq 0$), so the definition of $\det(D_\bullet^{\nabla_{\mathrm{Ad}(P)}})$ does not work, though it does work if ∇_P satisfies the appropriate instanton-type curvature condition. To get round this, we choose metrics on X and the E_i, so that we can take adjoints D_i^*, and replace (2.1) by the elliptic operator

$$\Gamma^\infty\bigl(\textstyle\bigoplus_{0\leqslant i\leqslant k/2}E_{2i}\bigr)\xrightarrow{\sum_i(D_{2i}+D_{2i-1}^*)}\Gamma^\infty\bigl(\textstyle\bigoplus_{0\leqslant i<k/2}E_{2i+1}\bigr), \tag{2.2}$$

and then Definition 1.2 works with (2.2) in place of $E_\bullet$.

Remark 2.3. In (1.4) we defined the n-orientation bundle $\check{O}_P^{E_\bullet}$ in terms of $O_{X\times G}^{E_\bullet}|_{[\nabla^0]}$, for which we gave a formula in (1.7) involving $\mathrm{Or}(\mathfrak{g})$, and said that choosing orientations on $\det D$ and $\mathfrak{g}$ gives an isomorphism $\check{O}_P^{E_\bullet}\cong O_P^{E_\bullet}$.

While all this makes sense, for it to be well behaved, *we need the orientation on $\mathfrak{g}$ to be invariant under the adjoint action of G on $\mathfrak{g}$*, and this is not true for all Lie groups G. For example, if $G=\mathrm{O}(2m)$ and $\gamma\in\mathrm{O}(2m)\setminus\mathrm{SO}(2m)$ then $\mathrm{Ad}(\gamma)$ is orientation-reversing on $\mathfrak{g}$, so no $\mathrm{Ad}(G)$-invariant orientation exists on $\mathfrak{g}$. If we restrict to *connected* Lie groups G then $\mathrm{Ad}(G)$ is automatically orientation-preserving on $\mathfrak{g}$, and this problem does not arise.

Let X and $E_\bullet$ be as in Definition 1.2. Take P to be the trivial principal $\mathrm{O}(2m)$-bundle over X. Consider the topological stack $\mathcal{M}_P$, determinant line bundle $L_P^{E_\bullet}\to\mathcal{M}_P$, and orientation bundle $O_P^{E_\bullet}\to\mathcal{M}_P$ from §1. The isotropy group of the stack $\mathcal{M}_P$ at $[\nabla^0]$ is $\mathrm{Iso}_{\mathcal{M}_P}([\nabla^0])=\mathrm{O}(2m)$, and its action on $L_P^{E_\bullet}|_{[\nabla^0]}$ in (1.6) is induced by the action of $\mathrm{Ad}(G)$ on $\mathfrak{g}$. Thus $\gamma\in\mathrm{O}(2m)\setminus\mathrm{SO}(2m)$ acts on $L_P^{E_\bullet}|_{[\nabla^0]}$ and $O_P^{E_\bullet}|_{[\nabla^0]}$ by multiplication by $(-1)^{\mathrm{ind}\,D}$.

Now suppose $\mathrm{ind}\,D$ is odd. Then $\gamma\in\mathrm{O}(2m)\setminus\mathrm{SO}(2m)$ acts on $O_P^{E_\bullet}|_{[\nabla^0]}$ by multiplication by -1. Any orientation on $\mathcal{M}_P$ must restrict at $[\nabla^0]$ to an $\mathrm{O}(2m)$-invariant trivialization of $O_P^{E_\bullet}|_{[\nabla^0]}$. Thus $\mathcal{M}_P$ *is not orientable.*

2.2 Elementary results on orientation bundles

We now give some results and constructions for orientation bundles $O_P^{E_\bullet}$ in Definition 1.2, and for answering Problem 1.3. Many of these are fairly obvious, or are already used in the references in Remark 1.4, but some are new.

2.2.1 Simply-connected moduli spaces $\mathcal{M}_P$ are orientable

As principal $\mathbb{Z}_2$-bundles on $\mathcal{M}_P$ are trivial if $H^1(\mathcal{M}_P, \mathbb{Z}_2) = 0$, we have:

Lemma 2.4. *In Definition* 1.2, *if $\mathcal{M}_P$ is simply-connected, or more generally if $H^1(\mathcal{M}_P, \mathbb{Z}_2) = 0$, then $\mathcal{M}_P$ is orientable, and n-orientable.*

Thus, if we can show $\pi_1(\mathcal{M}_P) = \{1\}$ using algebraic topology, then orientability in Problem 1.3(a) follows. This is used in Donaldson [15, Lem. 10], [18, §5.4], Cao and Leung [13, §10.4], [14, Th. 2.1], and Muñoz and Shahbazi [41].

2.2.2 Standard orientations for trivial connections

In Definition 1.2, let $P = X \times G$ be the trivial principal G-bundle over X, and write $\nabla^0 \in \mathcal{A}_P$ for the trivial connection. Then (1.7) gives a formula for $O^{E_\bullet}_{X\times G}|_{[\nabla^0]}$. Thus, if we fix an orientation for $\mathfrak{g}$ if $\operatorname{ind} D$ is odd, and an orientation for $\det D$ if $\dim \mathfrak{g}$ is odd, then we obtain an orientation on $\mathcal{M}_P = \mathcal{M}_{X\times G}$ at the trivial connection $[\nabla^0]$. We will call this the *standard orientation.* If $\mathcal{M}_P$ is orientable, the standard orientation determines an orientation on all of $\mathcal{M}_P$.

2.2.3 Natural orientations when G is abelian

In Definition 1.2, suppose the Lie group G is abelian (e.g. $G = \mathrm{U}(1)$). Then the adjoint action of G on $\mathfrak{g}$ is trivial, so $\mathrm{Ad}(P) \to X$ is the trivial vector bundle $X \times \mathfrak{g} \to X$, and $\nabla_{\mathrm{Ad}(P)}$ is the trivial connection. Thus as in (1.5)

$$\operatorname{Ker}(D^{\nabla_{\mathrm{Ad}(P)}}) = \mathfrak{g} \otimes \operatorname{Ker} D \quad \text{and} \quad \operatorname{Coker}(D^{\nabla_{\mathrm{Ad}(P)}}) = \mathfrak{g} \otimes \operatorname{Coker} D.$$

Hence as in (1.6), $L^{E_\bullet}_P \to \mathcal{M}_P$ is the trivial line bundle with fibre

$$(\det D)^{\otimes^{\dim \mathfrak{g}}} \otimes (\Lambda^{\dim \mathfrak{g}} \mathfrak{g})^{\otimes^{\operatorname{ind} D}},$$

so $\mathcal{M}_P$ is orientable. If we choose an orientation for $\mathfrak{g}$, and (if $\dim \mathfrak{g}$ is odd) an orientation for $\det D$ (equivalently, an orientation for $\operatorname{Ker} D \oplus \operatorname{Coker} D$) then we obtain a natural orientation on $\mathcal{M}_P$ for any principal G-bundle $P \to X$. Also $\mathcal{M}_P$ has a canonical n-orientation, independent of choices.

2.2.4 Natural orientations from complex structures on $E_\bullet$ or G

The next theorem is easy to prove, but very useful.

Theorem 2.5. *In Definition* 1.2, *suppose that E_0, E_1 have the structure of complex vector bundles, such that the symbol of D is complex linear. We will call this a* ***complex structure on*** $E_\bullet$. *Then for any Lie group G and principal G-bundle $P \to X$, we can define a canonical orientation ω_P and a canonical n-orientation $\check{\omega}_P$ on $\mathcal{M}_P$, that is, we define trivializations $\omega_P : O^{E_\bullet}_P \xrightarrow{\cong} \mathcal{M}_P \times \mathbb{Z}_2$ and $\check{\omega}_P : \check{O}^{E_\bullet}_P \xrightarrow{\cong} \mathcal{M}_P \times \mathbb{Z}_2$.*

Proof. As E_0, E_1 are complex vector bundles, $\Gamma^\infty(E_0), \Gamma^\infty(E_1)$ are complex vector spaces. First suppose $D : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$ is $\mathbb{C}$-linear. Then $\mathrm{Ad}(P) \otimes E_0, \mathrm{Ad}(P) \otimes E_1$ are also complex vector bundles, and $D^{\nabla_{\mathrm{Ad}(P)}}$ in (1.2) is $\mathbb{C}$-linear, so $\mathrm{Ker}(D^{\nabla_{\mathrm{Ad}(P)}})$ and $\mathrm{Coker}(D^{\nabla_{\mathrm{Ad}(P)}})$ are finite-dimensional complex vector spaces. With an appropriate orientation convention, the complex structures induce a natural orientation on $\det(D^{\nabla_{\mathrm{Ad}(P)}})$ in (1.3), which varies continuously with ∇_P in $\mathcal{M}_P$. This gives a canonical orientation for $\mathcal{M}_P$. To get a canonical n-orientation, combine the orientations for $\mathcal{M}_P$ and $\mathcal{M}_{X\times G}$ using (1.4).

If D is not $\mathbb{C}$-linear, though $\sigma(D)$ is, we can deform $D = D^0$ continuously through elliptic operators $D^t : \Gamma^\infty(E_0) \to \Gamma^\infty(E_1)$, $t \in [0,1]$ with symbols $\sigma(D^t) = \sigma(D)$ to D^1 which is $\mathbb{C}$-linear. As in Remark 2.2(i), the orientation bundle $O_P^{E_\bullet^t}$ deforms continuously with D^t, so $O_P^{E_\bullet} = O_P^{E_\bullet^0} \cong O_P^{E_\bullet^1}$, and the trivialization of $O_P^{E_\bullet^1}$ from D^1 complex linear induces a trivialization of $O_P^{E_\bullet}$. It is independent of choices, as the space of all D^t with $\sigma(D^t) = \sigma(D)$ is an infinite-dimensional affine space, and so contractible, and the subset of $\mathbb{C}$-linear D^1 is connected. $\square$

Example 2.6. Let (X, g) be a compact, oriented Riemannian manifold of dimension $4n+2$, and take $E_\bullet$ to be the elliptic operator on X

$$D = \mathrm{d} + \mathrm{d}^* : \Gamma^\infty\bigl(\textstyle\bigoplus_{i=0}^{2n+1} \Lambda^{2i} T^*X\bigr) \longrightarrow \Gamma^\infty\bigl(\textstyle\bigoplus_{i=0}^{2n} \Lambda^{2i+1} T^*X\bigr).$$

Using the Hodge star $*$ we can define complex structures on the bundles $E_0 = \Lambda^{\mathrm{even}} T^*X$, $E_1 = \Lambda^{\mathrm{odd}} T^*X$ such that the symbol of D is complex linear. So for these $X, E_\bullet$ we have canonical (n-)orientations on $\mathcal{M}_P$ for all G, P.

Example 2.7. **(a)** Let (X, g) be a compact, oriented Riemannian n-manifold with a spin structure with spin bundle $S \to X$, and let $E_\bullet$ be the Dirac operator $D : \Gamma^\infty(S) \to \Gamma^\infty(S)$. If $n \equiv 2, 3, 4, 5$ or $6 \bmod 8$ there is a complex structure on S with the symbol of D complex linear. So for these $X, E_\bullet$ we have canonical (n-)orientations on $\mathcal{M}_P$ for all G, P. This does not work if $n \equiv 0$ or $7 \bmod 8$.

(b) If $n \equiv 0$ or $4 \bmod 8$ then $S = S_+ \oplus S_-$, and we can take $E_\bullet$ to be the positive Dirac operator $D_+ : \Gamma^\infty(S_+) \to \Gamma^\infty(S_-)$. If $n \equiv 4 \bmod 8$ there are complex structures on $S_\pm$ with the symbol of D_+ complex linear, and again we get canonical (n-)orientations. This does not work if $n \equiv 0 \bmod 8$.

See Theorems 4.3, 4.4, 4.6 and 4.10 for more applications of Theorem 2.5.

In a similar way, if G is a *complex* Lie group, such as $\mathrm{SL}(m, \mathbb{C})$, then $\mathfrak{g}$ is a complex vector space, $\mathrm{Ad}(P)$ is a complex vector bundle, and $\nabla_{\mathrm{Ad}(P)}$ is complex linear, so $D^{\nabla_{\mathrm{Ad}(P)}}$ in (1.2) is complex linear, and as in Theorem 2.5 we obtain a canonical orientation on $\mathcal{M}_P$ for all $X, E_\bullet$ and principal G-bundles P.

2.2.5 Another case with natural orientations

In Definition 1.2, suppose that $E_\bullet$ is of the form $E_\bullet = \tilde{E}_\bullet \oplus \tilde{E}_\bullet^*$, where $\tilde{E}_\bullet$ is a real linear elliptic operator on X, and $\tilde{E}_\bullet^*$ is the formal adjoint of $\tilde{E}_\bullet$ under

some metrics on $X, \tilde{E}_0, \tilde{E}_1$. Then we have

$$L_P^{E_\bullet} \cong L_P^{\tilde{E}_\bullet} \otimes_{\mathbb{R}} L_P^{\tilde{E}_\bullet^*} \cong L_P^{\tilde{E}_\bullet} \otimes_{\mathbb{R}} (L_P^{\tilde{E}_\bullet})^* \cong \mathcal{M}_P \times \mathbb{R},$$
$$O_P^{E_\bullet} \cong O_P^{\tilde{E}_\bullet} \otimes_{\mathbb{Z}_2} O_P^{\tilde{E}_\bullet^*} \cong O_P^{\tilde{E}_\bullet} \otimes_{\mathbb{Z}_2} O_P^{\tilde{E}_\bullet} \cong \mathcal{M}_P \times \mathbb{Z}_2.$$

Thus $\mathcal{M}_P$ has a canonical orientation for any principal G-bundle $P \to X$, for any G. Since orientation bundles depend only on the symbol of $E_\bullet$, and this up to continuous isotopy, this is also true if $E_\bullet = \tilde{E}_\bullet \oplus \tilde{E}_\bullet^*$ holds only at the level of symbols, up to continuous isotopy.

In §4.2.5 we will use this method to show that moduli spaces $\mathcal{M}_P^{\mathrm{VW}}$ of solutions to the Vafa–Witten equations on 4-manifolds have canonical orientations.

2.2.6 Orientations on products of moduli spaces

Let X and $E_\bullet$ be fixed, and suppose G, H are Lie groups, and $P \to X$, $Q \to X$ are principal G- and H-bundles respectively. Then $P \times_X Q$ is a principal $G \times H$ bundle over X. There is a natural 1-1 correspondence between pairs (∇_P, ∇_Q) of connections ∇_P, ∇_Q on P, Q, and connections $\nabla_{P \times_X Q}$ on $P \times_X Q$. This induces an isomorphism of topological stacks $\Lambda_{P,Q} : \mathcal{M}_P \times \mathcal{M}_Q \to \mathcal{M}_{P \times_X Q}$.

For (∇_P, ∇_Q) and $\nabla_{P \times_X Q}$ as above, there are also natural isomorphisms

$$\mathrm{Ker}(D^{\nabla_{\mathrm{Ad}(P)}}) \oplus \mathrm{Ker}(D^{\nabla_{\mathrm{Ad}(Q)}}) \cong \mathrm{Ker}(D^{\nabla_{\mathrm{Ad}(P \times_X Q)}}),$$
$$\mathrm{Coker}(D^{\nabla_{\mathrm{Ad}(P)}}) \oplus \mathrm{Coker}(D^{\nabla_{\mathrm{Ad}(Q)}}) \cong \mathrm{Coker}(D^{\nabla_{\mathrm{Ad}(P \times_X Q)}}).$$

With an appropriate orientation convention (the same as that needed to define orientations on products of Kuranishi spaces in Fukaya et al. [22, §8]), these induce a natural isomorphism

$$\det(D^{\nabla_{\mathrm{Ad}(P)}}) \otimes \det(D^{\nabla_{\mathrm{Ad}(Q)}}) \cong \det(D^{\nabla_{\mathrm{Ad}(P \times_X Q)}}),$$

which is the fibre at (∇_P, ∇_Q) of an isomorphism of line bundles on $\mathcal{M}_P \times \mathcal{M}_Q$

$$L_P^{E_\bullet} \boxtimes L_Q^{E_\bullet} \cong \Lambda_{P,Q}^*(L_{P \times_X Q}^{E_\bullet}).$$

This induces an isomorphism of orientation bundles

$$\lambda_{P,Q} : O_P^{E_\bullet} \boxtimes_{\mathbb{Z}_2} O_Q^{E_\bullet} \xrightarrow{\cong} \Lambda_{P,Q}^*(O_{P \times_X Q}^{E_\bullet}).$$

Therefore $\mathcal{M}_{P \times_X Q}$ is orientable if and only if $\mathcal{M}_P, \mathcal{M}_Q$ are both orientable, and then there is a natural correspondence between pairs (ω_P, ω_Q) of orientations for $\mathcal{M}_P, \mathcal{M}_Q$, and orientations $\omega_{P \times_X Q}$ for $\mathcal{M}_{P \times_X Q}$.

By exchanging G, H and P, Q, we get an isomorphism on $\mathcal{M}_Q \times \mathcal{M}_P$:

$$\lambda_{Q,P} : O_Q^{E_\bullet} \boxtimes_{\mathbb{Z}_2} O_P^{E_\bullet} \xrightarrow{\cong} \Lambda_{Q,P}^*(O_{Q \times_X P}^{E_\bullet}).$$

Under the natural isomorphisms $\mathcal{M}_P \times \mathcal{M}_Q \cong \mathcal{M}_Q \times \mathcal{M}_P$, $\mathcal{M}_{P \times_X Q} \cong \mathcal{M}_{Q \times_X P}$, using the orientation convention we can show that

$$\lambda_{Q,P} = (-1)^{\mathrm{ind}_P^{E_\bullet} \cdot \mathrm{ind}_Q^{E_\bullet}} \cdot \lambda_{Q,P}. \tag{2.3}$$

This gives a *commutativity property* of the isomorphisms $\lambda_{P,Q}$.

If K is another Lie group and $R \to X$ a principal K-bundle, then we have

$$\begin{aligned} \Lambda_{P\times_X Q,R} \circ (\Lambda_{P,Q} \times \mathrm{id}_{\mathcal{M}_R}) = \Lambda_{P,Q\times_X R} \circ (\mathrm{id}_{\mathcal{M}_P} \times \Lambda_{Q,R}) : \\ \mathcal{M}_P \times \mathcal{M}_Q \times \mathcal{M}_R \longrightarrow \mathcal{M}_{P\times_X Q\times_X R}. \end{aligned} \tag{2.4}$$

Using this, we can show the following *associativity property* of the isomorphisms $\lambda_{P,Q}$ on $\mathcal{M}_P \times \mathcal{M}_Q \times \mathcal{M}_R$, where the sign is trivial:

$$\begin{aligned} &(\Lambda_{P,Q} \times \mathrm{id}_{\mathcal{M}_R})^*(\lambda_{P\times_X Q,R}) \circ (\pi^*_{\mathcal{M}_P\times\mathcal{M}_Q}(\lambda_{P,Q}) \otimes \mathrm{id}_{\pi^*_{\mathcal{M}_R}(O^{E_\bullet}_R)}) \\ &= (\mathrm{id}_{\mathcal{M}_P} \times \Lambda_{Q,R})^*(\lambda_{P,Q\times_X R}) \circ (\mathrm{id}_{\pi^*_{\mathcal{M}_P}(O^{E_\bullet}_P)} \otimes \pi^*_{\mathcal{M}_Q\times\mathcal{M}_R}(\lambda_{Q,R})) : \\ &O^{E_\bullet}_P \boxtimes_{\mathbb{Z}_2} O^{E_\bullet}_Q \boxtimes_{\mathbb{Z}_2} O^{E_\bullet}_R \xrightarrow{\cong} \big(\Lambda_{P\times_X Q,R} \circ (\Lambda_{P,Q} \times \mathrm{id}_{\mathcal{M}_R})\big)^*(O^{E_\bullet}_{P\times_X Q\times_X R}). \end{aligned} \tag{2.5}$$

Equations (2.3) and (2.5) are examples of the kind of explicit formula relating orientations referred to in Problem 1.3(c).

2.2.7 Relating moduli spaces for discrete quotients $H \twoheadrightarrow G$

Suppose G is a Lie group, $K \subset G$ a discrete (closed and dimension zero) normal subgroup, and set $H = G/K$ for the quotient Lie group. Let $X, E_\bullet$ be fixed. If $P \to X$ is a principal G-bundle, then $Q := P/K$ is a principal H-bundle over X. Each G-connection ∇_P on P induces a natural H-connection ∇_Q on Q, and the map $\nabla_P \mapsto \nabla_Q$ induces a natural morphism $\Delta^Q_P : \mathcal{M}_P \to \mathcal{M}_Q$ of topological stacks, which is an isomorphism. If ∇_P, ∇_Q are as above then the natural isomorphism $\mathfrak{g} \cong \mathfrak{h}$ induces an isomorphism $\mathrm{Ad}(P) \cong \mathrm{Ad}(Q)$ of vector bundles on X, which identifies the connections $\nabla_{\mathrm{Ad}(P)}, \nabla_{\mathrm{Ad}(Q)}$. Hence the twisted elliptic operators $D^{\nabla_{\mathrm{Ad}(P)}}, D^{\nabla_{\mathrm{Ad}(Q)}}$ are naturally isomorphic, and so are their determinants (1.3). This easily gives canonical isomorphisms

$$L^{E_\bullet}_P \cong (\Delta^Q_P)^*(L^{E_\bullet}_Q) \quad \text{and} \quad \delta^Q_P : O^{E_\bullet}_P \xrightarrow{\cong} (\Delta^Q_P)^*(O^{E_\bullet}_Q),$$

which induce a 1-1 correspondence between orientations on $\mathcal{M}_P, \mathcal{M}_Q$. For example, we can apply this when $G = \mathrm{SU}(2)$ and $H = \mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$.

Note however that not every principal H-bundle $Q \to X$ need come from a principal G-bundle $P \to X$ by $Q \cong P/K$. For example, a principal $\mathrm{SO}(3)$-bundle $Q \to X$ lifts to a principal $\mathrm{SU}(2)$-bundle $P \to X$ if and only if the second Stiefel-Whitney class $w_2(Q)$ is zero.

Example 2.8. Take $G = \mathrm{SU}(m) \times \mathrm{U}(1)$, and define $K \subset G$ by

$$K = \big\{(e^{2\pi ik/m}\mathrm{Id}_m, e^{-2\pi ik/m}) : k = 1, \ldots, m\big\} \cong \mathbb{Z}_m.$$

Then K lies in the centre $Z(G)$, so is normal in G, and $H = G/K \cong \mathrm{U}(m)$. To see this, note that the morphism $G = \mathrm{SU}(m) \times \mathrm{U}(1) \to \mathrm{U}(m) = H$ mapping $(A, e^{i\theta}) \mapsto e^{i\theta}A$ is surjective with kernel K.

For fixed $X, E_\bullet$, let $P \to X$ be a principal $\mathrm{SU}(m)$-bundle, and write $Q = X \times \mathrm{U}(1) \to X$ for the trivial $\mathrm{U}(1)$-bundle over X. Set $R = P \times_X Q$ for the

associated principal $\mathrm{SU}(m)\times\mathrm{U}(1)$-bundle over X, and define $S=R/K$ for the quotient principal $\mathrm{U}(m)$-bundle. We now have isomorphisms of moduli spaces

$$\mathcal{M}_P\times\mathcal{M}_Q\xrightarrow{\Lambda_{P,Q}}\mathcal{M}_{P\times_X Q}=\mathcal{M}_R\xrightarrow{\Delta_R^S}\mathcal{M}_S,$$

and isomorphisms of orientation bundles

$$O_P^{E_\bullet}\boxtimes_{\mathbb{Z}_2}O_Q^{E_\bullet}\xrightarrow{\lambda_{P,Q}}\Lambda_{P,Q}^*(O_R^{E_\bullet})\xrightarrow{\Lambda_{P,Q}^*(\delta_R^S)}(\Delta_R^S\circ\Lambda_{P,Q})^*(O_S^{E_\bullet}),$$

where $\Lambda_{P,Q},\lambda_{P,Q}$ are as in §2.2.6, and Δ_R^S,δ_R^S are as above.

As Q is the trivial $\mathrm{U}(1)$-bundle, it carries the trivial connection ∇_Q^0. Fixing an orientation for $\det D$, as in §2.2.2 we have the standard orientation for $\mathcal{M}_Q$ at $[\nabla_Q^0]$, giving an isomorphism $\sigma_Q:\mathbb{Z}_2\to O_Q^{E_\bullet}|_{[\nabla_Q^0]}$. Thus, we have a morphism

$$\mathrm{K}_P^S:\mathcal{M}_P\longrightarrow\mathcal{M}_S,\qquad \mathrm{K}_P^S:[\nabla_P]\longmapsto\Delta_R^S\circ\Lambda_{P,Q}\big([\nabla_P],[\nabla_Q^0]\big),$$

and an isomorphism of orientation bundles

$$\kappa_P^S:=\Lambda_{P,Q}^*(\delta_R^S)\circ\lambda_{P,Q}\circ(\mathrm{id}\boxtimes\sigma_Q):O_P^{E_\bullet}\longrightarrow(\mathrm{K}_P^S)^*(O_S^{E_\bullet}).$$

Hence orientations for the $\mathrm{U}(m)$-bundle moduli space $\mathcal{M}_S$ induce orientations for the $\mathrm{SU}(m)$-bundle moduli space $\mathcal{M}_P$. Our conclusion is:

> *For fixed $X,E_\bullet$, if we have orientability, or canonical orientations, on $\mathcal{M}_S$ for all principal $\mathrm{U}(m)$-bundles $S\to X$, then we have orientability, or canonical orientations, on $\mathcal{M}_P$ for all $\mathrm{SU}(m)$-bundles $P\to X$. The analogue holds for n-orientations.*

Example 2.12 will give a kind of converse to this.

The method of the next proposition was used by Donaldson and Kronheimer [18, §5.4.3] for simply-connected 4-manifolds X.

Proposition 2.9. *Let $X,E_\bullet$ be fixed as in Definition 1.2, and suppose that for all principal $\mathrm{U}(2)$-bundles $Q\to X$, the moduli space $\mathcal{M}_Q$ is orientable. Then for all principal $\mathrm{SO}(3)$-bundles $P\to X$ such that $w_2(P)\in H^2(X,\mathbb{Z}_2)$ lies in the image of $H^2(X,\mathbb{Z})\to H^2(X,\mathbb{Z}_2)$, the moduli space $\mathcal{M}_P$ is orientable. This holds for all $\mathrm{SO}(3)$-bundles $P\to X$ if $H^3(X,\mathbb{Z})$ has no 2-torsion.*

Proof. We apply the above construction with $G=\mathrm{U}(2)$, $K=\{\pm1\}\subset\mathrm{U}(2)$, and $H=\mathrm{U}(2)/\{\pm1\}\cong\mathrm{SO}(3)\times\mathrm{U}(1)$. Let $Q\to X$ be a principal $\mathrm{U}(2)$-bundle. Then $R=Q/\{\pm1\}$ is a principal $\mathrm{SO}(3)\times\mathrm{U}(1)$-bundle $R\to X$. Hence there are principal $\mathrm{SO}(3)$- and $\mathrm{U}(1)$-bundles $P,S\to X$ with $R\cong P\times_X S$. We now have isomorphisms of moduli spaces

$$\mathcal{M}_P\times\mathcal{M}_S\xrightarrow{\Lambda_{P,S}}\mathcal{M}_{P\times_X S}\cong\mathcal{M}_R\xrightarrow{\Delta_R^Q}\mathcal{M}_Q,$$

and isomorphisms of orientation bundles

$$O_P^{E_\bullet}\boxtimes_{\mathbb{Z}_2}O_S^{E_\bullet}\xrightarrow{\lambda_{P,S}}\Lambda_{P,S}^*(O_R^{E_\bullet})\xrightarrow{\Lambda_{P,S}^*(\delta_R^Q)}(\Delta_R^Q\circ\Lambda_{P,S})^*(O_Q^{E_\bullet}).$$

By assumption $O_Q^{E_\bullet}$ is orientable. Restricting to a point of $\mathcal{M}_S$ in the above equations, we see that $\mathcal{M}_P$ is orientable.

Since $\mathrm{U}(2) \cong \mathrm{Spin}^c(3)$, it is known from the theory of Spin^c-structures that an $\mathrm{SO}(3)$-bundle $P \to X$ extends to a $\mathrm{U}(2)$-bundle $Q \to X$ as above if and only if the second Stiefel–Whitney class $w_2(P) \in H^2(X, \mathbb{Z}_2)$ lies in the image of $H^2(X, \mathbb{Z}) \to H^2(Z, \mathbb{Z}_2)$, since $w_2(P)$ must be the image of $c_1(Q)$. The exact sequence $0 \to \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \to \mathbb{Z}_2 \to 0$ gives a long exact sequence in cohomology

$$\cdots \longrightarrow H^2(X, \mathbb{Z}) \longrightarrow H^2(X, \mathbb{Z}_2) \longrightarrow H^3(X, \mathbb{Z}) \xrightarrow{2\cdot} H^3(X, \mathbb{Z}) \longrightarrow \cdots.$$

This implies that $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z}_2)$ is surjective if and only if $H^3(X, \mathbb{Z})$ has no 2-torsion. The proposition follows. $\square$

2.2.8 Relating moduli spaces for Lie subgroups $G \subset H$

Let $X, E_\bullet$ be fixed, and let H be a Lie group and $G \subset H$ a Lie subgroup, with Lie algebras $\mathfrak{g} \subset \mathfrak{h}$. If $P \to X$ is a principal G-bundle, then $Q := (P \times H)/G$ is a principal H-bundle over X. Each G-connection ∇_P on P induces a natural H-connection ∇_Q on Q, and the map $\nabla_P \mapsto \nabla_Q$ induces a natural morphism $\Xi_P^Q : \mathcal{M}_P \to \mathcal{M}_Q$ of topological stacks. Thus, we can try to compare the line bundles $L_P^{E_\bullet}, (\Xi_P^Q)^*(L_Q^{E_\bullet})$ on $\mathcal{M}_P$, and the principal $\mathbb{Z}_2$-bundles $O_P^{E_\bullet}, (\Xi_P^Q)^*(O_Q^{E_\bullet})$.

Write $\mathfrak{m} = \mathfrak{h}/\mathfrak{g}$, and $\rho : G \to \mathrm{Aut}(\mathfrak{m})$ for the representation induced by the adjoint representation of $H \supset G$. Then we have an exact sequence

$$0 \longrightarrow \mathrm{Ad}(P) \longrightarrow \mathrm{Ad}(Q) \longrightarrow \rho(P) = (P \times \mathfrak{m})/G \longrightarrow 0 \tag{2.6}$$

of vector bundles on X, induced by $0 \to \mathfrak{g} \to \mathfrak{h} \to \mathfrak{m} \to 0$. If ∇_P, ∇_Q are as above, we have connections $\nabla_{\mathrm{Ad}(P)}, \nabla_{\mathrm{Ad}(Q)}, \nabla_{\rho(P)}$ on $\mathrm{Ad}(P), \mathrm{Ad}(Q), \rho(P)$ compatible with (2.6). Thus, twisting $E_\bullet$ by $\mathrm{Ad}(P), \mathrm{Ad}(Q), \rho(P)$ and their connections and taking determinants, we define an isomorphism

$$\det(D^{\nabla_{\mathrm{Ad}(P)}}) \otimes \det(D^{\nabla_{\rho(P)}}) \cong \det(D^{\nabla_{\mathrm{Ad}(Q)}}),$$

which is the fibre at ∇_P of an isomorphism of line bundles on $\mathcal{M}_P$

$$L_P^{E_\bullet} \otimes L_{P,\rho}^{E_\bullet} \cong (\Xi_P^Q)^*(L_Q^{E_\bullet}), \tag{2.7}$$

where $L_{P,\rho}^{E_\bullet} \to \mathcal{M}_P$ is the determinant line bundle associated to the family of elliptic operators $\nabla_P \mapsto D^{\nabla_{\rho(P)}}$ on $\mathcal{M}_P$. We will write $\mathrm{ind}_{P,\rho}^{E_\bullet} := \mathrm{ind}(D^{\nabla_{\rho(P)}})$ for the index of these operators, which is independent of $\nabla_P \in \mathcal{M}_P$.

Now suppose that we can give $\mathfrak{m}$ the structure of a *complex* vector space, such that $\rho : G \to \mathrm{Aut}(\mathfrak{m})$ is complex linear. This happens if H/G has an (almost) complex structure homogeneous under H. Then $\mathrm{ind}_{P,\rho}^{E_\bullet}$ is even, and as in §2.2.4 for complex G, we can define a natural orientation on $L_{P,\rho}^{E_\bullet}$, so taking orientations in (2.7) gives a natural isomorphism of principal $\mathbb{Z}_2$-bundles on $\mathcal{M}_P$:

$$\xi_P^Q : O_P^{E_\bullet} \longrightarrow (\Xi_P^Q)^*(O_Q^{E_\bullet}). \tag{2.8}$$

An easy special case is if $\mathfrak{m}=0$, e.g. for $\mathrm{SO}(m)\subset\mathrm{O}(m)$, when $L^{E_\bullet}_{P,\rho}$ is trivial.

This gives a method for proving orientability in Problem 1.3(a). Suppose we can show that $H^1(\mathcal{M}_Q,\mathbb{Z}_2)=0$, using homotopy-theoretic properties of X,H. Then $\mathcal{M}_Q$ is orientable by Lemma 2.4, so (2.8) shows that $\mathcal{M}_P$ is orientable, even if $H^1(\mathcal{M}_P,\mathbb{Z}_2)\neq 0$. The method is used by Donaldson [15, Lem. 10], [18, §5.4.2], and by Muñoz and Shahbazi [41] using the inclusion of Lie groups $\mathrm{SU}(9)/\mathbb{Z}_3\subset E_8$. Here are some examples of suitable $G\subset H$:

Example 2.10. We have an inclusion $G=\mathrm{U}(m_1)\times\mathrm{U}(m_2)\subset\mathrm{U}(m_1+m_2)=H$ for $m_1,m_2\geqslant 1$, with $\mathfrak{u}(m_1+m_2)/(\mathfrak{u}(m_1)\oplus\mathfrak{u}(m_2))=\mathfrak{m}\cong\mathbb{C}^{m_1}\otimes_{\mathbb{C}}\mathbb{C}^{m_2}$, where $G=\mathrm{U}(m_1)\times\mathrm{U}(m_2)$ acts on $\mathbb{C}^{m_1}\otimes_{\mathbb{C}}\mathbb{C}^{m_2}$ via the usual representations of $\mathrm{U}(m_1),\mathrm{U}(m_2)$ on $\mathbb{C}^{m_1},\mathbb{C}^{m_2}$, so the representation ρ is complex linear.

Suppose $X,E_\bullet$ are fixed, and $P_1\to X$, $P_2\to X$ are principal $\mathrm{U}(m_1)$- and $\mathrm{U}(m_2)$-bundles. Define a principal $\mathrm{U}(m_1+m_2)$-bundle $P_1\oplus P_2\to X$ by

$$P_1\oplus P_2=(\mathrm{U}(m_1+m_2)\times P_1\times_X P_2)/(\mathrm{U}(m_1)\times\mathrm{U}(m_2)). \tag{2.9}$$

Then combining the material of §2.2.6 for the product of $\mathrm{U}(m_1),\mathrm{U}(m_2)$ with the above, we have a morphism

$$\Phi_{P_1,P_2}:=\Xi^{P_1\oplus P_2}_{P_1\times_X P_2}\circ\Lambda_{P_1,P_2}:\mathcal{M}_{P_1}\times\mathcal{M}_{P_2}\longrightarrow\mathcal{M}_{P_1\oplus P_2}, \tag{2.10}$$

and a natural isomorphism of principal $\mathbb{Z}_2$-bundles on $\mathcal{M}_{P_1}\times\mathcal{M}_{P_2}$:

$$\begin{aligned}\phi_{P_1,P_2}&=\Lambda^*_{P_1,P_2}(\xi^{P_1\oplus P_2}_{P_1\times_X P_2})\circ\lambda_{P_1,P_2}:\\ &\qquad O^{E_\bullet}_{P_1}\boxtimes_{\mathbb{Z}_2}O^{E_\bullet}_{P_2}\xrightarrow{\cong}\Phi^*_{P_1,P_2}(O^{E_\bullet}_{P_1\oplus P_2}).\end{aligned} \tag{2.11}$$

As for (2.3)–(2.5), we can consider commutativity and associativity properties of the isomorphisms ϕ_{P_1,P_2}. For commutativity, under the natural isomorphisms $\mathcal{M}_{P_1}\times\mathcal{M}_{P_2}\cong\mathcal{M}_{P_2}\times\mathcal{M}_{P_1}$, $\mathcal{M}_{P_1\oplus P_2}\cong\mathcal{M}_{P_2\oplus P_1}$ we have

$$\phi_{P_2,P_1}=(-1)^{\mathrm{ind}^{E_\bullet}_{P_1}\cdot\mathrm{ind}^{E_\bullet}_{P_2}}\cdot(-1)^{\frac12\,\mathrm{ind}^{E_\bullet}_{P_1\times_X P_2,\rho}}\cdot\phi_{P_1,P_2}. \tag{2.12}$$

Here the first sign $(-1)^{\mathrm{ind}^{E_\bullet}_{P_1}\cdot\mathrm{ind}^{E_\bullet}_{P_2}}$ in (2.12) comes from (2.3), and exchanges the $\mathrm{U}(m_1)\times\mathrm{U}(m_2)$-bundle $P_1\times_X P_2$ with the $\mathrm{U}(m_2)\times\mathrm{U}(m_1)$-bundle $P_2\times_X P_1$.

The second sign $(-1)^{\frac12\,\mathrm{ind}^{E_\bullet}_{P_1\times_X P_2,\rho}}$ in (2.12) comes in as $\phi_{P_1,P_2},\phi_{P_2,P_1}$ in (2.11) depend on choices of complex structure on

$$\mathfrak{m}_{1,2}=\mathfrak{u}(m_1+m_2)/(\mathfrak{u}(m_1)\oplus\mathfrak{u}(m_2))\ \text{ and }\ \mathfrak{m}_{2,1}=\mathfrak{u}(m_2+m_1)/(\mathfrak{u}(m_2)\oplus\mathfrak{u}(m_1)).$$

Under the natural isomorphism $\mathfrak{m}_{1,2}\cong\mathfrak{m}_{2,1}$, these complex structures are *complex conjugate*. Because of this, under the natural isomorphism $L^{E_\bullet}_{P_1\times_X P_2,\rho_{12}}\cong L^{E_\bullet}_{P_2\times_X P_1,\rho_{21}}$, the orientations on $L^{E_\bullet}_{P_1\times_X P_2,\rho_{12}},L^{E_\bullet}_{P_2\times_X P_1,\rho_{21}}$ used to define ϕ_{P_1,P_2}, ϕ_{P_2,P_1} differ by a factor of $(-1)^{\mathrm{ind}_{\mathbb{C}}(D^{\nabla_{\rho(P_1\times_X P_2)}})}$, regarding $D^{\nabla_{\rho(P_1\times_X P_2)}}$ as a complex elliptic operator. As $\mathrm{ind}_{\mathbb{C}}(D^{\nabla_{\rho(P_1\times_X P_2)}})=\frac12\,\mathrm{ind}_{\mathbb{R}}(D^{\nabla_{\rho(P_1\times_X P_2)}})=\frac12\,\mathrm{ind}^{E_\bullet}_{P_1\times_X P_2,\rho}$, equation (2.12) follows.

For associativity, if $P_3 \to X$ is a principal $\mathrm{U}(m_3)$-bundle then we have

$$\Phi_{P_1\oplus P_2,P_3} \circ (\Phi_{P_1,P_2} \times \mathrm{id}_{\mathcal{M}_{P_3}}) = \Phi_{P_1,P_2\oplus P_3} \circ (\mathrm{id}_{\mathcal{M}_{P_1}} \times \Phi_{P_2,P_3}) : \\ \mathcal{M}_{P_1} \times \mathcal{M}_{P_2} \times \mathcal{M}_{P_3} \longrightarrow \mathcal{M}_{P_1\oplus P_2\oplus P_3},$$

as for (2.4), and then as for (2.5), we have

$$\begin{aligned} &(\Phi_{P_1,P_2} \times \mathrm{id}_{\mathcal{M}_{P_3}})^*(\phi_{P_1\oplus P_2,P_3}) \circ (\phi_{P_1,P_2} \boxtimes \mathrm{id}_{O^{E_\bullet}_{P_3}}) \\ &= (\mathrm{id}_{\mathcal{M}_{P_1}} \times \Phi_{P_2,P_3})^*(\phi_{P_1,P_2\oplus P_3}) \circ (\mathrm{id}_{O^{E_\bullet}_{P_1}} \boxtimes \phi_{P_2,P_3}) : \\ &O^{E_\bullet}_{P_1} \boxtimes_{\mathbb{Z}_2} O^{E_\bullet}_{P_2} \boxtimes_{\mathbb{Z}_2} O^{E_\bullet}_{P_3} \xrightarrow{\cong} \big(\Phi_{P_1\oplus P_2,P_3} \circ (\Phi_{P_1,P_2} \times \mathrm{id}_{\mathcal{M}_{P_3}})\big)^*(O^{E_\bullet}_{P_1\oplus P_2\oplus P_3}). \end{aligned} \tag{2.13}$$

The sign is trivial as there is no sign in (2.5), and the natural isomorphism $\mathfrak{m}_{12,3} \oplus \mathfrak{m}_{1,2} \cong \mathfrak{m}_{1,23} \oplus \mathfrak{m}_{2,3}$ is complex linear.

Remark 2.11. The analogue of Example 2.10 does not work for the families of groups $\mathrm{O}(m), \mathrm{SO}(m), \mathrm{Spin}(m)$ or $\mathrm{Sp}(m)$, since for example under the inclusion $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2) \hookrightarrow \mathrm{SO}(m_1 + m_2)$, there is no $\mathrm{SO}(m_1) \times \mathrm{SO}(m_2)$-invariant complex structure on $\mathfrak{m} = \mathfrak{so}(m_1 + m_2)/(\mathfrak{so}(m_1) \oplus \mathfrak{so}(m_2))$ unless $m_1 = 2$ or $m_2 = 2$. So the theory of §2.4 below works only for the unitary groups.

Example 2.12. Define an inclusion $\mathrm{U}(m) \hookrightarrow \mathrm{SU}(m + 1)$ by mapping

$$A \longmapsto \begin{pmatrix} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & (\det A)^{-1} \end{pmatrix}, \qquad A \in \mathrm{U}(m).$$

There is an isomorphism $\mathfrak{m} = \mathfrak{su}(m + 1)/\mathfrak{u}(m) \cong \mathbb{C}^m$, such that $A \in \mathrm{U}(m)$ acts on $\mathfrak{m} \cong \mathbb{C}^m$ by $A : \boldsymbol{x} \mapsto \det A \cdot A\boldsymbol{x}$, which is complex linear on $\mathfrak{m}$.

For fixed $X, E_\bullet$, let $P \to X$ be a principal $\mathrm{U}(m)$-bundle, and $Q = (P \times \mathrm{SU}(m + 1))/\mathrm{U}(m)$ the associated principal $\mathrm{SU}(m + 1)$-bundle. Then as above we have a morphism of moduli spaces $\Xi^Q_P : \mathcal{M}_P \to \mathcal{M}_Q$ and an isomorphism of orientation bundles $\xi^Q_P : O^{E_\bullet}_P \to (\Xi^Q_P)^*(O^{E_\bullet}_Q)$. Hence orientations for the $\mathrm{SU}(m + 1)$-bundle moduli space $\mathcal{M}_Q$ induce orientations for the $\mathrm{U}(m)$-bundle moduli space $\mathcal{M}_P$. In a converse to Example 2.8, our conclusion is:

> *For fixed $X, E_\bullet$, if we have orientability, or canonical orientations, on $\mathcal{M}_Q$ for all principal $\mathrm{SU}(m + 1)$-bundles $Q \to X$, then we have orientability, or canonical orientations, on $\mathcal{M}_P$ for all principal $\mathrm{U}(m)$-bundles $P \to X$. The analogue holds for n-orientations.*

Example 2.13. Define an inclusion $\mathrm{U}(m) \hookrightarrow \mathrm{Sp}(m)$ by mapping complex matrices to quaternionic matrices using the inclusion $\mathbb{C} = \langle 1, i\rangle_{\mathbb{R}} \hookrightarrow \mathbb{H} = \langle 1, i, j, k\rangle_{\mathbb{R}}$. There is an isomorphism of $\mathfrak{m} = \mathfrak{sp}(m)/\mathfrak{u}(m)$ with the complex vector space of $m \times m$ complex symmetric matrices B, such that $A \in \mathrm{U}(m)$ acts on $\mathfrak{m}$ by $A : B \mapsto ABA^t$, which is complex linear on $\mathfrak{m}$.

For fixed $X, E_\bullet$, let $P \to X$ be a $\mathrm{U}(m)$-bundle, and $Q = (P \times \mathrm{Sp}(m))/\mathrm{U}(m)$ the associated $\mathrm{Sp}(m)$-bundle. Then as above we have a morphism of moduli spaces $\Xi_P^Q : \mathcal{M}_P \to \mathcal{M}_Q$ and an isomorphism of orientation bundles $\xi_P^Q : O_P^{E_\bullet} \to (\Xi_P^Q)^*(O_Q^{E_\bullet})$. Hence orientations for the $\mathrm{Sp}(m)$-bundle moduli space $\mathcal{M}_Q$ induce orientations for the $\mathrm{U}(m)$-bundle moduli space $\mathcal{M}_P$. Our conclusion is:

> *For fixed $X, E_\bullet$, if we have orientability, or canonical orientations, on $\mathcal{M}_Q$ for all principal $\mathrm{Sp}(m)$-bundles $Q \to X$, then we have orientability, or canonical orientations, on $\mathcal{M}_P$ for all principal $\mathrm{U}(m)$-bundles $P \to X$. The analogue holds for n-orientations.*

Example 2.14. We have an inclusion $\mathrm{SO}(m) \times \mathrm{SO}(2) \subset \mathrm{SO}(m+2)$ for $m \geqslant 1$. There is a natural identification $\mathfrak{so}(m+2)/(\mathfrak{so}(m) \oplus \mathfrak{so}(2)) = \mathfrak{m} \cong \mathbb{R}^m \otimes_{\mathbb{R}} \mathbb{R}^2$, where $G = \mathrm{SO}(m) \times \mathrm{SO}(2)$ acts on $\mathbb{R}^m \otimes_{\mathbb{R}} \mathbb{R}^2$ by the tensor product of the obvious representations of $\mathrm{SO}(m), \mathrm{SO}(2)$ on $\mathbb{R}^m, \mathbb{R}^2$. Identifying $\mathbb{R}^2 \cong \mathbb{C}$ and $\mathrm{SO}(2) \cong \mathrm{U}(1)$ gives $\mathfrak{m} \cong \mathbb{R}^m \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}^m$, where ρ is complex linear.

Example 2.15. We have an inclusion $G = \mathrm{Sp}(m) \times \mathrm{U}(1) \subset \mathrm{Sp}(m+1) = H$ for $m \geqslant 1$, by combining $\mathrm{U}(1) \subset \mathrm{Sp}(1)$ and $\mathrm{Sp}(m) \times \mathrm{Sp}(1) \subset \mathrm{Sp}(m+1)$. There is a natural identification $\mathfrak{sp}(m+1)/(\mathfrak{sp}(m) \oplus \mathfrak{u}(1)) = \mathfrak{m} \cong \mathbb{H}^m \oplus \mathbb{C}$, where $G = \mathrm{Sp}(m) \times \mathrm{U}(1)$ acts on $\mathbb{H}^m \oplus \mathbb{C}$ by

$$\rho(A, e^{i\theta}) : (\boldsymbol{x}, y) \longmapsto (A\boldsymbol{x}e^{i\theta}, ye^{i\theta})$$

for $A \in \mathrm{Sp}(m)$, $e^{i\theta} \in \mathrm{U}(1)$, $\boldsymbol{x} \in \mathbb{H}^m$ and $y \in \mathbb{C}$, regarding $A, \boldsymbol{x}, e^{i\theta}$ as $m \times m$, $m \times 1$ and 1×1 matrices over $\mathbb{H}$ to define $A\boldsymbol{x}e^{i\theta}$. Identifying $\mathbb{H}^m \cong \mathbb{C}^{2m}$ using right multiplication by $i \in \mathbb{H}$, we see that ρ is complex linear on $\mathfrak{m} \cong \mathbb{C}^{2m+1}$.

2.2.9 Stabilizing $\mathrm{U}(m), \mathrm{SU}(m), \mathrm{SO}(m), \mathrm{Sp}(m)$ moduli spaces as $m \to \infty$

Let $X, E_\bullet$ be fixed, and fix an orientation on $\det D$, so that as in §2.2.3 we have a canonical orientation on $\mathcal{M}_Q$ for any principal $\mathrm{U}(1)$-bundle $Q \to M$.

Suppose $P \to X$ is a principal $\mathrm{U}(m)$-bundle for $m \geqslant 1$. Taking Q to be the trivial $\mathrm{U}(1)$-bundle $X \times \mathrm{U}(1) \to X$, equation (2.9) defines a principal $\mathrm{U}(m+1)$-bundle $P \oplus Q$, which we will write as $P \oplus \mathbb{C}$. Define a morphism $\Psi_P^{P\oplus\mathbb{C}} : \mathcal{M}_P \to \mathcal{M}_{P\oplus\mathbb{C}}$ to be the restriction of the morphism $\Xi_{P\times_X Q}^{P\oplus Q} \circ \Lambda_{P,Q} : \mathcal{M}_P \times \mathcal{M}_Q \to \mathcal{M}_{P\oplus Q} = \mathcal{M}_{P\oplus\mathbb{C}}$ in Example 2.10 to $\mathcal{M}_P \cong \mathcal{M}_P \times \{[\nabla^0]\} \subset \mathcal{M}_P \times \mathcal{M}_Q$, where ∇^0 is the trivial connection on the trivial $\mathrm{U}(1)$-bundle Q. Then restricting (2.11) to $\mathcal{M}_P \times \{[\nabla^0]\}$ and using the canonical orientation on $\mathcal{M}_Q$ at $[\nabla^0]$ gives a natural isomorphism of principal $\mathbb{Z}_2$-bundles on $\mathcal{M}_P$:

$$\psi_P^{P\oplus\mathbb{C}} : O_P^{E_\bullet} \xrightarrow{\cong} (\Psi_P^{P\oplus\mathbb{C}})^*(O_{P\oplus\mathbb{C}}^{E_\bullet}). \tag{2.14}$$

We can iterate this construction: write $P \oplus \mathbb{C}^k = (\cdots(P \oplus \mathbb{C})\cdots \oplus \mathbb{C}) \oplus \mathbb{C}$ for the principal $\mathrm{U}(m+k)$-bundle on X obtained by applying this construction

$k \geqslant 1$ times, and set $\Psi_P^{P\oplus\mathbb{C}^k} = \Psi_{P\oplus\mathbb{C}^{k-1}}^{P\oplus\mathbb{C}^k} \circ \cdots \circ \Psi_P^{P\oplus\mathbb{C}} : \mathcal{M}_P \to \mathcal{M}_{P\oplus\mathbb{C}^k}$. Then composing (2.14) for $P, \ldots, P \oplus \mathbb{C}^{k-1}$ gives a natural isomorphism

$$\psi_P^{P\oplus\mathbb{C}^k} : O_P^{E_\bullet} \xrightarrow{\cong} (\Psi_P^{P\oplus\mathbb{C}^k})^*(O_{P\oplus\mathbb{C}^k}^{E_\bullet}). \tag{2.15}$$

Take the direct limit $\mathcal{M}_{P\oplus\mathbb{C}^\infty} := \varinjlim_{k\to\infty} \mathcal{M}_{P\oplus\mathbb{C}^k}$ in topological stacks using the gluing maps $\Psi_{P\oplus\mathbb{C}^k}^{P\oplus\mathbb{C}^{k+1}} : \mathcal{M}_{P\oplus\mathbb{C}^k} \to \mathcal{M}_{P\oplus\mathbb{C}^{k+1}}$, and the direct limit principal $\mathbb{Z}_2$-bundle $O_{P\oplus\mathbb{C}^\infty}^{E_\bullet} \to \mathcal{M}_{P\oplus\mathbb{C}^\infty}$ using (2.14)–(2.15). This is called *stabilization*. We have a morphism $\Psi_P^{P\oplus\mathbb{C}^\infty} : \mathcal{M}_P \to \mathcal{M}_{P\oplus\mathbb{C}^\infty}$, and an isomorphism

$$\psi_P^{P\oplus\mathbb{C}^\infty} : O_P^{E_\bullet} \xrightarrow{\cong} (\Psi_P^{P\oplus\mathbb{C}^\infty})^*(O_{P\oplus\mathbb{C}^\infty}^{E_\bullet}).$$

The point of this is that $\mathcal{M}_{P\oplus\mathbb{C}^\infty}$ may be easier to understand using homotopy theory than $\mathcal{M}_P$, since $\mathcal{M}_P$ is homotopy-equivalent to a connected component of $\mathrm{Map}_{C^0}(X, BU(m))$, but $\mathcal{M}_{P\oplus\mathbb{C}^\infty}$ is homotopy-equivalent to a component of $\mathrm{Map}_{C^0}(X, BU)$, where BU is simpler than $BU(m)$ in some ways.

If $2m + 2k \geqslant \dim X$, one can show that $\Psi_{P\oplus\mathbb{C}^k}^{P\oplus\mathbb{C}^{k+1}} : \mathcal{M}_{P\oplus\mathbb{C}^k} \to \mathcal{M}_{P\oplus\mathbb{C}^{k+1}}$ induces an isomorphism on fundamental groups, and thus a 1-1 correspondence between orientations on $\mathcal{M}_{P\oplus\mathbb{C}^k}$ and orientations on $\mathcal{M}_{P\oplus\mathbb{C}^{k+1}}$. So for questions about orientations, it is not really necessary to take the limit $k \to \infty$.

We can apply similar ideas on stabilization as $m \to \infty$ to study orientations for principal $\mathrm{SU}(m)$-bundles (these may also be understood by relating them to orientations for $\mathrm{U}(m)$-bundles, and stabilizing the $\mathrm{U}(m)$-bundles as above), and for principal $\mathrm{SO}(m)$-bundles, using Example 2.14 to relate orientations for an $\mathrm{SO}(m)$-bundle P and the $\mathrm{SO}(m+2)$-bundle $P \oplus \mathbb{R}^2$, and for principal $\mathrm{Sp}(m)$-bundles, using Example 2.15 to relate orientations for an $\mathrm{Sp}(m)$-bundle P and the $\mathrm{Sp}(m+1)$-bundle $P \oplus \mathbb{H}$. All the above also works for n-orientation bundles, without choosing an orientation for $\det D$.

2.3 Studying $\mathrm{U}(m)$-bundles via stabilization and K-theory

Let X and $E_\bullet$ be fixed. We now explain a useful framework for studying orientations on $\mathcal{M}_P$ simultaneously for all principal $\mathrm{U}(m)$-bundles $P \to X$ and all $m \geqslant 1$, using the complex K-theory groups $K^0(X), K^1(X)$. This can then be used to study orientations on $\mathcal{M}_Q$ for all principal $\mathrm{SU}(m)$-bundles $Q \to X$. Parts of the theory also work for $\mathrm{SO}(m)$-bundles and $\mathrm{Sp}(m)$-bundles.

2.3.1 Background on K-theory

We briefly summarize some notation and results from topological K-theory. Some references are Atiyah [2], Karoubi [34] and Switzer [48, §11].

Let X be a topological space. Write $K^0(X)$ for the abelian group generated by isomorphism classes $[\![E]\!]$ of complex vector bundles $E \to X$ (which may have different ranks on different components of X) with the relation that $[\![E \oplus F]\!] = [\![E]\!] + [\![F]\!]$ in $K^0(X)$ for all complex vector bundles $E, F \to X$. If $f : X \to Y$

is continuous, define a group morphism $K^0(f) : K^0(Y) \to K^0(X)$ with $K^0(f) : [\![E]\!] \mapsto [\![f^*(E)]\!]$ for all $E \to Y$. This defines a functor $K^0 : \mathbf{Top} \to \mathbf{AbGp}$.

If $P \to X$ is a principal $\mathrm{U}(m)$-bundle, it has an associated complex vector bundle $E \to X$ with fibre $\mathbb{C}^m$ given by $E = (P \times \mathbb{C}^m)/\mathrm{U}(m)$. We write $[\![P]\!] = [\![E]\!]$ in $K^0(X)$.

If (X, x) is a topological space with base-point $x \in X$, define $\tilde{K}^0(X, x) = \mathrm{Ker}\bigl(K^0(x) : K^0(X) \to K^0(*)\bigr)$, regarding x as a map $* \to X$. We can make any topological space X into a space with basepoint $(X \amalg \{\infty\}, \infty)$ by adding a disjoint extra point ∞, and then $K^0(X) \cong \tilde{K}^0(X \amalg \{\infty\}, \infty)$.

Define $K^{-n}(X) = \tilde{K}^0\bigl(S^n(X \amalg \{\infty\}, \infty)\bigr)$ for $n = 1, 2, \ldots$, where $S^n(-)$ is the n-fold suspension of pointed topological spaces. Then *Bott periodicity* gives canonical isomorphisms $K^{-n}(X) \cong K^{-n-2}(X)$, so we can extend to $K^n(X)$ for $n \in \mathbb{Z}$ periodic of period 2. Reducing from $\mathbb{Z}$ to $\mathbb{Z}_2$, we have the *complex K-theory* $K^*(X) = K^0(X) \oplus K^1(X)$, graded over $\mathbb{Z}_2$.

Write $\mathbb{1}_X \in K^0(X)$ for the class $[\![X \times \mathbb{C}]\!]$ of the trivial line bundle $X \times \mathbb{C} \to X$. Tensor product induces a product $\cdot : K^0(X) \times K^0(X) \to K^0(X)$ with $[\![E]\!] \cdot [\![F]\!] = [\![E \otimes_{\mathbb{C}} F]\!]$, which extends to a graded product on $K^*(X)$, and is commutative and associative with identity $\mathbb{1}_X$, making $K^*(X)$ into a $\mathbb{Z}_2$-graded commutative ring. All this is contravariantly functorial under continuous maps $f : X \to Y$.

For compact X, the Chern character gives isomorphisms

$$\mathrm{Ch}^0 : K^0(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^{\mathrm{even}}(X, \mathbb{Q}), \quad \mathrm{Ch}^1 : K^1(X) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow H^{\mathrm{odd}}(X, \mathbb{Q}).$$

There is an *Atiyah–Hirzebruch spectral sequence* $H^{i+2\mathbb{Z}}(X, \mathbb{Z}) \Rightarrow K^i(X)$, which can be used to compute $K^*(X)$.

Now suppose X is a compact, connected manifold. Then there is a morphism $\mathrm{rank} : K^0(X) \to \mathbb{Z}$ mapping $[\![E]\!] \mapsto \mathrm{rank}\, E$. If $\alpha \in K^0(X)$ and $N \in \mathbb{Z}$ with $2(\mathrm{rank}\, \alpha + N) \geqslant \dim X$ (the *stable range*) then there exists a complex vector bundle $E \to X$ with $\mathrm{rank}\, E = m = \mathrm{rank}\, \alpha + N$ and $\alpha = [\![E]\!] - N\, \mathbb{1}_X$ in $K^0(X)$, determined uniquely up to isomorphism by $\alpha + N\, \mathbb{1}_X$. Choosing a metric h on the fibres of E gives a principal $\mathrm{U}(m)$-bundle $P \to X$ with $[\![P]\!] = \alpha + N\, \mathbb{1}_X$, also determined uniquely up to isomorphism by $\alpha + N\, \mathbb{1}_X$.

Instead of working with complex vector bundles $E \to X$, we can work with real vector bundles, giving *real K-theory* $KO^*(X)$, or with quaternionic vector bundles, yielding *quaternionic K-theory* $KSp^*(X)$. In these cases *Bott periodicity* gives isomorphisms $KO^{-n}(X) \cong KO^{-n-8}(X)$, $KSp^{-n}(X) \cong KSp^{-n-8}(X)$, so $KO^*(X), KSp^*(X)$ are both graded over $\mathbb{Z}_8$. There are canonical isomorphisms $KSp^n(X) \cong KO^{n+4}(X)$ for all X, n. Here $KO^*(X)$ is a $\mathbb{Z}_8$-graded commutative ring, but we do not have a natural graded product on $KSp^*(X)$, as there is no good notion of tensor product of quaternionic vector bundles.

2.3.2 The moduli spaces $\mathcal{M}_\alpha^{\mathrm{U}}$ and orientation bundles $O_\alpha^{E_\bullet}$

Here are the moduli spaces $\mathcal{M}_\alpha^{\mathrm{U}}$ that will be our main tool:

Definition 2.16. Let X be a compact, connected manifold of dimension $n > 0$, and $E_\bullet$ an elliptic operator on X, as in Definition 1.2. As in §2.3.1 we define the complex K-theory group $K^0(X)$, with a morphism $\mathrm{rank} : K^0(X) \to \mathbb{Z}$.

For each $\alpha \in K^0(X)$, choose N_α in $\mathbb{Z}$ with $2(\operatorname{rank}\alpha + N_\alpha) \geqslant n$ (we call this the *stable range*), for example we could take N_α minimal under this condition. Set $m_\alpha = \operatorname{rank}\alpha + N_\alpha$, and choose a principal $\mathrm{U}(m_\alpha)$-bundle $P_\alpha \to X$ with $[\![P_\alpha]\!] = \alpha + N_\alpha \mathbb{1}_X$ in $K^0(X)$. As in §2.3.1, this determines P_α uniquely up to isomorphism. Now using the ideas on stabilization in §2.2.9, define a topological stack $\mathcal{M}_\alpha^{\mathrm{U}}$ by

$$\mathcal{M}_\alpha^{\mathrm{U}} = \varinjlim_{k\to\infty} \mathcal{M}_{P_\alpha\oplus\mathbb{C}^k}, \tag{2.16}$$

taking the direct limit using the maps $\Psi_{P_\alpha\oplus\mathbb{C}^k}^{P_\alpha\oplus\mathbb{C}^{k+1}} : \mathcal{M}_{P_\alpha\oplus\mathbb{C}^k} \to \mathcal{M}_{P_\alpha\oplus\mathbb{C}^{k+1}}$. Then $\mathcal{M}_\alpha^{\mathrm{U}}$ is independent of the choices of N_α, P_α up to isomorphism, and the isomorphisms are unique up to isotopy. Here the superscript 'U' stands for 'unitary', as we could define similar direct limit stacks $\mathcal{M}_\alpha^{\mathrm{SU}}, \mathcal{M}_\alpha^{\mathrm{SO}}, \mathcal{M}_\alpha^{\mathrm{Sp}}$ for principal bundles over $\mathrm{SU}(m), \mathrm{SO}(m), \mathrm{Sp}(m)$.

Note that $\mathcal{M}_\alpha^{\mathrm{U}}$ is connected, as each $\mathcal{M}_{P_\alpha\oplus\mathbb{C}^k}$ is connected.

By definition of direct limits, for each $k \geqslant 0$ we have a morphism

$$\Psi_{P_\alpha\oplus\mathbb{C}^k}^{P_\alpha\oplus\mathbb{C}^\infty} = \varinjlim_{l\to\infty} \Psi_{P_\alpha\oplus\mathbb{C}^k}^{P_\alpha\oplus\mathbb{C}^l} : \mathcal{M}_{P_\alpha\oplus\mathbb{C}^k} \longrightarrow \mathcal{M}_\alpha^U. \tag{2.17}$$

Define a principal $\mathbb{Z}_2$-bundle $\pi : O_\alpha^{E\bullet} \to \mathcal{M}_\alpha^{\mathrm{U}}$ by $O_\alpha^{E\bullet} = \varinjlim_{k\to\infty} O_{P_\alpha\oplus\mathbb{C}^k}^{E\bullet}$, taking the direct limit using (2.14). Then we have natural isomorphisms

$$(\Psi_{P_\alpha\oplus\mathbb{C}^k}^{P_\alpha\oplus\mathbb{C}^\infty})^*(O_\alpha^{E\bullet}) \cong O_{P_\alpha\oplus\mathbb{C}^k}^{E\bullet} \tag{2.18}$$

for all $k \geqslant 0$. In the obvious way, we say that $\mathcal{M}_\alpha^{\mathrm{U}}$ is *orientable* if $O_\alpha^{E\bullet}$ is trivializable, and an *orientation* for $\mathcal{M}_\alpha^{\mathrm{U}}$ is a trivialization $O_\alpha^{E\bullet} \cong \mathcal{M}_\alpha^{\mathrm{U}} \times \mathbb{Z}_2$.

Now the morphisms $\psi_P^{P\oplus\mathbb{C}}$ in (2.14) required a choice of orientation for $\det D$. However, we can make $O_\alpha^{E\bullet}$ independent of this choice by noting that if $N_\alpha + k$ is even then (2.18) is independent of the orientation on $\det D$, and restricting to k with $N_\alpha + k$ even in the limit defining $O_\alpha^{E\bullet}$.

We define the *n-orientation bundle* $\check{O}_\alpha^{E\bullet} \to \mathcal{M}_\alpha^{\mathrm{U}}$ by $\check{O}_\alpha^{E\bullet} = \varinjlim_{k\to\infty} \check{O}_{P_\alpha\oplus\mathbb{C}^k}^{E\bullet}$.

If $f_0, f_1 : S \to T$ are morphisms of topological spaces or topological stacks, we will use the notation '$f_0 \simeq f_1$' to mean that f_0, f_1 are *isotopic*, that is, there exists a morphism $F : S \times [0,1] \to T$ with $F|_{S\times\{i\}} = f_i$ for $i = 0, 1$. This is an equivalence relation.

We can think of $\mathcal{M}_\alpha^{\mathrm{U}}$ as a 'moduli space of complexes in class α in $K^0(X)$', at least up to homotopy. The definition depends on the choice of a principal $\mathrm{U}(m_\alpha)$-bundle $P_\alpha \to X$, which is unique up to isomorphism, but *not* up to canonical isomorphism, and the isomorphisms $\iota : P_\alpha \to P'_\alpha$ between two choices P_α, P'_α are natural only up to isotopy of such isomorphisms. Because of this, *most morphisms we construct to $\mathcal{M}_\alpha^{\mathrm{U}}$ are unique only up to isotopy.* This will not matter to us, as questions about orientations are independent of isotopies, and we only really care about $\mathcal{M}_\alpha^{\mathrm{U}}$ up to homotopy anyway.

Definition 2.17. In the situation of Definition 2.16, suppose $P \to X$ is a principal $\mathrm{U}(m)$-bundle for $m \geqslant 1$ with $[\![P]\!] = \alpha$ in $K^0(X)$. Choose $l \in \mathbb{Z}$ with

$l \geqslant \max(N_\alpha, 0)$, and set $k = l - N_\alpha$. Then $P \oplus \mathbb{C}^l$ and $P_\alpha \oplus \mathbb{C}^k$ are both principal $\mathrm{U}(l+m)$-bundles with $[\![P \oplus \mathbb{C}^l]\!] = [\![P_\alpha \oplus \mathbb{C}^k]\!] = \alpha + l\,\mathbb{1}_X$ in $K^0(X)$, so as we are in the stable range there is an isomorphism $\mu_{P,l} : P \oplus \mathbb{C}^l \to P_\alpha \oplus \mathbb{C}^k$, which is unique up to isotopy of such isomorphisms. Define a 'stabilization' morphism $\Sigma_P^{\mathrm{U}} : \mathcal{M}_P \to \mathcal{M}_\alpha^{\mathrm{U}}$ by the commutative diagram

$$\begin{tikzcd} \mathcal{M}_P \arrow[r, "\Sigma_P^{\mathrm{U}}"] \arrow[d, "\Psi_P^{P\oplus\mathbb{C}^l}"] & \mathcal{M}_\alpha^{\mathrm{U}} \\ \mathcal{M}_{P\oplus\mathbb{C}^l} \arrow[r, "(\mu_{P,l})_*"] & \mathcal{M}_{P_\alpha\oplus\mathbb{C}^k}. \arrow[u, "\Psi_{P_\alpha\oplus\mathbb{C}^k}^{P_\alpha\oplus\mathbb{C}^\infty}"'] \end{tikzcd} \tag{2.19}$$

Then Σ_P^{U} is only unique up to isotopy, as $\mu_{P,l}$ is. We have an isomorphism of principal $\mathbb{Z}_2$-bundles

$$\begin{aligned} \sigma_P^{E_\bullet} = ((\mu_{P,l})_* \circ \Psi_P^{P\oplus\mathbb{C}^l})^*(\psi_{P_\alpha\oplus\mathbb{C}^k}^{P_\alpha\oplus\mathbb{C}^\infty}) \circ (\Psi_P^{P\oplus\mathbb{C}^l})^*((\mu_{P,l})_*) \circ \psi_P^{P\oplus\mathbb{C}^l} : \\ O_P^{E_\bullet} \xrightarrow{\cong} (\Sigma_P^{\mathrm{U}})^*(O_\alpha^{E_\bullet}). \end{aligned} \tag{2.20}$$

Up to isotopies of Σ_P^{U}, this is independent of choices. The analogue works for n-orientation bundles, giving $\check{\sigma}_P^{E_\bullet} : \check{O}_P^{E_\bullet} \xrightarrow{\cong} (\Sigma_P^{\mathrm{U}})^*(\check{O}_\alpha^{E_\bullet})$.

The importance of (2.20) is that it shows that if $\mathcal{M}_\alpha^{\mathrm{U}}$ is orientable then $\mathcal{M}_P$ is orientable for any principal $\mathrm{U}(m)$-bundle $P \to X$ with $[\![P]\!] = \alpha$ in $K^0(X)$, and an orientation for $\mathcal{M}_\alpha^{\mathrm{U}}$ induces orientations on $\mathcal{M}_P$ for all such P. Hence, if we can construct orientations on $\mathcal{M}_\alpha^{\mathrm{U}}$ for all $\alpha \in K^0(X)$, we obtain orientations on $\mathcal{M}_P$ for all $\mathrm{U}(m)$-bundles $P \to X$, for all $m \geqslant 1$.

2.3.3 The Euler form $\chi^{E_\bullet} : K^0(X) \times K^0(X) \to \mathbb{Z}$

Definition 2.18. In the situation of Definition 2.16, let $P_1 \to X$ and $P_2 \to X$ be principal $\mathrm{U}(m_1)$- and $\mathrm{U}(m_2)$-bundles with $[\![P_1]\!] = \alpha$ and $[\![P_2]\!] = \beta$ in $K^0(X)$. Choose connections $\nabla_{P_1}, \nabla_{P_2}$ on P_1, P_2. Write $\mathbb{C}^{m_1 m_2} = \mathbb{C}^{m_1} \otimes \overline{\mathbb{C}^{m_2}}$, and let $\rho_{12} : \mathrm{U}(m_1) \times \mathrm{U}(m_2) \to \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^{m_1 m_2})$ be the tensor product of the natural representation of $\mathrm{U}(m_1)$ on $\mathbb{C}^{m_1}$ and the complex conjugate natural representation of $\mathrm{U}(m_2)$ on $\mathbb{C}^{m_2}$. Then we have a complex vector bundle $\rho_{12}(P_1 \times_X P_2) = (P_1 \times_X P_2 \times \mathbb{C}^{m_1 m_2})/\mathrm{U}(m_1) \times \mathrm{U}(m_2)$ over X with fibre $\mathbb{C}^{m_1 m_2}$, with a connection $\nabla_{\rho_{12}(P_1 \times_X P_2)}$ induced by $\nabla_{P_1}, \nabla_{P_2}$. Thus as in Definition 1.2 may form the twisted complex elliptic operator

$$\begin{aligned} D^{\nabla_{\rho_{12}(P_1\times_X P_2)}} &: \Gamma^\infty(\rho_{12}(P_1 \times_X P_2) \otimes E_0) \longrightarrow \Gamma^\infty(\rho_{12}(P_1 \times_X P_2) \otimes E_1), \\ D^{\nabla_{\rho_{12}(P_1\times_X P_2)}} &: e \longmapsto \sum_{i=0}^{d} (\mathrm{id}_{\rho_{12}(P_1\times_X P_2)} \otimes a_i) \cdot \nabla^i_{\rho_{12}(P_1\times_X P_2)\otimes E_0} e. \end{aligned}$$

The complex index $\mathrm{ind}_{\mathbb{C}}(D^{\nabla_{\rho_{12}(P_1\times_X P_2)}})$ is independent of $\nabla_{P_1}, \nabla_{P_2}$, and so depends only on $X, E_\bullet, \alpha, \beta$. Replacing P_1 or P_2 by a direct sum $P_1' \oplus P_1''$, $P_2' \oplus P_2''$ gives a direct sum of the corresponding elliptic operators. Hence

$\mathrm{ind}_{\mathbb{C}}(D^{\nabla_{\rho_{12}(P_1\times_X P_2)}})$ is biadditive in α,β. Therefore there exists a unique biadditive map $\chi^{E_\bullet}:K^0(X)\times K^0(X)\to\mathbb{Z}$ which we call the *Euler form*, such that $\mathrm{ind}_{\mathbb{C}}(D^{\nabla_{\rho_{12}(P_1\times_X P_2)}})=\chi^{E_\bullet}(\alpha,\beta)$ for all P_1,P_2,α,β as above.

Swapping round P_1,P_2, m_1,m_2 and α,β replaces $\mathbb{C}^{m_1}\otimes\overline{\mathbb{C}^{m_2}}$ by $\mathbb{C}^{m_2}\otimes\overline{\mathbb{C}^{m_1}}$, and so complex conjugates $\rho_{12}(P_1\times_X P_2)$ and $D^{\nabla_{\rho_{12}(P_1\times_X P_2)}}$, which does not change the index. Hence $\chi^{E_\bullet}(\alpha,\beta)=\chi^{E_\bullet}(\beta,\alpha)$ for all $\alpha,\beta\in K^0(X)$.

When $P_1=P_2=P$ and $\alpha=\beta$, we have $\rho_{12}(P\times_X P)\cong\mathrm{Ad}(P)\otimes_{\mathbb{R}}\mathbb{C}$, so the complex index $\mathrm{ind}_{\mathbb{C}}(D^{\nabla_{\rho_{12}(P\times_X P)}})$ equals the real index $\mathrm{ind}_{\mathbb{R}}(D^{\nabla_{\mathrm{Ad}(P)}})$. Therefore for $\mathrm{ind}_P^{E_\bullet}$ as in Definition 1.2, we have

$$\mathrm{ind}_P^{E_\bullet}=\chi^{E_\bullet}([\![P]\!],[\![P]\!]). \tag{2.21}$$

If $P_1\to X$ and $P_2\to X$ are principal $\mathrm{U}(m_1)$- and $\mathrm{U}(m_2)$-bundles with $[\![P_1]\!]=\alpha$ and $[\![P_2]\!]=\beta$, then Example 2.10 defines an isomorphism $\phi_{P_1,P_2}:O_{P_1}^{E_\bullet}\boxtimes_{\mathbb{Z}_2}O_{P_2}^{E_\bullet}\to\Phi^*_{P_1,P_2}(O_{P_1\oplus P_2}^{E_\bullet})$, and (2.12) relates $\phi_{P_2,P_1},\phi_{P_1,P_2}$ under the isomorphism $\mathcal{M}_{P_2}\times\mathcal{M}_{P_1}\cong\mathcal{M}_{P_1}\times\mathcal{M}_{P_2}$. The first sign in (2.12) is written in terms of $\chi^{E_\bullet}$ by (2.21), and for the second we have

$$\tfrac12\,\mathrm{ind}^{E_\bullet}_{P_1\times_X P_2,\rho}=\tfrac12\,\mathrm{ind}_{\mathbb{R}}(D^{\nabla_{\rho_{12}(P_1\times_X P_2)}})=\mathrm{ind}_{\mathbb{C}}(D^{\nabla_{\rho_{12}(P_1\times_X P_2)}})=\chi^{E_\bullet}(\alpha,\beta).$$

Hence (2.12) may be rewritten

$$\phi_{P_2,P_1}=(-1)^{\chi^{E_\bullet}(\alpha,\beta)+\chi^{E_\bullet}(\alpha,\alpha)\chi^{E_\bullet}(\beta,\beta)}\cdot\phi_{P_1,P_2}. \tag{2.22}$$

2.3.4 The direct sum morphisms $\Phi^{\mathrm{U}}_{\alpha,\beta}:\mathcal{M}^{\mathrm{U}}_\alpha\times\mathcal{M}^{\mathrm{U}}_\beta\to\mathcal{M}^{\mathrm{U}}_{\alpha+\beta}$

Definition 2.19. In the situation of Definition 2.16, let $\alpha,\beta\in K^0(X)$. We will relate $\mathcal{M}^{\mathrm{U}}_\alpha,\mathcal{M}^{\mathrm{U}}_\beta,\mathcal{M}^{\mathrm{U}}_{\alpha+\beta}$ and $O^{E_\bullet}_\alpha,O^{E_\bullet}_\beta,O^{E_\bullet}_{\alpha+\beta}$. Choose $k_\alpha,k_\beta\geqslant0$ such that $k_{\alpha+\beta}:=N_\alpha+k_\alpha+N_\beta+k_\beta-N_{\alpha+\beta}\geqslant0$. Then $(P_\alpha\oplus\mathbb{C}^{k_\alpha})\oplus(P_\beta\oplus\mathbb{C}^{k_\beta})$ and $P_{\alpha+\beta}\oplus\mathbb{C}^{k_{\alpha+\beta}}$ are principal $\mathrm{U}(m)$-bundles on X for $m=\mathrm{rank}(\alpha+\beta)+N_{\alpha+\beta}+k_{\alpha+\beta}$, with class $\alpha+\beta+(N_{\alpha+\beta}+k_{\alpha+\beta})\mathbb{1}_X$ in $K^0(X)$. Thus there is an isomorphism of principal $\mathrm{U}(m)$-bundles

$$\nu^{k_\alpha,k_\beta}_{\alpha,\beta}:(P_\alpha\oplus\mathbb{C}^{k_\alpha})\oplus(P_\beta\oplus\mathbb{C}^{k_\beta})\longrightarrow P_{\alpha+\beta}\oplus\mathbb{C}^{k_{\alpha+\beta}},$$

which is unique up to isotopy. So by (2.10) we have a morphism

$$(\nu^{k_\alpha,k_\beta}_{\alpha,\beta})_*\circ\Phi_{P_\alpha\oplus\mathbb{C}^{k_\alpha},P_\beta\oplus\mathbb{C}^{k_\beta}}:\mathcal{M}_{P_\alpha\oplus\mathbb{C}^{k_\alpha}}\times\mathcal{M}_{P_\beta\oplus\mathbb{C}^{k_\beta}}\longrightarrow\mathcal{M}_{P_{\alpha+\beta}\oplus\mathbb{C}^{k_{\alpha+\beta}}}.$$

For each $k\geqslant0$ we then define isomorphisms

$$\nu^{k_\alpha+k,k_\beta+k}_{\alpha,\beta}:(P_\alpha\oplus\mathbb{C}^{k_\alpha+k})\oplus(P_\beta\oplus\mathbb{C}^{k_\beta+k})\longrightarrow P_{\alpha+\beta}\oplus\mathbb{C}^{k_{\alpha+\beta+2k}},$$

by starting with $\nu^{k_\alpha,k_\beta}_{\alpha,\beta}$, and identifying the additional $\mathbb{C}^k,\mathbb{C}^k,\mathbb{C}^{2k}$ factors in the vector bundles associated to $P_\alpha\oplus\mathbb{C}^{k_\alpha+k},P_\beta\oplus\mathbb{C}^{k_\beta+k},P_{\alpha+\beta}\oplus\mathbb{C}^{k_{\alpha+\beta}+2k}$ by

mapping $(x_1,\ldots,x_k)_\alpha,(y_1,\ldots,y_k)_\beta$ to $(x_1,y_1,\ldots,x_k,y_k)_{\alpha+\beta}$. This is compatible with the gluing maps $\Psi^{P_\alpha\oplus\mathbb{C}^{k+1}}_{P_\alpha\oplus\mathbb{C}^k}$ used to define (2.16) for $\alpha,\beta,\alpha+\beta$. Hence properties of the direct limits $\mathcal{M}^{\mathrm{U}}_\alpha,\mathcal{M}^{\mathrm{U}}_\beta$ give a unique morphism

$$\Phi^{\mathrm{U}}_{\alpha,\beta}:\mathcal{M}^{\mathrm{U}}_\alpha\times\mathcal{M}^{\mathrm{U}}_\beta\longrightarrow\mathcal{M}^{\mathrm{U}}_{\alpha+\beta} \tag{2.23}$$

such that the following commutes for all $k\geqslant 0$:

$$\begin{tikzcd}
\mathcal{M}_{P_\alpha\oplus\mathbb{C}^{k_\alpha+k}}\times\mathcal{M}_{P_\beta\oplus\mathbb{C}^{k_\beta+k}} \arrow[r, "\Phi_{P_\alpha\oplus\mathbb{C}^{k_\alpha+k},\,P_\beta\oplus\mathbb{C}^{k_\beta+k}}"'] \arrow[d, "\Psi^{P_\alpha\oplus\mathbb{C}^\infty}_{P_\alpha\oplus\mathbb{C}^{k_\alpha+k}}\times\Psi^{P_\beta\oplus\mathbb{C}^\infty}_{P_\beta\oplus\mathbb{C}^{k_\beta+k}}"] & \mathcal{M}_{(P_\alpha\oplus\mathbb{C}^{k_\alpha+k})\oplus(P_\beta\oplus\mathbb{C}^{k_\beta+k})} \arrow[r, "(\nu^{k_\alpha+k,k_\beta+k}_{\alpha,\beta})_*"'] & \mathcal{M}_{P_{\alpha+\beta}\oplus\mathbb{C}^{k_{\alpha+\beta+2k}}} \arrow[d, "\Psi^{P_{\alpha+\beta}\oplus\mathbb{C}^\infty}_{P_{\alpha+\beta}\oplus\mathbb{C}^{k_{\alpha+\beta+2k}}}"] \\
\mathcal{M}^{\mathrm{U}}_\alpha\times\mathcal{M}^{\mathrm{U}}_\beta \arrow[rr, "\Phi^{\mathrm{U}}_{\alpha,\beta}"] & & \mathcal{M}^{\mathrm{U}}_{\alpha+\beta}.
\end{tikzcd} \tag{2.24}$$

This $\Phi^{\mathrm{U}}_{\alpha,\beta}$ is independent of choices up to isotopy of such morphisms.

Similarly, by properties of the direct limits $O^{E_\bullet}_\alpha,O^{E_\bullet}_\beta$ there is a unique isomorphism of principal $\mathbb{Z}_2$-bundles on $\mathcal{M}^{\mathrm{U}}_\alpha\times\mathcal{M}^{\mathrm{U}}_\beta$

$$\phi^{E_\bullet}_{\alpha,\beta}:O^{E_\bullet}_\alpha\boxtimes_{\mathbb{Z}_2}O^{E_\bullet}_\beta\xrightarrow{\cong}(\Phi^{\mathrm{U}}_{\alpha,\beta})^*(O^{E_\bullet}_{\alpha+\beta}), \tag{2.25}$$

such that the following commutes on $\mathcal{M}_{P_\alpha\oplus\mathbb{C}^{k_\alpha+k}}\times\mathcal{M}_{P_\beta\oplus\mathbb{C}^{k_\beta+k}}$ for $k\geqslant 0$, where '$(\cdots)^*$' is pullback to $\mathcal{M}_{P_\alpha\oplus\mathbb{C}^{k_\alpha+k}}\times\mathcal{M}_{P_\beta\oplus\mathbb{C}^{k_\beta+k}}$ by the morphisms in (2.24):

$$\begin{tikzcd}
O^{E_\bullet}_{P_\alpha\oplus\mathbb{C}^{k_\alpha+k}}\boxtimes_{\mathbb{Z}_2}O^{E_\bullet}_{P_\beta\oplus\mathbb{C}^{k_\beta+k}} \arrow[r, "\phi_{P_\alpha\oplus\mathbb{C}^{k_\alpha+k},\,P_\beta\oplus\mathbb{C}^{k_\beta+k}}"'] \arrow[d, "\psi^{P_\alpha\oplus\mathbb{C}^\infty}_{P_\alpha\oplus\mathbb{C}^{k_\alpha+k}}\boxtimes\psi^{P_\beta\oplus\mathbb{C}^\infty}_{P_\beta\oplus\mathbb{C}^{k_\beta+k}}"] & (\cdots)^*(O^{E_\bullet}_{(P_\alpha\oplus\mathbb{C}^{k_\alpha+k})\oplus(P_\beta\oplus\mathbb{C}^{k_\beta+k})}) \arrow[r, "(\cdots)^*((\nu^{k_\alpha+k,k_\beta+k}_{\alpha,\beta})_*)"'] & (\cdots)^*(O^{E_\bullet}_{P_{\alpha+\beta}\oplus\mathbb{C}^{k_{\alpha+\beta+2k}}}) \arrow[d, "(\cdots)^*(\psi^{P_{\alpha+\beta}\oplus\mathbb{C}^\infty}_{P_{\alpha+\beta}\oplus\mathbb{C}^{k_{\alpha+\beta+2k}}})"] \\
(\cdots)^*(O^{E_\bullet}_\alpha\boxtimes_{\mathbb{Z}_2}O^{E_\bullet}_\beta) \arrow[rr, "(\cdots)^*(\phi^{E_\bullet}_{\alpha,\beta})"] & & (\cdots)^*(O^{E_\bullet}_{\alpha+\beta}).
\end{tikzcd}$$

Up to isotopies of $\Phi^{\mathrm{U}}_{\alpha,\beta}$, this $\phi^{E_\bullet}_{\alpha,\beta}$ is independent of choices. The analogue works for n-orientation bundles.

Now suppose $P_1\to X$, $P_2\to X$ are principal $\mathrm{U}(m_1)$- and $\mathrm{U}(m_2)$-bundles with $[\![P_1]\!]=\alpha$ and $[\![P_2]\!]=\beta$ in $K^0(X)$. Consider the diagram

$$\begin{tikzcd}
\mathcal{M}_{P_1}\times\mathcal{M}_{P_2} \arrow[rr, "\Phi_{P_1,P_2}"'] \arrow[d, "\Sigma^{\mathrm{U}}_{P_1}\times\Sigma^{\mathrm{U}}_{P_2}"] \arrow[dr, phantom, "\simeq"] & & \mathcal{M}_{P_1\oplus P_2} \arrow[d, "\Sigma^{\mathrm{U}}_{P_1\oplus P_2}"'] \\
\mathcal{M}^{\mathrm{U}}_\alpha\times\mathcal{M}^{\mathrm{U}}_\beta \arrow[rr, "\Phi^{\mathrm{U}}_{\alpha,\beta}"] & & \mathcal{M}^{\mathrm{U}}_{\alpha+\beta},
\end{tikzcd} \tag{2.26}$$

using the notation of Example 2.10 and Definition 2.17. This need not commute, but it does commute up to isotopy, that is, $\Phi^{\mathrm{U}}_{\alpha,\beta}\circ(\Sigma^{\mathrm{U}}_{P_1}\times\Sigma^{\mathrm{U}}_{P_2})\simeq\Sigma^{\mathrm{U}}_{P_1\oplus P_2}\circ\Phi_{P_1,P_2}$,

which we indicate by the symbol '$\simeq$' in the centre of (2.26). To see this, note that $\Sigma^{\mathrm{U}}_{P_1}, \Sigma^{\mathrm{U}}_{P_2}, \Sigma^{\mathrm{U}}_{P_1\oplus P_2}, \Phi^{\mathrm{U}}_{\alpha,\beta}$ are natural up to isotopy, and there are possible choices of them which make (2.26) strictly commute: we take $P_\alpha = P_1 \oplus \mathbb{C}^{N_\alpha}$, $P_\beta = P_2 \oplus \mathbb{C}^{N_\beta}$, $P_{\alpha+\beta} = P_1 \oplus P_2 \oplus \mathbb{C}^{N_{\alpha+\beta}}$ where $N_\alpha, N_\beta \geqslant 0$ and $N_{\alpha+\beta} = N_\alpha + N_\beta$, and then define $\Sigma^{\mathrm{U}}_{P_1}, \Sigma^{\mathrm{U}}_{P_2}, \Sigma^{\mathrm{U}}_{P_1\oplus P_2}$ using $\mu_{P,l} = \mathrm{id}$ in (2.19), and for $\Phi^{\mathrm{U}}_{\alpha,\beta}$ above we take $k_\alpha = k_\beta = 0$ and $\nu^{0,0}_{\alpha,\beta} = \mathrm{id}$.

Next consider the diagram of principal $\mathbb{Z}_2$-bundles on $\mathcal{M}_{P_1} \times \mathcal{M}_{P_2}$:

$$\begin{tikzcd}
O^{E_\bullet}_{P_1} \boxtimes_{\mathbb{Z}_2} O^{E_\bullet}_{P_2} \arrow[rr, "\phi_{P_1,P_2}"'] \arrow[dd, "\sigma^{E_\bullet}_{P_1}\boxtimes\sigma^{E_\bullet}_{P_2}"] \arrow[drr, phantom, "\simeq"] & & \Phi^*_{P_1,P_2}(O^{E_\bullet}_{P_1\oplus P_2}) \arrow[d, "(\Phi_{P_1,P_2})^*(\sigma^{E_\bullet}_{P_1\oplus P_2})"] \\
& & (\Sigma^{\mathrm{U}}_{P_1\oplus P_2}\circ\Phi_{P_1,P_2})^*(O^{E_\bullet}_{\alpha+\beta}) \arrow[d, dotted, "\simeq"'] \\
(\Sigma^{\mathrm{U}}_{P_1})^*(O^{E_\bullet}_\alpha) \boxtimes_{\mathbb{Z}_2} (\Sigma^{\mathrm{U}}_{P_1})^*(O^{E_\bullet}_\beta) \arrow[rr, "(\Sigma^{\mathrm{U}}_{P_1}\times\Sigma^{\mathrm{U}}_{P_2})^*(\phi^{E_\bullet}_{\alpha,\beta})"] & & (\Phi^{\mathrm{U}}_{\alpha,\beta}\circ(\Sigma^{\mathrm{U}}_{P_1}\times\Sigma^{\mathrm{U}}_{P_2}))^*(O^{E_\bullet}_{\alpha+\beta}).
\end{tikzcd} \tag{2.27}$$

Here the arrow '$\dashrightarrow$' links pullbacks of $O^{E_\bullet}_{P_1\oplus P_2}$ by the isotopic morphisms in (2.26). If we deform $\Sigma^{\mathrm{U}}_{P_1}, \ldots, \Phi^{\mathrm{U}}_{\alpha,\beta}$ by isotopies to the explicit choices above making (2.26) strictly commute, then comparing definitions shows that (2.27) also strictly commutes. We denote this by the symbol '$\simeq$' in the centre of (2.27), which means that the diagram is isotopic to a strictly commutative diagram via isotopies of the morphisms involved, and say that (2.27) *isotopy-commutes*.

Equation (2.27), and similar isotopy-commutative diagrams, have useful consequences for orientations. If $\omega_{\alpha+\beta}$ is an orientation on $\mathcal{M}^{\mathrm{U}}_{\alpha+\beta}$ then as (2.27) isotopy-commutes, and orientations are constant under isotopies, we have

$$\begin{aligned}
&(\sigma^{E_\bullet}_{P_1} \boxtimes \sigma^{E_\bullet}_{P_2})^* \circ [(\Sigma^{\mathrm{U}}_{P_1} \times \Sigma^{\mathrm{U}}_{P_2})^*(\phi^{E_\bullet}_{\alpha,\beta})]^*[(\Phi^{\mathrm{U}}_{\alpha,\beta} \circ (\Sigma^{\mathrm{U}}_{P_1} \times \Sigma^{\mathrm{U}}_{P_2}))^*(\omega_{\alpha+\beta})] \\
&\quad = \phi^*_{P_1,P_2} \circ [(\Phi_{P_1,P_2})^*(\sigma^{E_\bullet}_{P_1\oplus P_2})]^*[(\Phi^{\mathrm{U}}_{\alpha,\beta} \circ (\Sigma^{\mathrm{U}}_{P_1} \times \Sigma^{\mathrm{U}}_{P_2}))^*(\omega_{\alpha+\beta})].
\end{aligned}$$

2.3.5 Commutativity and associativity properties of the $\Phi^{\mathrm{U}}_{\alpha,\beta}$

Work in the situation of Definition 2.16. Let $\alpha, \beta, \gamma \in K^0(X)$, and consider the diagrams, where $\sigma : \mathcal{M}^{\mathrm{U}}_\alpha \times \mathcal{M}^{\mathrm{U}}_\beta \to \mathcal{M}^{\mathrm{U}}_\beta \times \mathcal{M}^{\mathrm{U}}_\alpha$ exchanges the two factors:

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{U}}_\alpha \times \mathcal{M}^{\mathrm{U}}_\beta \arrow[d, "\sigma"] \arrow[drr, bend left, "\Phi^{\mathrm{U}}_{\alpha,\beta}"] & & \\
\mathcal{M}^{\mathrm{U}}_\beta \times \mathcal{M}^{\mathrm{U}}_\alpha \arrow[rr, "\Phi^{\mathrm{U}}_{\beta,\alpha}"] \arrow[rr, phantom, shift right=3ex, "\simeq"] & & \mathcal{M}^{\mathrm{U}}_{\alpha+\beta},
\end{tikzcd} \tag{2.28}$$

$$\begin{tikzcd}
\mathcal{M}^{\mathrm{U}}_\alpha \times \mathcal{M}^{\mathrm{U}}_\beta \times \mathcal{M}^{\mathrm{U}}_\gamma \arrow[rr, "\mathrm{id}_{\mathcal{M}^{\mathrm{U}}_\alpha}\times\Phi^{\mathrm{U}}_{\beta,\gamma}"'] \arrow[d, "\Phi^{\mathrm{U}}_{\alpha,\beta}\times\mathrm{id}_{\mathcal{M}^{\mathrm{U}}_\gamma}"] \arrow[drr, phantom, "\simeq"] & & \mathcal{M}^{\mathrm{U}}_\alpha \times \mathcal{M}^{\mathrm{U}}_{\beta+\gamma} \arrow[d, "\Phi^{\mathrm{U}}_{\alpha,\beta+\gamma}"'] \\
\mathcal{M}^{\mathrm{U}}_{\alpha+\beta} \times \mathcal{M}^{\mathrm{U}}_\gamma \arrow[rr, "\Phi^{\mathrm{U}}_{\alpha+\beta,\gamma}"] & & \mathcal{M}^{\mathrm{U}}_{\alpha+\beta+\gamma}.
\end{tikzcd} \tag{2.29}$$

Equations (2.28)–(2.29) need not commute, as $\Phi^{\mathrm{U}}_{\alpha,\beta}, \Phi^{\mathrm{U}}_{\beta,\alpha}, \ldots$ are only natural up to isotopy. But since the morphisms Φ_{P_1,P_2} of Example 2.10 are commutative

and associative, as for (2.26) we can make explicit choices in the definitions of $\Phi^{\mathrm{U}}_{\alpha,\beta}, \Phi^{\mathrm{U}}_{\beta,\alpha}, \ldots$ such that (2.28)–(2.29) strictly commute. Hence (2.28)–(2.29) are isotopy-commutative.

We can also consider how the orientation bundle isomorphisms $\phi^{E_\bullet}_{\alpha,\beta}$ in (2.25) behave under (2.29), through the isotopy equations

$$\sigma^*(\phi^{E_\bullet}_{\beta,\alpha}) \simeq (-1)^{\chi^{E_\bullet}(\alpha,\beta)+\chi^{E_\bullet}(\alpha,\alpha)\chi^{E_\bullet}(\beta,\beta)} \cdot \phi^{E_\bullet}_{\alpha,\beta}, \tag{2.30}$$

$$\begin{aligned}(\Phi^{\mathrm{U}}_{\alpha,\beta} \times \mathrm{id}_{\mathcal{M}^{\mathrm{U}}_\gamma})^*(\phi^{E_\bullet}_{\alpha+\beta,\gamma}) \circ (\phi^{E_\bullet}_{\alpha,\beta} \boxtimes \mathrm{id}_{O^{E_\bullet}_\gamma}) \\ \simeq (\mathrm{id}_{\mathcal{M}^{\mathrm{U}}_\alpha} \times \Phi^{\mathrm{U}}_{\beta,\gamma})^*(\phi^{E_\bullet}_{\alpha,\beta+\gamma}) \circ (\mathrm{id}_{O^{E_\bullet}_\alpha} \boxtimes \phi^{E_\bullet}_{\beta,\gamma}).\end{aligned} \tag{2.31}$$

Here '$\simeq$' in (2.30) means that there exists a choice of isotopy between the two routes round (2.28) which deforms the orientation bundle isomorphisms in (2.30) so that they become equal with the given sign, and (2.31) is interpreted using isotopies in (2.29) in the same way, with trivial sign. We prove (2.30)–(2.31) by noting that for the explicit choices in the definitions of $\Phi^{\mathrm{U}}_{\alpha,\beta}, \ldots$ above making (2.28)–(2.29) strictly commute, then (2.12), (2.22) and (2.27) show (2.30) holds strictly, and (2.13) and (2.27) show (2.31) holds strictly.

As for (2.27), equations (2.30)–(2.31) imply identities on pullbacks of orientations, which are unchanged by isotopies. The analogues hold for n-orientations.

2.3.6 The isomorphism $\pi_1(\mathcal{M}^{\mathrm{U}}_\alpha) \cong K^1(X)$, and orientability

Proposition 2.20. *In Definition* 2.16, *for each* $\alpha \in K^0(X)$, *we have:*

- **(a)** $\mathcal{M}^{\mathrm{U}}_\alpha$ *is homotopic to* $\mathcal{M}^{\mathrm{U}}_0$.
- **(b)** $\mathcal{M}^{\mathrm{U}}_\alpha$ *is orientable if and only if* $\mathcal{M}^{\mathrm{U}}_0$ *is orientable.*
- **(c)** *The fundamental group is* $\pi_1(\mathcal{M}^{\mathrm{U}}_\alpha) \cong K^1(X)$, *for* $K^1(X)$ *as in* §2.3.1.

Proof. For (a), let $\alpha \in K^0(X)$, choose points $p \in \mathcal{M}^{\mathrm{U}}_\alpha$, $q \in \mathcal{M}^{\mathrm{U}}_{-\alpha}$, and set $r = \Phi^{U}_{\alpha,-\alpha}(p,q)$ and $s = \Phi^{U}_{-\alpha,\alpha}(p,q)$ in $\mathcal{M}^{\mathrm{U}}_0$. Then by (2.23) we have morphisms

$$\begin{aligned}\Phi^{\mathrm{U}}_{0,\alpha}|_{\mathcal{M}^{\mathrm{U}}_0\times\{p\}} : \mathcal{M}^{\mathrm{U}}_0 \cong \mathcal{M}^{\mathrm{U}}_0 \times \{p\} \longrightarrow \mathcal{M}^{\mathrm{U}}_\alpha, \\ \Phi^{\mathrm{U}}_{\alpha,-\alpha}|_{\mathcal{M}^{\mathrm{U}}_\alpha\times\{q\}} : \mathcal{M}^{\mathrm{U}}_\alpha \cong \mathcal{M}^{\mathrm{U}}_\alpha \times \{q\} \longrightarrow \mathcal{M}^{\mathrm{U}}_0.\end{aligned}$$

By isotopy-commutativity of (2.29), we have isotopies

$$\begin{aligned}\Phi^{\mathrm{U}}_{\alpha,-\alpha}|_{\mathcal{M}^{\mathrm{U}}_\alpha\times\{q\}} \circ \Phi^{\mathrm{U}}_{0,\alpha}|_{\mathcal{M}^{\mathrm{U}}_0\times\{p\}} \simeq \Phi_{0,0}|_{\mathcal{M}^{\mathrm{U}}_0\times\{r\}} : \mathcal{M}^{\mathrm{U}}_0 \longrightarrow \mathcal{M}^{\mathrm{U}}_0, \\ \Phi^{\mathrm{U}}_{0,\alpha}|_{\mathcal{M}^{\mathrm{U}}_0\times\{p\}} \circ \Phi^{\mathrm{U}}_{\alpha,-\alpha}|_{\mathcal{M}^{\mathrm{U}}_\alpha\times\{q\}} \simeq \Phi_{\alpha,0}|_{\mathcal{M}^{\mathrm{U}}_\alpha\times\{s\}} : \mathcal{M}^{\mathrm{U}}_\alpha \longrightarrow \mathcal{M}^{\mathrm{U}}_\alpha.\end{aligned}$$

But by isotoping $\Phi_{0,0}|_{\mathcal{M}^{\mathrm{U}}_0\times\{r\}}, \Phi_{\alpha,0}|_{\mathcal{M}^{\mathrm{U}}_\alpha\times\{s\}}$ to the operations of direct sum with the zero vector bundle with trivial connection, we see that $\Phi_{0,0}|_{\mathcal{M}^{\mathrm{U}}_0\times\{r\}} \simeq \mathrm{id}_{\mathcal{M}^{\mathrm{U}}_0}$ and $\Phi_{\alpha,0}|_{\mathcal{M}^{\mathrm{U}}_\alpha\times\{s\}} \simeq \mathrm{id}_{\mathcal{M}^{\mathrm{U}}_\alpha}$. Therefore $\Phi^{\mathrm{U}}_{0,\alpha}|_{\mathcal{M}^{\mathrm{U}}_0\times\{p\}}$ and $\Phi^{\mathrm{U}}_{\alpha,-\alpha}|_{\mathcal{M}^{\mathrm{U}}_\alpha\times\{q\}}$ are homotopy inverses, and $\mathcal{M}^{\mathrm{U}}_\alpha$ is homotopic to $\mathcal{M}^{\mathrm{U}}_0$.

For (b), if ω_α is an orientation for $\mathcal{M}^{\mathrm{U}}_\alpha$ then $(\phi^{E_\bullet}_{0,\alpha})^{-1}|_{\mathcal{M}^{\mathrm{U}}_0\times\{p\}}(\omega_\alpha)$ from (2.25) is an orientation for $O^{E_\bullet}_0 \otimes_{\mathbb{Z}_2} \bigl(O^{E_\bullet}_\alpha|_p\bigr)$, so choosing an identification $O^{E_\bullet}_\alpha|_p \cong \mathbb{Z}_2$

gives an orientation for $\mathcal{M}_0^{\mathrm{U}}$. Thus, if $\mathcal{M}_\alpha^{\mathrm{U}}$ is orientable, then $\mathcal{M}_0^{\mathrm{U}}$ is orientable. Similarly, if $\mathcal{M}_0^{\mathrm{U}}$ is orientable then $\mathcal{M}_\alpha^{\mathrm{U}}$ is orientable.

For (c), by (a) it is enough to show that $\pi_1(\mathcal{M}_0^{\mathrm{U}}) \cong K^1(X)$. Write $\mathcal{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ with base point $1 \in \mathcal{S}^1$. Then $K^1(X) \cong K^0(X \times \mathcal{S}^1, X \times \{1\})$. Thus, using ideas on stabilization in §2.2.9, after taking direct sum with the trivial bundle with fibre $\mathbb{C}^{N_0}$ for $2N_0 \geqslant \dim X + 1$, we can identify classes in $K^1(X)$ with isomorphism classes of pairs (P, ι), where $P \to X \times \mathcal{S}^1$ is a principal $\mathrm{U}(N_0)$-bundle, and $\iota : P|_{X\times\{1\}} \to X \times \mathrm{U}(N_0)$ is a trivialization of P on $X \times \{1\}$.

Consider partial connections ∇_P^{pa} on P giving a connection in the X direction in $X \times \mathcal{S}^1$, such that ι identifies $\nabla_P^{\mathrm{pa}}|_{\mathcal{S}^1}$ with the trivial connection ∇^0 on $X \times \mathrm{U}(N_0) \to X$. Now there is a natural isomorphism between isomorphism classes of such triples $(P, \iota, \nabla_P^{\mathrm{pa}})$, and smooth loops in $\mathcal{M}_{X\times\mathrm{U}(N_0)}$ based at ∇^0. As the family of such ∇_P^{pa} is an affine space, and so contractible, this induces an bijection between isomorphism classes of pairs (P, ι), which correspond to elements of $K^1(X)$, and isotopy classes of smooth loops in $\mathcal{M}_{X\times\mathrm{U}(N_0)}$ based at ∇^0, which correspond to elements of the fundamental group $\pi_1(\mathcal{M}_{X\times\mathrm{U}(N_0)})$.

This defines a bijection $K^1(X) \to \pi_1(\mathcal{M}_{X\times\mathrm{U}(N_0)})$. It is not immediately obvious this is a group morphism, as composition in $K^1(X)$ and $\pi_1(\mathcal{M}_{X\times\mathrm{U}(N_0)})$ is defined in different ways, but by considering isotopies of connections on principal $\mathrm{U}(2N_0)$-bundles $P \oplus Q$, we can show it is a group isomorphism.

Let us define $\mathcal{M}_0^{\mathrm{U}}$ in Definition 2.16 with $m_0 = N_0$ and $P_0 = X \times \mathrm{U}(N_0)$. Then as in (2.17) we have a morphism $\Psi_{P_0}^{P_0\oplus\mathbb{C}^\infty} : \mathcal{M}_{X\times\mathrm{U}(N_0)} \to \mathcal{M}_0^{\mathrm{U}}$, which induces an isomorphism of fundamental groups as we are in the stable range. Combining this with the isomorphism $K^1(X) \to \pi_1(\mathcal{M}_{X\times\mathrm{U}(N_0)})$ above gives an isomorphism $K^1(X) \to \pi_1(\mathcal{M}_0^{\mathrm{U}})$, as we want. □

Combining Proposition 2.20(c) with Lemma 2.4 yields a criterion for orientability of the $\mathcal{M}_\alpha^{\mathrm{U}}, \mathcal{M}_P$, as in Problem 1.3(a):

Corollary 2.21. *In the situation of Definition* 2.16, *if* $K^1(X) \otimes_{\mathbb{Z}} \mathbb{Z}_2 = 0$ *then* $\mathcal{M}_\alpha^{\mathrm{U}}$ *is orientable for all* $\alpha \in K^0(X)$, *and hence by* (2.20), $\mathcal{M}_P$ *is orientable for all principal* $\mathrm{U}(m)$*-bundles* $P \to X$. *By the Atiyah–Hirzebruch spectral sequence, a sufficient condition for* $K^1(X) \otimes_{\mathbb{Z}} \mathbb{Z}_2 = 0$ *is that* $H^{\mathrm{odd}}(X, \mathbb{Z}_2) = 0$.

A version of this corollary is used by Cao and Leung [13, §10.4], [14, Th. 2.1].

2.3.7 An index-theoretic expression for obstruction to orientability

In Definition 2.16, there is a group morphism $\pi_1(\mathcal{M}_0^{\mathrm{U}}) \to \{\pm 1\}$ mapping $[\gamma]$ in $\pi_1(\mathcal{M}_0^{\mathrm{U}})$ to the monodromy of the principal $\mathbb{Z}_2$-bundle $O_0^{E^\bullet} \to \mathcal{M}_0^{\mathrm{U}}$ round the loop γ, and $\mathcal{M}_0^{\mathrm{U}}$ is orientable if and only if this morphism is the constant map 1. So by Proposition 2.20(b),(c), this gives a natural morphism $\Omega : K^1(X) \to \{\pm 1\}$, such that $\mathcal{M}_\alpha^{\mathrm{U}}$ is orientable for all $\alpha \in K^0(X)$ if and only if $\Omega \equiv 1$.

By a calculation in index theory following Atiyah and Singer [4–8], we will

show that Ω is given by the commutative diagram:

$$\begin{tikzcd} K^1(X) \arrow[r, equal, "\text{Bott periodicity}"'] \arrow[d, "\Omega"] & K^{-1}(X) \arrow[r, "\mathrm{Ad}^{-1}"'] & KO^{-1}(X) \arrow[d, "\beta\mapsto\pi^*(\beta)\cup\sigma(E_\bullet)"'] \\ \{\pm1\} & KO^{-1}(*) \arrow[l, "\cong"'] & KO^{-1}_{\mathrm{cs}}(TX). \arrow[l, "\text{t-ind}^{-1}"'] \end{tikzcd} \tag{2.32}$$

Here $\mathrm{Ad}^i : K^i(X) \to KO^i(X)$ is a natural quadratic map which when $i = 0$ maps $\mathrm{Ad}^0 : [\![P]\!] \mapsto [\![\mathrm{Ad}(P)]\!]$ for any principal $\mathrm{U}(m)$-bundle $P \to X$. Also $KO^i_{\mathrm{cs}}(TX)$ is the compactly-supported real K-theory of the tangent bundle TX, and $\sigma(E_\bullet) \in KO^0_{\mathrm{cs}}(TX)$ is defined using the symbol of $E_\bullet$, and $\pi^* : KO^{-1}(X) \to KO^{-1}(TX)$ is pullback by $\pi : TX \to X$, and $\cup : KO^{-1}(TX) \times KO^0_{\mathrm{cs}}(X) \to KO^{-1}_{\mathrm{cs}}(X)$ is the cup product, and $\text{t-ind}^i : KO^i_{\mathrm{cs}}(TX) \to KO^i(*)$ is the *topological index* morphism of Atiyah and Singer [4, §3].

To prove (2.32) commutes, given an element κ of $K^1(X) \cong \pi_1(\mathcal{M}^{\mathrm{U}}_0)$, as in the proof of Proposition 2.20(c) we construct a principal $\mathrm{U}(N_0)$-bundle $P \to X \times \mathcal{S}^1$ with a partial connection ∇^{pa}_P. From this we define a family of twisted elliptic operators $(D^{\nabla^{\mathrm{pa}}_P|_{X\times\{e^{i\theta}\}}})_{e^{i\theta}\in\mathcal{S}^1}$ on X over the base $\mathcal{S}^1$. This has a family index in $KO^0(\mathcal{S}^1)$, which lies in $\widetilde{KO}^0(\mathcal{S}^1, 1)$ as the operator on $X \times \{1\}$ is trivial, where $\widetilde{KO}^0(\mathcal{S}^1, 1) = KO^{-1}(*) \cong \{1, -1\}$, and this family index coincides with the monodromy $\Omega(\kappa)$ of $O^{E_\bullet}_0$ round the corresponding loop in $\mathcal{M}^{\mathrm{U}}_0$. Computing the family index in real K-theory using the Atiyah–Singer Index Theorem for families [7] shows that (2.32) commutes at $\kappa \in K^1(X)$.

2.4 Comparing orientations under direct sums

We define the *orientation group* of $X, E_\bullet$:

Definition 2.22. In the situation of Definition 2.16, suppose that $\mathcal{M}^{\mathrm{U}}_0$ is orientable, so that $\mathcal{M}^{\mathrm{U}}_\alpha$ is orientable for all $\alpha \in K^0(X)$ by Proposition 2.20(b). Thus, each $\mathcal{M}^{\mathrm{U}}_\alpha$ has two possible orientations, as it is connected.

There is a natural orientation $\bar\omega_0$ on $\mathcal{M}^{\mathrm{U}}_0$, defined as follows. Let $P = X \times \mathrm{U}(0) \to X$ be the trivial $\mathrm{U}(0)$-bundle with $[\![P]\!] = 0$ in $K^0(X)$. Then $\mathcal{M}_P$ in Definition 1.1 is a point, and $O^{E_\bullet}_P = \{+1, -1\}$ is naturally trivial. We fix $\bar\omega_0$ by requiring that $(\sigma^{E_\bullet}_P)_*(+1) = (\Sigma^U_P)^*(\bar\omega_0)$, for $\sigma^{E_\bullet}_P$ as in (2.20).

Define the *orientation group* $\Omega(X)$, initially just as a set, by

$$\Omega(X) = \bigl\{(\alpha, \omega_\alpha) : \alpha \in K^0(X),\ \omega_\alpha \text{ is an orientation on } \mathcal{M}^{\mathrm{U}}_\alpha\bigr\}.$$

Define a map $\pi : \Omega(X) \to K^0(X)$ by $\pi : (\alpha, \omega_\alpha) \mapsto \alpha$. Define an action $\cdot : \{\pm1\} \times \Omega(X) \to \Omega(X)$ by $\epsilon \cdot (\alpha, \omega_\alpha) = (\alpha, \epsilon \cdot \omega_\alpha)$ for $\epsilon = \pm1$. Then $\pi, \cdot$ make $\Omega(X)$ into a principal $\mathbb{Z}_2$-bundle over $K^0(X)$.

Define a multiplication $\star : \Omega(X) \times \Omega(X) \to \Omega(X)$ by

$$(\alpha, \omega_\alpha) \star (\beta, \omega_\beta) = (\alpha + \beta, \omega_{\alpha+\beta}), \quad \text{where } \omega_{\alpha+\beta} \text{ is uniquely}$$
$$\text{determined by} \quad \phi^{E_\bullet}_{\alpha,\beta}(\omega_\alpha \boxtimes \omega_\beta) = (\Phi^{\mathrm{U}}_{\alpha,\beta})^*(\omega_{\alpha+\beta}),$$

using the notation of Definition 2.19. Equation (2.31) implies that $\star$ is associative. From the definition of $\bar\omega_0$ we see that $(0,\bar\omega_0)\star(\alpha,\omega_\alpha)=(\alpha,\omega_\alpha)\star(0,\bar\omega_0)=(\alpha,\omega_\alpha)$, so $(0,\bar\omega_0)$ is the identity in $\Omega(X)$. For any (α,ω_α) in $\Omega(X)$, we can easily show that some $\omega_{-\alpha}$, one of the two possible orientations on $\mathcal{M}^{\mathrm{U}}_{-\alpha}$, satisfies $(-\alpha,\omega_{-\alpha})\star(\alpha,\omega_\alpha)=(\alpha,\omega_\alpha)\star(-\alpha,\omega_{-\alpha})=(0,\bar\omega_0)$, so inverses exist in $\Omega(X)$. Thus $\Omega(X)$ is a group, which depends on $X,E_\bullet$. The multiplication in $\Omega(X)$ compares orientations on $\mathcal{M}^{\mathrm{U}}_\alpha,\mathcal{M}^{\mathrm{U}}_\beta,\mathcal{M}^{\mathrm{U}}_{\alpha+\beta}$ under the direct sum morphisms $\Phi^{\mathrm{U}}_{\alpha,\beta},\phi^{\mathrm{U}}_{\alpha,\beta}$ of §2.3.4.

Clearly the map $\pi:\Omega(X)\to K^0(X)$ is a surjective group morphism, with kernel $\{(0,\bar\omega_0),(0,-\bar\omega_0)\}\cong\{\pm1\}$, so we have an exact sequence of groups

$$0\longrightarrow\{1,-1\}\longrightarrow\Omega(X)\xrightarrow{\ \pi\ }K^0(X)\longrightarrow 0. \tag{2.33}$$

Equation (2.30) implies that for all $(\alpha,\omega_\alpha),(\beta,\omega_\beta)$ in $\Omega(X)$ we have

$$(\beta,\omega_\beta)\star(\alpha,\omega_\alpha)=(-1)^{\chi^{E_\bullet}(\alpha,\beta)+\chi^{E_\bullet}(\alpha,\alpha)\chi^{E_\bullet}(\beta,\beta)}\cdot\big((\alpha,\omega_\alpha)\star(\beta,\omega_\beta)\big). \tag{2.34}$$

So in general $\Omega(X)$ may not be abelian.

The orientation group $\Omega(X)$ is closely related to the problem of choosing canonical orientations on moduli spaces $\mathcal{M}^{\mathrm{U}}_\alpha$ for all $\alpha\in K^0(X)$, and hence canonical orientations on $\mathcal{M}_P$ for all principal $\mathrm{U}(m)$-bundles $P\to X$ as in Definition 2.17, with relations between these canonical orientations under direct sums, as in Problem 1.3(c).

Observe that choosing an orientation $\tilde\omega_\alpha$ on $\mathcal{M}^{\mathrm{U}}_\alpha$ for all $\alpha\in K^0(X)$ is equivalent to choosing a bijection $\Lambda:\Omega(X)\xrightarrow{\cong}K^0(X)\times\{\pm1\}$ compatible with (2.33). Then there are $\tilde\epsilon_{\alpha,\beta}\in\{\pm1\}$ such that $(\alpha,\tilde\omega_\alpha)\star(\beta,\tilde\omega_\beta)=\tilde\epsilon_{\alpha,\beta}\cdot(\alpha+\beta,\tilde\omega_{\alpha+\beta})$, which encode the multiplication $\star$ on $\Omega(X)$, and the signs ϵ_{P_1,P_2} in Problem 1.3(c) are $\epsilon_{P_1,P_2}=\tilde\epsilon_{[\![P_1]\!],[\![P_2]\!]}$. So, Problem 1.3(c) is really about understanding the group $\Omega(X)$ and writing the multiplication $\star$ in an explicit form under a suitable trivialization Λ of the principal $\mathbb{Z}_2$-bundle $\pi:\Omega(X)\to K^0(X)$.

The next theorem is just an exercise in group theory: it classifies groups $\Omega(X)$ in an exact sequence (2.33) satisfying (2.34), and uses no further properties of $\Omega(X)$. It shows that $\Omega(X)$ depends up to isomorphism only on the finitely generated abelian group $K^0(X)$, the Euler form $\chi^{E_\bullet}:K^0(X)\times K^0(X)\to\mathbb{Z}$, and a certain group morphism $\Xi:G\to\{\pm1\}$, where $G=\{\gamma\in K^0(X):2\gamma=0\}$ is the 2-torsion subgroup of $K^0(X)$. It provides explicit signs $\tilde\epsilon_{\alpha,\beta}$ as above, which are the signs $(-1)^{\sum_{1\leqslant h<i\leqslant 1}(\chi^{E_\bullet}_{hi}+\chi^{E_\bullet}_{hh}\chi^{E_\bullet}_{ii})a'_ha_i}\cdot\Xi(\gamma)$ in the third line of (2.38).

Theorem 2.23. *Let X be a compact n-manifold and $E_\bullet$ an elliptic operator on X, and use the notation of §2.3.1–§2.3.5. Suppose $\mathcal{M}^{\mathrm{U}}_0$ is orientable, so Definition 2.22 defines the orientation group $\Omega(X)$, and a natural orientation $\bar\omega_0$ for $\mathcal{M}^{\mathrm{U}}_0$. Since $K^0(X)$ is a finitely generated abelian group, we may choose an isomorphism*

$$K^0(X)\cong\mathbb{Z}^r\times\textstyle\prod_{j\in J}\mathbb{Z}_{2^{p_j}}\times\prod_{k\in K}\mathbb{Z}_{q_k}, \tag{2.35}$$

where J, K are finite indexing sets, and $p_j > 0$, $q_k > 1$ for $j \in J$, $k \in K$ with q_k odd. Under the isomorphism (2.35) *we write elements of $K^0(X)$ as $\big((a_1,\ldots,a_r),(b_j)_{j\in J},(c_k)_{k\in K}\big)$ for $a_i \in \mathbb{Z}$, $b_j \in \mathbb{Z}_{2^{p_j}}$, $c_k \in \mathbb{Z}_{q_k}$. We may write*

$$\begin{aligned}&\chi^{E_\bullet}\big[\big((a_1,\ldots,a_r),(b_j)_{j\in J},(c_k)_{k\in K}\big),\big((a'_1,\ldots,a'_r),(b'_j)_{j\in J},(c'_k)_{k\in K}\big)\big]\\&\quad=\textstyle\sum_{h,i=1}^r \chi^{E_\bullet}_{hi}\, a_h a'_i,\end{aligned}$$

where $\chi^{E_\bullet}_{hi} \in \mathbb{Z}$ with $\chi^{E_\bullet}_{ih} = \chi^{E_\bullet}_{hi}$. Write $G = \{\gamma \in K^0(X) : 2\gamma = 0\}$ for the 2-torsion subgroup of $K^0(X)$, so that in the representation (2.35) *we have*

$$G = \big\{\big((0,\ldots,0),(b_j)_{j\in J},(0)_{k\in K}\big) : b_j = 0 + 2^{p_j}\mathbb{Z} \text{ or } 2^{p_j-1} + 2^{p_j}\mathbb{Z}\big\}. \tag{2.36}$$

Then:

(a) *There is a unique group morphism $\Xi : G \to \{1,-1\}$ depending on $X, E_\bullet$, such that if $\gamma \in G$ then for any orientation ω_γ on $\mathcal{M}^{\mathrm{U}}_\gamma$ we have*

$$(\gamma,\omega_\gamma)\star(\gamma,\omega_\gamma) = \Xi(\gamma)\cdot(0,\bar\omega_0). \tag{2.37}$$

(b) *There exists a bijection $\Lambda : \Omega(X) \xrightarrow{\cong} K^0(X)\times\{\pm1\}$ such that using Λ and* (2.35) *to identify $\Omega(X)$ with $\mathbb{Z}^r \times \prod_{j\in J}\mathbb{Z}_{2^{p_j}} \times \prod_{k\in K}\mathbb{Z}_{q_k} \times \{\pm1\}$, the multiplication $\star$ in $\Omega(X)$ is given explicitly by*

$$\begin{aligned}&\big[\big((a_1,\ldots,a_r),(b_j)_{j\in J},(c_k)_{k\in K}\big),\epsilon\big]\star\big[\big((a'_1,\ldots,a'_r),(b'_j)_{j\in J},(c'_k)_{k\in K}\big),\epsilon'\big]\\&=\big[\big((a_1+a'_1,\ldots,a_r+a'_r),(b_j+b'_j)_{j\in J},(c_k+c'_k)_{k\in K}\big),\\&\qquad(-1)^{\sum_{1\leqslant h<i\leqslant 1}(\chi^{E_\bullet}_{hi}+\chi^{E_\bullet}_{hh}\chi^{E_\bullet}_{ii})a'_h a_i}\cdot\Xi(\gamma)\cdot\epsilon\epsilon'\big],\end{aligned} \tag{2.38}$$

where $\gamma \in G$ is constructed from $(b_j)_{j\in J},(b'_j)_{j\in J}$ as follows: write $b_j = \bar b_j + 2^{p_j}\mathbb{Z}$, $b'_j = \bar b'_j + 2^{p_j}\mathbb{Z}$ for $\bar b_j, \bar b'_j \in \{0,1,\ldots,2^{p_j}-1\}$. Then under (2.36) *we set $\gamma = \big((a_1,\ldots,a_r),(\tilde b_j)_{j\in J},(c_k)_{k\in K}\big)$, where $\tilde b_j = 0 + 2^{p_j}\mathbb{Z}$ if $\bar b_j + \bar b'_j < 2^{p_j}$ and $\tilde b_j = 2^{p_j-1} + 2^{p_j}\mathbb{Z}$ if $\bar b_j + \bar b'_j \geqslant 2^{p_j}$, for $j \in J$.*

(c) *Suppose $\Lambda, \tilde\Lambda$ both satisfy* **(b)**. *Then there exist unique signs $\eta_i, \zeta_j \in \{\pm1\}$ for $i = 1,\ldots,r$ and $j \in J$ such that for all $[((a_1,\ldots,a_r),\ldots),\epsilon]$ we have*

$$\begin{aligned}&\tilde\Lambda\circ\Lambda^{-1}\big[\big((a_1,\ldots,a_r),(b_j)_{j\in J},(c_k)_{k\in K}\big),\epsilon\big]\\&=\big[\big((a_1,\ldots,a_r),(b_j)_{j\in J},(c_k)_{k\in K}\big),\textstyle\prod_{i=1}^r\eta_i^{a_i}\cdot\prod_{j\in J}\zeta_j^{b_j}\cdot\epsilon\big].\end{aligned} \tag{2.39}$$

Conversely, if Λ satisfies **(b)** *and $\eta_i, \zeta_j \in \{\pm1\}$ are given for all i, j, and we define $\tilde\Lambda : \Omega(X) \xrightarrow{\cong} K^0(X)\times\{\pm1\}$ by* (2.39), *then $\tilde\Lambda$ satisfies* **(b)**.

Proof. The first part of the theorem is immediate. For (a), if $\gamma \in G$ and ω_γ is an orientation on $\mathcal{M}^{\mathrm{U}}_\gamma$ then there is a unique $\Xi(\gamma) = \pm1$ satisfying (2.37), so as $(\gamma,\omega_\gamma)\star(\gamma,\omega_\gamma) = (\gamma,-\omega_\gamma)\star(\gamma,-\omega_\gamma)$, the map $\Xi : G \to \{\pm1\}$ is well defined.

To see that Ξ is a group morphism, note that if $\gamma,\gamma'\in G$ with $\gamma''=\gamma+\gamma'$ and $\omega_\gamma,\omega_{\gamma'}$ are orientations on $\mathcal{M}^{\mathrm{U}}_\gamma,\mathcal{M}^{\mathrm{U}}_{\gamma'}$ with $(\gamma,\omega_\gamma)\star(\gamma',\omega_{\gamma'})=(\gamma'',\omega_{\gamma''})$, then

$$\begin{aligned}&\Xi(\gamma+\gamma')\cdot(0,\bar\omega_0)=(\gamma'',\omega_{\gamma''})\star(\gamma'',\omega_{\gamma''})=(\gamma,\omega_\gamma)\star(\gamma',\omega_{\gamma'})\star(\gamma,\omega_\gamma)\star(\gamma',\omega_{\gamma'})\\&=(\gamma,\omega_\gamma)\star(\gamma,\omega_\gamma)\star(\gamma',\omega_{\gamma'})\star(\gamma',\omega_{\gamma'})=\bigl[\Xi(\gamma)\cdot(0,\bar\omega_0)\bigr]\star\bigl[\Xi(\gamma')\cdot(0,\bar\omega_0)\bigr]\\&=\Xi(\gamma)\Xi(\gamma')\cdot(0,\bar\omega_0),\end{aligned}$$

using (2.37) in the fourth step and (2.34) with $\chi^{E_\bullet}|_{G\times G}=0$ in the third. Hence $\Xi(\gamma+\gamma')=\Xi(\gamma)\Xi(\gamma')$, and Ξ is a morphism, proving (a).

For (b), define elements λ_i,μ_j,ν_k in $K^0(X)$ for $i=1,\ldots,r$, $j\in J$, $k\in K$ which are identified by (2.35) with elements which have $a_k=1$, and $b_j=1$, and $c_k=1$ respectively, and all other entries zero, so that (2.35) identifies $\bigl((a_1,\ldots,a_r),(b_j)_{j\in J},(c_k)_{k\in K}\bigr)$ with $\sum_i a_i\lambda_i+\sum_j b_j\mu_j+\sum_k c_k\nu_k$ in $K^0(X)$.

For all $i=1,\ldots,r$ choose an arbitrary orientation $\tilde\omega_{\lambda_i}$ for $\mathcal{M}^U_{\lambda_i}$. For all $j\in J$ choose an arbitrary orientation $\tilde\omega_{\mu_j}$ for $\mathcal{M}^U_{\mu_j}$. For each $k\in K$, let $\tilde\omega_{\nu_k}$ be the unique orientation for $\mathcal{M}^U_{\nu_k}$ satisfying

$$(\nu_k,\tilde\omega_{\nu_k})^{q_k}=\overbrace{(\nu_k,\tilde\omega_{\nu_k})\star\cdots\star(\nu_k,\tilde\omega_{\nu_k})}^{q_k\text{ copies}}=(0,\bar\omega_0).\tag{2.40}$$

This is well defined as $q_k\nu_k=0$ in $K^0(X)$, and replacing $\tilde\omega_{\nu_k}$ by $-\tilde\omega_{\nu_k}$ multiplies the left hand side of (2.40) by $(-1)^{q_k}=-1$, as q_k is odd, so exactly one of the two orientations on $\mathcal{M}^U_{\nu_k}$ satisfies (2.40).

Let $\alpha\in K^0(X)$. Then α corresponds to under (2.35) to some $\bigl((a_1,\ldots,a_r),(b_j)_{j\in J},(c_k)_{k\in K}\bigr)$. Write $b_j=\bar b_j+2^{p_j}\mathbb{Z}$ for unique $\bar b_j=0,\ldots,2^{p_j}-1$ and $c_j=\bar c_j+q_j\mathbb{Z}$ for unique $\bar c_j=0,\ldots,q_j-1$. Define an orientation $\tilde\omega_\alpha$ on $\mathcal{M}^{\mathrm{U}}_\alpha$ by

$$(\alpha,\tilde\omega_\alpha)=(\lambda_1,\tilde\omega_{\lambda_1})^{a_1}\star\cdots\star(\lambda_r,\tilde\omega_{\lambda_r})^{a_r}\star\prod_{j\in J}(\mu_j,\tilde\omega_{\mu_j})^{\bar b_j}\star\prod_{k\in K}(\nu_k,\tilde\omega_{\nu_k})^{\bar c_k}.\tag{2.41}$$

Here we should be careful as $\Omega(X)$ may not be abelian by (2.34), and we have not specified orderings of J,K. But in fact $\chi^{E_\bullet}$ is zero on the torsion factors of $K^0(X)$, so the elements $(\mu_j,\tilde\omega_{\mu_j}),(\nu_k,\tilde\omega_{\nu_k})$ in (2.41) lie in the centre of $\Omega(X)$, and only the order of the factors $(\lambda_1,\tilde\omega_{\lambda_1})^{a_1},\ldots,(\lambda_r,\tilde\omega_{\lambda_r})^{a_r}$ matters. Define $\Lambda:\Omega(X)\to K^0(X)\times\{\pm1\}$ in (b) by, for all $\alpha\in K^0(X)$ and $\epsilon=\pm1$

$$\Lambda:(\alpha,\epsilon\cdot\tilde\omega_\alpha)\longmapsto\bigl[\bigl((a_1,\ldots,a_r),(b_j)_{j\in J},(c_k)_{k\in K}\bigr),\epsilon\bigr].\tag{2.42}$$

Equation (2.38) now follows from (2.41)–(2.42), the fact that $(\mu_j,\tilde\omega_{\mu_j})$, $(\nu_k,\tilde\omega_{\nu_k})$ lie in the centre of $\Omega(X)$, and the next three equations

$$\begin{aligned}&\bigl[(\lambda_1,\tilde\omega_{\lambda_1})^{a_1}\star\cdots\star(\lambda_r,\tilde\omega_{\lambda_r})^{a_r}\bigr]\star\bigl[(\lambda_1,\tilde\omega_{\lambda_1})^{a'_1}\star\cdots\star(\lambda_r,\tilde\omega_{\lambda_r})^{a'_r}\bigr]\\&=(-1)^{\sum_{1\leqslant h<i\leqslant1}(\chi^{E_\bullet}_{hi}+\chi^{E_\bullet}_{hh}\chi^{E_\bullet}_{ii})a'_ha_i}\cdot\bigl[(\lambda_1,\tilde\omega_{\lambda_1})^{a_1+a'_1}\star\cdots\star(\lambda_r,\tilde\omega_{\lambda_r})^{a_r+a'_r}\bigr],\end{aligned}\tag{2.43}$$

$$
\begin{aligned}
&\bigl[\textstyle\prod_{j\in J}(\mu_j,\tilde\omega_{\mu_j})^{\bar b_j}\bigr]\star\bigl[\textstyle\prod_{j\in J}(\mu_j,\tilde\omega_{\mu_j})^{\bar b'_j}\bigr]\\
&=\bigl[\textstyle\prod_{j\in J}(\mu_j,\tilde\omega_{\mu_j})^{\bar b_j+\bar b'_j-\overline{(b_j+b'_j)}}\bigr]\star\bigl[\textstyle\prod_{j\in J}(\mu_j,\tilde\omega_{\mu_j})^{\overline{(b_j+b'_j)}}\bigr]\\
&=(\gamma,\tilde\omega_\gamma)\star(\gamma,\tilde\omega_\gamma)\star\bigl[\textstyle\prod_{j\in J}(\mu_j,\tilde\omega_{\mu_j})^{\overline{(b_j+b'_j)}}\bigr]\\
&=\Xi(\gamma)\cdot\bigl[\textstyle\prod_{j\in J}(\mu_j,\tilde\omega_{\mu_j})^{\overline{(b_j+b'_j)}}\bigr],
\end{aligned}
\tag{2.44}
$$

$$
\begin{aligned}
&\bigl[\textstyle\prod_{k\in K}(\nu_k,\tilde\omega_{\nu_k})^{\bar c_k}\bigr]\star\bigl[\textstyle\prod_{k\in K}(\nu_k,\tilde\omega_{\nu_k})^{\bar c'_k}\bigr]\\
&=\bigl[\textstyle\prod_{k\in K}(\nu_k,\tilde\omega_{\nu_k})^{\bar c_k+\bar c'_k-\overline{(c_k+c'_k)}}\bigr]\star\bigl[\textstyle\prod_{k\in K}(\nu_k,\tilde\omega_{\nu_k})^{\overline{(c_k+c'_k)}}\bigr]\\
&=\bigl[\textstyle\prod_{k\in K}(\nu_k,\tilde\omega_{\nu_k})^{\overline{(c_k+c'_k)}}\bigr],
\end{aligned}
\tag{2.45}
$$

where $\gamma\in G$ in (2.44) is defined as in (b). Here (2.43) follows from (2.34). The first step of (2.44) is immediate as the $(\mu_j,\tilde\omega_{\mu_j})$ commute in $\Omega(X)$, the second holds as for each $j\in J$ either $\bar b_j+\bar b'_j-\overline{(b_j+b'_j)}=0$ in which case $\tilde b_j=0+2^{p_j}\mathbb{Z}$ in (b), or $\bar b_j+\bar b'_j-\overline{(b_j+b'_j)}=2^{p_j}$ in which case $\tilde b_j=2^{p_j-1}+2^{p_j}\mathbb{Z}$ in (b), and the third holds by (2.37) as $(0,\bar\omega_0)$ is the identity in $\Omega(X)$. The first step of (2.45) is immediate as the $(\nu_k,\tilde\omega_{\nu_k})$ commute in $\Omega(X)$, and the second step holds by (2.40) as $\bar c_k+\bar c'_k-\overline{(c_k+c'_k)}=0$ or q_k for each $k\in K$. This completes (b).

For (c), note that the only arbitrary choices we made in the proof of (b) were orientations $\tilde\omega_{\lambda_i}$ for $\mathcal{M}^U_{\lambda_i}$ for $i=1,\ldots,r$ and $\tilde\omega_{\mu_j}$ for $\mathcal{M}^U_{\mu_j}$ for all $j\in J$. Replacing $\tilde\omega_{\lambda_i}$ and $\tilde\omega_{\mu_j}$ by $\eta_i\cdot\tilde\omega_{\lambda_i}$ and $\zeta_j\cdot\tilde\omega_{\mu_j}$ for all i,j with $\eta_i,\zeta_j\in\{\pm1\}$ would yield an alternative bijection $\tilde\Lambda$ satisfying (b), where $\Lambda,\tilde\Lambda$ are related by (2.39). This proves the last part of (c).

For the first part, note that if $\tilde\Lambda$ satisfies (b) then we must have $\tilde\Lambda(\lambda_i,\tilde\omega_{\lambda_i})=(\lambda_i,\eta_i)$ for some $\eta_i=\pm1$, all $i=1,\ldots,r$, and $\tilde\Lambda(\mu_j,\tilde\omega_{\mu_j})=(\mu_j,\zeta_j)$ for some $\zeta_j=\pm1$, all $j\in J$. Using (2.38) and (2.40) we find that $\tilde\Lambda(\nu_k,\tilde\omega_{\nu_k})=(\nu_k,1)$ for all $k\in K$. Then for any $\alpha\in K^0(X)$, writing $(\alpha,\tilde\omega_\alpha)$ as in (2.41), we can use (2.38) to determine $\tilde\Lambda(\alpha,\tilde\omega_\alpha)$ from $\tilde\Lambda(\lambda_i,\tilde\omega_{\lambda_i})=(\lambda_i,\eta_i)$, $\tilde\Lambda(\mu_j,\tilde\omega_{\mu_j})=(\mu_j,\zeta_j)$ and $\tilde\Lambda(\nu_k,\tilde\omega_{\nu_k})=(\nu_k,1)$, and it must be the same as $\tilde\Lambda$ constructed above with $\eta_i\cdot\tilde\omega_{\lambda_i}$ and $\zeta_j\cdot\tilde\omega_{\mu_j}$ in place of $\tilde\omega_{\lambda_i},\tilde\omega_{\mu_j}$, so (2.39) holds. □

Remark 2.24. The material of this section does not extend from unitary groups $\mathrm{U}(m)$ and stabilized moduli spaces $\mathcal{M}^{\mathrm{U}}_\alpha$, to any of the families of Lie groups $\mathrm{O}(m),\mathrm{SO}(m),\mathrm{Spin}(m)$ or $\mathrm{Sp}(m)$, and the corresponding stabilized moduli spaces $\mathcal{M}^{\mathrm{O}}_\alpha,\mathcal{M}^{\mathrm{SO}}_\alpha,\mathcal{M}^{\mathrm{Spin}}_\alpha,\mathcal{M}^{\mathrm{Sp}}_\alpha$. This is because Example 2.10 does not extend to $\mathrm{O}(m),\ldots,\mathrm{Sp}(m)$, as in Remark 2.11, so we have no way to compare orientations for these groups under direct sums.

3 Constructing orientations by excision

We now explain a method for orienting moduli spaces using 'excision'. This was introduced by Donaldson [15, §II.4], [16, §3(b)], [18, §7.1.6] for moduli spaces of instantons on 4-manifolds.

3.1 The Excision Theorem

The next theorem is proved by the first and last authors [32], based on Donaldson [15, §II.4], [16, §3(b)], and [18, §7.1.6].

Theorem 3.1 (Excision Theorem)**.** *Suppose we are given the following data:*

- **(a)** *Compact n-manifolds X^+, X^-.*
- **(b)** *Elliptic complexes $E^\pm_\bullet$ on $X^\pm$.*
- **(c)** *A Lie group G, and principal G-bundles $P^\pm \to X^\pm$ with connections $\nabla_{P^\pm}$.*
- **(d)** *Open covers $X^+ = U^+ \cup V^+$, $X^- = U^- \cup V^-$.*
- **(e)** *A diffeomorphism $\iota : U^+ \to U^-$, such that $E^+_\bullet|_{U^+}$ and $\iota^*(E^-_\bullet|_{U^-})$ are isomorphic elliptic complexes on U^+.*
- **(f)** *An isomorphism $\sigma : P^+|_{U^+} \to \iota^*(P^-|_{U^-})$ of principal G-bundles over U^+, which identifies $\nabla_{P^+}|_{U^+}$ with $\iota^*(\nabla_{P^-}|_{U^-})$.*
- **(g)** *Trivializations of principal G-bundles $\tau^\pm : P^\pm|_{V^\pm} \to V^\pm \times G$ over $V^\pm$, which identify $\nabla_{P^\pm}|_{V^\pm}$ with the trivial connections ∇^0, and satisfy*

$$\iota|^*_{U^+\cap V^+}(\tau^-) \circ \sigma|_{U^+\cap V^+} = \tau^+|_{U^+\cap V^+}.$$

Then we have a canonical identification of n-orientation $\mathbb{Z}_2$-torsors from (1.4)*:*

$$\Omega^{+-} : \check{O}^{E^+_\bullet}_{P^+}\big|_{[\nabla_{P^+}]} \xrightarrow{\cong} \check{O}^{E^-_\bullet}_{P^-}\big|_{[\nabla_{P^-}]}. \tag{3.1}$$

The isomorphisms (3.1) *are functorial in a very strong sense. For example:*

- **(i)** *If we vary any of the data in* **(a)**–**(g)** *continuously in a family over $t \in [0,1]$, then the isomorphisms Ω^{+-} also vary continuously in $t \in [0,1]$.*
- **(ii)** *The isomorphisms Ω^{+-} are unchanged by shrinking the open sets $U^\pm, V^\pm$ such that $X^\pm = U^\pm \cup V^\pm$ still hold, and restricting $\iota, \sigma, \tau^\pm$.*
- **(iii)** *If we are also given a compact n-manifold $X^\times$, elliptic complex $E^\times_\bullet$, bundle $P^\times \to X^\times$, connection $\nabla_{P^\times}$, open cover $X^\times = U^\times \cup V^\times$, diffeomorphism $\iota' : U^- \to U^\times$, and isomorphisms $\sigma' : P^-|_{U^-} \to \iota'^*(P^\times|_{U^\times})$, $\tau^\times : P^\times|_{V^\times} \to V^\times \times G$ satisfying the analogues of* **(a)**–**(g)**, *then $\Omega^{+\times} = \Omega^{-\times} \circ \Omega^{+-}$, where $\Omega^{+\times}$ is defined using $\iota' \circ \iota : U^+ \to U^\times$ and $\iota^*(\sigma') \circ \sigma : P^+|_{U^+} \to (\iota' \circ \iota)^*(P^\times|_{U^\times})$.*

Sketch proof. On $X^\pm$, consider the elliptic operator $D^{\nabla_{\mathrm{Ad}(P^\pm)}} \oplus (D^{\nabla^0_{\mathrm{Ad}(X^\pm\times G)}})^*$, where $D^{\nabla^0_{\mathrm{Ad}(X^\pm\times G)}}$ is the twisted elliptic operator (1.2) from the trivial bundle $X^\pm \times G \to X^\pm$ with the trivial connection ∇^0, and $(D^{\nabla^0_{\mathrm{Ad}(X^\pm\times G)}})^*$ is its formal adjoint. This has determinant line

$$\det\bigl(D^{\nabla_{\mathrm{Ad}(P^\pm)}}\bigr) \otimes \det\bigl(D^{\nabla^0_{\mathrm{Ad}(X^\pm\times G)}}\bigr)^*,$$

and thus by (1.4) has orientation $\mathbb{Z}_2$-torsor

$$\check{O}^{E_\bullet^\pm}_{P^\pm}\big|_{[\nabla_{P^\pm}]} = O^{E_\bullet^\pm}_{P^\pm}\big|_{[\nabla_{P^\pm}]} \otimes_{\mathbb{Z}_2} O^{E_\bullet^\pm}_{X^\pm\times G}\big|_{[\nabla^0]},$$

as in the left and right hand sides of (3.1). Using the isomorphisms $\tau^\pm$ in (g), we may deform $D^{\nabla_{\mathrm{Ad}(P^\pm)}} \oplus (D^{\nabla^0_{\mathrm{Ad}(X^\pm\times G)}})^*$ continuously through elliptic *pseudo*-differential operators on $X^\pm$ to operators supported on $U^\pm$, and arrange that these operators on U^+ and U^- are identified by $\iota : U^+ \to U^-$. Since orientation torsors also work for elliptic pseudo-differential operators, and are unchanged under continuous deformations, the identification (3.1) follows. $\square$

Here is a refinement of Problem 1.3 for Gauge Orientation Problems:

Problem 3.2. *Suppose we are given a Gauge Orientation Problem as in Definition* 1.5 *and Example* 1.6. *Then for all compact n-manifolds X with geometric structure $\mathcal{T}$ of the prescribed kind, and all principal G-bundles $P \to X$ for $G \in \mathcal{G}$, we should construct a canonical n-orientation on $\mathcal{M}_P$, such that:*

- **(i)** *The n-orientations are functorial under isomorphisms of $(X, \mathcal{T}, P)$, and change continuously under continuous deformations of $\mathcal{T}$.*
- **(ii)** *In the situation of Theorem* 3.1, *if the diffeomorphism $\iota : U^+ \to U^-$ identifies the geometric structures $\mathcal{T}^+|_{U^+}$ and $\mathcal{T}^-|_{U^-}$, then Ω^{+-} in* (3.1) *identifies the canonical n-orientations on $\mathcal{M}_{P^+}$ and $\mathcal{M}_{P^-}$ at $[\nabla_{P^\pm}]$.*

Here part (ii) is a strong condition: in some cases it may determine the canonical n-orientations more-or-less uniquely for all $(X, \mathcal{T})$ and $P \to X$, though in other cases it can be overdetermined, so no such canonical n-orientations exist.

We have two powerful methods for trivializing bundles $\check{O}^{E_\bullet}_P \to \mathcal{M}_P$: when the symbol of $E_\bullet$ is complex linear as in Theorem 2.5, and excision, Theorem 3.1. The next theorem relates these methods. In Theorems 4.4, 4.6 and 4.10 below we will use the two methods in combination, in a way we believe is new.

Theorem 3.3. **(a)** *Suppose we are given data $X^+ = U^+ \cup V^+, E_\bullet^+, G, P^+, \nabla_{P^+}, \tau^+$ as in Theorem* 3.1(a)–(g), *and we are also given a complex structure on $E_\bullet^+|_{U^+}$, as in Theorem* 2.5. *Then there is a natural trivialization of $\mathbb{Z}_2$-torsors, depending on the choice of complex structure on $E_\bullet^+|_{U^+}$:*

$$\check{O}^{E_\bullet^+}_{P^+}\big|_{[\nabla_{P^+}]} \cong \mathbb{Z}_2. \tag{3.2}$$

These isomorphisms (3.2) *are strongly functorial as in Theorem* 3.1(i)–(iv).

(b) *In* **(a)**, *suppose the complex structure on $E_\bullet^+|_{U^+}$ is the restriction of a complex structure on $E_\bullet^+$. Then Theorem* 2.5 *gives a natural trivialization of $\check{O}^{E_\bullet^+}_{P^+}$, and* (3.2) *agrees with this at $[\nabla_{P^+}]$.*

(c) *In the situation of Theorem* 3.1, *suppose we have complex structures on $E_\bullet^\pm|_{U^\pm}$ which are identified by the isomorphism between $E_\bullet^+|_{U^+}$ and $\iota^*(E_\bullet^-|_{U^-})$ in Theorem* 3.1(e). *Then* (3.2) *for $X^\pm$ induce trivializations of the left and right hand sides of* (3.1), *and* (3.1) *identifies these trivializations.*

Proof. For (a), in the sketch proof of Theorem 3.1, we explained that $\check{O}^{E^+_\bullet}_{P^+}|_{[\nabla_{P^+}]}$ is the orientation $\mathbb{Z}_2$-torsor of the elliptic operator $D^{\nabla_{\mathrm{Ad}(P^\pm)}} \oplus (D^{\nabla^0_{\mathrm{Ad}(X^\pm\times G)}})^*$ on X^+, and we can deform this continuously through elliptic pseudo-differential operators on X^+ to an operator supported on U^+. Using the complex structure on $E^+_\bullet|_{U^+}$, we can make this operator supported on U^+ $\mathbb{C}$-linear. Then as in the proof of Theorem 2.5, we get a trivialization of $\check{O}^{E^+_\bullet}_{P^+}\big|_{[\nabla_{P^+}]}$, inducing the isomorphism (3.2). It is functorial as in the proofs of Theorems 2.5 and 3.1.

For (b), given a complex structure on $E^+_\bullet$, we can take the continuous deformation from $D^{\nabla_{\mathrm{Ad}(P^\pm)}} \oplus (D^{\nabla^0_{\mathrm{Ad}(X^\pm\times G)}})^*$ to a complex linear operator supported on U^+ to be the composition of two continuous deformations: first we deform $D^{\nabla_{\mathrm{Ad}(P^+)}}$ and $D^{\nabla^0_{\mathrm{Ad}(X^\pm\times G)}}$ through elliptic differential operators to $\mathbb{C}$-linear operators as used to construct the trivializations of $O^{E^+_\bullet}_{P^+}, O^{E^+_\bullet}_{X^+\times G}|_{[\nabla^0]}$ in the proof of Theorem 2.5. Secondly, we deform through $\mathbb{C}$-linear elliptic pseudo-differential operators on X^+ to a $\mathbb{C}$-linear operator supported on U^+. Throughout the second deformation we have canonical orientations by $\mathbb{C}$-linearity, and (b) follows.

For (c), the trivializations of the left and right hand sides of (3.1) from part (a) come from identifying them with the orientation $\mathbb{Z}$-torsors of $\mathbb{C}$-linear elliptic pseudo-differential operators supported on U^+ and U^-, where the $\mathbb{C}$-linearity is built using the complex structures on $E^\pm_\bullet|_{U^\pm}$. The isomorphism (3.1) was proved by identifying both sides with orientation $\mathbb{Z}$-torsors of elliptic pseudo-differential operators supported on U^+ and U^-, and identifying these elliptic pseudo-differential operators under $\iota : U^+ \to U^-$ using the isomorphism $E^+_\bullet|_{U^+} \cong \iota^*(E^-_\bullet|_{U^-})$. Since this isomorphism identifies the complex structures on $E^\pm_\bullet|_{U^\pm}$, we can take the isomorphism of elliptic pseudo-differential operators under ι to identify the $\mathbb{C}$-linear structures, so (3.1) identifies the corresponding trivializations from (3.2). $\square$

3.2 Trivializing principal bundles outside codimension d

Remark 3.4. **(a)** Suppose that G is a Lie group, and $d \geqslant 2$ with homotopy groups $\pi_i(G) = 0$ for $i = 0, \ldots, d-2$. Then for $k = 0, \ldots, d-1$, any principal G-bundle $P \to \mathcal{S}^k$ is trivial, as these are classified by $\pi_{k-1}(G)$. It follows that if Z is a manifold or CW-complex of dimension $\leqslant d-1$ then any principal G-bundle $P \to Z$ is trivial.

(b) Here are some facts about homotopy groups $\pi_i(G)$ for Lie groups G, which can be found in Borel [10]:

- **(i)** $\pi_0(G) = 0$ if G is connected.
- **(ii)** $\pi_1(G)$ is abelian, and $\pi_1(G) = 0$ if G is simply-connected.
- **(iii)** $\pi_2(G) = 0$ for any Lie group G.
- **(iv)** $\pi_3(G) \cong \mathbb{Z}^k$, where k is the number of simple Lie group factors of G.

(c) Combining **(a)**,**(b)**, we see that for a Lie group G:

(i) If G is connected then $\pi_i(G) = 0$ for $i = 0, \ldots, d-2$ with $d = 2$.

(ii) If G is connected and simply-connected then $\pi_i(G) = 0$ for $i = 0, \ldots, d-2$ with $d = 4$. This does not hold for $d > 4$ unless G is contractible.

We use this to show that we can trivialize principal G-bundles $P \to X$ outside the $(n-d)$-skeleton of X for $d = 2$ or 4.

Proposition 3.5. *Let X be a compact n-manifold, G a Lie group, and $P \to X$ a principal G-bundle. Suppose that either* **(i)** *G is connected and set $d = 2$, or* **(ii)** *G is connected and simply-connected and set $d = 4$. Then:*

(a) *Let τ be any triangulation of X into smooth n-simplices $\sigma : \Delta_n \to X$, and let Y be the $(n-d)$-skeleton of τ, that is, the union of all i-dimensional faces of simplices in τ for $i \leqslant n-d$. Then Y is a closed subset of X, and a finite CW complex of dimension $n-d$.*

We can find a trivialization $\Phi : P|_{X \setminus Y} \xrightarrow{\cong} (X \setminus Y) \times G$.

(b) *Suppose τ_0, Y_0, Φ_0 and τ_1, Y_1, Φ_1 are alternative choices in* **(a)**. *Then there exists a triangulation $\tilde{\tau}$ of $X \times [0,1]$ into smooth $(n+1)$-simplices, which restricts to τ_i on $X \times \{i\}$ for $i = 0, 1$. Let Z be the $(n+1-d)$-skeleton of $\tilde{\tau}$ relative to $X \times \{0,1\}$, i.e. the union of all i-dimensional faces of simplices in $\tilde{\tau}$ which have either $i \leqslant n-d$, or $i = n+1-d$ and do not lie wholly in $X \times \{0,1\}$. Then Z is closed in $X \times [0,1]$, and a finite CW complex of dimension $n+1-d$, with $Z \cap (X \times \{i\}) = Y_i \times \{i\}$ for $i = 0, 1$.*

We can find a trivialization $\Psi : \pi_X^*(P)|_{(X \times [0,1]) \setminus Z} \xrightarrow{\cong} ((X \times [0,1]) \setminus Z) \times G$, such that $\Psi|_{(X \setminus Y_i) \times \{i\}} = \Phi_i$ for $i = 0, 1$.

Proof. For (a), given a triangulation τ of X, let τ' be the *barycentric subdivision* of τ, that is, the subtriangulation that places an extra vertex at the barycentre of each i-simplex in τ for $i > 0$, and divides each k-simplex $\sigma : \Delta_k \to X$ in τ into $(k+1)!$ smaller k-simplices. Define $C \subset X$ to be the union of all i-simplices $\sigma_i'(\Delta_i) \subset X$ in τ' for $i = 0, \ldots, d-1$ which meet an $(n-i)$-simplex $\sigma_{n-i}(\Delta_{n-i})$ in τ transversely at the barycentre of $\sigma_{n-i}(\Delta_{n-i})$. Then C is closed in X, and is a CW complex of dimension $d-1$.

We can think of C as the $(d-1)$-skeleton of the 'dual triangulation' τ^* of τ, though τ^* divides X into polyhedra rather than simplices. For example, the icosahedron is a triangulation of $\mathcal{S}^2$ into twenty 2-simplices, thirty 1-simplices and twelve 0 simplices. The 'dual triangulation' is the dodecahedron, which divides $\mathcal{S}^2$ into twenty 0-simplices, thirty 1-simplices, and twelve pentagons.

The important facts we need are that $C \cap Z = \emptyset$, and $X \setminus Y$ retracts onto C, since $X \setminus Y$ is a union of interiors of i-simplices $\sigma_i'(\Delta_i) \subset X$ in τ' all of which have one face in C, and can be retracted onto C in a natural way. As C is a CW complex of dimension $d-1$, we see that $P|_C$ is trivial by Remark 3.4(a),(c). Since $X \setminus Y$ retracts onto C, it follows that $P|_{X \setminus Y}$ is trivial, proving (a).

For (b), by standard facts about triangulations we can choose $\tilde{\tau}$. Let $\tilde{\tau}'$ be the barycentric subdivision of $\tilde{\tau}$, and define $D \subset X \times [0,1]$ to be the union of all

i-simplices $\tilde\sigma'_i(\Delta_i) \subset X \times [0,1]$ in $\tilde\tau'$ for $i = 0, \ldots, d-1$ which meet an $(n+1-i)$-simplex $\tilde\sigma_{n+1-i}(\Delta_{n+1-i})$ in $\tilde\tau$ transversely at the barycentre of $\tilde\sigma_{n+1-i}(\Delta_{n+1-i})$. Then D is closed in $X \times [0,1]$, and is a CW complex of dimension $d-1$, with $D \cap X \times \{i\} = C_i \times \{i\}$ for $i = 0, 1$. As above we have $D \cap Z = \emptyset$, and $(X \times [0,1]) \setminus Z$ retracts onto D.

Since D is a CW complex of dimension $d-1$, $P|_D$ is trivial by Remark 3.4(a),(c). Furthermore, as $(C_0 \times \{0\}) \amalg (C_1 \times \{1\})$ is a CW-subcomplex of D, the trivializations $\Phi_i|_{C_i \times \{i\}}$ of $\pi_X^*(P)|_{C_i \times \{i\}}$ for $i = 0, 1$ can be extended to a single trivialization of $\pi_X^*(P)|_D$. As $(X \times [0,1]) \setminus Z$ retracts onto D, we can then extend the trivialization of $\pi_X^*(P)$ from D to $\Psi : \pi_X^*(P)|_{(X \times [0,1]) \setminus Z} \to ((X \times [0,1]) \setminus Z) \times G$, such that $\Psi|_{(X \setminus Y_i) \times \{i\}} = \Phi_i$ for $i = 0, 1$. $\square$

3.3 A general method for solving Problem 3.2

Suppose we are given a Gauge Orientation Problem, as in Definition 1.5 and Example 1.6. We will take the family $\mathcal{G}$ of Lie groups G to be either:

(a) all connected Lie groups G, so Proposition 3.5 applies with $d = 2$; or

(b) all connected, simply-connected Lie groups G, so Proposition 3.5 applies with $d = 4$.

(c) $\mathcal{G}$ is $\bigl\{\mathrm{SU}(m) : m = 1, 2, \ldots\bigr\}$, so Proposition 3.5 applies with $d = 4$, and we can use results on stabilization and K-theory.

We now outline a strategy for solving Problem 3.2:

Step 1. Suppose for simplicity that $\mathcal{S}^n$ admits a geometric structure $\mathcal{T}'$ of the prescribed kind, and that $\mathcal{T}'$ is unique up to isotopy. Prove that when $X = \mathcal{S}^n$, all moduli spaces $\mathcal{M}_P$ are n-orientable.

Step 2. Choose n-orientations for all moduli spaces $\mathcal{M}_P$ when $X = \mathcal{S}^n$.

Step 3. Let $(X, \mathcal{T})$, G and $P \to X$ be as in Definition 1.5, and let ∇_P be a connection on P. By Proposition 3.5(a) we can choose an $(n-d)$-skeleton $Y \subset X$ and a trivialization $\Phi : P|_{X \setminus Y} \to (X \setminus Y) \times G$. Choose a small open neighbourhood U of Y in X such that U retracts onto Y. Choose an open $V \subset X$ with $\overline{V} \subseteq X \setminus Y$ and $U \cup V = X$.

Choose a connection $\hat\nabla_P$ on $P \to X$ which is trivial over $V \subset X \setminus Y$, using the chosen trivialization Φ of $P|_{X \setminus Y}$. Choose an embedding $\iota : U \hookrightarrow \mathcal{S}^n$ of U as an open submanifold of $\mathcal{S}^n$, if this is possible, and a geometric structure $\mathcal{T}'$ on $\mathcal{S}^n$ of the prescribed kind, such that $\iota^*(\mathcal{T}') \cong \mathcal{T}|_U$. Set $U' = \iota(U)$ and $V' = \iota(U \cap V) \amalg (\mathcal{S}^n \setminus U')$. Then $V' \subset \mathcal{S}^n$ is open with $U' \cup V' = \mathcal{S}^n$.

Define a principal G-bundle $P' \to \mathcal{S}^n$, such that $P'|_{U'}$ is identified with $P|_U$ under $\iota : U \to U'$, and $P'|_{V'}$ is trivial, and on the overlap $U' \cap V'$, the identification matches the given trivialization $\Phi|_{U \cap V}$ of $P|_U$ on $U \cap V$ under ι.

Define a connection $\hat\nabla_{P'}$ on $P' \to \mathcal{S}^n$, such that $\hat\nabla_{P'}|_{U'}$ is identified with $\hat\nabla_P|_U$ under the identification of $P'|_{U'}$ with $P|_U$ under $\iota : U \to U'$, and $\hat\nabla_{P'}$ is trivial over V', using the chosen trivialization of $P'|_{V'}$.

Theorem 3.1 now gives an isomorphism of $\mathbb{Z}_2$-torsors

$$\Omega^{X\mathcal{S}^n} : \check{O}_P^{E\bullet}\big|_{[\hat{\nabla}_P]} \xrightarrow{\cong} \check{O}_{P'}^{E'\bullet}\big|_{[\hat{\nabla}_{P'}]}. \tag{3.3}$$

The right hand side has a chosen n-orientation by Step 2, and Problem 3.2(ii) requires $\Omega^{X\mathcal{S}^n}$ be n-orientation-preserving, so this gives an n-orientation of $O_P^{E\bullet}\big|_{[\hat{\nabla}_P]}$. We then determine the n-orientation of $O_P^{E\bullet}\big|_{[\nabla_P]}$ by choosing a smooth path from $\hat{\nabla}_P$ to ∇_P in the (contractible) space of connections on $P \to X$, and deforming the n-orientation continuously along this path.

This constructs an n-orientation at any point $[\nabla_P]$ in $\mathcal{M}_P$, for any $(X, \mathcal{T})$, Lie group G, and principal G-bundle $P \to X$, which is uniquely determined by Step 2 and Problem 3.2(ii).

Step 4. Prove that the n-orientation on $\mathcal{M}_P$ at $[\nabla_P]$ in Step 3 is independent of all arbitrary choices in its construction, and so is well defined.

Start with two sets of choices $Y_0, \Phi_0, U_0, V_0, \iota_0, \hat{\nabla}_{P,0}$ and $Y_1, \Phi_1, U_1, V_1, \iota_1, \hat{\nabla}_{P,1}$ in Step 3. We use Proposition 3.5(b) to get an $(n-d)$-skeleton $Z \subset X \times [0,1]$ interpolating between Y_0 and Y_1, and a trivialization $\Psi : \pi_X^*(P)|_{(X\times[0,1])\setminus Z} \to ((X \times [0,1]) \setminus Z) \times G$ interpolating between Φ_0 and Φ_1. We choose a small open neighbourhood W of Z in $X \times [0,1]$ which interpolates between U_0 and U_1 and retracts onto Z. Then we construct data on $X \times [0,1]$ and $\mathcal{S}^n \times [0,1]$ interpolating between $V_0, \iota_0, \hat{\nabla}_{P,0}$ and $V_1, \iota_1, \hat{\nabla}_{P,1}$, if this is possible. This gives a continuous family of excision problems from X to $\mathcal{S}^n$ parametrized by $t \in [0,1]$, yielding a 1-parameter family of isomorphisms (3.3). Theorem 3.1(i) says that these depend continuously on $t \in [0,1]$. Thus the n-orientations on $\mathcal{M}_P$ at $[\nabla_P]$ determined by $Y_0, \ldots, \hat{\nabla}_{P,0}$ and $Y_1, \ldots, \hat{\nabla}_{P,1}$ are joined by a continuous family over $t \in [0,1]$, so they are equal, and independent of choices.

Step 5. Steps 1–4 give canonical n-orientations $\check{\omega}_P$ on all moduli spaces $\mathcal{M}_P$ in the Gauge Orientation Problem. Finally, we show that these n-orientations satisfy any other properties that we want, e.g. Problem 3.2(i)–(ii), or comparison of n-orientations for $\mathrm{U}(m)$-bundles under direct sums with given signs, as in §2.4.

We will use versions of this method in the proofs of Theorems 4.3, 4.4, 4.6, and 4.10 below (though not using $\mathcal{S}^n$ as the model space in Steps 1 and 2), and in the sequels [12, 32].

4 Application to orientations in gauge theory

We now apply the ideas of §2–§3 to construct canonical orientations on several classes of gauge theory moduli spaces. Some of our results are new. We begin with a general discussion of gauge theory moduli spaces in §4.1.

4.1 Orienting moduli spaces in gauge theory

In gauge theory one studies moduli spaces $\mathcal{M}_P^{\mathrm{ga}}$ of (irreducible) connections ∇_P on a principal bundle $P \to X$ (perhaps plus some extra data, such as a Higgs field) satisfying a curvature condition. Under suitable genericity conditions,

these moduli spaces $\mathcal{M}_P^{\rm ga}$ will be smooth manifolds, and the ideas of §2–§3 can often be used to prove $\mathcal{M}_P^{\rm ga}$ is orientable, and construct a canonical orientation on $\mathcal{M}_P^{\rm ga}$. These orientations are important in defining enumerative invariants such as Casson invariants, Donaldson invariants, and Seiberg–Witten invariants. We illustrate this with the example of instantons on 4-manifolds, [18]:

Example 4.1. Let (X,g) be a compact, oriented Riemannian 4-manifold, and G a Lie group (e.g. $G={\rm SU}(2)$), and $P\to X$ a principal G-bundle. For each connection ∇_P on P, the curvature F^{∇_P} is a section of ${\rm Ad}(P)\otimes\Lambda^2T^*X$. We have $\Lambda^2T^*X=\Lambda^2_+T^*X\oplus\Lambda^2_-T^*X$, where $\Lambda^2_\pm T^*X$ are the subbundles of 2-forms α on X with $*\alpha=\pm\alpha$. Thus $F^{\nabla_P}=F_+^{\nabla_P}\oplus F_-^{\nabla_P}$, with $F_\pm^{\nabla_P}$ the component in ${\rm Ad}(P)\otimes\Lambda^2_\pm T^*X$. We call (P,∇_P) an (*anti-self-dual*) *instanton* if $F_+^{\nabla_P}=0$.

Write $\mathcal{M}_P^{\rm asd}$ for the moduli space of gauge isomorphism classes $[\nabla_P]$ of irreducible instanton connections ∇_P on P. The deformation theory of $[\nabla_P]$ in $\mathcal{M}_P^{\rm asd}$ is governed by the Atiyah–Hitchin–Singer complex [3]:

$$\begin{aligned}0\longrightarrow\Gamma^\infty({\rm Ad}(P)\otimes\Lambda^0T^*X)&\xrightarrow{\ {\rm d}^{\nabla_P}\ }\Gamma^\infty({\rm Ad}(P)\otimes\Lambda^1T^*X)\\&\xrightarrow{\ {\rm d}_+^{\nabla_P}\ }\Gamma^\infty({\rm Ad}(P)\otimes\Lambda^2_+T^*X)\longrightarrow 0,\end{aligned}\tag{4.1}$$

where ${\rm d}_+^{\nabla_P}\circ{\rm d}^{\nabla_P}=0$ as $F_+^{\nabla_P}=0$. Write $\mathcal{H}^0,\mathcal{H}^1,\mathcal{H}^2_+$ for the cohomology groups of (4.1). Then $\mathcal{H}^0$ is the Lie algebra of ${\rm Aut}(\nabla_P)$, so $\mathcal{H}^0=Z(\mathfrak{g})$, the Lie algebra of the centre $Z(G)$ of G, as ∇_P is irreducible. Also $\mathcal{H}^1$ is the Zariski tangent space of $\mathcal{M}_P^{\rm asd}$ at $[\nabla_P]$, and $\mathcal{H}^2_+$ is the obstruction space. If g is generic then for non-flat connections $\mathcal{H}^2_+=0$ for all ∇_P, as in [18, §4.3], and $\mathcal{M}_P^{\rm asd}$ is a smooth manifold, with tangent space $T_{[\nabla_P]}\mathcal{M}_P^{\rm asd}=\mathcal{H}^1$. Note that $\mathcal{M}_P^{\rm asd}\subset\mathcal{M}_P$ is a subspace of the topological stack $\mathcal{M}_P$ from Definition 1.1.

Take $E_\bullet$ to be the elliptic operator on X

$$D={\rm d}+{\rm d}_+^*:\Gamma^\infty(\Lambda^0T^*X\oplus\Lambda^2_+T^*X)\longrightarrow\Gamma^\infty(\Lambda^1T^*X).\tag{4.2}$$

Turning the complex (4.1) into a single elliptic operator as in Remark 2.2(ii) yields the twisted operator $D^{\nabla_{{\rm Ad}(P)}}$ from (1.2). Hence the line bundle $L_P^{E_\bullet}\to\mathcal{M}_P$ in Definition 1.2 has fibre at $[\nabla_P]$ the determinant line of (4.1), which (after choosing an isomorphism $\det Z(\mathfrak{g})\cong\mathbb{R}$) is $\det(\mathcal{H}^1)^*=\det T^*_{[\nabla_P]}\mathcal{M}_P^{\rm asd}$. It follows that $O_P^{E_\bullet}|_{\mathcal{M}_P^{\rm asd}}$ is the orientation bundle of the manifold $\mathcal{M}_P^{\rm asd}$, and an orientation on $\mathcal{M}_P$ in Definition 1.2 restricts to an orientation on the manifold $\mathcal{M}_P^{\rm asd}$ in the usual sense of differential geometry. This is a very useful way of defining orientations on $\mathcal{M}_P^{\rm asd}$.

There are several other important classes of gauge-theoretic moduli spaces $\mathcal{M}_P^{\rm ga}$ which have elliptic deformation theory, and so are generically smooth manifolds, for which orientations can be defined by pullback from $\mathcal{M}_P$. See Reyes Carrión [45] for a study of instanton-type equations governed by complexes generalizing (4.1). We can also generalize the programme above in three ways:

Remark 4.2. **(i)** For example, for 'G_2-instantons' on a 7-manifold (X,φ,g) with holonomy G_2 we replace (4.1) by a *four term* elliptic complex:

$$\begin{aligned} 0 \longrightarrow \Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^0T^*X) &\xrightarrow{\mathrm{d}^{\nabla_P}} \Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^1T^*X) \\ \xrightarrow{\mathrm{d}_7^{\nabla_P}} \Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^2_7T^*X) &\xrightarrow{*\varphi\wedge\mathrm{d}^{\nabla_P}} \Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^7T^*X) \longrightarrow 0, \end{aligned} \tag{4.3}$$

where exactness follows from $\pi^2_7(F^{\nabla_P})=0$, $\mathrm{d}(*\varphi)=0$, and the Bianchi identity. The cohomology at the fourth term is dual to the cohomology at the first term, and so is $Z(\mathfrak{g})^*$ for irreducible connections. Because of this, G_2-instanton moduli spaces $\mathcal{M}_P^{G_2}$ are generically manifolds with well-behaved orientations. Flat connections on 3-manifolds are similar.

(ii) Many interesting problems involve moduli spaces $\mathcal{M}_P^{\mathrm{Hi}}$ of pairs (∇_P,H), where ∇_P is a connection on $P\to X$, and H is some extra data, such as a *Higgs field*, a section of a vector bundle on X defined using P, where (∇_P,H) satisfy some p.d.e. Under good conditions $\mathcal{M}_P^{\mathrm{Hi}}$ is a manifold, and the orientation bundle of $\mathcal{M}_P^{\mathrm{Hi}}$ is the pullback of an orientation bundle $O_P^{E^\bullet}\to\mathcal{M}_P$ under the forgetful map $\mathcal{M}_P^{\mathrm{Hi}}\to\mathcal{M}_P$, $[\nabla_P,H]\mapsto[\nabla_P]$.

(iii) If we omit the genericness/transversality conditions, gauge theory moduli spaces $\mathcal{M}_P^{\mathrm{ga}}$ are generally not smooth manifolds. However, as long as their deformation theory is given by an elliptic complex similar to (4.1) or (4.3) whose cohomology is constant except at the second and third terms, $\mathcal{M}_P^{\mathrm{ga}}$ will still be a *derived smooth manifold* (*d-manifold*, or *m-Kuranishi space*) in the sense of Joyce [26, 28–30]. Orientations for derived manifolds are defined and well behaved, and we can define orientations on $\mathcal{M}_P^{\mathrm{ga}}$ by pullback of orientations on $\mathcal{M}_P$ exactly as in the case when $\mathcal{M}_P^{\mathrm{ga}}$ is a manifold.

4.2 Examples of orientation problems

We now give a series of examples of gauge theory moduli spaces we can orient using our techniques, in dimensions $n=2,\ldots,6$.

4.2.1 Flat connections on 2-manifolds

Let X be a compact 2-manifold, G a Lie group, and $P\to X$ a principal G-bundle. Consider the moduli space $\mathcal{M}_P^{\mathrm{fl}}$ of irreducible flat connections ∇_P on P. Then (at least if X is orientable) $\mathcal{M}_P^{\mathrm{fl}}$ is a smooth manifold. The deformation theory of $[\nabla_P]$ in $\mathcal{M}_P^{\mathrm{fl}}$ is controlled by the elliptic complex

$$\begin{aligned} 0 \longrightarrow \Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^0T^*X) &\xrightarrow{\mathrm{d}^{\nabla_P}} \Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^1T^*X) \\ &\xrightarrow{\mathrm{d}^{\nabla_P}} \Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^2T^*X) \longrightarrow 0, \end{aligned}$$

with $\det T^*\mathcal{M}_P^{\mathrm{fl}}$ the determinant line of this complex.

Choose a Riemannian metric g on X, and take $E_\bullet$ to be the elliptic operator

$$D = \mathrm{d} \oplus \mathrm{d}^* : \Gamma^\infty(\Lambda^0 T^*X \oplus \Lambda^2 T^*X) \longrightarrow \Gamma^\infty(\Lambda^1 T^*X).$$

Then the orientation bundle of the manifold $\mathcal{M}_P^{\mathrm{fl}}$ is the pullback under the inclusion $\mathcal{M}_P^{\mathrm{fl}} \hookrightarrow \mathcal{M}_P$ of the bundle $O_P^{E_\bullet} \to \mathcal{M}_P$ from Definition 1.2.

In the next theorem, part (a) is proved by Freed, Hopkins and Teleman [21, §3] as part of their construction of a 2-d TQFT, but (b) may be new.

Theorem 4.3. **(a)** *For compact, oriented 2-manifolds X, the moduli spaces $\mathcal{M}_P^{\mathrm{fl}}$ above have canonical orientations for all G and $P \to X$.*

(b) *If X is not oriented, then after choosing orientations for $\mathfrak{g}$ and $\det D$ we can define a canonical orientation on $\mathcal{M}_P^{\mathrm{fl}}$ if G is any connected, simply-connected Lie group, or if $G = \mathrm{U}(m)$. Also $\mathcal{M}_P^{\mathrm{fl}}$ is orientable if $G = \mathrm{SO}(3)$.*

Proof. Part (a) holds by Theorem 2.5, as if X is oriented then there are complex structures on $E_0 \cong X \times \mathbb{C}$ and $E_1 \cong T^{*(0,1)}X$ for which the symbol of $D \cong \bar\partial$ is complex linear. Part (b) for G connected and simply-connected works by the method of §3.3 with $d = 4$ in a trivial way, as $Y = Z = \emptyset$ in Steps 3 and 4 for dimensional reasons, so Steps 1 and 2 are unnecessary. Part (b) for $G = \mathrm{U}(m)$ then follows from Example 2.12, and for $G = \mathrm{SO}(3)$ from Proposition 2.9, noting that $H^3(X, \mathbb{Z}) = 0$ as $\dim X = 2$. $\square$

4.2.2 Flat connections on 3-manifolds, and Casson invariants

Let X be a compact 3-manifold, G a Lie group, and $P \to X$ a principal G-bundle. Consider the moduli space $\mathcal{M}_P^{\mathrm{fl}}$ of irreducible flat connections ∇_P on P. In contrast to the 2-dimensional case, $\mathcal{M}_P^{\mathrm{fl}}$ is generally *not* a smooth manifold. However, (at least if X is orientable) $\mathcal{M}_P^{\mathrm{fl}}$ is a *derived manifold* of virtual dimension 0, as in Remark 4.2(iii), so orientations for $\mathcal{M}_P^{\mathrm{fl}}$ make sense. As in Remark 4.2(i), the deformation theory of $[\nabla_P]$ in $\mathcal{M}_P^{\mathrm{fl}}$ is controlled by the four term elliptic complex

$$\begin{aligned} 0 \longrightarrow \Gamma^\infty(\mathrm{Ad}(P) \otimes \Lambda^0 T^*X) &\xrightarrow{\mathrm{d}^{\nabla_P}} \Gamma^\infty(\mathrm{Ad}(P) \otimes \Lambda^1 T^*X) \\ \xrightarrow{\mathrm{d}^{\nabla_P}} \Gamma^\infty(\mathrm{Ad}(P) \otimes \Lambda^2 T^*X) &\xrightarrow{\mathrm{d}^{\nabla_P}} \Gamma^\infty(\mathrm{Ad}(P) \otimes \Lambda^3 T^*X) \longrightarrow 0, \end{aligned}$$

which should have constant cohomology at the first and fourth terms for $\mathcal{M}_P^{\mathrm{fl}}$ to be a derived manifold.

The moduli spaces $\mathcal{M}_P^{\mathrm{fl}}$ are studied in connection with the Casson invariant of 3-manifolds, as in Akbulut and McCarthy [1]. Casson originally defined a $\mathbb{Z}$-valued invariant $\mathrm{Cass}(X)$ of an oriented integral homology 3-sphere X using a Heegard splitting of X. Later, Taubes [53] provided an alternative definition of $\mathrm{Cass}(X)$ as a virtual count of $\mathcal{M}_P^{\mathrm{fl}}$ for $P \to X$ the trivial $\mathrm{SU}(2)$-bundle.

The theory of Casson invariants has been generalized in several directions. As in Donaldson [17], $\mathrm{Cass}(X)$ is the Euler characteristic of the $\mathrm{SU}(2)$-instanton Floer homology groups of X. Boden and Herald [9] defined an invariant for

homology 3-spheres as a virtual count of $\mathcal{M}_P^{\mathrm{fl}}$ for $P \to X$ the trivial SU(3)-bundle, and there are extensions to 3-manifolds other than homology 3-spheres using flat connections on U(2)-bundles and SO(3)-bundles, [17, §5.6].

Choose a Riemannian metric g on X, and take $E_\bullet$ to be the elliptic operator

$$D = \mathrm{d} + \mathrm{d}^* : \Gamma^\infty(\Lambda^0 T^*X \oplus \Lambda^2 T^*X) \longrightarrow \Gamma^\infty(\Lambda^1 T^*X \oplus \Lambda^3 T^*X).$$

Then the orientation bundle of the derived manifold $\mathcal{M}_P^{\mathrm{fl}}$ is the pullback under the inclusion $\mathcal{M}_P^{\mathrm{fl}} \hookrightarrow \mathcal{M}_P$ of the bundle $O_P^{E_\bullet} \to \mathcal{M}_P$ from Definition 1.2.

Note that X need not be orientable in the next theorem.

Theorem 4.4. **(a)** *In the situation above, suppose $\alpha \in \Gamma^\infty(\Lambda^2 T^*X)$ is a non-vanishing 2-form with $|\alpha|_g \equiv 1$. Such α exist for any compact Riemannian 3-manifold (X, g). Then there are unique complex vector bundle structures on E_0, E_1 such that $i \cdot 1 = \alpha$ in E_0 and the symbol of D is complex linear.*

Thus for any Lie group G and principal G-bundle $P \to X$, Theorem 2.5 defines a canonical n-orientation on $\mathcal{M}_P$.

(b) *If G is connected then the n-orientation on $\mathcal{M}_P$ in* **(a)** *is independent of α.*

Proof. For (a), at a point $x \in X$, choose an orthonormal basis e_1, e_2, e_3 for T_x^*X with $\alpha|_x = e_1 \wedge e_2$. Then define $\mathbb{C}$-vector space structures on $E_0|_x, E_1|_x$ by

$$i \cdot 1 = e_1 \wedge e_2 = \alpha|_x, \;\; i \cdot e_1 \wedge e_3 = e_2 \wedge e_3, \;\; i \cdot e_1 = e_2, \;\; i \cdot e_3 = e_1 \wedge e_2 \wedge e_3.$$

It is easy to check that these are independent of the choice of (e_1, e_2, e_3), so over all $x \in X$ they extend to complex vector bundle structures on E_0, E_1, and the symbol of D is complex linear. Part (a) follows.

For (b), let (X, g) be a compact Riemannian 3-manifold, G a connected Lie group, and $P \to X$ a principal G-bundle. By Proposition 3.5(a) with $d = 2$ we can choose a 1-skeleton $Y \subset X$, a CW complex of dimension 1, such that $P|_{X \setminus Y}$ is trivial. As in Step 3 of §3.3, choose a small open neighbourhood U of Y in X such that U retracts onto Y, and an open $V \subset X$ with $\overline{V} \subseteq X \setminus Y$ and $U \cup V = X$, and a connection ∇_P on $P \to X$ which is trivial over $V \subset X \setminus Y$, using the chosen trivialization of $P|_{X \setminus Y}$.

Suppose $\alpha^+, \alpha^- \in \Gamma^\infty(\Lambda^2 T^*X)$ are nonvanishing 2-forms with $|\alpha^\pm|_g = 1$. In general, α^+, α^- are not isotopic through nonvanishing 2-forms on X. However, $\alpha^+|_Y, \alpha^-|_Y$ are isotopic through nonvanishing sections of $\Lambda^2 T^*X|_Y$ over the 1-skeleton Y, as $\Lambda^2 T^*X$ has rank 3. As U retracts onto Y, it follows that $\alpha^+|_U$ and $\alpha^-|_U$ are isotopic through nonvanishing sections of $\Lambda^2 T^*X|_U$. So after a continuous deformation of α^-, we can suppose that $\alpha^+|_U = \alpha^-|_U$.

Part (a) gives complex structures $J^\pm$ on $E_\bullet$ coming from $\alpha^\pm$, with $J^+|_U = J^-|_U$ as $\alpha^+|_U = \alpha^-|_U$. Write $\check{\omega}_P^+, \check{\omega}_P^-$ for the n-orientations on $\mathcal{M}_P$ given by Theorem 2.5 using J^+, J^-. Theorem 3.3(a) with $X, U, V, E_\bullet, P, \nabla_P$ in place of $X^+, U^+, V^+, P^+, \nabla_{P^+}$ now gives an isomorphism $\check{O}_P^{E_\bullet}|_{[\nabla_P]} \cong \mathbb{Z}_2$, depending only on the complex structure $J^+|_U = J^-|_U$ on $E_\bullet|_U$. Theorem 3.3(b) for J^+ and J^- implies that this agrees with $\check{\omega}_P^+|_{[\nabla_P]}$ and $\check{\omega}_P^-|_{[\nabla_P]}$, so $\check{\omega}_P^+ = \check{\omega}_P^-$ as $\mathcal{M}_P$ is connected. This proves (b). $\square$

To pass from an n-orientation of $\mathcal{M}_P$ to an orientation of $\mathcal{M}_P$, by (1.4) and (1.7) we need to choose an orientation for $\det D$, noting that as $\operatorname{ind} D = 0$ on any 3-manifold X we do not need an orientation on $\mathfrak{g}$. Since orientations on $\mathcal{M}_P$ pull back to orientations of $\mathcal{M}_P^{\mathrm{fl}}$, we deduce:

Theorem 4.5. *Let (X, g) be a compact Riemannian 3-manifold, and choose an orientation for $\det D$ (equivalently, an orientation on $\bigoplus_{k=0}^{3} H^k(X, \mathbb{R})$). Then for any connected Lie group G and principal G-bundle $P \to X$ we can construct a canonical orientation for the derived manifold $\mathcal{M}_P^{\mathrm{fl}}$.*

Here is how this relates to results in the literature: when X is oriented and $G = \mathrm{SU}(2)$, Taubes [53, Prop. 2.1] shows $\mathcal{M}_P$ is orientable, and then uses the standard orientation to get canonical orientations as any $\mathrm{SU}(2)$-bundle $P \to X$ is trivial, [53, p. 554-5]. Boden and Herald [9, §4] prove the analogue for $G = \mathrm{SU}(3)$. We believe Theorem 4.5 may be new for non-orientable X, and also for non-simply-connected G, when $P \to X$ need not be trivial, so standard orientations do not suffice to define canonical orientations.

4.2.3 Anti-self-dual instantons on 4-manifolds

Let (X, g) be a compact, oriented Riemannian 4-manifold, G a Lie group, and $P \to X$ a principal G-bundle. In Example 4.1 we defined the moduli space $\mathcal{M}_P^{\mathrm{asd}}$ of irreducible anti-self-dual instantons on P, and explained that for generic g it is a smooth manifold, whose orientation bundle is the pullback of $O_P^{E_\bullet} \to \mathcal{M}_P$ from Definition 1.2 under the inclusion $\mathcal{M}_P^{\mathrm{asd}} \hookrightarrow \mathcal{M}_P$, for $E_\bullet$ as in (4.2).

Instanton moduli spaces $\mathcal{M}_P^{\mathrm{asd}}$ are used to define Donaldson invariants of 4-manifolds X, as in [15,16,18], which can distinguish different smooth structures on homeomorphic 4-manifolds X_1, X_2. Most work in the area focusses on $G = \mathrm{SU}(2)$ and $G = \mathrm{SO}(3)$, although Kronheimer [35] extends the definition to $G = \mathrm{SU}(m)$ and $\mathrm{PSU}(m)$. Orientations on $\mathcal{M}_P^{\mathrm{asd}}$ are needed to determine the sign of the Donaldson invariants, and have been well studied. Many of the methods of §2–§3 were first used in Donaldson theory.

Theorem 4.6. *Let (X, g) be a compact, oriented Riemannian 4-manifold, and $D, E_\bullet$ be as in (4.2).*

(a) *Suppose J is an almost complex structure on X which is Hermitian with respect to g and compatible with the orientation. Then there are unique complex vector bundle structures on E_0, E_1 such that J acts on $E_1 = T^*X$ by multiplication by i, and the symbol of D is complex linear.*

Thus for any Lie group G and principal G-bundle $P \to X$, Theorem 2.5 defines a canonical n-orientation on $\mathcal{M}_P$.

(b) *Now let G be a connected Lie group, and choose a Spin^c-structure $\mathfrak{s}$ on X, noting that Spin^c-structures exist for any (X, g). Then for all principal G-bundles $P \to X$, we can construct a canonical n-orientation on $\mathcal{M}_P$.*

In the situation of **(a)**, *the almost complex structure J induces a Spin^c-structure $\mathfrak{s}_J$ on X, and the n-orientation on $\mathcal{M}_P$ from* **(a)** *agrees with the n-orientation constructed above from the Spin^c-structure $\mathfrak{s}_J$.*

(c) *If G is also simply-connected, or if $G = \mathrm{U}(m)$, then the n-orientation on $\mathcal{M}_P$ in* **(b)** *is independent of the choice of* Spin^c*-structure* $\mathfrak{s}$.

Proof. For (a), given J we can put complex structures on E_0, E_1, such that the symbol of $E_\bullet$ agrees with that of $\bar\partial + \bar\partial^* : \Gamma^\infty(\Lambda^{0,0} \oplus \Lambda^{0,2}) \to \Gamma^\infty(\Lambda^{0,1})$, which is complex linear. So (a) holds by Theorem 2.5.

For (b), choose $G, \mathfrak{s}, P \to X$ as in the theorem. Let ∇_P be any connection on P. Proposition 3.5(a) with $d = 2$ gives a 2-skeleton $Y \subset X$, which is a CW-complex of dimension 2, and a trivialization $\Phi : P|_{X\setminus Y} \to (X \setminus Y) \times G$. As in Step 3 of §3.3, choose a small open neighbourhood U of Y in X such that U retracts onto Y, an open $V \subset X$ with $\overline{V} \subseteq X \setminus Y$ and $U \cup V = X$, and a connection $\hat\nabla_P$ on $P \to X$ which is trivial over $V \subset X \setminus Y$, using the chosen trivialization Φ of $P|_{X\setminus Y}$.

Recall from [40, 42] that a Spin^c-structure $\mathfrak{s}$ on (X, g) consists of rank 2 Hermitian vector bundles $S^{\mathfrak{s}}_\pm \to X$ and a Clifford multiplication map $T^*X \to \mathrm{Hom}_{\mathbb{C}}(S^{\mathfrak{s}}_+, S^{\mathfrak{s}}_-)$. They are related to almost complex structures in the following way. The projective space bundle $\mathbb{P}_{\mathbb{C}}(S^{\mathfrak{s}}_+) \to X$ is naturally isomorphic to the bundle of oriented Hermitian almost complex structures J on (X, g), as used in (a). Hence, oriented Hermitian almost complex structures J on X correspond to complex line subbundles $L_J \subset S^{\mathfrak{s}}_+$. Such a J determines a Spin^c-structure $\mathfrak{s}_J$, unique up to isomorphism, for which L_J is the trivial line bundle. Thus, if $S^{\mathfrak{s}}_+$ has a nonvanishing section $\sigma \in \Gamma^\infty(S^{\mathfrak{s}}_+)$, this determines a unique oriented Hermitian almost complex structure J on (X, g), and $\mathfrak{s} \cong \mathfrak{s}_J$.

For general X there need not exist nonvanishing sections $\sigma \in \Gamma^\infty(S^{\mathfrak{s}}_+)$, and if they exist they need not be unique up to isotopy. But $S^{\mathfrak{s}}_+|_Y$ has nonvanishing sections which are unique up to isotopy over the 2-skeleton Y, as $S^{\mathfrak{s}}_+$ has real rank $4 > 2 + 1$. Since U retracts onto Y, it follows that $S^{\mathfrak{s}}_+|_U$ has nonvanishing sections $\sigma \in \Gamma^\infty(S^{\mathfrak{s}}_+|_U)$ which are unique up to isotopy. These yield oriented Hermitian almost complex structures J_σ on $(U, g|_U)$, which are unique up to isotopy, and determine $\mathfrak{s}|_U$ up to isomorphism.

Apply Theorem 3.3(a) with $X, U, V, E_\bullet, P, \hat\nabla_P, \Phi|_V$ in place of $X^+, U^+, V^+, E^+_\bullet, P^+, \nabla_{P^+}, \tau^+$, and using the complex structure on $E_\bullet|_U$ induced by J_σ for some nonvanishing $\sigma \in \Gamma^\infty(S^{\mathfrak{s}}_+|_U)$. This gives an isomorphism

$$\check{O}^{E_\bullet}_P\big|_{[\hat\nabla_P]} \cong \mathbb{Z}_2. \tag{4.4}$$

We then determine the isomorphism $\check{O}^{E_\bullet}_P|_{[\nabla_P]} \cong \mathbb{Z}_2$ by choosing a smooth path from $\hat\nabla_P$ to ∇_P in the (contractible) space of connections on $P \to X$, and deforming the n-orientation continuously along this path. This constructs an n-orientation at any point $[\nabla_P]$ in $\mathcal{M}_P$.

We will show this n-orientation at $[\nabla_P]$ is independent of choices in its construction, following Step 4 of §3.3. Suppose that $Y_0, \Phi_0, U_0, V_0, \hat\nabla_{P,0}, \sigma_0$ and $Y_1, \Phi_1, U_1, V_1, \hat\nabla_{P,1}, \sigma_1$ are alternative choices above. Then Proposition 3.5(b) with $d = 2$ gives a 3-skeleton $Z \subset X \times [0, 1]$ interpolating between $Y_0 \times \{0\}$ and $Y_1 \times \{1\}$, and a trivialization $\Psi : \pi^*_X(P)|_{(X\times[0,1])\setminus Z} \to ((X \times [0, 1]) \setminus Z) \times G$ interpolating between Φ_0 and Φ_1. Choose a small open neighbourhood W of Z in $X \times [0, 1]$ which interpolates between U_0 and U_1 and retracts onto Z.

As Z is a CW-complex of dimension 3, there exist nonvanishing sections of the rank 4 bundle $\pi_X^*(S_+^{\mathfrak{s}})|_Z$, and we can choose them to interpolate between given nonvanishing sections on Y_0 and Y_1. Since W retracts onto Z, it follows that there exist nonvanishing sections τ of $\pi_X^*(S_+^{\mathfrak{s}})|_W$, and we can choose τ with $\tau|_{Y_i\times\{i\}} = \sigma_i$ for $i = 0, 1$.

We can now choose data $\Phi_t, U_t, V_t, \hat{\nabla}_{P,t}, \sigma_t$ as above, depending smoothly on $t \in [0, 1]$, and interpolating between $\Phi_0, U_0, V_0, \hat{\nabla}_{P,0}, \sigma_0$ and $\Phi_1, U_1, V_1, \hat{\nabla}_{P,1}, \sigma_1$, where $\Phi_t(x) = \Psi(x, t)$ for $(x, t) \in (X\times[0, 1])\backslash Z$, and $U_t = \bigl\{x \in X : (x, t) \in W\bigr\}$, and $\sigma_t(x) = \tau(x, t)$ for $x \in U_t$. So as in (4.4) we get isomorphisms for $t \in [0, 1]$

$$\check{O}_P^{E\bullet}\big|_{[\hat{\nabla}_{P,t}]} \cong \mathbb{Z}_2, \tag{4.5}$$

which depend continuously on $t \in [0, 1]$ by strong functoriality in Theorem 3.3(a). We choose a smooth path from $\hat{\nabla}_{P,t}$ to ∇_P in the (contractible) space of connections on $P \to X$, depending smoothly on $t \in [0, 1]$, and interpolating between the previous choices when $t = 0$ and $t = 1$.

Deforming the trivialization of $\check{O}_P^{E\bullet}|_{[\hat{\nabla}_{P,t}]}$ along this path gives a continuous family of trivializations of $\check{O}_P^{E\bullet}|_{[\nabla_P]}$ for $t \in [0, 1]$ which interpolate between the two previous choices at $t = 0$ and $t = 1$. Hence the trivialization of $\check{O}_P^{E\bullet}|_{[\nabla_P]}$ defined in the first part of the proof is independent of choices in its construction, and is well defined. These trivializations clearly depend continuously on $[\nabla_P]$ in $\mathcal{M}_P$, and so define a canonical n-orientation on $\mathcal{M}_P$, for all principal G-bundles $P \to X$. This completes the first part of (b).

For the second part, let J be an almost complex structure on X, inducing a Spin^c-structure $\mathfrak{s}_J$. Then in the definition of the orientation on $\mathcal{M}_P$ above, we can take $J_\sigma = J|_U$. So by Theorem 3.3(b), the isomorphism (4.4) agrees with natural n-orientation defined by Theorem 2.5 at $[\hat{\nabla}_P]$, so deforming along the path from $[\hat{\nabla}_P]$ to $[\nabla_P]$, part (b) follows.

For (c), if G is simply-connected, then in the proof above we can apply Proposition 3.5(a) with $d = 4$ instead of $d = 2$. So $Y \subset X$ is a 0-skeleton, and the almost complex structure J_σ on U is unique up to isotopy, and independent of $\mathfrak{s}$. Thus the orientation on $\mathcal{M}_P$ is independent of $\mathfrak{s}$. The case $G = \mathrm{U}(m)$ follows from Example 2.12 and $G = \mathrm{SU}(m + 1)$, which is simply-connected. □

As in Definition 1.2, we can convert n-orientations on $\mathcal{M}_P$ to orientations on $\mathcal{M}_P$ by choosing orientations on $\det D$ and $\mathfrak{g}$. Since orientations on $\mathcal{M}_P$ pull back to orientations of $\mathcal{M}_P^{\mathrm{asd}}$, we deduce:

Theorem 4.7. *Let (X, g) be a compact, oriented Riemannian 4-manifold.*

- **(a)** *Let G be a connected Lie group, and choose an orientation on $\det D$ (equivalently, an orientation on $H^0(X) \oplus H^1(X) \oplus H^2_+(X)$) and on $\mathfrak{g}$, and a Spin^c-structure $\mathfrak{s}$ on X. Then for all principal G-bundles $P \to X$, we can construct a canonical orientation on $\mathcal{M}_P^{\mathrm{asd}}$.*
- **(b)** *If G is also simply-connected, or if $G = \mathrm{U}(m)$, then the orientation on $\mathcal{M}_P^{\mathrm{asd}}$ in* **(a)** *is independent of the choice of Spin^c-structure $\mathfrak{s}$.*

Here is how this relates to results in the literature: part (b) is proved for $G = \mathrm{SU}(m)$ and X simply-connected by Donaldson [15, II.4] using the methods of Lemma 2.4 and §2.2.9, for $G = \mathrm{U}(m), \mathrm{SU}(m)$ and X arbitrary by Donaldson [16, §3(d)] using the method of §3, and for X simply-connected and G a simply-connected, simple Lie group by Donaldson and Kronheimer [18, §5.4]. So far as the authors know, part (a) is new, both the orientability of $\mathcal{M}_P$, and the use of Spin^{c}-structures in constructing canonical orientations.

4.2.4 Seiberg–Witten theory on 4-manifolds

We will explain the Seiberg–Witten $\mathrm{U}(m)$-monopole equations, following Zentner [62]. Most of the literature on Seiberg–Witten theory (see e.g. Morgan [40] and Nicolaescu [42]) is concerned with the case $m = 1$ of these. Let (X, g) be a compact, oriented Riemannian 4-manifold. Fix a Spin^{c}-*structure* $\mathfrak{s}$ on (X, g), consisting of rank 2 Hermitian vector bundles $S^{\mathfrak{s}}_{\pm} \to X$ with a given connection $\nabla_{S^{\mathfrak{s}}_+}$ on $S^{\mathfrak{s}}_+$, and a Clifford multiplication map $T^*X \to \mathrm{Hom}_{\mathbb{C}}(S^{\mathfrak{s}}_+, S^{\mathfrak{s}}_-)$.

Let $P \to X$ be a principal $\mathrm{U}(m)$-bundle, and $E = (P \times \mathbb{C}^m)/\mathrm{U}(m)$ the associated rank m complex vector bundle $E \to X$. A *Seiberg–Witten* $\mathrm{U}(m)$-*monopole* on P is a pair (∇_P, Ψ) of a connection ∇_P on $P \to X$, and a section $\Psi \in \Gamma^\infty(S^{\mathfrak{s}}_+ \otimes_{\mathbb{C}} E)$, satisfying the *Seiberg–Witten equations*

$$\not{D}^{\nabla_P}\Psi = 0, \qquad F^{\nabla_P}_+ = q(\Psi), \tag{4.6}$$

where $\not{D}^{\nabla_P} : \Gamma^\infty(S^{\mathfrak{s}}_+ \otimes_{\mathbb{C}} E) \to \Gamma^\infty(S^{\mathfrak{s}}_- \otimes_{\mathbb{C}} E)$ is a twisted Dirac operator defined using $\nabla_{S^{\mathfrak{s}}_+}$ and ∇_P, and $q : S^{\mathfrak{s}}_+ \otimes_{\mathbb{C}} E \to \mathrm{Ad}(P) \otimes \Lambda^2_+ T^*X$ is a certain real quadratic bundle morphism.

The gauge group $\mathcal{G} = \mathrm{Map}_{C^\infty}(X, \mathrm{U}(m))$ acts on the family of Seiberg–Witten monopoles (∇_P, Ψ). We call (∇_P, Ψ) *irreducible* if its stabilizer group in $\mathcal{G}$ is trivial. Define $\mathcal{M}^{\mathrm{SW}}_{P,\mathfrak{s}}$ to be the moduli space of gauge equivalence classes $[\nabla_P, \Psi]$ of irreducible $\mathrm{U}(m)$-monopoles (∇_P, Ψ). If $g, \nabla_{S^{\mathfrak{s}}_+}$ are generic then $\mathcal{M}^{\mathrm{SW}}_{P,\mathfrak{s}}$ is a smooth manifold. There is a forgetful map $\mathcal{M}^{\mathrm{SW}}_{P,\mathfrak{s}} \to \mathcal{M}_P$ taking $[\nabla_P, \Psi] \mapsto [\nabla_P]$, for $\mathcal{M}_P$ as in Definition 1.1. The moduli spaces $\mathcal{M}^{\mathrm{SW}}_{P,\mathfrak{s}}$ for $m = 1$ (with orientations as below) are used to define the *Seiberg–Witten invariants* of X, [40, 42], which are closely related to Donaldson invariants in §4.2.3, but are technically less demanding.

To understand orientations for $\mathcal{M}^{\mathrm{SW}}_{P,\mathfrak{s}}$, observe that the linearization of the Seiberg–Witten equations (4.6) is isotopic to the direct sum of the Atiyah–Hitchin–Singer complex (4.1), and $\not{D}^{\nabla_P} : \Gamma^\infty(S^{\mathfrak{s}}_+ \otimes_{\mathbb{C}} E) \to \Gamma^\infty(S^{\mathfrak{s}}_- \otimes_{\mathbb{C}} E)$. Hence the orientation bundle of $\mathcal{M}^{\mathrm{SW}}_{P,\mathfrak{s}}$ is the tensor product of orientation bundles from these two factors. The first is the pullback of the orientation bundle $O^E_P{}^{\bullet} \to \mathcal{M}_P$ for anti-self-dual instantons in §4.2.3 under the forgetful map $\mathcal{M}^{\mathrm{SW}}_{P,\mathfrak{s}} \to \mathcal{M}_P$. The second is trivial by Theorem 2.5, as the symbol of $\not{D}^{\nabla_P}$ is complex linear. Thus, by Theorem 4.6(c), if we choose an orientation on $\det D$ (equivalently, an orientation on $H^0(X) \oplus H^1(X) \oplus H^2_+(X)$), we have canonical orientations on all moduli spaces $\mathcal{M}^{\mathrm{SW}}_{P,\mathfrak{s}}$. When $m = 1$ this is proved by Morgan [40, Cor. 6.6.3].

4.2.5 The Vafa–Witten equations on 4-manifolds

Let (X,g) be a compact, oriented Riemannian 4-manifold, G a Lie group, and $P\rightarrow X$ a principal G-bundle. The *Vafa–Witten equations* concern triples (∇_P,α,β) of a connection ∇_P on P and sections $\alpha\in\Gamma^\infty(\mathrm{Ad}(P)))$ and $\beta\in\Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^2_+T^*X)$ satisfying

$$\mathrm{d}_{\nabla_P}\alpha+\mathrm{d}^*_{\nabla_P}\beta=0,\qquad F_+^{\nabla_P}+\tfrac12[\beta,\alpha]+\tfrac18[\beta.\beta]=0,$$

where $[\beta.\beta]\in\Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^2_+T^*X)$. They were introduced by Vafa and Witten [55] in the study of the S-duality conjecture in $\mathcal{N}=4$ super Yang–Mills theory. See Mares [37, §A.1.6] or Tanaka [51, §2] for more details.

The gauge group $\mathcal{G}=\mathrm{Map}_{C^\infty}(X,G)$ acts on the family of Vafa–Witten solutions (∇_P,α,β). We call (∇_P,α,β) *irreducible* if its stabilizer group in $\mathcal{G}$ is trivial. Define $\mathcal{M}_P^{\mathrm{VW}}$ to be the moduli space of gauge equivalence classes $[\nabla_P,\alpha,\beta]$ of irreducible Vafa–Witten solutions (∇_P,α,β). It is a derived manifold. There is a forgetful map $\mathcal{M}_P^{\mathrm{VW}}\rightarrow\mathcal{M}_P$ taking $[\nabla_P,\alpha,\beta]\mapsto[\nabla_P]$, for $\mathcal{M}_P$ as in Definition 1.1.

The deformations of $[\nabla_P,\alpha,\beta]$ in $\mathcal{M}_P^{\mathrm{VW}}$ are controlled by the elliptic complex

$$\begin{aligned}0\longrightarrow\Gamma^\infty\bigl(\mathrm{Ad}(P)\otimes\Lambda^0T^*X\bigr)&\xrightarrow{(\mathrm{d}_{\nabla_P}\,0\,0)}\Gamma^\infty\bigl(\mathrm{Ad}(P)\otimes(\Lambda^1T^*X\oplus\Lambda^0T^*X\oplus\Lambda^2_+T^*X)\bigr)\\ &\xrightarrow{\begin{pmatrix}0&\mathrm{d}_{\nabla_P}&\mathrm{d}^*_{\nabla_P}\\ \mathrm{d}_{\nabla_P}&0&0\end{pmatrix}}\Gamma^\infty(\mathrm{Ad}(P)\otimes(\Lambda^1T^*X\oplus\Lambda^2_+T^*X))\longrightarrow 0,\end{aligned}$$

modulo degree 0 operators. As in §4.2.1–§4.2.4, the orientation bundle of $\mathcal{M}_P^{\mathrm{VW}}$ is the pullback of the orientation bundle $O_P^{E_\bullet}\rightarrow\mathcal{M}_P$ under the forgetful map $\mathcal{M}_P^{\mathrm{VW}}\rightarrow\mathcal{M}_P$, where $E_\bullet$ is the elliptic operator

$$\begin{aligned}\begin{pmatrix}\mathrm{d}&\mathrm{d}^*&0\\0&0&\mathrm{d}^*\\0&0&\mathrm{d}_+\end{pmatrix}:\Gamma^\infty\bigl(\Lambda^0T^*X\oplus\Lambda^2_+T^*X\oplus\Lambda^1T^*X\bigr)&\longrightarrow\\ \Gamma^\infty\bigl(\Lambda^1T^*X\oplus\Lambda^0T^*X\oplus\Lambda^2_+T^*X\bigr).&\end{aligned}$$

Then $E_\bullet=\tilde E_\bullet\oplus\tilde E_\bullet^*$, for $\tilde E_\bullet$ as in (4.2). Hence §2.2.5 shows that $O_P^{E_\bullet}$ is canonically trivial. This proves:

Theorem 4.8. *The Vafa–Witten moduli spaces $\mathcal{M}_P^{\mathrm{VW}}$ have canonical orientations for all G and $P\rightarrow X$.*

The authors believe Theorem 4.8 is new.

4.2.6 The Kapustin–Witten equations on 4-manifolds

Let (X,g) be a compact, oriented Riemannian 4-manifold, G a Lie group, and $P\rightarrow X$ a principal G-bundle. The *Kapustin–Witten equations* concern pairs (∇_P,ϕ) of a connection ∇_P on P and a section $\phi\in\Gamma^\infty(\mathrm{Ad}(P)\otimes T^*X)$ satisfying

$$\begin{aligned}\mathrm{d}^*_{\nabla_P}\phi=0\quad\text{in }\Gamma^\infty(\mathrm{Ad}(P)),\qquad\mathrm{d}^-_{\nabla_P}\phi=0\quad\text{in }\Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^2_-T^*X),\\ \text{and}\qquad F_+^{\nabla_P}+[\phi,\phi]_+=0\quad\text{in }\Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^2_+T^*X).\end{aligned}$$

They were introduced by Kapustin and Witten in [33, §3.3], coming from topologically twisted $\mathcal{N}=4$ super Yang–Mills theory, and are studied in Gagliardo and Uhlenbeck [23], Taubes [54], and Tanaka [52].

The gauge group $\mathcal{G}=\mathrm{Map}_{C^\infty}(X,G)$ acts on the family of Kapustin–Witten solutions (∇_P,ϕ). We call (∇_P,ϕ) *irreducible* if its stabilizer group in $\mathcal{G}$ is trivial. Define $\mathcal{M}_P^{\mathrm{KW}}$ to be the moduli space of gauge equivalence classes $[\nabla_P,\phi]$ of irreducible Kapustin–Witten solutions (∇_P,ϕ), as a derived manifold. There is a forgetful map $\mathcal{M}_P^{\mathrm{KW}}\rightarrow\mathcal{M}_P$ taking $[\nabla_P,\phi]\mapsto[\nabla_P]$, for $\mathcal{M}_P$ as in Definition 1.1.

The deformations of $[\nabla_P,\phi]$ in $\mathcal{M}_P^{\mathrm{KW}}$ are controlled by the elliptic complex

$$\begin{aligned}0\longrightarrow\Gamma^\infty\bigl(\mathrm{Ad}(P)\otimes\Lambda^0T^*X\bigr)&\xrightarrow{(\mathrm{d}_{\nabla_P}\ 0)}\Gamma^\infty\bigl(\mathrm{Ad}(P)\otimes(\Lambda^1T^*X\oplus\Lambda^1T^*X)\bigr)\\&\xrightarrow{\begin{pmatrix}0&\mathrm{d}^*_{\nabla_P}\\ \mathrm{d}^+_{\nabla_P}&0\\ 0&\mathrm{d}^-_{\nabla_P}\end{pmatrix}}\Gamma^\infty(\mathrm{Ad}(P)\otimes(\Lambda^0T^*X\oplus\Lambda^2_+T^*X\oplus\Lambda^2_-T^*X))\longrightarrow 0,\end{aligned}$$

modulo degree 0 operators. As in §4.2.1–§4.2.5, the orientation bundle of $\mathcal{M}_P^{\mathrm{KW}}$ is the pullback of the orientation bundle $O_P^{E_\bullet}\rightarrow\mathcal{M}_P$ under the forgetful map $\mathcal{M}_P^{\mathrm{KW}}\rightarrow\mathcal{M}_P$, where $E_\bullet$ is the elliptic operator

$$\begin{aligned}\begin{pmatrix}0&\mathrm{d}_+&0&\mathrm{d}^*\\ \mathrm{d}^*&0&\mathrm{d}_-&0\end{pmatrix}:\Gamma^\infty\bigl(\Lambda^0T^*X\oplus\Lambda^2_+T^*X\oplus\Lambda^2_-T^*X\oplus\Lambda^0T^*X\bigr)\\ \longrightarrow\Gamma^\infty\bigl(\Lambda^1T^*X\oplus\Lambda^1T^*X\bigr).\end{aligned}$$

This is a direct sum $E_\bullet=E_\bullet^+\oplus E_\bullet^-$, where $E_\bullet^+$ is as in (4.2), and $E_\bullet^-$ is $E_\bullet^+$ for the opposite orientation on X. Hence $O_P^{E_\bullet}\cong O_P^{E_\bullet^+}\otimes_{\mathbb{Z}_2}O_P^{E_\bullet^-}$. As Theorem 4.6 describes orientations for $O_P^{E_\bullet^\pm}$, we deduce:

Theorem 4.9. *Let (X,g) be a compact, oriented Riemannian 4-manifold.*

- **(a)** *Let G be a connected Lie group, and choose an orientation on $\det D$ (equivalently, an orientation on $H^2(X)$) and on $\mathfrak{g}$, and a Spin^c-structure $\mathfrak{s}$ on X. Then for all principal G-bundles $P\rightarrow X$, we can construct a canonical orientation on $\mathcal{M}_P^{\mathrm{KW}}$, as a derived manifold.*
- **(b)** *If G is also simply-connected, or if $G=\mathrm{U}(m)$, then the orientation on $\mathcal{M}_P^{\mathrm{KW}}$ in* **(a)** *is independent of the choice of Spin^c-structure $\mathfrak{s}$.*

The authors believe Theorem 4.9 is new.

4.2.7 The Haydys–Witten equations on 5-manifolds

The *Haydys–Witten equations* are a 5-dimensional gauge theory introduced independently by Haydys [25, §3] and Witten [61, §5.2.6]. Both Haydys and Witten work on oriented 5-manifolds. We will give a different presentation of the Haydys–Witten equations to [25, 61], which also works for non-oriented 5-manifolds, and is equivalent to [25, 61] in the oriented case.

Let (X,g) be a compact Riemannian 5-manifold, and $\alpha\in\Gamma^\infty(\Lambda^4T^*X)$ be a nonvanishing 4-form with $|\alpha|_g\equiv 1$. Then there are orthogonal splittings

$$\begin{gathered}\Lambda^1T^*X=\Lambda^1_\times T^*X\oplus\Lambda^1_0T^*X,\ \ \Lambda^2T^*X=\Lambda^2_+T^*X\oplus\Lambda^2_-T^*X\oplus\Lambda^2_0T^*X,\\ \Lambda^3T^*X=\Lambda^3_+T^*X\oplus\Lambda^3_-T^*X\oplus\Lambda^3_0T^*X,\end{gathered}\tag{4.7}$$

where $\Lambda^1_\times T^*X$ has rank 1, $\Lambda^j_\pm T^*X$ have rank 3, and $\Lambda^j_0T^*X$ have rank 4. They may be described explicitly as follows: if $x\in X$ and $(e_1,\ldots,e_5)$ is an orthonormal basis of T^*_xX with $\alpha|_x=e_1\wedge e_2\wedge e_3\wedge e_4$, then

$$\begin{aligned}&\Lambda^1_\times T^*_xX=\langle e_5\rangle_{\mathbb{R}},\ \ \Lambda^1_0T^*_xX=\langle e_1,e_2,e_3,e_4\rangle_{\mathbb{R}},\ \ \Lambda^2_0T^*_xX=\langle e_{15},e_{25},e_{35},e_{45}\rangle_{\mathbb{R}},\\ &\Lambda^2_+T^*_xX=\langle e_{12}+e_{34},e_{13}+e_{42},e_{14}+e_{23}\rangle_{\mathbb{R}},\\ &\Lambda^2_-T^*_xX=\langle e_{12}-e_{34},e_{13}-e_{42},e_{14}-e_{23}\rangle_{\mathbb{R}},\\ &\Lambda^3_+T^*_xX=\langle e_{125}+e_{345},e_{135}+e_{425},e_{145}+e_{235}\rangle_{\mathbb{R}},\\ &\Lambda^3_-T^*_xX=\langle e_{125}+e_{345},e_{135}+e_{425},e_{145}+e_{235}\rangle_{\mathbb{R}},\\ &\Lambda^3_0T^*_xX=\langle e_{234},e_{341},e_{412},e_{123}\rangle_{\mathbb{R}},\end{aligned}\tag{4.8}$$

where $e_{ij\cdots l}$ means $e_i\wedge e_j\wedge\cdots\wedge e_l$.

Let G be a Lie group and $P\to X$ be a principal G-bundle. The *Haydys–Witten equations* concern pairs (∇_P,ψ) of a connection ∇_P of P and a section $\psi\in\Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^3_+T^*X)$ satisfying

$$\begin{aligned}F_+^{\nabla_P}+(\mathrm{d}^*_{\nabla_P}\psi)_++q(\psi)&=0\quad\text{in }\Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^2_+T^*X),\\ F_0^{\nabla_P}+(\mathrm{d}^*_{\nabla_P}\psi)_0&=0\quad\text{in }\Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^2_0T^*X),\end{aligned}$$

for $q:\mathrm{Ad}(P)\otimes\Lambda^3_+T^*X\to\mathrm{Ad}(P)\otimes\Lambda^2_+T^*X$ a certain quadratic bundle map. They were introduced independently by Haydys [25, §3] and Witten [61, §5.2.6].

The gauge group $\mathcal{G}=\mathrm{Map}_{C^\infty}(X,G)$ acts on the family of Haydys–Witten solutions (∇_P,ψ). We call (∇_P,ψ) *irreducible* if its stabilizer group in $\mathcal{G}$ is trivial. Define $\mathcal{M}_P^{\mathrm{HW}}$ to be the moduli space of gauge equivalence classes $[\nabla_P,\psi]$ of irreducible Haydys–Witten solutions (∇_P,ϕ). It is a derived manifold of virtual dimension 0. There is a forgetful map $\mathcal{M}_P^{\mathrm{HW}}\to\mathcal{M}_P$ taking $[\nabla_P,\psi]\mapsto[\nabla_P]$, for $\mathcal{M}_P$ as in Definition 1.1.

The deformations of $[\nabla_P,\psi]$ in $\mathcal{M}_P^{\mathrm{HW}}$ are controlled by the elliptic complex

$$\begin{aligned}0\longrightarrow\Gamma^\infty\bigl(\mathrm{Ad}(P)\otimes\Lambda^0T^*X\bigr)&\xrightarrow{(\mathrm{d}_{\nabla_P}\ 0)}\Gamma^\infty\bigl(\mathrm{Ad}(P)\otimes(\Lambda^1T^*X\oplus\Lambda^3_+T^*X)\bigr)\\ &\xrightarrow{\begin{pmatrix}(\mathrm{d}_{\nabla_P})_+&(\mathrm{d}^*_{\nabla_P})_+\\(\mathrm{d}_{\nabla_P})_0&(\mathrm{d}^*_{\nabla_P})_0\end{pmatrix}}\Gamma^\infty(\mathrm{Ad}(P)\otimes(\Lambda^2_+T^*X\oplus\Lambda^2_0T^*X))\longrightarrow 0,\end{aligned}$$

modulo degree 0 operators. As in §4.2.1–§4.2.6, the orientation bundle of $\mathcal{M}_P^{\mathrm{HW}}$ is the pullback of the orientation bundle $O_P^{E_\bullet}\to\mathcal{M}_P$ under the forgetful map $\mathcal{M}_P^{\mathrm{HW}}\to\mathcal{M}_P$, where $E_\bullet$ is the elliptic operator

$$\begin{aligned}D=\begin{pmatrix}\mathrm{d}&\mathrm{d}_0^*&\mathrm{d}_+^*\\0&\mathrm{d}_+&\mathrm{d}_+\end{pmatrix}:&\Gamma^\infty\bigl(\Lambda^0T^*X\oplus\Lambda^2_0T^*X\oplus\Lambda^2_+T^*X\bigr)\\ &\longrightarrow\Gamma^\infty\bigl(\Lambda^1T^*X\oplus\Lambda^3_+T^*X\bigr).\end{aligned}\tag{4.9}$$

Theorem 4.10. *Let (X,g) be a compact Riemannian 5-manifold, and $\alpha \in \Gamma^\infty(\Lambda^4 T^*X)$ a unit length 4-form.*

(a) *Suppose there exists a Hermitian complex structure J on the fibres of $\Lambda^1_0 T^*X$ in (4.7) compatible with the natural orientation on $\Lambda^1_0 T^*X$. (This is equivalent to choosing an* ***almost CR structure*** *on X.) Then using J we can define a complex structure on $E_\bullet$, as in Theorem 2.5.*

Thus for any Lie group G and principal G-bundle $P \to X$, Theorem 2.5 defines a canonical n-orientation on $\mathcal{M}_P$.

(b) *Now let G be a connected, simply-connected Lie group. Then for all principal G-bundles $P \to X$, we can construct a canonical n-orientation on $\mathcal{M}_P$. It agrees with that defined in* **(a)** *when part* **(a)** *applies.*

Proof. For (a), at a point $x \in X$, choose an orthonormal basis $(e_1,\ldots,e_5)$ for $T^*_x X$ with $\alpha|_x = e_1 \wedge e_2 \wedge e_3 \wedge e_4$ and $J(e_1) = e_2$, $J(e_3) = e_4$ in the basis (4.8) for $\Lambda^1_0 T^*_x X$. Define complex vector space structures on $E_0|_x, E_1|_x$ in (4.9) by

$$i \cdot 1 = e_{12} + e_{34},\ i \cdot e_{15} = e_{25},\ i \cdot e_{35} = e_{45},\ i \cdot (e_{13} + e_{42}) = e_{14} + e_{23},$$
$$i \cdot e_5 = e_{125} + e_{345},\ i \cdot e_1 = e_2,\ i \cdot e_3 = e_4,\ i \cdot (e_{135} + e_{425}) = e_{145} + e_{235}.$$

It is easy to check that these are independent of the choice of $(e_1 \ldots, e_5)$, so over all $x \in X$ they extend to complex vector bundle structures on E_0, E_1, and the symbol of D is complex linear. Part (a) follows.

For (b), let G be a connected, simply-connected Lie group, and $P \to X$ a principal G-bundle. By Proposition 3.5(a) with $d = 4$ we can choose a 1-skeleton $Y \subset X$, a CW complex of dimension 1, and a trivialization $\Phi : P|_{X \setminus Y} \to (X \setminus Y) \times G$. As in Step 3 of §3.3, choose a small open neighbourhood U of Y in X such that U retracts onto Y, and an open $V \subset X$ with $\overline{V} \subseteq X \setminus Y$ and $U \cup V = X$, and a connection $\hat{\nabla}_P$ on $P \to X$ which is trivial over $V \subset X \setminus Y$, using the chosen trivialization Φ of $P|_{X \setminus Y}$.

For general X there need not exist oriented Hermitian complex structures J on $\Lambda^1_0 T^*X$, and if they exist they need not be unique up to isotopy. But such complex structures exist on $\Lambda^1_0 T^*X|_Y$ over the 1-skeleton Y, and are unique up to isotopy. As U retracts onto Y, it follows that $\Lambda^1_0 T^*X|_U$ admits oriented Hermitian complex structures J, uniquely up to isotopy. Thus as in (a) we can define a complex structure on $E_\bullet|_U$, unique up to isotopy.

The rest of the proof of (b) closely follows the proof of Theorem 4.6(b) from the paragraph containing (4.4), except that in the paragraph before that containing (4.5), as Z is now a CW-complex of dimension 2, there exist oriented Hermitian complex structures on $\pi_X^*(\Lambda^1_0 T^*X)|_Z$, and we can choose them to interpolate between given complex structures over Y_0 and Y_1. □

To pass from an n-orientation of $\mathcal{M}_P$ to an orientation of $\mathcal{M}_P$, by (1.4) and (1.7) we need to choose an orientation for $\det D$, noting that as $\operatorname{ind} D = 0$ on any 5-manifold X we do not need an orientation on $\mathfrak{g}$. Since orientations on $\mathcal{M}_P$ pull back to orientations of $\mathcal{M}_P^{\mathrm{HW}}$, we deduce:

Theorem 4.11. *Let (X,g) be a compact Riemannian 5-manifold, and $\alpha \in \Gamma^\infty(\Lambda^4 T^*X)$ a unit length 4-form. Suppose G is a connected, simply-connected Lie group, and choose an orientation on $\det D$. Then for all principal G-bundles $P \to X$, we can construct a canonical orientation on $\mathcal{M}_P^{\mathrm{HW}}$.*

The authors believe Theorems 4.10 and 4.11 are new.

4.2.8 Donaldson–Thomas instantons on symplectic 6-manifolds

Let (X,ω) be a symplectic 6-manifold, J a compatible almost complex structure on X, and $g = \omega(-, J-)$ the associated Hermitian metric. The second author [50] introduced 'Donaldson–Thomas instantons' [50], a gauge theory on X, in the hope of defining 'analytic Donaldson–Thomas invariants'.

Let G be a Lie group and $P \to X$ a principal G-bundle. A *Donaldson–Thomas instanton* on P is a pair (∇_P, ξ) of a connection ∇_P on P and a section $\xi \in \Gamma^\infty(\mathrm{Ad}(P) \otimes_{\mathbb{R}} \Lambda^{0,3}T^*X)$ satisfying

$$F_{0,2}^{\nabla_P} = \bar\partial^*_{\nabla_P}\xi, \qquad F_{1,1}^{\nabla_P} \wedge \omega = 0.$$

The gauge group $\mathcal{G} = \mathrm{Map}_{C^\infty}(X,G)$ acts on the family of Donaldson–Thomas instantons (∇_P,ξ). We call (∇_P,ξ) *irreducible* if its stabilizer group in $\mathcal{G}$ is trivial. Write $\mathcal{M}_P^{\mathrm{DT}}$ for the moduli space of gauge equivalence classes $[\nabla_P,\xi]$ of irreducible Donaldson–Thomas instantons (∇_P,ξ). It is a derived manifold. There is a forgetful map $\mathcal{M}_P^{\mathrm{DT}} \to \mathcal{M}_P$ taking $[\nabla_P,\xi] \mapsto [\nabla_P]$.

The deformations of $[\nabla_P,\xi]$ in $\mathcal{M}_P^{\mathrm{DT}}$ are controlled by the elliptic complex

$$\begin{aligned} 0 \longrightarrow \Gamma^\infty\big(\mathrm{Ad}(P)\otimes\Lambda^0T^*X\big) &\xrightarrow{(\mathrm{d}_{\nabla_P}\ 0)} \Gamma^\infty\big(\mathrm{Ad}(P)\otimes(\Lambda^1T^*X\oplus\Lambda^{0,3}T^*X)\big) \\ &\xrightarrow{\begin{pmatrix}\bar\partial^{0,2}_{\nabla_P} & \bar\partial^*_{\nabla_P}\\ \mathrm{d}^{\langle\omega\rangle}_{\nabla_P} & 0\end{pmatrix}} \Gamma^\infty(\mathrm{Ad}(P)\otimes(\Lambda^{0,2}T^*X\oplus\langle\omega\rangle)) \longrightarrow 0, \end{aligned}$$

modulo degree 0 operators. The orientation bundle of $\mathcal{M}_P^{\mathrm{DT}}$ is the pullback of the orientation bundle $O_P^{E_\bullet} \to \mathcal{M}_P$ under the forgetful map $\mathcal{M}_P^{\mathrm{DT}} \to \mathcal{M}_P$, where up to isotopy we may take $E_\bullet$ to be the elliptic operator

$$D = \bar\partial + \bar\partial^* : \Gamma^\infty\big(\Lambda^{0,0}T^*X \oplus \Lambda^{0,2}T^*X\big) \longrightarrow \Gamma^\infty\big(\Lambda^{0,1}T^*X \oplus \Lambda^{0,3}T^*X\big).$$

As D is complex linear, Theorem 2.5 gives canonical orientations on all moduli spaces $\mathcal{M}_P$ and $\mathcal{M}_P^{\mathrm{DT}}$.

4.2.9 G_2-instantons on G_2-manifolds

In the sequel [32] we solve Problem 3.2 for the Dirac operator on 7-manifolds and $G = \mathrm{U}(m)$ or $\mathrm{SU}(m)$, using a variation on the method of §3.3. Here *flag structures* [27, §3.1] are an algebro-topological structure on 7-manifolds X, related to 'linking numbers' of disjoint homologous 3-submanifolds $Y_1, Y_2 \subset X$.

Theorem 4.12 (Joyce and Upmeier [32]). *Suppose X is a compact, oriented, spin Riemannian 7-manifold, and take $E_\bullet$ to be the Dirac operator $D : \Gamma^\infty(S) \to \Gamma^\infty(S)$ on X. Fix an orientation on $\det D$ and a* ***flag structure*** *on X, as in Joyce* [27, §3.1]. *Let G be $\mathrm{U}(m)$ or $\mathrm{SU}(m)$ and $P \to X$ be a principal G-bundle. Then we can construct a canonical orientation on $O_P^{E_\bullet} \to \mathcal{M}_P$.*

The orientability of $O_P^{E_\bullet}$ was previously proved by Walpuski [56, §6.1], but the canonical orientations are new.

Theorem 4.12 is related to a 7-dimensional gauge theory discussed by Donaldson and Thomas [20] and Donaldson and Segal [19]. Let (X, φ, g) be a compact G_2-manifold with $\mathrm{d}(*\varphi) = 0$, G a Lie group, and $P \to X$ a principal G-bundle. A G_2-*instanton* on P is a connection ∇_P on P with $F^{\nabla_P} \wedge *\varphi = 0$ in $\Gamma^\infty(\mathrm{Ad}(P) \otimes \Lambda^6 T^*X)$. Write $\mathcal{M}_P^{G_2}$ for the moduli space of irreducible G_2-instantons on P. Then $\mathcal{M}_P^{G_2}$ is a derived manifold of virtual dimension 0. Examples and constructions of G_2-instantons are given in [38, 46, 47, 57–59].

As in §4.1, we may orient $\mathcal{M}_P^{G_2}$ by restricting orientations on $O_P^{E_\bullet} \to \mathcal{M}_P$, for $E_\bullet$ the Dirac operator of the spin structure on X induced by (φ, g). Thus Theorem 4.12 implies:

Corollary 4.13. *Let (X, φ, g) be a compact G_2-manifold with $\mathrm{d}(*\varphi) = 0$, and fix an orientation on $\det D$ and a flag structure on X. Then for any principal G-bundle $P \to X$ for $G = \mathrm{U}(m)$ or $\mathrm{SU}(m)$, we can construct a canonical orientation on $\mathcal{M}_P^{G_2}$.*

This confirms a conjecture of the first author [27, Conj. 8.3].

Donaldson and Segal [19] propose defining enumerative invariants of (X, φ, g) by counting $\mathcal{M}_P^{G_2}$, *with signs*, and adding correction terms from associative 3-folds in X. To determine the signs we need an orientation on $\mathcal{M}_P^{G_2}$. Thus, Corollary 4.13 contributes to the Donaldson–Segal programme.

4.2.10 Spin(7)-instantons on Spin(7)-manifolds

In the sequel [12], using a variation on the method of §3.3 in which $P \to X$ is the trivial bundle, we prove:

Theorem 4.14 (Cao and Joyce [12]). *Let X be a compact, oriented, spin Riemannian 8-manifold, and $E_\bullet$ be the positive Dirac operator $D_+ : \Gamma^\infty(S_+) \to \Gamma^\infty(S_-)$ on X. Suppose $P \to X$ is a principal G-bundle for $G = \mathrm{U}(m)$ or $\mathrm{SU}(m)$. Then $\mathcal{M}_P$ is orientable.*

This extends results of Cao and Leung [14, Th. 1.2], who proved Theorem 4.14 if $G = \mathrm{U}(m)$ and $H_{\mathrm{odd}}(X, \mathbb{Z}) = 0$, and Muñoz and Shahbazi [41], who proved Theorem 4.14 if $G = \mathrm{SU}(m)$ and $\mathrm{Hom}(H^3(X, \mathbb{Z}), \mathbb{Z}) = 0$.

Again, Theorem 4.12 is related to an 8-dimensional gauge theory discussed by Donaldson and Thomas [20]. Let (X, Ω, g) be a compact Spin(7)-manifold. Then there is a natural splitting $\Lambda^2 T^*X = \Lambda^2_7 T^*X \oplus \Lambda^2_{21} T^*X$ into vector subbundles of ranks 7 and 21. Suppose G is a Lie group and $P \to X$ a principal G-bundle. A Spin(7)-*instanton* on P is a connection ∇_P on P with $\pi^2_7(F^{\nabla_P}) = 0$

in $\Gamma^\infty(\mathrm{Ad}(P)\otimes\Lambda^2_7T^*X)$. Write $\mathcal{M}_P^{\mathrm{Spin}(7)}$ for the moduli space of irreducible Spin(7)-instantons on P. Then $\mathcal{M}_P^{\mathrm{Spin}(7)}$ is a derived manifold. Examples of Spin(7)-instantons were given by Lewis [36], Tanaka [50], and Walpuski [60].

As in §4.1, we may orient $\mathcal{M}_P^{\mathrm{Spin}(7)}$ by restricting orientations on $O_P^{E_\bullet}\to\mathcal{M}_P$, for $E_\bullet$ the positive Dirac operator of the spin structure on X induced by (Ω,g). Thus Theorem 4.14 implies:

Corollary 4.15. *Let (X,Ω,g) be a compact* Spin(7)*-manifold. Then $\mathcal{M}_P^{\mathrm{Spin}(7)}$ is orientable for any principal G-bundle $P\to X$ with $G=\mathrm{U}(m)$ or $\mathrm{SU}(m)$.*

Borisov and Joyce [11] and Cao and Leung [13] set out a programme to define Donaldson–Thomas type invariants 'counting' moduli spaces of (semi)stable coherent sheaves $\mathcal{M}_\alpha^{\mathrm{coh}}$ on a Calabi–Yau 4-fold X. To do this requires an 'orientation' on $\mathcal{M}_\alpha^{\mathrm{coh}}$ in the sense of [11, §2.4]. Gross and Joyce [24] use Theorem 4.14 for $G=\mathrm{U}(m)$ to prove that all such moduli spaces $\mathcal{M}_\alpha^{\mathrm{coh}}$ are orientable.

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