Complex manifolds and Kähler Geometry

Lecture 9 of 16: Vanishing theorems and the Kodaira Embedding Theorem

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These slides available at
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Plan of talk:

9.1. Vanishing theorems

Let (X, J) be a compact complex manifold, and $E \to X$ a holomorphic vector bundle. A *vanishing theorem* says that under some assumptions $H^q(E) = 0$ for $q > 0$. Then

$$\chi(X, E) = \sum_{q=0}^{n} (-1)^q \dim_{\mathbb{C}} H^q(E) = \dim_{\mathbb{C}} H^0(E),$$

so the Hirzebruch–Riemann–Roch Theorem in §8.2 gives

$$\dim_{\mathbb{C}} H^0(E) = \int_X \mathrm{ch}(E)\,\mathrm{td}(X).$$

So we can compute the number of holomorphic sections of E. This can be a powerful tool.

Positive line bundles

Definition

Let (X, J) be a complex manifold, and L a holomorphic line bundle. We call L *positive* if $c_1(L)$ in $H^2_{\mathrm{dR}}(X;\mathbb{R})$ can be represented by a positive closed real (1,1)-form η, where η *positive* means that $\eta(v, Jv) > 0$ for nonzero vectors v.
Then $g(v, w) = \eta(v, Jw)$ is a Kähler metric on X, with Kähler form η. So if X has a positive line bundle then X admits Kähler metrics.
The converse is not true, e.g. there exist Kähler $K3$ surfaces with no positive line bundles. We call L *negative* if L^{-1} is positive.

Recall from §5.3 that the *Kähler cone* $\mathcal{K}$ of X is the set of Kähler classes of Kähler metrics on X, an open convex cone in $H^2_{\rm dR}(X;\mathbb{R})\cap H^{1,1}(X)$. A holomorphic line bundle $L\to X$ is positive iff $c_1(L)\in\mathcal{K}$.

From this and §7, we see that if (X,J) is compact and admits Kähler metrics, then X has positive line bundles L iff

$$H^2(X;\mathbb{Z})\cap\mathcal{K}\neq\emptyset,$$

with intersection in $H^2_{\rm dR}(X;\mathbb{C})$.

As any element of $H^2(X;\mathbb{Q})$ has a positive multiple in $H^2(X;\mathbb{Z})$, this is equivalent to

$$H^2(X;\mathbb{Q})\cap\mathcal{K}\neq\emptyset.$$

The tautological line bundle $\mathcal{O}(1)$ on $\mathbb{CP}^n$ is positive. If (X, J) is a projective complex manifold then X is (isomorphic to) a complex submanifold of some $\mathbb{CP}^n$, and then $\mathcal{O}(1)|_X$ is a positive line bundle on X. Thus, all projective complex manifolds admit positive line bundles. The Kodaira Embedding Theorem (later) shows that if a compact complex manifold (X, J) admits positive line bundles, then it is projective.

9.2. The Kodaira and Serre Vanishing Theorems

Theorem 9.1 (Kodaira Vanishing Theorem)

Let N be a positive line bundle on a compact complex manifold (X, J) of complex dimension n. Then

$$H^q(N \otimes \Lambda^p T^*X) = 0 \ \text{ for } p + q > n.$$

We sketch a proof. As N is positive, we may choose a positive closed real (1,1)-form ω with $[\omega] = 2\pi c_1(N)$. Let g be the Kähler metric on X with Kähler form ω.

From §6.2 we may choose a Hermitian metric h on N, such that if ∇ is the connection on N preserving h and inducing the $\bar\partial$-operator $\bar\partial_N$ of N, then $F_\nabla = -i\omega$. Write $\nabla = \partial_N + \bar\partial_N$ and $\nabla^* = \partial_N^* + \bar\partial_N^*$.

From §8.1 we have

$$H^q(N \otimes \Lambda^p T^*X) \cong \mathcal{H}^{p,q}(N),$$

where $\mathcal{H}^{p,q}(N) = \mathrm{Ker}\, \Delta_N^{p,q}$.

As in §4.4 we have operators $L, \Lambda, \partial, \bar\partial, \partial^*, \bar\partial^*$ on (p,q)-forms on X satisfying the Kähler identities. Another identity that we did not mention is that $[\Lambda, L] = (n-(p+q))\cdot\mathrm{id}$ on (p,q)-forms. This extends to N-valued (p,q)-forms using $\partial_N, \bar\partial_N$. Also $[\Lambda, \bar\partial] = -i\partial^*$ extends to N-valued (p,q)-forms. If α is an N-valued (p,q)-form we have

$$\begin{aligned} L(\alpha) &= \omega\wedge\alpha = iF_\nabla\wedge\alpha \\ &= i(\nabla\wedge\nabla)\alpha = i(\bar\partial_N+\partial_N)^2\alpha \\ &= i(\bar\partial_N\partial_N+\partial_N\bar\partial_N)\alpha. \end{aligned}$$

Suppose $\alpha \in \mathcal{H}^{p,q}(N)$, so that $\bar\partial_N\alpha = \bar\partial_N^*\alpha = 0$. Then

$$\begin{aligned}\langle L\circ\Lambda\alpha,\alpha\rangle_{L^2} &= \langle i(\bar\partial_N\partial_N + \partial_N\bar\partial_N)\Lambda\alpha,\alpha\rangle_{L^2}\\ &= \langle \partial_N\Lambda\alpha, -i\bar\partial_N^*\alpha\rangle_{L^2} + \langle\bar\partial_N\Lambda\alpha, -i\partial_N^*\alpha\rangle_{L^2}\\ &= 0 + \langle\bar\partial_N\Lambda\alpha, [\Lambda,\bar\partial_N]\alpha\rangle_{L^2}\\ &= \langle\bar\partial_N\Lambda\alpha, \Lambda\bar\partial_N\alpha\rangle_{L^2} - \langle\bar\partial_N\Lambda\alpha,\bar\partial_N\Lambda\alpha\rangle_{L^2}\\ &= 0 - \|\bar\partial_N\Lambda\alpha\|_{L^2}^2,\end{aligned}$$

using $\bar\partial_N^*\alpha = \bar\partial_N\alpha = 0$ and $[\Lambda,\bar\partial_N] = -i\partial_N^*$. Similarly

$$\langle\Lambda\circ L\alpha,\alpha\rangle_{L^2} = \|\partial_N\alpha\|_{L^2}^2.$$

But $[\Lambda, L] = (n-(p+q))\cdot\mathrm{id}$ on N-valued (p,q)-forms. Hence

$$\begin{aligned}(n-(p+q))\|\alpha\|_{L^2}^2 &= \langle[\Lambda,L]\alpha,\alpha\rangle_{L^2}\\ &= \|\partial_N\alpha\|_{L^2}^2 + \|\bar\partial_N\Lambda\alpha\|_{L^2}^2.\end{aligned}$$

If $p+q>n$ then the l.h.s. is $\leqslant 0$ and the r.h.s. $\geqslant 0$, so $\alpha=0$, and $\mathcal{H}^{p,q}(N)=0$, giving $H^q(N\otimes\Lambda^pT^*X)=0$ as we want.

In the case $p = n$ we have $\Lambda^p T^*X = K_X$ and $p + q > n$ becomes $q > 0$, giving:

Corollary 9.2

Suppose L is a positive line bundle on a compact complex manifold (X, J). Then

$$H^q(L \otimes K_X) = 0 \ \text{for all } q > 0.$$

Equivalently, if L is a line bundle with $L \otimes K_X^{-1}$ positive then $H^q(L) = 0$ for $q > 0$, so that

$$\dim_{\mathbb{C}} H^0(L) = \int_X \mathrm{ch}(L)\,\mathrm{td}(X)$$

by the Hirzebruch–Riemann–Roch Theorem.

The Serre Vanishing Theorem

A similar proof to the Kodaira Vanishing Theorem yields:

Theorem 9.3 (Serre Vanishing Theorem)

Let L be a positive line bundle on a compact complex manifold (X, J), and E any holomorphic vector bundle on X. Then there exists $m_0 \in \mathbb{Z}$ such that $H^q(E \otimes L^m) = 0$ for all $q > 0$ and $m \geqslant m_0$.

This also holds for coherent sheaves E, using sheaf cohomology.

Let E be a holomorphic vector bundle of rank $k>0$, and consider $\chi(X,E\otimes L^m)$ as a function of m in $\mathbb{Z}$. The H–R–R Theorem gives

$$\begin{aligned}\chi(X,E\otimes L^m)&=\int_X \mathrm{ch}(E\otimes L^m)\,\mathrm{td}(X)\\&=\int_X \mathrm{ch}(E)\exp(m\,c_1(L))\,\mathrm{td}(X).\end{aligned}$$

Here $\exp(m\,c_1(L))=1+mc_1(L)+\frac{m^2}{2!}c_1(L)^2+\cdots+\frac{m^n}{n!}c_1(L)^n$, where $n=\dim_{\mathbb{C}}X$. Thus $\chi(X,E\otimes L^m)$ is a polynomial in m of degree n, with leading term

$$\chi(X,E\otimes L^m)=\frac{k}{n!}\int_X c_1(L)^n\,m^n+\cdots.$$

As L is positive, $c_1(L)$ is represented by the Kähler form ω of a Kähler metric g on X, and then $\int_X c_1(L)^n = \int_X \omega^n = n!\,\mathrm{vol}_g(X) > 0$. Thus the leading term of $\chi(X, E \otimes L^m)$ is positive, proving:

Lemma 9.4

Let (X, J) be a compact complex manifold, L a positive line bundle on X, and E a holomorphic vector bundle on X of positive rank. Then $\chi(X, E \otimes L^m) \gg 0$ for $m \gg 0$. Hence $\dim H^0(E \otimes L^m) \gg 0$ for $m \gg 0$ by the Serre Vanishing Theorem.

9.3. Application to line bundles and divisors

Recall from §7 that if (X, J) is a compact complex manifold then the Picard group $\mathrm{Pic}(X)$ is the group of holomorphic line bundles up to isomorphism, and $\mathrm{Div}(X)/\sim$ is the group of divisors on X up to equivalence, and there is an injective morphism $\mu : (\mathrm{Div}(X)/\sim) \to \mathrm{Pic}(X)$ whose image is the subgroup of $[L] \in \mathrm{Pic}(X)$ for which L admits meromorphic sections.

Suppose X has a positive line bundle $\tilde{L}$. We will show that any line bundle L on X has a meromorphic section. Applying Lemma 9.4 to L and $\mathcal{O}_X$ shows that $\dim H^0(L \otimes \tilde{L}^m) \gg 0$ and $\dim H^0(\mathcal{O}_X \otimes \tilde{L}^m) \gg 0$ when $m \gg 0$. So we can choose $m \gg 0$ and $0 \neq s \in H^0(L \otimes \tilde{L}^m)$, $0 \neq t \in H^0(\mathcal{O}_X \otimes \tilde{L}^m)$. Then $s \otimes t^{-1}$ is a meromorphic section of $(L \otimes \tilde{L}^m) \otimes (\mathcal{O}_X \otimes \tilde{L}^m)^* \cong L$.

This proves:

Theorem 9.5

Suppose (X, J) is a compact complex manifold which admits positive line bundles (equivalently, (X, J) is projective). Then $\mu : (\mathrm{Div}(X)/\sim) \to \mathrm{Pic}(X)$ in §7.4 is an isomorphism.

As in §7.2, we can describe $\mathrm{Pic}(X)$ very precisely in terms of $H_1(X;\mathbb{Z})$, $H^2(X;\mathbb{Z})$, and $H^{1,1}(X)$. So we get a description of $\mathrm{Div}(X)/\sim$. In particular, this proves the existence of many (possibly singular) complex hypersurfaces in projective complex manifolds. This proves the case $k = n - 1$ of the Hodge Conjecture in §5.4.

The base locus, morphisms to projective spaces

Definition

Let (X, J) be a compact complex manifold, and L a holomorphic line bundle on X. Then $H^0(L)$ is a finite-dimensional vector space. The *base locus* of L is

$$B = \{x \in X : s(x) = 0 \;\; \forall s \in H^0(L)\}.$$

It is a closed subset of X, algebraic when X is algebraic.

Theorem 9.6 (Bertini's Theorem)

Let (X, J) be a compact complex manifold, and L a holomorphic line bundle on X. Then for generic $s \in H^0(L)$, the zeroes $s^{-1}(0)$ are a smooth hypersurface in X away from B.

In particular, if $B=\emptyset$, which is often true, then $Y=s^{-1}(0)$ is a compact complex submanifold of X of dimension $\dim_{\mathbb C} Y=\dim_{\mathbb C} X-1$, whose homology class $[Y]$ is Poincaré dual to $c_1(L)$. So we can prove the existence of many compact hypersurfaces in X, and by induction, of many compact submanifolds of any codimension.

If L has base locus B, we can define a natural holomorphic map $\Phi_L: X\setminus B\rightarrow \mathbb P\bigl(H^0(L)^*\bigr)$ as follows: for $x\in X\setminus B$, choose an isomorphism $\phi_x:L_x\rightarrow\mathbb C$, and define $\psi_x:H^0(L)\rightarrow\mathbb C$ by $\psi_x(s)=\phi_x(s(x))$. Then $\psi_x\in H^0(L)^*$, with $\psi_x\ne 0$ as $x\notin B$, so $[\psi_x]\in\mathbb P\bigl(H^0(L)^*\bigr)$. We define $\Phi_L(x)=[\psi_x]$. This is independent of the choice of ϕ_x.

9.4. The Kodaira Embedding Theorem

Definition

Let L be a holomorphic line bundle on a compact complex manifold (X, J). We call L *very ample* if the base locus B of L is $\emptyset$, and the map $\Phi_L : X \to \mathbb{P}\big(H^0(L)^*\big)$ is an embedding of complex manifolds. We call L *ample* if L^k is very ample for some positive integer k.

If L is very ample then choosing a basis for $H^0(L)$ gives an embedding $\Phi_L : X \to \mathbb{CP}^N$, where $N + 1 = \dim H^0(L)$, which identifies X with a complex submanifold of $\mathbb{CP}^N$. One can show that $L \cong \Phi_L^*(\mathcal{O}(1))$, where $\mathcal{O}(1)$ is the usual line bundle on $\mathbb{CP}^N$. But $\mathcal{O}(1)$ is a positive line bundle on $\mathbb{CP}^N$, so $\Phi_L^*(\mathcal{O}(1))$ is positive. So any very ample line bundle on X is positive.

Also if L^k is positive for $k > 0$, so that $c_1(L^k)$ is represented by a positive (1,1)-form ω, then $c_1(L)$ is represented by $\frac{1}{k}\omega$, so L is positive. Thus, if L is ample, then L is positive. The important Kodaira Embedding Theorem is a converse to this:

Theorem 9.7 (Kodaira Embedding Theorem)

Let (X, J) be a compact complex manifold, and L a positive line bundle on X. Then L is ample.

The proof is complicated. A partial explanation is that as $\dim H^0(L^k) \gg 0$ for $k \gg 0$ by Lemma 9.4, when k is large there are many sections of L^k, and these are enough both to force $B = \emptyset$, and to embed X in $\mathbb{P}\big(H^0(L^k)^*\big) \cong \mathbb{CP}^N$.

Consequences of Kodaira Embedding

Given a positive line bundle L, a multiple L^k induces an embedding of X in a projective space, giving:

Corollary 9.8

Suppose (X, J) is a compact complex manifold admitting positive line bundles. Then X is projective, that is, X is isomorphic to a complex submanifold of $\mathbb{CP}^N$ for some $N \gg 0$.
Conversely, if $X \subset \mathbb{CP}^N$ is projective then it admits positive line bundles, e.g. $\mathcal{O}(1)|_X$ is positive.

From §9.1, if (X, J) is a compact complex manifold admitting Kähler metrics, then X admits positive line bundles if and only if

$$H^2(X;\mathbb{Q}) \cap \mathcal{K} \neq \emptyset,$$

with intersection in $H^2_{\mathrm{dR}}(X;\mathbb{C})$. So we deduce:

Corollary 9.9

Let (X, J) be a compact complex manifold admitting Kähler metrics. Then X is projective if and only if

$$H^2(X;\mathbb{Q}) \cap \mathcal{K} \neq \emptyset.$$

In particular, if $H^{2,0}(X)=0$ then $H^{1,1}(X)=H^2_{\mathrm{dR}}(X;\mathbb{C})$, so

$$H^2(X;\mathbb{Q})\cap H^{1,1}(X)=H^2(X;\mathbb{Q}),$$

which is dense in $H^2(X;\mathbb{R})$. Also $\mathcal{K}$ is a nonempty open set in $H^{1,1}(X)\cap H^2(X;\mathbb{R})=H^2(X;\mathbb{R})$, so $H^2(X;\mathbb{Q})\cap\mathcal{K}\neq\emptyset$. Thus we have:

Corollary 9.10

Let (X,J) be a compact complex manifold admitting Kähler metrics with $H^{2,0}(X)=0$. Then X is projective.

So under mild conditions, compact Kähler manifolds are projective, and can be studied using complex algebraic geometry.

Complex manifolds and Kähler Geometry

Lecture 10 of 16: Topics on line bundles and divisors

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Plan of talk:

10.1. Finite covers

From the Kodaira Embedding Theorem, a compact complex manifold is projective iff it admits positive line bundles. We use this to prove:

Proposition 10.1

Let (X, J) be a compact complex manifold, and $(\tilde{X}, \tilde{J})$ a finite cover of X, with covering map $\pi : \tilde{X} \to X$. Then $\tilde{X}$ is projective iff X is projective.

Proof of Proposition 10.1

Suppose X is projective. Then there exists a positive line bundle L on X, so $c_1(L)$ is represented by a positive, closed, real (1,1)-form η. The pullback $\pi^*(L)$ has $c_1(\pi^*(L))$ represented by $\pi^*(\eta)$, which is positive as π is a local diffeomorphism, so $\pi^*(L)$ is positive, and $\tilde X$ is projective.

Conversely, suppose $\tilde X$ is projective, so there exists $\tilde L$ on $\tilde X$ positive, with $c_1(\tilde L)$ represented by $\tilde\eta$ positive.

Define a line bundle L on X to have fibre $L|_x = \bigotimes_{\tilde x \in \tilde X : \pi(\tilde x) = x} \tilde L|_{\tilde x}$.

Then L is holomorphic (it is the determinant line bundle of the push-forward sheaf $\pi_*(\tilde L)$) and $c_1(L)$ is represented by η, where

$$\eta|_x = \sum_{\tilde x \in X : \pi(\tilde x) = x} \mathrm{d}\pi_*(\tilde\eta|_{\tilde x}).$$

This is locally a sum of positive forms, so is positive, and L is positive, and X is projective. □

Example: complex tori

Let $n \geqslant 2$, and consider the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, where $\mathbb{R}^{2n}$ has coordinates $(x_1, \ldots, x_{2n})$. Let $J = J_a^b$ be a complex structure and $g = g_{ab}$ a compatible Kähler metric on $\mathbb{R}^{2n}$ (not necessarily the standard ones), where J_a^b and g_{ab} are constant in coordinates $(x_1, \ldots, x_{2n})$. That is, J is an element of $\mathrm{GL}(2n, \mathbb{R})$ with $J^2 = -1$. The set of such J is $\mathcal{M}_n \cong \mathrm{GL}(2n; \mathbb{R}) / \mathrm{GL}(n; \mathbb{C})$, a complex manifold with $\dim_{\mathbb{C}} \mathcal{M}_n = n^2$. Then J, g both descend to T^{2n}, to make (T^{2n}, J, g) a compact Kähler manifold.

Under what conditions is (T^{2n}, J) projective? Well, if $\alpha \in H^2(T^{2n}; \mathbb{Z}) \cong \mathbb{Z}^{n(2n-1)}$ then α is $c_1(L)$ for a holomorphic line bundle L iff $\pi_{2,0}(\alpha) = 0$, where $\pi_{2,0} : H^2(T^{2n}; \mathbb{Z}) \rightarrow H^{2,0}(T^{2n})$ is projection to the (2,0)-component in $H^2(T^{2n}; \mathbb{C})$.

We have $H^{2,0}(T^{2n}) \cong \mathbb{C}^{n(n-1)/2}$. So the subset of J for which (T^{2n}, J) has a holomorphic line bundle L with $c_1(L) = \alpha$ is a subvariety $\mathcal{N}_\alpha$ in $\mathcal{M}_n$ of codimension $\frac{1}{2}n(n-1)$.

Example: complex tori

In particular, $\mathcal{M}_n \setminus \bigcup_{0\neq\alpha\in\mathbb{Z}^{n(2n-1)}} \mathcal{N}_\alpha$ is nonempty, and if J lies in this subset of $\mathcal{M}_n$ then (T^{2n}, J) has no holomorphic line bundles L with $c_1(L) \neq 0$, so no positive line bundles, and (T^{2n}, J) is not projective. Thus, generic complex tori (T^{2n}, J) for $n \geqslant 2$ are not projective; the family of projective complex tori are of complex codimension $\frac{1}{2}n(n-1)$ in the family of all complex tori.

10.2. The Lefschetz Hyperplane Theorem

Let (X, J) be a compact complex manifold, and Y a hypersurface in X, that is, Y is a closed, embedded complex submanifold of X with $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X - 1$. Then Y is a *divisor* in X. (We assume Y is nonsingular, though divisors can be singular). By the correspondence between line bundles and divisors in §7, there exists a line bundle L_Y, and $s \in H^0(L_Y)$ with $Y = s^{-1}(0)$, and $s = 0$ with multiplicity 1 on Y.

How are the cohomologies of X and Y related? Well, restriction of k-forms on X to Y induces a map $\rho : H^k_{\mathrm{dR}}(X;\mathbb{C}) \to H^k_{\mathrm{dR}}(Y;\mathbb{C})$. If X admits Kähler metrics then so does Y, and $H^k_{\mathrm{dR}}(X;\mathbb{C})$ splits into $H^{p,q}(X)$. As the restriction of a (p,q)-form on X to Y is a (p,q)-form on Y, we see that ρ maps $H^{p,q}(X) \to H^{p,q}(Y)$.

The Lefschetz Hyperplane Theorem gives conditions for these ρ to be isomorphisms.

The Lefschetz Hyperplane Theorem

Theorem 10.2 (Lefschetz Hyperplane Theorem)

Let (X,J) be a compact complex manifold with $\dim_{\mathbb{C}} X = n$, and Y a smooth hypersurface in X. Suppose the induced line bundle L_Y on X is positive. Then the restriction maps $\rho : H^k_{\rm dR}(X;\mathbb{C}) \to H^k_{\rm dR}(Y;\mathbb{C})$ are isomorphisms for $k \leqslant n-2$ and injective for $k = n-1$. Hence $\rho : H^{p,q}(X) \to H^{p,q}(Y)$ is an isomorphism for $p+q \leqslant n-2$ and injective for $p+q = n-1$. Also, if $n \geqslant 3$ then $\pi_1(X) \cong \pi_1(Y)$.

Sketch proof.

In the case $p = 0$, using sheaf cohomology ideas, one can show that there is a long exact sequence

$$\cdots \to H^q(L_Y^*) \to H^{0,q}(X) \xrightarrow{\rho} H^{0,q}(Y) \to H^{q+1}(L_Y^*) \to \cdots .$$

By Serre duality in §8.3 we have $H^q(L_Y^*) \cong H^{n-q}(L_Y \otimes \Lambda^n T^*X)^*$. So by the Kodaira Vanishing Theorem in §9.2 and L_Y positive we have $H^q(L_Y^*) = 0$ for $q < n$. Hence $\rho : H^{0,q}(X) \to H^{0,q}(Y)$ is an isomorphism for $q < n - 1$, and injective for $q = n - 1$. The case $p > 0$ is more complicated, with two long exact sequences. □

The Lefschetz Hyperplane Theorem is a useful computational tool. Usually we use it when we understand the topology of X well, e.g. $X = \mathbb{CP}^n$, and we want to compute $H^*(Y)$. The Lefschetz Hyperplane Theorem gives $H^k(Y) \cong H^k(X)$ for $k < n-1$. Then Poincaré duality gives $H^k(Y)$ for $k > n-1$. It remains only to compute $H^{n-1}(Y)$, the middle dimension. For instance, if we can compute $\chi(Y)$ then as we know $b^k(Y)$ for $k \neq n-1$, we can deduce $b^{n-1}(Y)$.

Example 10.3

Consider the line bundle $\mathcal{O}(k)$ on $\mathbb{CP}^n$ for $k > 0$. For every $0 \neq \boldsymbol{z} \in \mathbb{C}^{n+1}$, there is a homogeneous order k polynomial p with $p(\boldsymbol{z}) \neq 0$. This corresponds to $s \in H^0(\mathcal{O}(k))$ with $s([\boldsymbol{z}]) \neq 0$. Hence the base locus of $\mathcal{O}(k)$ is empty. Let $s \in H^0(\mathcal{O}(k))$ be generic. Then $s^{-1}(0)$ is smooth by Bertini's theorem in §9.3. Let $X = \mathbb{CP}^n$ and $Y = s^{-1}(0)$. The line bundle L_Y is $\mathcal{O}(k)$, which is positive, so the Lefschetz Hyperplane Theorem applies. Hence $H^j(Y;\mathbb{C}) = \mathbb{C}$ if $0 \leqslant j < n-1$ is even, and $H^j(Y;\mathbb{C}) = 0$ if $0 \leqslant j < n-1$ is odd, and $\pi_1(Y) = \{1\}$ if $n \geqslant 3$.

Example 10.4

Let $X = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $Y = \{[1,0],[0,1]\} \times \mathbb{CP}^1$. Then the line bundle L_Y on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is $\mathcal{O}(2,0)$, where $\mathcal{O}(k,l) = \pi_1^*(\mathcal{O}(k)) \otimes \pi_2^*(\mathcal{O}(l))$. Here $\mathcal{O}(k,l)$ is positive iff $k,l > 0$, so $\mathcal{O}(2,0)$ is not positive, and the Lefschetz Hyperplane Theorem does not apply. In fact $H^0(X;\mathbb{C}) \cong \mathbb{C}$ and $H^0(Y;\mathbb{C}) \cong \mathbb{C}^2$, so $\rho : H^0(X;\mathbb{C}) \to H^0(Y;\mathbb{C})$ is not an isomorphism, and the conclusions of the Lefschetz Hyperplane Theorem do not hold.

10.3. The adjunction formula

Let (X, J) be a compact complex manifold, and Y a hypersurface in X, that is, Y is a closed complex submanifold of X with $\dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X - 1$. Then Y induces a holomorphic line bundle L_Y on X, with a holomorphic section s vanishing on Y. The *normal bundle* ν_Y of Y in X is $TX|_Y / TY$, a holomorphic line bundle on Y. As $s|_Y \equiv 0$ but $\nabla s \neq 0$ on Y, the derivative of s in the normal directions to Y gives an isomorphism of line bundles $\mathrm{d}s|_Y : \nu_Y \to L_Y|_Y$.

The adjunction formula

We have an exact sequence of holomorphic vector bundles on Y

$$0 \longrightarrow TY \longrightarrow TX|_Y \longrightarrow \nu_Y \longrightarrow 0.$$

Using $\nu_Y \cong L_Y|_Y$ and dualizing gives

$$0 \longrightarrow L_Y^*|_Y \longrightarrow T^*X|_Y \longrightarrow T^*Y \longrightarrow 0.$$

Thus taking top exterior powers gives an isomorphism

$$\Lambda^n T^*X|_Y \cong \Lambda^{n-1} T^*Y \otimes L_Y^*|_Y,$$

where $n = \dim_{\mathbb{C}} X$. Therefore

$$K_Y \cong (K_X \otimes L_Y)|_Y. \tag{10.1}$$

This is the *adjunction formula*.

We often use the adjunction formula when we understand X and K_X – e.g. $X = \mathbb{CP}^n$ – and we want to compute K_Y.

Example 10.5

Suppose Y is a smooth degree k hypersurface in $X = \mathbb{CP}^n$. That is, $Y = s^{-1}(0)$ for $s \in H^0(\mathcal{O}(k))$. Then $L_Y \cong \mathcal{O}(k)$. Also $K_{\mathbb{CP}^n} \cong \mathcal{O}(-n-1)$, as in §7.2. So the adjunction formula gives

$$K_Y \cong (\mathcal{O}(-n-1) \otimes \mathcal{O}(k))|_Y = \mathcal{O}(k-n-1)|_Y.$$

In particular, if $k = n+1$ then $K_Y \cong \mathcal{O}(0)|_Y \cong \mathcal{O}_Y$, that is, the canonical bundle of Y is trivial. Then Y is called *Calabi–Yau*. So, for example, a smooth quartic in $\mathbb{CP}^3$ is a Calabi–Yau 2-fold ($K3$ surface), and a smooth quintic in $\mathbb{CP}^4$ is a Calabi–Yau 3-fold.
If $k < n+1$ then K_Y is a negative line bundle (Y is a *Fano manifold*). If $k > n+1$ then K_Y is a positive line bundle (Y is of *general type*).

10.4. Blow-ups

Let (X, J) be a complex n-manifold, and Y a closed, embedded complex k-submanifold in X. The *blow-up of X along Y* is a complex manifold $\tilde{X}$ with a proper holomorphic map $\pi : \tilde{X} \to X$, such that $\pi^{-1}(Y)$ is a smooth, closed hypersurface D in $\tilde{X}$ called the *exceptional divisor*, and $\pi : \tilde{X} \setminus D \to X \setminus Y$ is a biholomorphism. Thus, $\tilde{X}$ is made by cutting the k-submanifold Y out of X and replacing it by the $(n-1)$-submanifold D. If X is compact then $\tilde{X}$ is compact.

Blow-ups also work in the worlds of varieties and schemes – basically, singular complex manifolds. One can define the blow-up of a scheme at a closed subscheme, which is another scheme.

Blow-ups are often used to resolve singularities. That is, if X is a singular complex manifold (scheme), then by (repeatedly) blowing up X at its singularities, we can define a nonsingular complex manifold $\tilde{X}$.

The next example defines the blow-up of $\mathbb{C}^n$ at 0.

Example 10.6

Let $\tilde X$ be the subset of points $\bigl((x_1,\ldots,x_n),[y_1,\ldots,y_n]\bigr)$ in $\mathbb{C}^n\times\mathbb{CP}^{n-1}$ such that $x_j=\lambda y_j$ for $j=1,\ldots,n$, for some $\lambda\in\mathbb{C}$. That is, either $(x_1,\ldots,x_n)\neq(0,\ldots,0)$ and $[y_1,\ldots,y_n]=[x_1,\ldots,x_n]$, or $(x_1,\ldots,x_n)=(0,\ldots,0)$ and $[y_1,\ldots,y_n]$ is arbitrary. Then $\tilde X$ is a complex submanifold of $\mathbb{C}^n\times\mathbb{CP}^{n-1}$, with complex dimension n.

Define $\pi:\tilde X\to\mathbb{C}^n$ by

$$\pi:\bigl((x_1,\ldots,x_n),[y_1,\ldots,y_n]\bigr)\longmapsto(x_1,\ldots,x_n).$$

Then π is holomorphic. If $(x_1,\ldots,x_n)\neq(0,\ldots,0)$ then $\pi^{-1}(x_1,\ldots,x_n)$ is the point $\bigl((x_1,\ldots,x_n),[x_1,\ldots,x_n]\bigr)$. Also $\pi^{-1}(0)=\{0\}\times\mathbb{CP}^{n-1}$ is a smooth hypersurface D in $\tilde X$, and $\pi:\tilde X\setminus D\to\mathbb{C}^n\setminus\{0\}$ is biholomorphic. The other projection $\pi_2:\tilde X\to\mathbb{CP}^{n-1}$ identifies $\tilde X$ with the total space of the line bundle $\mathcal{O}(-1)$ over $\mathbb{CP}^{n-1}$.

In the same way, the blow-up $\tilde X$ of a complex manifold X at a point x replaces x by the projective space $D=\mathbb{P}(T_xX)$. The blow-up $\tilde X$ of X along a complex submanifold Y replaces Y by $D=\mathbb{P}(\nu)$, where $\nu=TX|_Y/TY$ is the normal bundle of Y in X. That is, we have $\pi:D\to Y$ with $\pi^{-1}(y)=\mathbb{P}(T_yX/T_yY)$ for $y\in Y$.

We can consider holomorphic line bundles on blow-ups. If $\tilde{X}$ is the blow-up of X along Y, with exceptional divisor D, and $L \to X$ is a holomorphic line bundle on X, then $\pi^*(L)$ is a holomorphic line bundle on $\tilde{X}$.

We also have the holomorphic line bundle L_D on $\tilde{X}$ associated to D. A calculation similar to the adjunction formula shows that

$$K_{\tilde{X}} \cong L_D^{n-k-1} \otimes \pi^*(K_X),$$

where $n = \dim_{\mathbb{C}} X$, $k = \dim_{\mathbb{C}} Y$.

Proposition 10.7

Suppose (X, J) is a compact complex manifold, Y a closed complex submanifold in X, $\pi : \tilde{X} \to X$ the blow-up of X along Y with exceptional divisor D, and L a positive line bundle on X. Then $L_D^{-1} \otimes \pi^*(L)^k$ is a positive line bundle on $\tilde{X}$ for $k \gg 0$.

Sketch proof.

The projection $\pi : D \to Y$ has fibre $\mathbb{CP}^{n-k-1}$ over $y \in Y$, where $n = \dim_{\mathbb{C}} X$, $k = \dim_{\mathbb{C}} Y$. One can show that $L_D|_{\pi^{-1}(y)}$ is the line bundle $\mathcal{O}(-1) \to \mathbb{CP}^{n-k-1}$. Thus $L_D^{-1}|_{\pi^{-1}(y)}$ is $\mathcal{O}(1)$, which is positive. We can choose a closed real (1,1)-form η on $\tilde{X}$ representing $c_1(L_D^{-1})$, such that $\eta|_{\pi^{-1}(y)}$ is positive on $\pi^{-1}(y) \cong \mathbb{CP}^{n-k-1}$ for each $y \in Y$.
As L is positive on X we can choose a closed, real, positive (1,1)-form ζ on X representing $c_1(L)$. Then $\pi^*(\zeta)$ represents $c_1(\pi^*(L))$, and $\eta + k\pi^*(\zeta)$ represents $c_1\bigl(L_D^{-1} \otimes \pi^*(L)^k\bigr)$.
We claim $\eta + k\pi^*(\zeta)$ is a positive (1,1)-form for $k \gg 0$, so that $L_D^{-1} \otimes \pi^*(L)^k$ is positive. To see this, note that $\pi^*(\zeta)$ is nonnegative on $\tilde{X}$, and zero only on the tangent bundles of $\pi^{-1}(y)$ for $y \in Y$; also, η is positive on the tangent bundles of $\pi^{-1}(y)$, though it may be negative in other directions. □

By a corollary of the Kodaira Embedding Theorem, a compact complex manifold is projective iff it admits positive line bundles. So we deduce.

Corollary 10.8

Let (X, J) be a projective complex manifold, Y a closed complex submanifold of X, and $\tilde{X}$ the blow-up of X along Y. Then $\tilde{X}$ is projective.