New Donaldson–Thomas style counting invariants for Calabi–Yau 4-folds

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Joint work with Dennis Borisov.
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These slides available at
`http://people.maths.ox.ac.uk/~joyce/`.
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Plan of talk:

1. Introduction

If Y is a Calabi–Yau 3-fold (say over $\mathbb{C}$), then the *Donaldson-Thomas invariants* $DT^\alpha(\tau)$ in $\mathbb{Z}$ or $\mathbb{Q}$ 'count' τ-(semi)stable coherent sheaves on Y with Chern character $\alpha \in H^{\mathrm{even}}(Y,\mathbb{Q})$, for τ a (say Gieseker) stability condition. The $DT^\alpha(\tau)$ are unchanged under continuous deformations of Y, and transform by a wall-crossing formula under change of stability condition τ.
We have τ-(semi)stable moduli schemes $\mathcal{M}^\alpha_{\rm st}(\tau) \subseteq \mathcal{M}^\alpha_{\rm ss}(\tau)$, where $\mathcal{M}^\alpha_{\rm ss}(\tau)$ is proper, and $\mathcal{M}^\alpha_{\rm st}(\tau)$ has a symmetric obstruction theory.
The easy case (Thomas 1998) is when $\mathcal{M}^\alpha_{\rm ss}(\tau) = \mathcal{M}^\alpha_{\rm st}(\tau)$. Then $DT^\alpha(\tau) \in \mathbb{Z}$ is the *virtual cycle* (which has dimension zero) of the proper scheme with obstruction theory $\mathcal{M}^\alpha_{\rm st}(\tau)$.

In joint work with Dennis Borisov, I am developing a similar story for Calabi–Yau 4-folds. We want to define invariants 'counting' τ-(semi)stable coherent sheaves on Calabi–Yau 4-folds. If CY3 Donaldson–Thomas invariants are 'holomorphic Casson invariants', as in Thomas 1998, these should be thought of as 'holomorphic Donaldson invariants'.
The idea for doing this goes back to Donaldson–Thomas 1998, using gauge theory: one wants to 'count' moduli spaces of Spin(7)-instantons on a Calabi–Yau 4-fold (or more generally a Spin(7)-manifold). However, it has not gone very far, as compactifying such higher-dimensional gauge-theoretic moduli spaces in a nice way is too difficult. (See Cao arXiv:1309.4230 and Cao and Leung arXiv:1407.7659.)

Virtual cycles using algebraic geometry?

Rather than using gauge theory, we stay within algebraic geometry, so we get compactness of moduli spaces more-or-less for free. So, suppose Y is a Calabi–Yau 4-fold, and $\alpha \in H^{\rm even}(Y,\mathbb{Q})$ such that $\mathcal{M}_{\rm ss}^\alpha(\tau)=\mathcal{M}_{\rm st}^\alpha(\tau)$ (the easy case).
There is a natural obstruction theory $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$ on $\mathcal{M}_{\rm st}^\alpha(\tau)$, but $\mathcal{E}^\bullet$ is perfect in $[-2,0]$ not $[-1,0]$, so **the usual Behrend–Fantechi virtual cycles do not work**.
Instead, we will use a **completely new method** to define a virtual cycle, which is special to the Calabi–Yau 4-fold case, and for the moment works only over $\mathbb{C}$. It uses heavy machinery from Derived Algebraic Geometry — the 'shifted symplectic derived schemes' of Pantev–Toën–Vaquié–Vezzosi (PTVV) — and Derived Differential Geometry — 'derived smooth manifolds'.

Let $\boldsymbol{\mathcal{M}}_{\rm st}^\alpha(\tau)$ be the derived moduli scheme corresponding to the classical moduli scheme $\mathcal{M}_{\rm st}^\alpha(\tau)$. Then PTVV show that $\boldsymbol{\mathcal{M}}_{\rm st}^\alpha(\tau)$ has a '-2-shifted symplectic structure' ω, a geometric structure which roughly encodes Serre duality of sheaves on Y.
We show that given any -2-shifted symplectic derived $\mathbb{C}$-scheme $(\mathbf{X},\omega)$, we can construct a 'derived smooth manifold' $X_{\rm dm}$ with the same underlying topological space. The virtual dimension is $\mathrm{vdim}_{\mathbb{R}} X_{\rm dm}=\frac{1}{2}\mathrm{vdim}_{\mathbb{R}}\mathbf{X}$, which is half what one would expect. Roughly, this is because $X_{\rm dm}$ is the base of a 'real Lagrangian fibration' $\pi:\mathbf{X}\to X_{\rm dm}$ of the -2-shifted symplectic derived scheme $\mathbf{X}$. If $\mathbf{X}$ is proper, so that $X_{\rm dm}$ is compact, and we can find an orientation on $X_{\rm dm}$, then $X_{\rm dm}$ has a deformation-invariant virtual cycle, in bordism or homology. Using this, we can define our new Donaldson–Thomas style invariants of Calabi–Yau 4-folds.

2. PTVV's shifted symplectic geometry

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, e.g. $\mathbb{K}=\mathbb{C}$. Work in the context of Toën and Vezzosi's theory of *derived algebraic geometry*. This gives ∞-categories of *derived $\mathbb{K}$-schemes* $\mathbf{dSch}_{\mathbb{K}}$ and *derived stacks* $\mathbf{dSt}_{\mathbb{K}}$, including *derived Artin stacks* $\mathbf{dArt}_{\mathbb{K}}$. Think of a derived $\mathbb{K}$-scheme $\mathbf{X}$ as a geometric space which can be covered by Zariski open sets $\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A$ for $A=(A,\mathrm{d})$ a commutative differential graded algebra (cdga) over $\mathbb{K}$.

Cotangent complexes of derived schemes and stacks

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a notion of *k-shifted symplectic structure* on a derived $\mathbb{K}$-scheme or derived $\mathbb{K}$-stack $\mathbf{X}$, for $k\in\mathbb{Z}$. This is complicated, but here is the basic idea. The *cotangent complex* $\mathbb{L}_{\mathbf{X}}$ of $\mathbf{X}$ is an element of a derived category $L_{\mathrm{qcoh}}(\mathbf{X})$ of quasicoherent sheaves on $\mathbf{X}$. It has exterior powers $\Lambda^p\mathbb{L}_{\mathbf{X}}$ for $p=0,1,\ldots$. The *de Rham differential* $\mathrm{d}_{dR}:\Lambda^p\mathbb{L}_{\mathbf{X}}\to\Lambda^{p+1}\mathbb{L}_{\mathbf{X}}$ is a morphism of complexes, though not of $\mathcal{O}_{\mathbf{X}}$-modules. Each $\Lambda^p\mathbb{L}_{\mathbf{X}}$ is a complex, so has an internal differential $\mathrm{d}:(\Lambda^p\mathbb{L}_{\mathbf{X}})^k\to(\Lambda^p\mathbb{L}_{\mathbf{X}})^{k+1}$. We have $\mathrm{d}^2=\mathrm{d}_{dR}^2=\mathrm{d}\circ\mathrm{d}_{dR}+\mathrm{d}_{dR}\circ\mathrm{d}=0$.

p-forms and closed p-forms

A *p-form of degree k* on $\mathbf{X}$ for $k \in \mathbb{Z}$ is an element $[\omega^0]$ of $H^k\bigl(\Lambda^p\mathbb{L}_{\mathbf{X}}, \mathrm{d}\bigr)$. A *closed p-form of degree k* on $\mathbf{X}$ is an element

$$[(\omega^0, \omega^1, \ldots)] \in H^k\bigl(\textstyle\prod_{i=0}^{\infty} \Lambda^{p+i}\mathbb{L}_{\mathbf{X}}[i], \mathrm{d} + \mathrm{d}_{dR}\bigr).$$

There is a projection $\pi : [(\omega^0, \omega^1, \ldots)] \mapsto [\omega^0]$ from closed p-forms $[(\omega^0, \omega^1, \ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.
Note that a closed p-form *is not a special example of a p-form*, but a p-form with an extra structure. The map π from closed p-forms to p-forms can be neither injective nor surjective.

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\mathbf{X}$. Then $[\omega^0]$ induces a morphism $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$, where $\mathbb{T}_{\mathbf{X}} = \mathbb{L}_{\mathbf{X}}^\vee$ is the tangent complex of $\mathbf{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ is a quasi-isomorphism.
If $\mathbf{X}$ is a derived scheme then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, \infty)$. So $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ can be a quasi-isomorphism only if $k \leqslant 0$, and then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, -k]$. If $k = 0$ then $\mathbf{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k = -1$ then $\mathbf{X}$ is quasi-smooth.
A closed 2-form $\omega = [(\omega^0, \omega^1, \ldots)]$ of degree k on $\mathbf{X}$ is called a *k-shifted symplectic structure* if $[\omega^0] = \pi(\omega)$ is nondegenerate.

Calabi–Yau moduli schemes and moduli stacks

Pantev et al. prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a natural $(2-m)$-shifted symplectic structure ω. So Calabi–Yau 3-folds give -1-shifted derived schemes or stacks.

We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E]\in\boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{i-1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong \mathrm{Ext}^{1-i}(E,E)^*$.

The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E)\cong\mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]}\cong h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism $\omega^0:\mathbb{T}_{\boldsymbol{\mathcal{M}}}\to\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

3. A Darboux theorem for shifted symplectic schemes

Theorem 3.1 (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$. If $k\not\equiv 2 \mod 4$, then each $x\in\mathbf{X}$ admits a Zariski open neighbourhood $\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A$ for (A,d) an explicit cdga over $\mathbb{K}$ generated by graded variables x_j^{-i},y_j^{k+i} for $0\leqslant i\leqslant -k/2$, and $\omega|_{\mathbf{Y}}=[(\omega^0,0,0,\ldots)]$ where x_j^l,y_j^l have degree l, and

$$\omega^0=\textstyle\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d in (A,d) is given by Poisson bracket with a Hamiltonian H in A of degree $k+1$.

If $k\equiv 2 \mod 4$, we have two statements, one étale local with ω^0 standard, and one Zariski local with the components of ω^0 in the degree $k/2$ variables depending on some invertible functions.

See also Bouaziz and Grojnowski arXiv:1309.2197.

Sketch of the proof of Theorem 3.1

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $x\in\mathbf{X}$. Then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k,0]$. We first show that we can build Zariski open $x\in\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A$, for $A=\bigoplus_{i\leqslant 0}A^i$ a cdga over $\mathbb{K}$ with A^0 a smooth $\mathbb{K}$-algebra, and such that A is freely generated over A^0 by graded variables x_j^{-i},y_j^{k+i} in degrees $-1,-2,\ldots,k$. We take $\dim A^0$ and the number of x_j^{-i},y_j^{k+i} to be minimal at x.
Using theorems about periodic cyclic cohomology, we show that on $Y\simeq\operatorname{Spec}A$ we can write $\omega|_Y=[(\omega^0,0,0,\ldots)]$, for ω^0 a 2-form of degree k with $\mathrm{d}\omega^0=\mathrm{d}_{dR}\omega^0=0$. Minimality at x implies ω^0 is strictly nondegenerate near x, so we can change variables to write $\omega^0=\sum_{i,j}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}$. Finally, we show d in (A,d) is a symplectic vector field, which integrates to a Hamiltonian H.

The case of −1-shifted symplectic derived schemes

When $k=-1$ (the Calabi–Yau 3-fold case) the Hamiltonian H in the theorem has degree 0. Then the theorem reduces to:

Corollary 3.2

Suppose $(\mathbf{X},\omega)$ is a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then $(\mathbf{X},\omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H:U\to\mathbb{A}^1)$, for U a smooth classical $\mathbb{K}$-scheme and $H:U\to\mathbb{A}^1$ a regular function. Hence, the underlying classical $\mathbb{K}$-scheme $X=t_0(\mathbf{X})$ is Zariski locally isomorphic to a classical critical locus $\mathrm{Crit}(H:U\to\mathbb{A}^1)$.

Combining this with results of Pantev et al. from §1 gives interesting consequences in classical algebraic geometry:

Corollary 3.3

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

Here we note that $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$ for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli scheme, which is -1-shifted symplectic by PTVV.
A complex analytic analogue of this for moduli of coherent sheaves was proved using gauge theory by Joyce and Song arXiv:0810.5645, and for moduli of complexes was claimed by Behrend and Getzler.
Note that the proof of the corollary is wholly algebro-geometric.

The case of −2-shifted symplectic derived schemes

Let $(\mathbf{X},\omega)$ be a -2-shifted symplectic derived $\mathbb{K}$-scheme. Then the Zariski local models for $(\mathbf{X},\omega)$ given by the 'Darboux Theorem' depend on the following data:

- A smooth $\mathbb{K}$-scheme U
- An algebraic vector bundle $E \to U$
- A section $s \in H^0(E)$
- A nondegenerate quadratic form Q on E with $Q(s,s)=0$.

The underlying classical $\mathbb{K}$-scheme X of $\mathbf{X}$ is locally $s^{-1}(0) \subset U$.
The virtual dimension of $\mathbf{X}$ is $\mathrm{vdim}_{\mathbb{K}} \mathbf{X} = 2\dim_{\mathbb{K}} U - \mathrm{rank}_{\mathbb{K}} E$.
The cotangent complex $\mathbb{L}_{\mathbf{X}}|_X$ of $\mathbf{X}$ is locally given by

$$\bigl[\underset{-2}{TU}|_{s^{-1}(0)} \xrightarrow{Q\circ \mathrm{d}s} \underset{-1}{E^*}|_{s^{-1}(0)} \xrightarrow{\mathrm{d}s} \underset{0}{T^*U}|_{s^{-1}(0)}\bigr].$$

This is the local model for (derived) moduli schemes of (simple) coherent sheaves F on a Calabi–Yau 4-fold Y. Think of U as an étale open neighbourhood of 0 in $\mathrm{Ext}^1(F,F)$, and $E \to U$ as a trivial vector bundle with fibre $\mathrm{Ext}^2(F,F)$, and Q as the nondegenerate quadratic form on $\mathrm{Ext}^2(F,F)$

$$\mathrm{Ext}^2(F,F)\times\mathrm{Ext}^2(F,F) \xrightarrow{\wedge} \mathrm{Ext}^4(F,F) \xrightarrow{\text{Serre duality}} \mathrm{Ext}^0(F,F)^* \xrightarrow{\mathrm{id}_F^*} \mathbb{K},$$

and s as a Kuranishi map $s : \mathrm{Ext}^1(E,E) \supseteq U \to \mathrm{Ext}^2(E,E)$. The special thing the theorem tells us is that we can choose U, E, s, Q such that $Q(s,s) = 0$, rather than just $Q(s,s) = 0$ modulo s^3, for instance.

4. Virtual cycles for −2-shifted symplectic derived schemes

Here is the first part of what we prove:

Theorem 4.1 (Borisov–Joyce, arXiv:1504.00690)

Let $(\mathbf{X},\omega)$ be a -2-shifted symplectic derived scheme over $\mathbb{C}$. Then one can construct a **d-manifold**, *or* **M-Kuranishi space**, *or* **Kuranishi space with trivial isotropy**, *or* **Spivak derived manifold** *(all forms of* **derived smooth manifolds**, *more-or-less equivalent) X_{dm} which has the same underlying topological space X as $(\mathbf{X},\omega)$, with the complex analytic topology.*
The construction involves arbitrary choices, but X_{dm} is unique up to bordisms which fix the topological space X.
The (real) virtual dimension of X_{dm} is

$$\mathrm{vdim}_{\mathbb{R}} X_{\mathrm{dm}} = \mathrm{vdim}_{\mathbb{C}} \mathbf{X} = \tfrac{1}{2}\,\mathrm{vdim}_{\mathbb{R}} \mathbf{X},$$

which is half what one would have expected.

Derived manifolds and bordisms

I haven't time to explain derived smooth manifolds properly – see `people.maths.ox.ac.uk/∼joyce/dmanifolds.html`, and papers arXiv:1206.4207, arXiv:1208.4948 on d-manifolds, and arXiv:1409.6908 on (M-)Kuranishi spaces. Some useful facts:

- Moduli spaces of solutions of nonlinear elliptic equations on compact manifolds have the structure of derived manifolds.
- A derived manifold $X_{\rm dm}$ is locally modelled by a 'Kuranishi neighbourhood' (V,E,s) of a real manifold V, real vector bundle $E\to V$ and smooth section $s:V\to E$, where the topological space of $X_{\rm dm}$ is locally homeomorphic to $s^{-1}(0)\subset V$. Think of $X_{\rm dm}$ as locally the (homotopy) fibre product $V\times_{s,E,0}V$.
- Any (compact) derived manifold $\mathbf{X}$ can be perturbed to a (compact) ordinary manifold $\tilde X$, which is unique up to bordism. In a Kuranishi neighbourhood (V,E,s), perturb s to a generic, transverse $\tilde s:V\to E$, so that $\tilde s^{-1}(0)\subset V$ is a manifold.

Orientations of −2-shifted symplectic derived schemes

Thus, if $(\mathbf{X},\omega)$ is a proper -2-shifted symplectic derived $\mathbb{C}$-scheme, Theorem 4.1 gives us a bordism class of (unoriented) compact manifolds $\tilde X$, which is basically a virtual cycle over $\mathbb{Z}_2$. To lift this to a virtual cycle over $\mathbb{Z}$, we need to include orientations of $(\mathbf{X},\omega)$ and $X_{\rm dm}$.

Recall that if $(\mathbf{X},\omega)$ is a -1-shifted symplectic derived scheme (the CY3 case), an *orientation* of $(\mathbf{X},\omega)$ is a square root line bundle $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$. These were introduced by Kontsevich and Soibelman, for motivic and categorified D–T theory. Here is the CY4 analogue:

Definition

Let $(\mathbf{X},\omega)$ be a -2-shifted symplectic derived scheme. There is a natural isomorphism $\iota:\det(\mathbb{L}_{\mathbf{X}})^{\otimes^2}\longrightarrow\mathcal{O}_{\mathbf{X}}$. An *orientation* of $(\mathbf{X},\omega)$ is an isomorphism $\alpha:\det(\mathbb{L}_{\mathbf{X}})\longrightarrow\mathcal{O}_{\mathbf{X}}$ with $\alpha\otimes\alpha=\iota$.

Note that this is simpler, one categorical level down from the CY3 case: a morphism in a category, not an object in a category.
The next two results are easy, given Theorem 4.1.

Lemma 4.2

In Theorem 4.1, there is a natural 1-1 correspondence between orientations on $(\mathbf{X},\omega)$ and orientations on the d-manifold $X_{\rm dm}$.

Corollary 4.3

Let $(\mathbf{X},\omega)$ be a proper, oriented -2-shifted symplectic derived $\mathbb{C}$-scheme. Then we construct a bordism class $[X_{\rm dm}]$ of compact oriented manifolds. We consider this a **virtual cycle** *for $(\mathbf{X},\omega)$.*

Observe that though all the input data is strictly complex algebraic, the 'virtual cycle' can have odd real dimension, which is weird, and very unlike Behrend–Fantechi style virtual cycles.

Sketch proof of Theorem 4.1

Let $(\mathbf{X},\omega)$ be a -2-shifted symplectic derived $\mathbb{C}$-scheme. Then the BBJ 'Darboux Theorem', Theorem 3.1, gives local models for $(\mathbf{X},\omega)$ in the Zariski topology. As in §2, in the -2-shifted case, the local models reduce to the following data:

- A smooth $\mathbb{C}$-scheme U
- A vector bundle $E \to U$
- A section $s \in H^0(E)$
- A nondegenerate quadratic form Q on E with $Q(s,s)=0$.

The underlying topological space of $\mathbf{X}$ is $\{x \in U : s(x)=0\}$. The virtual dimension of $\mathbf{X}$ is $\mathrm{vdim}_{\mathbb{C}}\,\mathbf{X} = 2\dim_{\mathbb{C}} U - \mathrm{rank}_{\mathbb{C}} E$. The cotangent complex $\mathbb{L}_{\mathbf{X}}|_X$ of $\mathbf{X}$ is

$$\big[\underset{-2}{TU}\,|_{s^{-1}(0)} \xrightarrow{Q\circ\mathrm{d}s} \underset{-1}{E^*}\,|_{s^{-1}(0)} \xrightarrow{\mathrm{d}s} \underset{0}{T^*U}\,|_{s^{-1}(0)}\big].$$

The local model for $X_{\rm dm}$

Here is how to build the derived manifold $X_{\rm dm}$ locally: regard $E \to U$ as a real vector bundle over the real manifold U. Choose a splitting $E = E_+ \oplus E_-$, where $Q|_{E_+}$ is real and positive definite, and $E_- = iE_+$ so that $Q|_{E_-}$ is real and negative definite. Write $s = s_+ \oplus s_-$ with $s_\pm \in C^\infty(E_\pm)$. Then $X_{\rm dm}$ is locally the derived fibre product $U \times_{0,E_+,s_+} U$, given by the 'Kuranishi neighbourhood' (U, E_+, s_+). It has real virtual dimension

$$\dim_{\mathbb R} U - \mathrm{rank}_{\mathbb R} E_+ = 2\dim_{\mathbb C} U - \mathrm{rank}_{\mathbb C} E = \mathrm{vdim}_{\mathbb C} \mathbf{X}.$$

Observe that $Q(s,s) = 0$ implies that $|s_+|^2 = |s_-|^2$, where norms $|\,.\,|$ on E_+, E_- are defined using $\pm \mathrm{Re}\, Q$. Hence as sets we have

$$\{x \in U : s(x) = 0\} = \{x \in U : s_+(x) = 0\} \subseteq U.$$

This is why $\mathbf{X}$ and $X_{\rm dm}$ have the same topological space X.

The difficult bit is to show we can choose compatible splittings $E = E_+ \oplus E_-$ on an open cover of $\mathbf{X}$, and glue the local models to make a global derived manifold $X_{\rm dm}$.

5. D–T style invariants for Calabi–Yau 4-folds

Suppose Y is a Calabi–Yau 4-fold over $\mathbb{C}$, and $\alpha \in H^{\rm even}(Y,\mathbb{Q})$ such that $\mathcal{M}^\alpha_{\rm ss}(\tau) = \mathcal{M}^\alpha_{\rm st}(\tau)$. Write $\boldsymbol{\mathcal{M}}^\alpha_{\rm st}(\tau)$ for the corresponding derived moduli scheme. Then $\boldsymbol{\mathcal{M}}^\alpha_{\rm st}(\tau)$ has a -2-shifted symplectic structure by PTVV. Suppose we can choose an orientation. Then Theorem 4.1 constructs a compact, oriented derived manifold $\mathcal{M}^\alpha_{\rm st}(\tau)_{\rm dm}$ with the same topological space, of dimension

$$\mathrm{vdim}_{\mathbb R} \mathcal{M}^\alpha_{\rm st}(\tau)_{\rm dm} = \mathrm{vdim}_{\mathbb C} \boldsymbol{\mathcal{M}}^\alpha_{\rm st}(\tau) = 2 - \deg(\alpha \cup \bar\alpha \cup \mathrm{td}(TY))_8 = d.$$

The derived manifold has a virtual cycle $[\mathcal{M}^\alpha_{\rm st}(\tau)_{\rm dm}]_{\rm vir}$ in bordism, or in homology $H_d(\mathcal{M}^\alpha_{\rm st}(\tau);\mathbb{Z})$. If $d = 0$ this virtual cycle is an integer, and we define $DT^\alpha_4(\tau) = [\mathcal{M}^\alpha_{\rm st}(\tau)_{\rm dm}]_{\rm vir} \in \mathbb{Z}$.

If $d > 0$ then as for Donaldson invariants, we hope to find cohomology classes on $\mathcal{M}^{\alpha}_{\rm st}(\tau)$ using the Chern characters of the universal sheaf $\mathcal{E} \to \mathcal{M}^{\alpha}_{\rm st}(\tau) \times Y$, and make integer invariants by integrating products of these classes over $[\mathcal{M}^{\alpha}_{\rm st}(\tau)_{\rm dm}]_{\rm vir}$.

These invariants will have the nice property of being unchanged under continuous deformations of the complex structure of the Calabi–Yau 4-fold Y. There are still lots of interesting open questions – computation in examples such as Hilbert schemes of points, wall-crossing formulae, use for curve-counting, and so on.

Motivation from gauge theory

We can explain using gauge theory why one should pass from the -2-shifted symplectic derived scheme $\boldsymbol{\mathcal{M}}^{\alpha}_{\rm st}(\tau)$ to the derived manifold $\mathcal{M}^{\alpha}_{\rm st}(\tau)_{\rm dm}$, which was part of our original motivation. Consider a moduli space $\mathcal{M}$ of stable rank r holomorphic vector bundles $F \to Y$ over Y, with $c_1(F) = 0$ for simplicity. By the Hitchin–Kobayashi correspondence, each F has a unique connection ∇_F with curvature R_F satisfying the Hermitian–Einstein equations

$$R_F^{2,0} = 0, \qquad R_F^{1,1} \wedge \omega^3 = 0. \tag{5.1}$$

These equations are overdetermined ($13r^2$ equations plus r^2 gauge rescalings on $8r^2$ unknowns), which corresponds to the fact that $\boldsymbol{\mathcal{M}}^{\alpha}_{\rm st}(\tau)$ does not have cotangent complex in $[-1, 0]$ (is not 'quasi-smooth'), and so does not have a virtual cycle.

In the CY4 case, there is a splitting $R_F^{2,0} = R_F^{2,0\,+} \oplus R_F^{2,0\,-}$ into real 'self-dual' and 'anti-self-dual' components. So we can impose instead the weaker 'Spin(7) instanton equations'

$$R_F^{2,0\,+} = 0, \qquad R_F^{1,1} \wedge \omega^3 = 0. \tag{5.2}$$

These equations are determined ($7r^2$ equations plus r^2 gauge rescalings on $8r^2$ unknowns), and elliptic, and form a moduli space $\mathcal{M}_{\rm st}^{\alpha}(\tau)_{\rm dm}$ with a virtual cycle, at least if compact and oriented. Since (5.2) is a subset of (5.1), one would expect the moduli space $\mathcal{M}_{\rm HE}$ of solutions of (5.1) to be a subset of the moduli space $\mathcal{M}_{\rm Spin(7)}$ of solutions of (5.2). But using L^2 norms of components of R_F, one can show that (for suitable Chern characters α) we have $\mathcal{M}_{\rm HE} = \mathcal{M}_{\rm Spin(7)}$ as sets, they differ only as non-reduced spaces. Thus it is reasonable to expect a derived scheme $\mathcal{M}_{\rm HE} = \boldsymbol{\mathcal{M}}_{\rm st}^{\alpha}(\tau)$ and a derived manifold $\mathcal{M}_{\rm Spin(7)} = \mathcal{M}_{\rm st}^{\alpha}(\tau)_{\rm dm}$ with the same underlying topological space.