Complex manifolds and Kähler Geometry

Lecture 1 of 16: Complex manifolds

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Plan of talk:

1.1. Complex manifolds

We will give two definitions of complex manifolds. This lecture, we use complex charts and holomorphic transition functions. Next lecture, in a more Differential Geometric style, we use (almost) complex structures on a real manifold. The two points of view are equivalent, by the Newlander–Nirenberg Theorem.

Recall the definition of a (smooth, real) manifold: a topological space X with an atlas of charts (U_i, ϕ_i) with transition functions ϕ_{ij} diffeomorphisms between open sets in $\mathbb{R}^n$. We can instead require other conditions on ϕ_{ij}, e.g. ϕ_{ij} continuous gives you *topological manifolds*, or we could require ϕ_{ij} to be C^k, or real analytic. Requiring the ϕ_{ij} to be holomorphic gives you *complex manifolds*.

Definition

Let X be a topological space, and fix $n \geqslant 0$. A (*complex*) *chart* on X is (U, ϕ), where $U \subseteq \mathbb{C}^n$ is open and $\phi : U \to X$ is a homeomorphism from U to an open subset $\phi(U)$ in X. Let $(U, \phi), (V, \psi)$ be charts. The *transition function* between them is

$$\psi^{-1} \circ \phi : \phi^{-1}\bigl(\phi(U) \cap \psi(V)\bigr) \longrightarrow \psi^{-1}\bigl(\phi(U) \cap \psi(V)\bigr).$$

It is automatically a homeomorphism between open subsets of $\mathbb{C}^n$. We call $(U, \phi), (V, \psi)$ *compatible* if $\psi^{-1} \circ \phi$ is a *biholomorphism* between open subsets of $\mathbb{C}^n$, i.e. holomorphic with holomorphic inverse.

A (*complex*) *atlas* on X is a system $\bigl\{(U_i, \phi_i) : i \in I\bigr\}$ of pairwise compatible charts on X with $X = \bigcup_{i \in I} \phi_i(U_i)$. We may write ϕ_{ij} for the transition function $\phi_j^{-1} \circ \phi_i$.

Definition (Continued)

An atlas is called *maximal* if it is not a proper subset of any other atlas. Every atlas $\{(U_i, \phi_i) : i \in I\}$ is contained in a unique maximal atlas, the set of all charts (U, ϕ) compatible with (U_i, ϕ_i) for all $i \in I$.

An (*n-dimensional*) *complex manifold* is a second countable, Hausdorff topological space X together with a maximal atlas $\{(U_i, \phi_i) : i \in I\}$ of n-dimensional complex charts (U_i, ϕ_i). Here *second countable* is to avoid pathological examples from topology; sometimes one asks for *paracompact* instead.

Usually we refer to X as the complex manifold, and suppress the atlas. Taking the atlas *maximal* makes it independent of choices.

What a complex atlas on X gives you is a notion of *local holomorphic coordinates*. Let $x \in X$. Then we can choose a chart (U_i, ϕ_i) with $x \in \phi_i(U_i)$, since $X = \bigcup_{i \in I} \phi_i(U_i)$. Then we think of $\phi_i^{-1} : \phi_i(U_i) \to \mathbb{C}^n$ as holomorphic coordinates $(z_1, \ldots, z_n)$ defined on an open neighbourhood $\phi_i(U_i)$ of x. We can do a lot of definitions and proofs using local holomorphic coordinates.

Example

The simplest complex manifold is $\mathbb{C}^n$. $(U, \phi) = (\mathbb{C}^n, \mathrm{id}_{\mathbb{C}^n})$ is a chart on $\mathbb{C}^n$, and $\{(\mathbb{C}^n, \mathrm{id}_{\mathbb{C}^n})\}$ is an atlas on $\mathbb{C}^n$. This is contained in a unique maximal atlas, which makes $\mathbb{C}^n$ into a complex manifold.

Example

Complex projective space $\mathbb{CP}^n$ is a compact n-dimensional complex manifold. We use homogeneous coordinates $[z_0, \ldots, z_n]$ on $\mathbb{CP}^n$. For $i = 0, \ldots, n$, define a chart (U_i, ϕ_i) on $\mathbb{CP}^n$ by $U_i = \mathbb{C}^n$ and $\phi_i : \mathbb{C}^n \to \mathbb{CP}^n$ given by

$$\phi_i : (w_1, \ldots, w_n) \longmapsto [w_1, \ldots, w_i, 1, w_{i+1}, \ldots, w_n].$$

This is a homeomorphism with the open subset $\phi_i(U_i) = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n : z_i \neq 0\}$ in $\mathbb{CP}^n$.
For $0 \leqslant i < j \leqslant n$, the transition function $\phi_{ij} = \phi_j^{-1} \circ \phi_i$ maps $\{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_j \neq 0\}$ to $\{(y_1, \ldots, y_n) \in \mathbb{C}^n : y_{i+1} \neq 0\}$ by

$$(x_1, \ldots, x_n) \longmapsto \left(\tfrac{x_1}{x_j}, \ldots, \tfrac{x_i}{x_j}, \tfrac{1}{x_j}, \tfrac{x_{i+1}}{x_j}, \ldots, \tfrac{x_{j-1}}{x_j}, \tfrac{x_{j+1}}{x_j}, \ldots, \tfrac{x_n}{x_j}\right).$$

This is a biholomorphism. So $(U_i, \phi_i), (U_j, \phi_j)$ are compatible, and $\{(U_i, \phi_i) : i = 0, \ldots, n\}$ is an atlas. It is contained in a unique maximal atlas, which makes $\mathbb{CP}^n$ into a complex manifold.

1.2. Holomorphic functions and holomorphic maps

Let X be a complex manifold, and $f : X \to \mathbb{C}$ a function. We call f *holomorphic* if for all charts (U, ϕ) in the (maximal) atlas on X, $f \circ \phi$ is a holomorphic function $U \to \mathbb{C}$, where $U \subseteq \mathbb{C}^n$ is open. It is enough to check this on the charts of any atlas on X.

Let X, Y be complex manifolds of dimensions m, n, and $f : X \to Y$ a continuous function. We call f *holomorphic* if whenever (U, ϕ) and (V, ψ) are charts from the atlases on X, Y, the map

$$\psi^{-1} \circ f \circ \phi : (f \circ \phi)^{-1}\bigl(f(\phi(U)) \cap \psi(V)\bigr) \longrightarrow V$$

is a holomorphic map from an open subset of $\mathbb{C}^m$ to an open subset of $\mathbb{C}^n$. Complex manifolds and holomorphic maps form a *category*.

A *biholomorphism* $f : X \to Y$ is a holomorphic map with a holomorphic inverse.

1.3. Complex submanifolds

Let X be a complex manifold of dimension n, and $Y \subseteq X$. We call Y an (*embedded*) *complex submanifold of X of dimension k*, for $0 \leqslant k \leqslant n$, if for each $y \in Y$ there exist local holomorphic coordinates $(z_1, \ldots, z_n)$ on X such that Y is locally of the form $z_{k+1} = \cdots = z_n = 0$. That is, we have a chart (U, ϕ) on X with $y \in \phi(U)$ such that $Y \cap \phi(U) = \phi(\mathbb{C}^k \cap U)$, where $\mathbb{C}^k = \{(z_1, \ldots, z_k, 0, \ldots, 0) \in \mathbb{C}^n\}$. Usually we want Y closed in X.

We can give a complex submanifold Y of X the structure of a complex k-manifold: for (U, ϕ) as above, $(\mathbb{C}^k \cap U, \phi|_{\mathbb{C}^k \cap U})$ is a k-dimensional chart on Y, and the set of such charts is an atlas on Y. The inclusion $i_Y : Y \hookrightarrow X$ is holomorphic.

Conversely, a holomorphic map $f : Y \to X$ is called an *embedding* if it is injective, locally closed, and on tangent spaces $\mathrm{d}f|_y : T_y Y \to T_{f(y)} X$ is injective for all $y \in Y$. If f is an embedding then $f(Y)$ is a complex submanifold of X biholomorphic to Y.

1.4. Projective complex manifolds

Let $\mathbb{CP}^n$ have homogeneous coordinates $[z_0, \ldots, z_n]$. Let $p(z_0, \ldots, z_n)$ be a complex polynomial in $n+1$ variables, which is homogeneous of order k. Then $p(\lambda z_0, \ldots, \lambda z_n) = \lambda^k p(z_0, \ldots, z_n)$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Hence $p(\lambda z_0, \ldots, \lambda z_n) = 0$ if and only if $p(z_0, \ldots, z_n) = 0$. Thus, for $[z_0, \ldots, z_n] \in \mathbb{CP}^n$, the condition $p(z_0, \ldots, z_n) = 0$ is independent of the choice of representative $(z_0, \ldots, z_n)$ for $[z_0, \ldots, z_n]$.

A *projective variety* is a subset X of $\mathbb{CP}^n$ which is defined by the vanishing of finitely many homogeneous polynomials $p_1(z_0, \ldots, z_n), \ldots, p_d(z_0, \ldots, z_n)$, that is,

$$X = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n : p_i(z_0, \ldots, z_n) = 0, \;\; i = 1, \ldots, d\}.$$

Then X is closed in $\mathbb{CP}^n$, and so compact. We call X a *projective complex manifold* if X is also a complex submanifold of $\mathbb{CP}^n$.

Example

Let $p(z_0, \ldots, z_n)$ be a nonzero homogeneous complex polynomial, and define

$$X = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n : p(z_0, \ldots, z_n) = 0\}.$$

Then X is a complex submanifold of $\mathbb{CP}^n$, of dimension $n-1$, provided the following condition holds: let $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ with $p(z_0, \ldots, z_n) = 0$. Then $\frac{\partial p}{\partial z_i}(z_0, \ldots, z_n) \neq 0$ for some $i = 0, \ldots, n$. This holds for generic homogeneous polynomials p.

Example

For $d = 1, 2, \ldots$, $X = \{[z_0, z_1, z_2] \in \mathbb{CP}^2 : z_0^d + z_1^d + z_2^d = 0\}$ is a projective complex 1-manifold, a Riemann surface of genus $g = \frac{1}{2}(d-1)(d-2)$.

Example

$X = \{[z_0, z_1, z_2, z_3] \in \mathbb{CP}^3 : z_0^2 + \cdots + z_3^2 = 0\}$ is a projective complex 2-manifold biholomorphic to $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Example

Let $p_1, \ldots, p_k(z_0, \ldots, z_n)$ be homogeneous polynomials for $k \leqslant n$. Suppose that whenever $(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ with $p_i(z_0, \ldots, z_n) = 0$ for all i, then $\mathrm{d}p_1(z_0, \ldots, z_n), \ldots, \mathrm{d}p_k(z_0, \ldots, z_n)$ are linearly independent in $(\mathbb{C}^{n+1})^*$. Then

$$X = \{[z_0, \ldots, z_n] \in \mathbb{CP}^n : p_i(z_0, \ldots, z_n) = 0, \ \ i = 1, \ldots, k\}$$

is a projective complex manifold of dimension $n - k$, called a *complete intersection.*
Most projective complex manifolds are not complete intersections.

Projective complex manifolds give a huge number of interesting examples of complex manifolds. As they are defined using polynomials, one can study and classify them using algebraic techniques – Complex Algebraic Geometry.
Also, under some conditions one can guarantee that a compact complex manifold X has an embedding $X \hookrightarrow \mathbb{CP}^n$ making it into a projective complex manifold. This is due to two important results, Chow's Theorem and the Kodaira Embedding Theorem.

Theorem 1.1 (Chow's Theorem)

Suppose X is a compact complex submanifold in $\mathbb{CP}^n$. Then X is a projective complex manifold, that is, X may be defined as a subset of $\mathbb{CP}^n$ by the vanishing of homogeneous polynomials $p_1(z_0, \ldots, z_n), \ldots, p_k(z_0, \ldots, z_n)$.

Thus, compact submanifolds of $\mathbb{CP}^n$ are algebraic objects. For a proof, see Griffiths and Harris, *Principles of Algebraic Geometry*.
As $\mathbb{CP}^n$ is compact, X compact is equivalent to X closed.
We will cover the Kodaira Embedding Theorem later in the course.
In brief, it says that if X is a compact complex manifold and $L \to X$ is an 'ample line bundle' then we can use L to construct an embedding $f : X \hookrightarrow \mathbb{CP}^n$ for some $n \gg 0$. Then X is biholomorphic to $f(X)$, which is a compact complex submanifold of $\mathbb{CP}^n$, so by Chow's Theorem, $f(X)$ is algebraic, and X is biholomorphic to a projective complex manifold.

Projective complex manifolds are also closely connected to compact Kähler manifolds (next week).

Every projective complex manifold is Kähler. But also, if X is a compact Kähler manifold, then under mild topological conditions on X one can show that X possesses many ample line bundles $L \hookrightarrow X$, and then the Kodaira Embedding Theorem applies, and X is biholomorphic to a projective complex manifold.

Complex manifolds and Kähler Geometry

Lecture 2 of 16: Complex manifolds as real manifolds; almost complex structures

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Plan of talk:

2.1. Almost complex structures

We now explain a second way to define complex manifolds. To see the point simply, suppose V is a complex vector space, of complex dimension n. Underlying V is a real vector space $V_{\mathbb{R}}$, of real dimension $2n$. Given $V_{\mathbb{R}}$, what extra information do we need to reconstruct V? The only thing we are missing is multiplication by $i \in \mathbb{C}$. This induces a real linear map $J : V_{\mathbb{R}} \to V_{\mathbb{R}}$ with $J^2 = -\mathrm{id}_{V_{\mathbb{R}}}$.

Conversely, given a real vector space $V_{\mathbb{R}}$ and $J \in \mathrm{End}(V_{\mathbb{R}})$ with $J^2 = -\mathrm{id}_{V_{\mathbb{R}}}$, we make $V_{\mathbb{R}}$ into a complex vector space by setting $(a + ib) \cdot v = a \cdot v + b \cdot J(v)$, for $a, b \in \mathbb{R}$ and $v \in V_{\mathbb{R}}$; note that $\dim_{\mathbb{R}} V_{\mathbb{R}}$ must be even. So, complex vector spaces are equivalent to real vector spaces with an endomorphism J with $J^2 = -\mathrm{id}$.

If X is a complex n-manifold in the sense of §1, then underlying X is a real $2n$-manifold $X_{\mathbb{R}}$. It has a tangent bundle $TX_{\mathbb{R}}$, whose fibres $T_xX_{\mathbb{R}}$ for $x \in X$ are real vector spaces of real dimension $2n$. Since X is a complex n-manifold, they are also complex vector spaces of dimension n. So they have $J_x \in \mathrm{End}(T_xX_{\mathbb{R}})$ with $J_x^2 = -\mathrm{id}_{T_xX_{\mathbb{R}}}$. Over all $x \in X_{\mathbb{R}}$, these J_x form a tensor J_a^b with $J_a^b J_b^c = -\delta_a^c$, using index notation.

Definition

Let X be a real $2n$-manifold. An *almost complex structure* J on X is a tensor J_a^b in $C^\infty(T^*X \otimes TX)$ with $J_a^b J_b^c = -\delta_a^c$. For a vector field $v \in C^\infty(TX)$, define $(Jv)^b = J_a^b v^a$. Then $J^2 = -1$, so J makes the tangent spaces T_xX into *complex vector spaces*.

Any complex manifold in the sense of §1 yields a real manifold X with an almost complex structure J. But not all (X, J) come from complex manifolds: we must impose extra conditions on J.

Holomorphic functions

Definition

Suppose X is a $2n$-manifold, and J an almost complex structure on X. Let $f : X \to \mathbb{C}$ be smooth, and write $f = u + iv$. Then $\mathrm{d}u, \mathrm{d}v$ are 1-forms on X, so in index notation $\mathrm{d}u = \mathrm{d}u_a$, $\mathrm{d}v = \mathrm{d}v_b$. We call f *holomorphic* if $\mathrm{d}u_a = J_a^b \mathrm{d}v_b$. Since $J^2 = -\mathrm{id}$, this is equivalent to $\mathrm{d}v_a = -J_a^b \mathrm{d}u_b$. Hence in complex 1-forms we have

$$J_a^b(\mathrm{d}u_b + i\mathrm{d}v_b) = i(\mathrm{d}u_a + i\mathrm{d}v_a),$$

that is, $J_a^b \mathrm{d}f_b = i\,\mathrm{d}f_a$.

Example

Let $\mathbb{R}^2$ have coordinates (x, y), and let $J = \mathrm{d}x \otimes \frac{\partial}{\partial y} - \mathrm{d}y \otimes \frac{\partial}{\partial x}$ in $C^\infty(T^*\mathbb{R}^2 \otimes T\mathbb{R}^2)$. Then the equation $\mathrm{d}u_a = J_a^b \mathrm{d}v_b$ becomes

$$\frac{\partial u}{\partial x} \cdot \mathrm{d}x + \frac{\partial u}{\partial y} \cdot \mathrm{d}y = -\frac{\partial v}{\partial x} \cdot \mathrm{d}y + \frac{\partial v}{\partial y} \cdot \mathrm{d}x,$$

or equivalently

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

the *Cauchy–Riemann equations* for $u(x, y) + iv(x, y)$ to be a holomorphic function of $x + iy$.

2.2. The Nijenhuis tensor

It turns out that when $n > 1$, for some almost complex structures on X there may be few holomorphic functions locally on X — in extreme cases, all holomorphic functions are constant. This is because the equations are *overdetermined*: there are $2n$ equations on 2 functions. We can express this in terms of an *obstruction* to the existence of holomorphic functions locally on X, called the *Nijenhuis tensor*.

Definition

Write $[v, w]$ for the *Lie bracket* of vector fields v, w on X. The *Nijenhuis tensor* $N = N^a_{bc}$ of J satisfies

$$N^a_{bc} v^b w^c = \big([v, w] + J\big([Jv, w] + [v, Jw]\big) - [Jv, Jw]\big)^a \qquad (2.1)$$

for all $v, w \in C^\infty(TX)$.

The point is that the r.h.s. of (2.1) is *pointwise* linear in v, w (exercise): if we replace v, w by $f \cdot v$, $g \cdot w$ for smooth $f, g : X \to \mathbb{R}$, then the r.h.s. is multiplied by fg, with no terms in derivatives of f, g.

Let $s + it : X \to \mathbb{C}$ be holomorphic. Then using (2.1) one can show that for all vector fields v, w we have $N_{bc}^a v^b w^c \mathrm{d}s_a \equiv N_{bc}^a v^b w^c \mathrm{d}t_a \equiv 0$ (exercise). Hence $N_{bc}^a \mathrm{d}s_a \equiv N_{bc}^a \mathrm{d}t_a \equiv 0$ in $C^\infty(\Lambda^2 T^* X)$. Thus, the Nijenhuis tensor constrains the possible first derivatives of holomorphic functions.

For (X, J) to be a complex manifold, we want there to exist a system of holomorphic coordinates $(z_1, \ldots, z_n)$ near each point x in X, that is, $(z_1, \ldots, z_n)$ are complex coordinates defined on open $x \in U \subseteq X$, and $z_j : U \to \mathbb{C}$ is holomorphic. If $z_j = s_j + it_j$ then $\mathrm{d}s_1, \ldots, \mathrm{d}s_n, \mathrm{d}t_1, \ldots, \mathrm{d}t_n$ span T^*X on U. So $N^a_{bc}(\mathrm{d}s_j)_a \equiv N^a_{bc}(\mathrm{d}t_j)_a \equiv 0$ imply that $N \equiv 0$. Thus, holomorphic coordinates $(z_1, \ldots, z_n)$ can exist locally on X only if the Nijenhuis tensor $N \equiv 0$.

The converse is the difficult *Newlander–Nirenberg Theorem*:

Theorem 2.1 (Newlander–Nirenberg)

Suppose J is an almost complex structure on X with Nijenhuis tensor $N \equiv 0$. Then near each $x \in X$ there exist holomorphic coordinates $(z_1, \ldots, z_n)$.

The point is to show that the first derivatives of holomorphic functions near x span T_x^*X; then choosing any $(z_1, \ldots, z_n)$ whose derivatives span T_x^*X, they will be holomorphic coordinates in a small open neighbourhood of x.

Think of the Nijenhuis tensor N as being like the 'curvature' of J, and the condition $N \equiv 0$ as a 'flatness condition'. If $g = g_{ab}$ is a Riemannian metric, the Riemann curvature R^i_{jkl} is a tensor defined using g and its derivatives, in a similar way to N^a_{bc}, and $R^i_{jkl} \equiv 0$ if g is flat. (Actually, N is a *torsion* rather than a curvature, as it depends on one derivative of J, not two.)

2.3. Another definition of complex manifolds

Here is our second definition of complex manifold:

Definition

Let X be a $2n$-manifold, and J an almost complex structure on X with Nijenhuis tensor N. We call J an *integrable almost complex structure*, or just a *complex structure*, if $N \equiv 0$, and then we call (X, J) a *complex manifold*.

This is equivalent to the definition of complex manifolds using complex atlases in §1. Here is why.

Suppose (X, J) is a complex manifold in the sense above. Then by the Newlander–Nirenberg theorem, there exist holomorphic coordinates $(z_1, \ldots, z_n)$ near each $x \in X$. Using these we define an atlas of charts (U, ϕ) on X. The transition functions are automatically holomorphic. Extending to the unique maximal atlas defines a complex structure on X in the sense of §1.

Conversely, given a complex manifold $X_{\mathbb{C}}$ in the sense of §1, there is a natural underlying real manifold $X_{\mathbb{R}}$, and a unique almost complex structure J on $X_{\mathbb{R}}$ for which all local coordinate functions z_j are holomorphic, and $N \equiv 0$, so J is a complex structure.

Holomorphic maps

Definition

Let (X, I) and (Y, J) be complex manifolds, and $f : X \to Y$ a smooth map. We call f *holomorphic* if for all $x \in X$ with $y = f(x) \in Y$, so that $\mathrm{d}f|_x : T_xX \to T_yY$ is a linear map, we have $\mathrm{d}f|_x \circ I|_x = J|_y \circ \mathrm{d}f|_x$. That is, $\mathrm{d}f|_x : T_xX \to T_yY$ is a complex linear map, regarding T_xX, T_yY as complex vector spaces using $I|_x, J|_y$.

This agrees with the definition of holomorphic maps in §1, under the correspondence between the two definitions of complex manifold. If $g : Y \to \mathbb{C}$ is a holomorphic function then $g \circ f : X \to \mathbb{C}$ is a holomorphic function. In fact, a smooth map $f : X \to Y$ is holomorphic if and only if for all local holomorphic functions $g : V \to \mathbb{C}$ for $V \subseteq Y$ open, $g \circ f : U = f^{-1}(V) \to \mathbb{C}$ is a local holomorphic function on X.

Complex submanifolds

Definition

Let (X, J) be a complex manifold, and Y a submanifold of X. We call Y a *complex submanifold* if for each $y \in Y$ we have $J(T_y Y) = T_y Y$, as subspaces of $T_y X$.

Then $J_Y = J|_{TY}$ is an almost complex structure on Y. The Nijenhuis tensor N_Y of J_Y is the restriction to Y of the Nijenhuis tensor N of J, so it is zero, J_Y is a complex structure, and (Y, J_Y) is a complex manifold.

Real dimension two

Let J be an almost complex structure on X, with Nijenhuis tensor $N = N^a_{bc}$. Then N has natural symmetries $N^a_{bc} = -N^a_{cb}$, and $J^d_b J^e_c N^a_{de} = -N^a_{bc}$ (exercise). Using these one can show that $N \equiv 0$ when $\dim_{\mathbb{R}} X = 2$. So almost complex 2-manifolds are complex, that is, they are Riemann surfaces. This corresponds to the fact that for $f : X \to \mathbb{C}$ to be holomorphic is $2n$ equations on 2 functions, which is overdetermined when $n > 1$, but determined when $n = 1$.

2.4. More on almost complex geometry

Consider the question: how much of complex geometry also works for non-integrable almost complex structures J on X with $\dim_{\mathbb{R}} X > 2$?

We already know there are few holomorphic functions $f : X \to \mathbb{C}$ even locally. There are also few complex submanifolds $Y \subset X$ with $2 < \dim_{\mathbb{R}} Y < \dim_{\mathbb{R}} X$. However, 2-real-dimensional complex submanifolds Y in X (*J-holomorphic curves*) are well-behaved. This is important in *Symplectic Geometry*.

Definition

Let X be a $2n$-manifold. A *symplectic form* ω on X is a 2-form ω with $\mathrm{d}\omega \equiv 0$, such that $\omega|_x^n$ is nonzero in $\Lambda^{2n} T_x^* X$ for all $x \in X$. Then (X, ω) is a *symplectic manifold*.

Symplectic manifolds

Darboux' Theorem says that near each point x in a symplectic manifold (X,ω) we can choose coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ on X with $\omega=\sum_{j=1}^n \mathrm{d}x_j\wedge\mathrm{d}y_j$. So all symplectic manifolds are locally the same as the standard model $(\mathbb{R}^{2n},\omega_0)$.

Similarly, the Newlander–Nirenberg Theorem shows that if J is an almost complex structure on X with Nijenhuis tensor $N\equiv 0$, then near each $x\in X$ we can choose coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ on X with $J=\sum_{j=1}^n \mathrm{d}x_j\otimes\frac{\partial}{\partial y_j}-\mathrm{d}y_j\otimes\frac{\partial}{\partial x_j}$.
Thus, all complex manifolds are locally the same as the standard model $(\mathbb{R}^{2n},J_0)$.

Let (X, ω) be symplectic. An almost complex structure J on X is *compatible with* ω if $\omega(Jv, Jw) = \omega(v, w)$ for all vector fields v, w on X, and $\omega(v, Jv) > 0$ if $v \neq 0$. Every symplectic manifold admits compatible almost complex structures.

Many important areas of Symplectic Geometry — Gromov-Witten invariants, Lagrangian Floer cohomology, Fukaya categories, . . . — depend on choosing a compatible J on (X, ω) and then 'counting' J-holomorphic curves in X. Often one can make the 'number' independent of the choice of J.