What is a Kuranishi space?

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Based on arXiv:1409.6908, arXiv:1510.07444, and
multiple-volume book project in progress.
Preliminary versions of first two volumes available at
`people.maths.ox.ac.uk/~joyce/Kuranishi.html`.

These slides available at
`people.maths.ox.ac.uk/~joyce/talks.html`.

Introduction

In 1962, Kuranishi proved that if (X, J) is a compact complex manifold, then the moduli space $\mathfrak{M}$ of complex structures on X can be described near J using the zeroes $\Phi^{-1}(0)$ of a holomorphic map $\Phi : U \to H^2_J(TX)$, where $0 \in U \subset H^1_J(TX)$ is open. Here $H^1_J(TX)$ is the space of *infinitesimal deformations*, $H^2_J(TX)$ the *obstructions*, and $H^0_J(TX)$ the Lie algebra of the *automorphism group* of (X, J). This may have been the first major result on deformation theory in the presence of obstructions. Later, many other deformation theory problems in real and complex geometry turned out to have this basic structure identified by Kuranishi. 'Kuranishi spaces' were introduced by Fukaya–Ono 1999 and Fukaya–Oh–Ohta–Ono 2008 as the geometric structure on moduli spaces of J-holomorphic curves $\bar{\mathcal{M}}_{g,k}(J, \beta)$ in symplectic geometry. Roughly, a Kuranishi space is a topological space X covered by an atlas of 'Kuranishi neighbourhoods' (V, E, Γ, s, ψ), each arising from deformation theory à la Kuranishi.

Fukaya–Ono's main goal was to define virtual cycles/chains for these, to be able to define Gromov–Witten invariants, Lagrangian Floer theory, etc. Though their definitions work for their purposes, they are not very satisfactory as geometric spaces in their own right. For example, in the FOOO theory, there is no good notion of morphism $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ (or even isomorphism) between Kuranishi spaces, only of smooth maps $\boldsymbol{f} : \boldsymbol{X} \to Y$ to a manifold Y.

One would like such morphisms for applications (e.g. forgetful maps between moduli spaces), to make sense of 'fibre products' of Kuranishi spaces, and as Kuranishi spaces are interesting for their own sake. Kuranishi spaces should be a differential-geometric analogue of (derived) schemes in Algebraic Geometry. I will explain an alternative definition of Kuranishi spaces (Joyce 2014), which form a 2-*category*, with well-behaved 1- and 2-morphisms.

The FOOO definition of Kuranishi space $\boldsymbol{X}$ is a topological space X with an 'atlas of charts', like the definition of manifolds. The charts, called 'Kuranishi neighbourhoods', are $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ for V_i a manifold, $E_i \to V_i$ a vector bundle, Γ_i a finite group acting on V_i, E_i, $s_i : V_i \to E_i$ a Γ_i-equivariant section, and $\psi_i : s_i^{-1}(0)/\Gamma_i \to X$ a homeomorphism with an open set $\mathrm{Im}\,\psi_i \subset X$. 'Coordinate changes' $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ involve embeddings $V_i \overset{\text{open}}{\supset} V_{ij} \xrightarrow{\phi_{ij}} V_j$, $E_i|_{V_{ij}} \xrightarrow{\hat\phi_{ij}} E_j$, and exist only if $\dim V_i \leqslant \dim V_j$, so they are generally not invertible. Coordinate changes must be strictly associative on triple overlaps, $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$.

In my definition coordinate changes are weaker — $\phi_{ij}, \hat\phi_{ij}$ need only be smooth maps, not embeddings. We introduce a notion of 2-*isomorphism* $\Lambda : \Phi_{ij} \Rightarrow \Phi'_{ij}$ of coordinate changes, making Kuranishi neighbourhoods into a 2-category. Coordinate changes need only be associative up to 2-isomorphisms $\Lambda_{ijk} : \Phi_{jk} \circ \Phi_{ij} \Rightarrow \Phi_{ik}$. Coordinate changes Φ_{ij} are invertible up to 2-isomorphism, there exist Φ_{ji} and 2-isomorphisms $\Lambda_{ii} : \Phi_{ji} \circ \Phi_{ij} \to \mathrm{id}_{(V_i,\ldots)}$, $\Lambda_{jj} : \Phi_{ij} \circ \Phi_{ji} \to \mathrm{id}_{(V_j,\ldots)}$.

Kuranishi spaces and Derived Differential Geometry

The inspiration for this definition came from the Derived Algebraic Geometry of Jacob Lurie and Toën–Vezzosi. We should understand Kuranishi spaces as *derived smooth orbifolds*, where 'derived' is in the sense of DAG. Definitions of ∞-categories / 2-categories of derived manifolds modelled on the definition of derived schemes were given by Lurie 2009 (sketch), Spivak 2010, Borisov–Noel 2011, and Joyce 2012, $\mathbf{dMan}, \mathbf{dOrb}$. They are topological spaces with ∞-sheaves/2-sheaves of derived C^∞-rings. My definition (2014) of Kuranishi space is an 'atlas of charts' definition, but constructed to give an equivalent 2-category **Kur** to my 2-category **dOrb**.

One lesson from DAG is that higher categories (∞- or 2-categories) are key: truncating to ordinary categories loses too much information.

FOOO Kuranishi spaces (1999) predate DAG (2006). This is one reason for problems with the original definition: some essential ideas were missing.

1. The category of μ-Kuranishi spaces
1.1. μ-Kuranishi neighbourhoods

As a warm-up exercise, I first explain how to define an ordinary category of 'μ-Kuranishi spaces', a simplified version of the Kuranishi space construction without quotients by finite groups, and using ordinary category rather than 2-category methods.

Definition

Let X be a topological space. A *μ-Kuranishi neighbourhood* on X is a quadruple (V, E, s, ψ) such that:

(a) V is a smooth manifold.
(b) $E \to V$ is a vector bundle over V, the *obstruction bundle*.
(c) $s \in C^\infty(E)$ is a smooth section of E, the *Kuranishi section*.
(d) ψ is a homeomorphism from $s^{-1}(0)$ to an open subset $\mathrm{Im}\,\psi$ in X, where $\mathrm{Im}\,\psi$ is called the *footprint* of (V, E, s, ψ).

Morphisms of μ-Kuranishi neighbourhoods

Definition

Let $f : X \to Y$ be a continuous map of topological spaces, (V_i, E_i, s_i, ψ_i), (W_j, F_j, t_j, χ_j) be μ-Kuranishi neighbourhoods on X, Y, and $S \subseteq \mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j) \subseteq X$ be an open set. Consider triples $(V_{ij}, f_{ij}, \hat{f}_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.

(b) $f_{ij} : V_{ij} \to W_j$ is smooth, with $f \circ \psi_i = \chi_j \circ f_{ij}$ on $s_i^{-1}(0) \cap V_{ij}$.

(c) $\hat{f}_{ij} : E_i|_{V_{ij}} \to f_{ij}^*(F_j)$ is a morphism of vector bundles on V_{ij}, with $\hat{f}_{ij}(s_i|_{V_{ij}}) = f_{ij}^*(t_j) + O(s_i^2)$.

Define an equivalence relation $\sim$ by $(V_{ij}, f_{ij}, \hat{f}_{ij}) \sim (V'_{ij}, f'_{ij}, \hat{f}'_{ij})$ if there are open $\psi_i^{-1}(S) \subseteq \dot{V}_{ij} \subseteq V_{ij} \cap V'_{ij}$ and $\Lambda : E_i|_{\dot{V}_{ij}} \to f_{ij}^*(TW_j)|_{\dot{V}_{ij}}$ with $f'_{ij} = f_{ij} + \Lambda \cdot s_i + O(s_i^2)$ and $\hat{f}'_{ij} = \hat{f}_{ij} + \Lambda \cdot f_{ij}^*(\mathrm{d}t_j) + O(s_i)$. We write $[V_{ij}, f_{ij}, \hat{f}_{ij}]$ for the $\sim$-equivalence class of $(V_{ij}, f_{ij}, \hat{f}_{ij})$, and call $[V_{ij}, f_{ij}, \hat{f}_{ij}] : (V_i, E_i, s_i, \psi_i) \to (W_j, F_j, t_j, \chi_j)$ a *morphism over* S, f.

Here the equivalence relation $\sim$ is weird, but crucial for later. Given continuous maps $f : X \to Y$ and $g : Y \to Z$, open $S \subseteq X$, $T \subseteq Y$, morphisms $[U_{ij}, \phi_{ij}, \hat\phi_{ij}] : (U_i, D_i, r_i, \phi_i) \to (V_j, E_j, s_j, \psi_j)$ over S, f and $[V_{jk}, \psi_{jk}, \hat\psi_{jk}] : (V_j, E_j, s_j, \psi_j) \to (W_k, F_k, t_k, \chi_k)$ over T, g, the *composition* over $S \cap f^{-1}(T)$, $g \circ f$ is

$$[V_{jk}, \psi_{jk}, \hat\psi_{jk}] \circ [U_{ij}, \phi_{ij}, \hat\phi_{ij}] = \bigl[\phi_{ij}^{-1}(V_{jk}), \psi_{jk} \circ \phi_{ij}|_{\dots}, \phi_{ij}^{-1}(\hat\psi_{jk}) \circ \hat\phi_{ij}|_{\dots}\bigr] : (U_i, D_i, r_i, \phi_i) \longrightarrow (W_k, F_k, t_k, \chi_k).$$

Theorem 1.1 (Sheaf property of μ-Kuranishi morphisms.)

Let (V_i, E_i, s_i, ψ_i), (W_j, F_j, t_j, χ_j) be μ-Kuranishi neighbourhoods on X, Y, and $f : X \to Y$ be continuous. Then morphisms from (V_i, E_i, s_i, ψ_i) to (W_j, F_j, t_j, χ_j) over f form a sheaf $\mathcal{H}om_f\bigl((V_i, E_i, s_i, \psi_i), (W_j, F_j, t_j, \chi_j)\bigr)$ on $\mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j)$.

This will be essential for defining compositions of morphisms of μ-Kuranishi spaces. The lack of such a sheaf property in the FOOO theory is why FOOO Kuranishi spaces are not a category.

Coordinate changes of μ-Kuranishi neighbourhoods

Take $Y = X$ and $f = \mathrm{id}_X$. A morphism $\Phi_{ij} = [V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ over id_X is called a *coordinate change* if there exists $\Phi_{ji} = [V_{ji}, \phi_{ji}, \hat\phi_{ji}] : (V_j, E_j, s_j, \psi_j) \to (V_i, E_i, s_i, \psi_i)$ such that $\Phi_{ji} \circ \Phi_{ij} = [V_i, \mathrm{id}_{V_i}, \mathrm{id}_{E_i}]$ and $\Phi_{ij} \circ \Phi_{ji} = [V_j, \mathrm{id}_{V_j}, \mathrm{id}_{E_j}]$.

This does not require $\phi_{ji} \circ \phi_{ij} = \mathrm{id}_{V_i}$, $\hat\phi_{ji} \circ \hat\phi_{ij} = \mathrm{id}_{E_i}$, but only that $\phi_{ji} \circ \phi_{ij} = \mathrm{id}_{V_i} + \Lambda \cdot s_i + O(s_i^2)$ and $\hat\phi_{ji} \circ \hat\phi_{ij} = \mathrm{id}_{E_i} + \Lambda \cdot f_{ij}^*(\mathrm{d}t_j) + O(s_i)$.

Coordinate changes exist even if $\dim V_i \neq \dim V_j$.

Theorem

A morphism $[V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a coordinate change over S if and only if for all $x \in S$ with $v_i = \psi_i^{-1}(x)$ and $v_j = \psi_j^{-1}(x)$, the following sequence is exact:

$$0 \longrightarrow T_{v_i} V_i \xrightarrow{\mathrm{d}s_i|_{v_i} \oplus T_{v_i}\phi_{ij}} E_i|_{v_i} \oplus T_{v_j} V_j \xrightarrow{\hat\phi_{ij}|_{v_i} \oplus -\mathrm{d}s_j|_{v_j}} E_j|_{v_j} \longrightarrow 0.$$

1.2. The definition of μ-Kuranishi space

Definition

Let X be a Hausdorff, second countable topological space, and $n \in \mathbb{Z}$. A *μ-Kuranishi structure $\mathcal{K}$ on X of virtual dimension n* is data $\mathcal{K} = \bigl(I, (V_i, E_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\ i,j\in I}\bigr)$, where:

(a) I is an indexing set.

(b) (V_i, E_i, s_i, ψ_i) is a μ-Kuranishi neighbourhood on X for each $i \in I$, with $\dim V_i - \operatorname{rank} E_i = n$.

(c) $\Phi_{ij} = [V_{ij}, \phi_{ij}, \hat\phi_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a coordinate change over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$ for all $i, j \in I$.

(d) $\bigcup_{i\in I} \operatorname{Im}\psi_i = X$.

(e) $\Phi_{ii} = \mathrm{id}_{(V_i,E_i,s_i,\psi_i)}$ for all $i \in I$.

(f) $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$ for all $i, j, k \in I$ over $S = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j \cap \operatorname{Im}\psi_k$.

We call $\boldsymbol{X} = (X, \mathcal{K})$ a *μ-Kuranishi space*, of *virtual dimension* $\operatorname{vdim}\boldsymbol{X} = n$.

Definition 1.2

Let $\boldsymbol{X}=(X,\mathcal{K})$ with $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},\Phi_{ii',\,i,i'\in I}\bigr)$ and $\boldsymbol{Y}=(Y,\mathcal{L})$ with $\mathcal{L}=\bigl(J,(W_j,F_j,t_j,\chi_j)_{j\in J},\Psi_{jj',\,j,j'\in J}\bigr)$ be μ-Kuranishi spaces. A *morphism* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is $\boldsymbol{f}=\bigl(f,\boldsymbol{f}_{ij,\,i\in I,\,j\in J}\bigr)$, where $f:X\to Y$ is a continuous map, and $\boldsymbol{f}_{ij}=[V_{ij},f_{ij},\hat f_{ij}]:(V_i,E_i,s_i,\psi_i)\to(W_j,F_j,t_j,\chi_j)$ is a morphism of μ-Kuranishi neighbourhoods over $S=\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f for all $i\in I$, $j\in J$, satisfying the conditions:

(a) If $i,i'\in I$ and $j\in J$ then $\boldsymbol{f}_{i'j}\circ\Phi_{ii'}|_S=\boldsymbol{f}_{ij}|_S$ over $S=\mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_{i'}\cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f.

(b) If $i\in I$ and $j,j'\in J$ then $\Psi_{jj'}\circ\boldsymbol{f}_{ij}|_S=\boldsymbol{f}_{ij'}|_S$ over $S=\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j\cap\mathrm{Im}\,\chi_{j'})$ and f.

When $\boldsymbol{Y}=\boldsymbol{X}$, so that $J=I$, define the identity morphism $\mathbf{id}_{\boldsymbol{X}}:\boldsymbol{X}\to\boldsymbol{X}$ by $\mathbf{id}_{\boldsymbol{X}}=\bigl(\mathrm{id}_X,\Phi_{ij,\,i,j\in I}\bigr)$.

1.3. Composition of morphisms in μKur

Let $\boldsymbol{X}=(X,\mathcal{I})$ with $\mathcal{I}=\bigl(I,(U_i,D_i,r_i,\phi_i)_{i\in I},\Phi_{ii',\,i,i'\in I}\bigr)$ and $\boldsymbol{Y}=(Y,\mathcal{J})$ with $\mathcal{J}=\bigl(J,(V_j,E_j,s_j,\psi_j)_{j\in J},\Psi_{jj',\,j,j'\in J}\bigr)$ and $\boldsymbol{Z}=(Z,\mathcal{K})$ with $\mathcal{K}=\bigl(K,(W_k,F_k,t_k,\xi_k)_{k\in K},\Xi_{kk',\,k,k'\in K}\bigr)$ be μ-Kuranishi spaces, and $\boldsymbol{f}=(f,\boldsymbol{f}_{ij}):\boldsymbol{X}\to\boldsymbol{Y}$, $\boldsymbol{g}=(g,\boldsymbol{g}_{jk}):\boldsymbol{Y}\to\boldsymbol{Z}$ be morphisms. Consider the problem of how to define the composition $\boldsymbol{g}\circ\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$.

For all $i\in I$ and $k\in K$, $\boldsymbol{g}\circ\boldsymbol{f}$ must contain a morphism $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}:(U_i,D_i,r_i,\phi_i)\to(W_k,F_k,t_k,\xi_k)$ defined over $S_{ik}=\mathrm{Im}\,\phi_i\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g\circ f$.

For each $j\in J$, we have a morphism $\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij}:(U_i,D_i,r_i,\phi_i)\to(W_k,F_k,t_k,\xi_k)$, but it is defined over $S_{ijk}=\mathrm{Im}\,\phi_i\cap f^{-1}(\mathrm{Im}\,\psi_j)\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$ and $g\circ f$, not over the whole of $S_{ik}=\mathrm{Im}\,\phi_i\cap(g\circ f)^{-1}(\mathrm{Im}\,\xi_k)$.

Composition of morphisms in $\boldsymbol{\mu}$**Kur**

The solution is to use the sheaf property of morphisms, Theorem 1.1. The sets S_{ijk} for $j \in J$ form an open cover of S_{ik}. Using Definition 1.2(a),(b) we can show that $\boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij}|_{S_{ijk} \cap S_{ij'k}} = \boldsymbol{g}_{j'k} \circ \boldsymbol{f}_{ij'}|_{S_{ijk} \cap S_{ij'k}}$. Therefore by Theorem 1.1 there is a unique morphism of μ-Kuranishi neighbourhoods $(\boldsymbol{g} \circ \boldsymbol{f})_{ik} : (U_i, D_i, r_i, \phi_i) \to (W_k, F_k, t_k, \xi_k)$ defined over S_{ik} and $g \circ f$ with $(\boldsymbol{g} \circ \boldsymbol{f})_{ik}|_{S_{ijk}} = \boldsymbol{g}_{jk} \circ \boldsymbol{f}_{ij}$ for all $j \in J$. We show that $\boldsymbol{g} \circ \boldsymbol{f} := \bigl(g \circ f, (\boldsymbol{g} \circ \boldsymbol{f})_{ik,\ i \in I,\ k \in K}\bigr)$ is a morphism $\boldsymbol{g} \circ \boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Z}$ of μ-Kuranishi spaces, which we call *composition*.

Composition is associative, and makes μ-Kuranishi spaces into a category $\boldsymbol{\mu}$**Kur**.

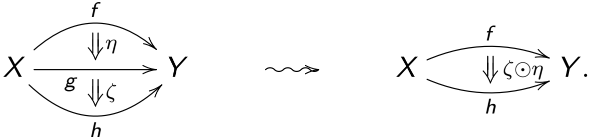
2. The 2-category of Kuranishi spaces
2.1. 2-categories

A 2-category $\boldsymbol{\mathcal{C}}$ has *objects* $X, Y, \ldots$, 1-*morphisms* $f, g : X \to Y$ (morphisms), and 2-*morphisms* $\eta : f \Rightarrow g$ (morphisms between morphisms). Here are some examples to bear in mind:

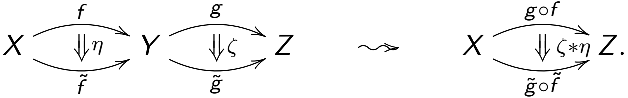
Example

(a) The strict 2-category $\mathfrak{Cat}$ has objects categories $\mathscr{C}, \mathscr{D}, \ldots$, 1-morphisms functors $F, G : \mathscr{C} \to \mathscr{D}$, and 2-morphisms natural transformations $\eta : F \Rightarrow G$.
(b) The strict 2-category $\mathbf{Top^{ho}}$ of *topological spaces up to homotopy* has objects topological spaces $X, Y, \ldots$, 1-morphisms continuous maps $f, g : X \to Y$, and 2-morphisms isotopy classes $[H] : f \Rightarrow g$ of homotopies H from f to g.

There are three kinds of composition in a 2-category. If $f : X \to Y$ and $g : Y \to Z$ are 1-morphisms we have *composition of 1-morphisms*, $g \circ f : X \to Z$. If $f, g, h : X \to Y$ are 1-morphisms and $\eta : f \Rightarrow g$, $\zeta : g \Rightarrow h$ are 2-morphisms we have *vertical composition of 2-morphisms* $\zeta \odot \eta : f \Rightarrow h$, as a diagram



If $f, \tilde f : X \to Y$ and $g, \tilde g : Y \to Z$ are 1-morphisms and $\eta : f \Rightarrow \tilde f$, $\zeta : g \Rightarrow \tilde g$ are 2-morphisms twe have *horizontal composition of 2-morphisms* $\zeta * \eta : g \circ f \Rightarrow \tilde g \circ \tilde f$, as a diagram



There are *identity 1-morphisms* $\mathrm{id}_X : X \to X$ and *identity 2-morphisms* $\mathrm{id}_f : f \Rightarrow f$. *2-isomorphisms* are invertible under vertical composition.

2.2. Kuranishi neighbourhoods

Definition

Let X be a topological space. A *Kuranishi neighbourhood* on X is a quintuple (V, E, Γ, s, ψ) such that:

(a) V is a smooth manifold.

(b) $\pi : E \to V$ is a vector bundle over V, the *obstruction bundle*.

(c) Γ is a finite group with compatible smooth actions on V and E preserving the vector bundle structure.

(d) $s : V \to E$ is a Γ-equivariant smooth section of E, the *Kuranishi section*.

(e) $\psi : s^{-1}(0)/\Gamma \to X$ is a homeomorphism with an open $\mathrm{Im}\,\psi \subseteq X$.

If $S \subseteq X$ is open, we call (V, E, Γ, s, ψ) a *Kuranishi neighbourhood over* S if $S \subseteq \mathrm{Im}\,\psi \subseteq X$.

This is the same as Fukaya-Oh-Ohta-Ono Kuranishi neighbourhoods.

Let $f : X \to Y$ be a continuous map of topological spaces, and $(V_i, E_i, \Gamma_i, s_i, \psi_i)$, $(W_j, F_j, \Delta_j, t_j, \chi_j)$ be Kuranishi neighbourhoods on X, Y. Then we define 1-*morphisms* $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (W_j, F_j, \Delta_j, t_j, \chi_j)$ over f, and 2-*morphisms* $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$ between 1-morphisms. We define compositions of 1- and 2-morphisms, and identity 1- and 2-morphisms. Here 1-morphisms are an orbifold version of maps $\phi_{ij} : V_i \supset V_{ij} \to W_j$, $\hat{\phi}_{ij} : E_i|_{V_{ij}} \to \phi_{ij}^*(F_j)$ in the μ-Kuranishi case, and 2-morphisms generalize the equivalence relation $\sim$.

Let $Y = X$ and $f = \mathrm{id}_X$. We call a 1-morphism $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ a *coordinate change* if it is invertible up to 2-isomorphism. That is, there exist $\Phi_{ji} : (V_j, E_j, \Gamma_j, s_j, \psi_j) \to (V_i, E_i, \Gamma_i, s_i, \psi_i)$ and 2-isomorphisms $\Lambda_{ii} : \Phi_{ji} \circ \Phi_{ij} \Rightarrow \mathrm{id}_{(V_i,E_i,\Gamma_i,s_i,\psi_i)}$ and $\Lambda_{jj} : \Phi_{ij} \circ \Phi_{ji} \Rightarrow \mathrm{id}_{(V_j,E_j,\Gamma_j,s_j,\psi_j)}$.

Theorem 2.1 (2-sheaf property of Kuranishi neighbourhoods.)

Let $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ and $(W_j, F_j, \Delta_j, t_j, \chi_j)$ be Kuranishi neighbourhoods on X, Y, and $f : X \to Y$ be continuous. Then 1-morphisms $\Phi_{ij}, \Phi'_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (W_j, F_j, \Delta_j, t_j, \chi_j)$ over f and 2-morphisms $\Lambda_{ij} : \Phi_{ij} \Rightarrow \Phi'_{ij}$, on open subsets $S \subseteq \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$, form a 2-sheaf (stack) on $\mathrm{Im}\,\psi_i \cap f^{-1}(\mathrm{Im}\,\chi_j)$, that is, they glue well on open covers, in a 2-categorical sense. When $Y = X$ and $f = \mathrm{id}_X$, coordinate changes Φ_{ij} are a 2-subsheaf.

This will be crucial for defining compositions of 1-morphisms of Kuranishi spaces. It is not obvious. It depends on the weird definition of 2-morphisms.

2.3. The definition of Kuranishi space

Definition

Let X be a Hausdorff, second countable topological space. A *Kuranishi structure* $\mathcal{K}$ on X of *virtual dimension* $n \in \mathbb{Z}$ is data $\mathcal{K} = \bigl(I, (V_i, E_i, \Gamma_i, s_i, \psi_i)_{i\in I}, \Phi_{ij,\ i,j\in I}, \Lambda_{ijk,\ i,j,k\in I}\bigr)$, where:

(a) I is an indexing set.

(b) $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ is a Kuranishi neighbourhood on X for $i \in I$, with $\dim V_i - \operatorname{rank} E_i = n$. Write $S_{ij} = \operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$, etc.

(c) $\Phi_{ij} : (V_i, E_i, \Gamma_i, s_i, \psi_i) \to (V_j, E_j, \Gamma_j, s_j, \psi_j)$ is a coordinate change over S_{ij} for $i, j \in I$.

(d) $\Lambda_{ijk} : \Phi_{jk} \circ \Phi_{ij} \Rightarrow \Phi_{ik}$ is a 2-morphism over S_{ijk} for $i, j, k \in I$.

(e) $\bigcup_{i\in I} \operatorname{Im}\psi_i = X$. (f) $\Phi_{ii} = \mathrm{id}_{(V_i,E_i,\Gamma_i,s_i,\psi_i)}$ for $i \in I$.

(g) $\Lambda_{iij} = \Lambda_{ijj} = \mathrm{id}_{\Phi_{ij}}$ for $i, j \in I$.

(h) $\Lambda_{ikl} \odot (\mathrm{id}_{\Phi_{kl}} * \Lambda_{ijk})|_{S_{ijkl}} = \Lambda_{ijl} \odot (\Lambda_{jkl} * \mathrm{id}_{\Phi_{ij}})|_{S_{ijkl}} :$ $\Phi_{kl} \circ \Phi_{jk} \circ \Phi_{ij}|_{S_{ijkl}} \Longrightarrow \Phi_{il}|_{S_{ijkl}}$ for $i, j, k, l \in I$.

We call $\boldsymbol{X} = (X, \mathcal{K})$ a *Kuranishi space*, with $\operatorname{vdim} \boldsymbol{X} = n$.

Definition

Let $\boldsymbol{X}=(X,\mathcal{K})$ and $\boldsymbol{Y}=(Y,\mathcal{L})$ be Kuranishi spaces, with $\mathcal{K}=\bigl(I,(V_i,E_i,\Gamma_i,s_i,\psi_i)_{i\in I},\ \Phi_{ii',\,i,i'\in I},\ \Lambda_{ii'i'',\,i,i',i''\in I}\bigr)$ and $\mathcal{L}=\bigl(J,(W_j,F_j,\Delta_j,t_j,\chi_j)_{j\in J},\ \Psi_{jj',\,j,j'\in J},\ \mathrm{M}_{jj'j'',\,j,j',j''\in J}\bigr)$. A *1-morphism* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is $\boldsymbol{f}=\bigl(f,\boldsymbol{f}_{ij,\,i\in I,j\in J},\ F_{ii',\,i,i'\in I}^{j,\,j\in J},\ F_{i,\,i\in I}^{jj',\,j,j'\in J}\bigr)$, with: (a) $f:X\to Y$ is a continuous map.

(b) $\boldsymbol{f}_{ij}:(V_i,E_i,\Gamma_i,s_i,\psi_i)\to(W_j,F_j,\Delta_j,t_j,\chi_j)$ is a 1-morphism of Kuranishi neighbourhoods over $S=\mathrm{Im}\,\psi_i\cap f^{-1}(\mathrm{Im}\,\chi_j)$ and f for $i\in I$, $j\in J$.

(c) $F_{ii'}^j:\boldsymbol{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\boldsymbol{f}_{ij}$ is a 2-morphism over f for $i,i'\in I$, $j\in J$.

(d) $F_i^{jj'}:\Psi_{jj'}\circ\boldsymbol{f}_{ij}\Rightarrow\boldsymbol{f}_{ij'}$ is a 2-morphism over f for $i\in I$, $j,j'\in J$.

(e) $F_{ii}^j=F_i^{jj}=\mathrm{id}_{\boldsymbol{f}_{ij}}$.

(f) $F_{ii''}^j\odot(\mathrm{id}_{\boldsymbol{f}_{i''j}}*\Lambda_{ii'i''})=F_{ii'}^j\odot(F_{i'i''}^j*\mathrm{id}_{\Phi_{ii'}}):\boldsymbol{f}_{i''j}\circ\Phi_{i'i''}\circ\Phi_{ii'}\Rightarrow\boldsymbol{f}_{i''j}$.

(g) $F_i^{jj'}\odot(\mathrm{id}_{\Psi_{jj'}}*F_{ii'}^j)=F_{ii'}^{j'}\odot(F_{i'}^{jj'}*\mathrm{id}_{\Phi_{ii'}}):\Psi_{jj'}\circ\boldsymbol{f}_{i'j}\circ\Phi_{ii'}\Rightarrow\boldsymbol{f}_{ij'}$.

(h) $F_i^{j'j''}\odot(\mathrm{id}_{\Psi_{j'j''}}*F_i^{jj'})=F_i^{jj''}\odot(\mathrm{M}_{jj'j''}*\mathrm{id}_{\boldsymbol{f}_{ij}}):\Psi_{j'j''}\circ\Psi_{jj'}\circ\boldsymbol{f}_{ij}\Rightarrow\boldsymbol{f}_{ij''}$.

Here (c)–(h) hold for all $i, j, \ldots$, restricted to appropriate domains.

Definition

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms of Kuranishi spaces, with $\boldsymbol{f} = \bigl(f, \boldsymbol{f}_{ij,\, i\in I,\, j\in J},\ F_{ii',\, i,i'\in I}^{j,\, j\in J},\ F_{i,\, i\in I}^{jj',\, j,j'\in J}\bigr)$, $\boldsymbol{g} = \bigl(g, \boldsymbol{g}_{ij,\, i\in I,\, j\in J},\ G_{ii',\, i,i'\in I}^{j,\, j\in J},\ G_{i,\, i\in I}^{jj',\, j,j'\in J}\bigr)$. Suppose the continuous maps $f, g : X \to Y$ satisfy $f = g$. A *2-morphism* $\boldsymbol{\Lambda} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is data $\boldsymbol{\Lambda} = (\Lambda_{ij,\, i\in I,\, j\in J})$, where $\Lambda_{ij} : \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij}$ is a 2-morphism of Kuranishi neighbourhoods over $f = g$, satisfying:

(a) $G_{ii'}^{j} \odot (\Lambda_{i'j} * \mathrm{id}_{\Phi_{ii'}}) = \Lambda_{ij} \odot F_{ii'}^{j} : \boldsymbol{f}_{i'j} \circ \Phi_{ii'} \Rightarrow \boldsymbol{g}_{ij}$ for $i, i' \in I$, $j \in J$.

(b) $G_{i}^{jj'} \odot (\mathrm{id}_{\Psi_{jj'}} * \Lambda_{ij}) = \Lambda_{ij'} \odot F_{i}^{jj'} : \Psi_{jj'} \circ \boldsymbol{f}_{ij} \Rightarrow \boldsymbol{g}_{ij'}$ for $i \in I$, $j, j' \in J$.

We can then define composition of 1- and 2-morphisms, identity 1- and 2-morphisms, and so on, making Kuranishi spaces into a 2-category **Kur**. Composition of 1-morphisms involves an arbitray choice, and needs the 2-sheaf property of 1- and 2-morphisms of Kuranishi neighbourhoods, as in Theorem 2.1.

3. Properties of Kuranishi spaces

Kuranishi spaces include manifolds and orbifolds as full (2-)subcategories, **Man** $\subset$ **Orb** $\subset$ **Kur**. Orbifolds should also be defined as a 2-category for their differential geometry to work well.

Lots of differential geometry of manifolds has good extensions to Kuranishi spaces: orientations, immersions, submersions, tangent spaces T_xX, transversality and transverse fibre products, ...

If $\boldsymbol{X}$ is a compact oriented Kuranishi space it has a *virtual class* $[\boldsymbol{X}]_{\rm virt}$ in (Čech) homology $\check{H}_{\mathrm{vdim}\,\boldsymbol{X}}(X,\mathbb{Q})$.

There are versions of Kuranishi spaces with boundary and corners.

3.1. Tangent and obstruction spaces of Kuranishi spaces

For a Kuranishi space $\boldsymbol{X}$ we can define the *tangent space* $T_x\boldsymbol{X}$ and *obstruction space* $O_x\boldsymbol{X}$ and *isotropy group* $G_x\boldsymbol{X}$ for any $x \in \boldsymbol{X}$, where if $(V_i, E_i, \Gamma_i, s_i, \psi_i)$ is a Kuranishi chart on $\boldsymbol{X}$ with $x = \psi_i(v_i)$ then $G_x\boldsymbol{X} = \mathrm{Stab}_{\Gamma_i}(v_i)$ and we have an exact sequence

$$0 \longrightarrow T_x\boldsymbol{X} \longrightarrow T_{v_i}V_i \xrightarrow{\mathrm{d}s_i|_{v_i}} E_i|_{v_i} \longrightarrow O_x\boldsymbol{X} \longrightarrow 0.$$

If $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a 1-morphism in **Kur** and $x \in \boldsymbol{X}$ with $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$ we get functorial morphisms $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$, $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$ and $G_x\boldsymbol{f} : G_x\boldsymbol{X} \to G_y\boldsymbol{Y}$. If $\boldsymbol{\eta} : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a 2-morphism in **Kur** then $T_x\boldsymbol{f} = T_x\boldsymbol{g}$, $O_x\boldsymbol{f} = O_x\boldsymbol{g}$, $G_x\boldsymbol{f} = G_x\boldsymbol{g}$.

Theorem

(a) *A Kuranishi space* $\boldsymbol{X}$ *is an orbifold iff* $O_x\boldsymbol{X} = 0$ *for all* $x \in \boldsymbol{X}$.
(b) *A* 1*-morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ *in* **Kur** *is étale (a local equivalence) iff* $T_x\boldsymbol{f} : T_x\boldsymbol{X} \to T_y\boldsymbol{Y}$, $O_x\boldsymbol{f} : O_x\boldsymbol{X} \to O_y\boldsymbol{Y}$, $G_x\boldsymbol{f} : G_x\boldsymbol{X} \to G_y\boldsymbol{Y}$ *are isomorphisms for all* $x \in \boldsymbol{X}$ *with* $\boldsymbol{f}(x) = y \in \boldsymbol{Y}$. *And* $\boldsymbol{f}$ *is an equivalence in* **Kur** *if also* $f : X \to Y$ *is a bijection.*

3.2. Transversality and fibre products

Recall that smooth maps of manifolds $g : X \to Z$, $h : Y \to Z$ are *transverse* if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z \in Z$, then $T_x g \oplus T_y h : T_x X \oplus T_y Y \to T_z Z$ is surjective. If g, h are transverse then a fibre product $W = X \times_{g,Z,h} Y$ exists in **Man**, with $\dim W = \dim X + \dim Y - \dim Z$.

We give two derived analogues of transversality, weak and strong:

Definition

Let $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$, $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ be 1-morphisms in **Kur**. We call $\boldsymbol{g}, \boldsymbol{h}$ *weakly transverse* if for all $x \in \boldsymbol{X}$, $y \in \boldsymbol{Y}$ with $\boldsymbol{g}(x) = \boldsymbol{h}(y) = z$ in $\boldsymbol{Z}$, then $O_x\boldsymbol{g} \oplus O_y\boldsymbol{h} : O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y} \to O_z\boldsymbol{Z}$ is surjective.

We call $\boldsymbol{g}, \boldsymbol{h}$ *strongly transverse* if for all $x \in X$, $y \in Y$ with $g(x) = h(y) = z \in Z$, then $T_x\boldsymbol{g} \oplus T_y\boldsymbol{h} : T_x\boldsymbol{X} \oplus T_y\boldsymbol{Y} \to T_z\boldsymbol{Z}$ is surjective, and $O_x\boldsymbol{g} \oplus O_y\boldsymbol{h} : O_x\boldsymbol{X} \oplus O_y\boldsymbol{Y} \to O_z\boldsymbol{Z}$ is an isomorphism.

If $\boldsymbol{Z}$ is an orbifold then $O_z\boldsymbol{Z} = 0$, so any $\boldsymbol{g}, \boldsymbol{h}$ are weakly transverse.

Theorem

Let $\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\rightarrow\boldsymbol{Z}$ be weakly transverse 1-morphisms in **Kur**. *Then a fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exists in the 2-category* **Kur**, *with $\mathrm{vdim}\,\boldsymbol{W}=\mathrm{vdim}\,\boldsymbol{X}+\mathrm{vdim}\,\boldsymbol{Y}-\mathrm{vdim}\,\boldsymbol{Z}$.*
This $\boldsymbol{W}$ is an orbifold if and only if $\boldsymbol{g},\boldsymbol{h}$ are strongly transverse.
For $w\in W$ with $\boldsymbol{e}(w)=x$, $\boldsymbol{f}(w)=y$, $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ in $\boldsymbol{Z}$, there is an exact sequence

$$
\begin{array}{ccccccc}
0\longrightarrow T_w\boldsymbol{W} & \xrightarrow[T_w\boldsymbol{e}\oplus-T_w\boldsymbol{f}]{} & T_x\boldsymbol{X}\oplus T_y\boldsymbol{Y} & \xrightarrow[T_x\boldsymbol{g}\oplus T_y\boldsymbol{h}]{} & T_z\boldsymbol{Z} \\
 & & & & \downarrow \\
0\longleftarrow O_z\boldsymbol{Z} & \xleftarrow{O_x\boldsymbol{g}\oplus O_y\boldsymbol{h}} & O_x\boldsymbol{X}\oplus O_y\boldsymbol{Y} & \xleftarrow{O_w\boldsymbol{e}\oplus-O_w\boldsymbol{f}} & O_w\boldsymbol{W},
\end{array}
$$

where $\boldsymbol{e}:\boldsymbol{W}\rightarrow\boldsymbol{X}$, $\boldsymbol{f}:\boldsymbol{W}\rightarrow\boldsymbol{Y}$ are the projections.

The definition of fibre product in a 2-category uses 2-morphisms in an essential way – the theorem would be false in ordinary categories. Fibre products over manifolds or orbifolds $\boldsymbol{Z}$ always exist.