'Darboux theorems' for shifted symplectic derived schemes and stacks

Lecture 1 of 3

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Plan of talk:

1. PTVV's shifted symplectic geometry

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, e.g. $\mathbb{K} = \mathbb{C}$. Work in the context of Toën and Vezzosi's theory of *derived algebraic geometry*. This gives ∞-categories of *derived $\mathbb{K}$-schemes* $\mathbf{dSch}_{\mathbb{K}}$ and *derived stacks* $\mathbf{dSt}_{\mathbb{K}}$, including *derived Artin stacks* $\mathbf{dArt}_{\mathbb{K}}$. Think of a derived $\mathbb{K}$-scheme $\mathbf{X}$ as a geometric space which can be covered by Zariski open sets $\mathbf{Y} \subseteq \mathbf{X}$ with $\mathbf{Y} \simeq \operatorname{Spec} A$ for $A = (A, \mathrm{d})$ a commutative differential graded algebra (cdga) over $\mathbb{K}$.

Cotangent complexes of derived schemes and stacks

Pantev, Toën, Vaquié and Vezzosi (arXiv:1111.3209) defined a notion of *k-shifted symplectic structure* on a derived $\mathbb{K}$-scheme or derived $\mathbb{K}$-stack $\mathbf{X}$, for $k \in \mathbb{Z}$. This is complicated, but here is the basic idea. The *cotangent complex* $\mathbb{L}_{\mathbf{X}}$ of $\mathbf{X}$ is an element of a derived category $L_{\mathrm{qcoh}}(\mathbf{X})$ of quasicoherent sheaves on $\mathbf{X}$. It has exterior powers $\Lambda^p \mathbb{L}_{\mathbf{X}}$ for $p = 0, 1, \ldots$. The *de Rham differential* $\mathrm{d}_{dR} : \Lambda^p \mathbb{L}_{\mathbf{X}} \to \Lambda^{p+1} \mathbb{L}_{\mathbf{X}}$ is a morphism of complexes, though not of $\mathcal{O}_{\mathbf{X}}$-modules. Each $\Lambda^p \mathbb{L}_{\mathbf{X}}$ is a complex, so has an internal differential $\mathrm{d} : (\Lambda^p \mathbb{L}_{\mathbf{X}})^k \to (\Lambda^p \mathbb{L}_{\mathbf{X}})^{k+1}$. We have $\mathrm{d}^2 = \mathrm{d}_{dR}^2 = \mathrm{d} \circ \mathrm{d}_{dR} + \mathrm{d}_{dR} \circ \mathrm{d} = 0$.

p-forms and closed p-forms

A *p-form of degree k* on $\mathbf{X}$ for $k \in \mathbb{Z}$ is an element $[\omega^0]$ of $H^k(\Lambda^p \mathbb{L}_{\mathbf{X}}, \mathrm{d})$. A *closed p-form of degree k* on $\mathbf{X}$ is an element

$$[(\omega^0, \omega^1, \ldots)] \in H^k\big(\textstyle\bigoplus_{i=0}^{\infty} \Lambda^{p+i} \mathbb{L}_{\mathbf{X}}[i], \mathrm{d} + \mathrm{d}_{dR}\big).$$

There is a projection $\pi : [(\omega^0, \omega^1, \ldots)] \mapsto [\omega^0]$ from closed p-forms $[(\omega^0, \omega^1, \ldots)]$ of degree k to p-forms $[\omega^0]$ of degree k.
Note that a closed p-form *is not a special example of a p-form*, but a p-form with an extra structure. The map π from closed p-forms to p-forms can be neither injective nor surjective.

Nondegenerate 2-forms and symplectic structures

Let $[\omega^0]$ be a 2-form of degree k on $\mathbf{X}$. Then $[\omega^0]$ induces a morphism $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$, where $\mathbb{T}_{\mathbf{X}} = \mathbb{L}_{\mathbf{X}}^{\vee}$ is the tangent complex of $\mathbf{X}$. We call $[\omega^0]$ *nondegenerate* if $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ is a quasi-isomorphism.
If $\mathbf{X}$ is a derived scheme then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, \infty)$. So $\omega^0 : \mathbb{T}_{\mathbf{X}} \to \mathbb{L}_{\mathbf{X}}[k]$ can be a quasi-isomorphism only if $k \leqslant 0$, and then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k, 0]$ and $\mathbb{T}_{\mathbf{X}}$ in degrees $[0, -k]$. If $k = 0$ then $\mathbf{X}$ is a smooth classical $\mathbb{K}$-scheme, and if $k = -1$ then $\mathbf{X}$ is quasi-smooth.
A closed 2-form $\omega = [(\omega^0, \omega^1, \ldots)]$ of degree k on $\mathbf{X}$ is called a *k-shifted symplectic structure* if $[\omega^0] = \pi(\omega)$ is nondegenerate.

Calabi–Yau moduli schemes and moduli stacks

Pantev et al. prove that if Y is a Calabi–Yau m-fold over $\mathbb{K}$ and $\boldsymbol{\mathcal{M}}$ is a derived moduli scheme or stack of (complexes of) coherent sheaves on Y, then $\boldsymbol{\mathcal{M}}$ has a natural $(2-m)$-shifted symplectic structure ω. So Calabi–Yau 3-folds give -1-shifted derived schemes or stacks.
We can understand the associated nondegenerate 2-form $[\omega^0]$ in terms of *Serre duality*. At a point $[E] \in \boldsymbol{\mathcal{M}}$, we have $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{i-1}(E,E)$ and $h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong \mathrm{Ext}^{1-i}(E,E)^*$.
The Calabi–Yau condition gives $\mathrm{Ext}^i(E,E) \cong \mathrm{Ext}^{m-i}(E,E)^*$, which corresponds to $h^i(\mathbb{T}_{\boldsymbol{\mathcal{M}}})|_{[E]} \cong h^i(\mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m])|_{[E]}$. This is the cohomology at $[E]$ of the quasi-isomorphism
$\omega^0 : \mathbb{T}_{\boldsymbol{\mathcal{M}}} \to \mathbb{L}_{\boldsymbol{\mathcal{M}}}[2-m]$.

Lagrangians and Lagrangian intersections

Let $(\mathbf{X},\omega)$ be a k-shifted symplectic derived scheme or stack. Then Pantev et al. define a notion of *Lagrangian* $\mathbf{L}$ in $(\mathbf{X},\omega)$, which is a morphism $\mathbf{i} : \mathbf{L} \to \mathbf{X}$ of derived schemes or stacks together with a homotopy $i^*(\omega) \sim 0$ satisfying a nondegeneracy condition, implying that $\mathbb{T}_{\mathbf{L}} \simeq \mathbb{L}_{\mathbf{L}/\mathbf{X}}[k-1]$.
If $\mathbf{L},\mathbf{M}$ are Lagrangians in $(\mathbf{X},\omega)$, then the fibre product $\mathbf{L} \times_{\mathbf{X}} \mathbf{M}$ has a natural $(k-1)$-shifted symplectic structure.
If (S,ω) is a classical smooth symplectic scheme, then it is a 0-shifted symplectic derived scheme in the sense of PTVV, and if $L, M \subset S$ are classical smooth Lagrangian subschemes, then they are Lagrangians in the sense of PTVV. Therefore the (derived) Lagrangian intersection $L \cap M = L \times_S M$ is a -1-shifted symplectic derived scheme.

Examples of Lagrangians

Let $(\mathbf{X},\omega)$ be k-shifted symplectic, and $\mathbf{i}_a:\mathbf{L}_a\to\mathbf{X}$ be Lagrangian in $\mathbf{X}$ for $a=1,\ldots,d$. Then Ben-Bassat (arXiv:1309.0596) shows

$$\mathbf{L}_1\times_{\mathbf{X}}\mathbf{L}_2\times_{\mathbf{X}}\cdots\times_{\mathbf{X}}\mathbf{L}_d\longrightarrow(\mathbf{L}_1\times_{\mathbf{X}}\mathbf{L}_2)\times\cdots\times(\mathbf{L}_{d-1}\times_{\mathbf{X}}\mathbf{L}_d)\times(\mathbf{L}_d\times_{\mathbf{X}}\mathbf{L}_1)$$

is Lagrangian, where the r.h.s. is $(k-1)$-shifted symplectic by PTVV. This is relevant to defining 'Fukaya categories' of complex symplectic manifolds.

Let Y be a Calabi–Yau m-fold, so that the derived moduli stack $\boldsymbol{\mathcal{M}}$ of coherent sheaves (or complexes) on Y is $(2-m)$-shifted symplectic by PTVV, with symplectic form ω. We expect (Oren Ben-Bassat, work in progress) that

$$\boldsymbol{\mathcal{E}}\mathbf{xact}\xrightarrow{\pi_1\times\pi_2\times\pi_3}(\boldsymbol{\mathcal{M}},\omega)\times(\boldsymbol{\mathcal{M}},-\omega)\times(\boldsymbol{\mathcal{M}},\omega)$$

is Lagrangian, where $\boldsymbol{\mathcal{E}}\mathbf{xact}$ is the derived moduli stack of short exact sequences in $\mathrm{coh}(Y)$ (or distinguished triangles in $D^b\mathrm{coh}(Y)$). This is relevant to Cohomological Hall Algebras.

2. A 'Darboux theorem' for shifted symplectic schemes

Theorem (Brav, Bussi and Joyce arXiv:1305.6302)

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$. If $k\not\equiv 2 \mod 4$, then each $x\in\mathbf{X}$ admits a Zariski open neighbourhood $\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\mathrm{Spec}\,A$ for (A,d) an explicit cdga over $\mathbb{K}$ generated by graded variables x_j^{-i},y_j^{k+i} for $0\leqslant i\leqslant -k/2$, and $\omega|_{\mathbf{Y}}=[(\omega^0,0,0,\ldots)]$ where x_j^l,y_j^l have degree l, and

$$\omega^0=\textstyle\sum_{i=0}^{[-k/2]}\sum_{j=1}^{m_i}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}.$$

Also the differential d in (A,d) is given by Poisson bracket with a Hamiltonian H in A of degree $k+1$.

If $k\equiv 2 \mod 4$, we have two statements, one étale local with ω^0 standard, and one Zariski local with the components of ω^0 in the degree $k/2$ variables depending on some invertible functions.

Sketch of the proof of the theorem

Suppose $(\mathbf{X},\omega)$ is a k-shifted symplectic derived $\mathbb{K}$-scheme for $k<0$, and $x\in\mathbf{X}$. Then $\mathbb{L}_{\mathbf{X}}$ lives in degrees $[k,0]$. We first show that we can build Zariski open $x\in\mathbf{Y}\subseteq\mathbf{X}$ with $\mathbf{Y}\simeq\operatorname{Spec}A$, for $A=\bigoplus_{i\leqslant 0}A^i$ a cdga over $\mathbb{K}$ with A^0 a smooth $\mathbb{K}$-algebra, and such that A is freely generated over A^0 by graded variables x_j^{-i},y_j^{k+i} in degrees $-1,-2,\ldots,k$. We take $\dim A^0$ and the number of x_j^{-i},y_j^{k+i} to be minimal at x.
Using theorems about periodic cyclic cohomology, we show that on $Y\simeq\operatorname{Spec}A$ we can write $\omega|_Y=[(\omega^0,0,0,\ldots)]$, for ω^0 a 2-form of degree k with $\mathrm{d}\omega^0=\mathrm{d}_{dR}\omega^0=0$. Minimality at x implies ω^0 is strictly nondegenerate near x, so we can change variables to write $\omega^0=\sum_{i,j}\mathrm{d}_{dR}y_j^{k+i}\mathrm{d}_{dR}x_j^{-i}$. Finally, we show d in (A,d) is a symplectic vector field, which integrates to a Hamiltonian H.

The case of -1-shifted symplectic derived schemes

When $k=-1$ the Hamiltonian H in the theorem has degree 0. Then the theorem reduces to:

Corollary

Suppose $(\mathbf{X},\omega)$ is a -1-shifted symplectic derived $\mathbb{K}$-scheme. Then $(\mathbf{X},\omega)$ is Zariski locally equivalent to a derived critical locus $\mathbf{Crit}(H:U\rightarrow\mathbb{A}^1)$, for U a smooth classical $\mathbb{K}$-scheme and $H:U\rightarrow\mathbb{A}^1$ a regular function. Hence, the underlying classical $\mathbb{K}$-scheme $X=t_0(\mathbf{X})$ is Zariski locally isomorphic to a classical critical locus $\operatorname{Crit}(H:U\rightarrow\mathbb{A}^1)$.

Combining this with results of Pantev et al. from §1 gives interesting consequences in classical algebraic geometry:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y. Then $\mathcal{M}$ is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

Here we note that $\mathcal{M} = t_0(\boldsymbol{\mathcal{M}})$ for $\boldsymbol{\mathcal{M}}$ the corresponding derived moduli scheme, which is -1-shifted symplectic by PTVV.
A complex analytic analogue of this for moduli of coherent sheaves was proved using gauge theory by Joyce and Song arXiv:0810.5645, and for moduli of complexes was claimed by Behrend and Getzler. Note that the proof of the corollary is wholly algebro-geometric.

As intersections of algebraic Lagrangians are -1-shifted symplectic, we also deduce:

Corollary

Let (S, ω) be a classical smooth symplectic $\mathbb{K}$-scheme, and $L, M \subseteq S$ be smooth algebraic Lagrangians. Then the intersection $L \cap M$, as a $\mathbb{K}$-subscheme of S, is Zariski locally isomorphic to the critical locus $\mathrm{Crit}(H : U \to \mathbb{A}^1)$ of a regular function on a smooth $\mathbb{K}$-scheme.

In real or complex symplectic geometry, where the Darboux Theorem holds, the analogue of the corollary is easy to prove, but in classical algebraic symplectic geometry we do not have a Darboux Theorem, so the corollary is not obvious.

The case of −2-shifted symplectic derived schemes

Let $(\mathbf{X},\omega)$ be a -2-shifted symplectic derived $\mathbb{K}$-scheme. Then the Zariski local models for $(\mathbf{X},\omega)$ given by the 'Darboux Theorem' depend on the following data:

- A smooth $\mathbb{K}$-scheme U
- An algebraic vector bundle $E \to U$
- A section $s \in H^0(E)$
- A nondegenerate quadratic form Q on E with $Q(s,s)=0$.

The underlying classical $\mathbb{K}$-scheme X of $\mathbf{X}$ is locally $s^{-1}(0) \subset U$. The virtual dimension of $\mathbf{X}$ is $\mathrm{vdim}_{\mathbb{K}}\,\mathbf{X} = 2\dim_{\mathbb{K}} U - \mathrm{rank}_{\mathbb{K}}\, E$. The cotangent complex $\mathbb{L}_{\mathbf{X}}|_X$ of $\mathbf{X}$ is locally given by

$$\big[\underset{-2}{TU}|_{s^{-1}(0)} \xrightarrow{Q\circ ds} \underset{-1}{E^*}|_{s^{-1}(0)} \xrightarrow{ds} \underset{0}{T^*U}|_{s^{-1}(0)}\big].$$

We will use this in lecture 3 to define Donaldson–Thomas style invariants 'counting' coherent sheaves on Calabi–Yau 4-folds.

3. Extension to shifted symplectic derived Artin stacks

In Ben-Bassat, Bussi, Brav and Joyce arXiv:1312.0090 we extend the material of §2 from (derived) schemes to (derived) Artin stacks. We call a derived stack $\mathbf{X}$ a *derived Artin stack* $\mathbf{X}$ if it is 1-geometric, and the associated classical (higher) stack $X = t_0(\mathbf{X})$ is 1-truncated, all in the sense of Toën and Vezzosi. Then the cotangent complex $\mathbb{L}_{\mathbf{X}}$ lives in degrees $(-\infty,1]$, and $X = t_0(\mathbf{X})$ is a classical Artin stack (in particular, not a higher stack).
A derived Artin stack $\mathbf{X}$ admits a smooth atlas $\varphi : \mathbf{U} \to \mathbf{X}$ with $\mathbf{U}$ a derived scheme. If Y is a smooth projective scheme and $\boldsymbol{\mathcal{M}}$ is a derived moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b\,\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet,F^\bullet)=0$, then $\boldsymbol{\mathcal{M}}$ is a derived Artin stack.

A 'Darboux Theorem' for atlases of derived stacks

Theorem (Ben-Bassat, Bussi, Brav, Joyce, arXiv:1312.0090)

Let $(\mathbf{X},\omega_{\mathbf{X}})$ be a k-shifted symplectic derived Artin stack for $k<0$, and $p\in\mathbf{X}$. Then there exist 'standard form' affine derived schemes $\mathbf{U}=\operatorname{Spec} A$, $\mathbf{V}=\operatorname{Spec} B$, points $u\in\mathbf{U}$, $v\in\mathbf{V}$ with A,B minimal at u,v, morphisms $\varphi:\mathbf{U}\to\mathbf{X}$ and $\mathbf{i}:\mathbf{U}\to\mathbf{V}$ with $\varphi(u)=p$, $\mathbf{i}(u)=v$, such that φ is smooth of relative dimension $\dim H^1(\mathbb{L}_{\mathbf{X}}|_p)$, and $t_0(\mathbf{i}):t_0(\mathbf{U})\to t_0(\mathbf{V})$ is an isomorphism on classical schemes, and $\mathbb{L}_{\mathbf{U}/\mathbf{V}}\simeq\mathbb{T}_{\mathbf{U}/\mathbf{X}}[1-k]$, and a 'Darboux form' k-shifted symplectic form ω_B on $\mathbf{V}=\operatorname{Spec} B$ such that $\mathbf{i}^*(\omega_B)\sim\varphi^*(\omega_{\mathbf{X}})$ in k-shifted closed 2-forms on $\mathbf{U}$.

Discussion of the 'Darboux Theorem' for stacks

Let $(\mathbf{X},\omega_{\mathbf{X}})$ be a k-shifted symplectic derived Artin stack for $k<0$, and $p\in\mathbf{X}$. Although we do not know how to give a complete, explicit 'standard model' for $(\mathbf{X},\omega_{\mathbf{X}})$ near p, we can give standard models for a smooth atlas $\varphi:\mathbf{U}\to\mathbf{X}$ for $\mathbf{X}$ near p with $\mathbf{U}=\operatorname{Spec} A$ a derived scheme, and for the pullback 2-form $\varphi^*(\omega_{\mathbf{X}})$. We may think of $\varphi:\mathbf{U}\to\mathbf{X}$ as an open neighbourhood of p in the smooth topology, rather than the Zariski topology. Now $(\mathbf{U},\varphi^*(\omega_{\mathbf{X}}))$ is not k-shifted symplectic, as $\varphi^*(\omega_{\mathbf{X}})$ is closed, but not nondegenerate. However, there is a way to modify $\mathbf{U},A$ to get another derived scheme $\mathbf{V}=\operatorname{Spec} B$, where A has generators in degrees $0,-1,\ldots,-k-1$, and $B\subseteq A$ is the dg-subalgebra generated by the generators in degrees $0,-1,\ldots,-k$ only.

Then $\mathbf{V}$ has a natural k-shifted symplectic form ω_B, which we may take to be in 'Darboux form' as in §2, with $\mathbf{i}^*(\omega_B) \sim \boldsymbol{\varphi}^*(\omega_{\mathbf{X}})$. In terms of cotangent complexes, $\mathbb{L}_{\mathbf{U}}$ is obtained from $\boldsymbol{\varphi}^*(\mathbb{L}_{\mathbf{X}})$ by deleting a vector bundle $\mathbb{L}_{\mathbf{U}/\mathbf{X}}$ in degree 1. Also $\mathbb{L}_{\mathbf{V}}$ is obtained from $\mathbb{L}_{\mathbf{U}}$ by deleting the dual vector bundle $\mathbb{T}_{\mathbf{U}/\mathbf{X}}$ in degree $k-1$. As these two deletions are dual under $\boldsymbol{\varphi}^*(\omega_{\mathbf{X}})$, the symplectic form descends to $\mathbf{V}$.
An example in which we have this picture $(\mathbf{V}, \omega_B) \xleftarrow{\mathbf{i}} \mathbf{U} \xrightarrow{\boldsymbol{\varphi}} (\mathbf{X}, \omega_{\mathbf{X}})$ is a 'k-shifted symplectic quotient', when an algebraic group G acts on a k-shifted symplectic derived scheme $(\mathbf{V}, \omega_B)$ with 'moment map' $\mu \in H^k(\mathbf{V}, \mathfrak{g}^* \otimes \mathcal{O}_{\mathbf{V}})$, and $\mathbf{U} = \mu^{-1}(0)$, and $X = [\mathbf{U}/G]$. (See Safronov arXiv:1311.6429.)

-1-shifted symplectic derived stacks

When $k = -1$, $(\mathbf{V}, \omega_B)$ is a derived critical locus $\mathbf{Crit}(f : S \to \mathbb{A}^1)$ for S a smooth scheme. Then $t_0(\mathbf{V}) \cong t_0(\mathbf{U})$ is the classical critical locus $\mathrm{Crit}(f : S \to \mathbb{A}^1)$, and $U = t_0(\mathbf{U})$ is a smooth atlas for the classical Artin stack $X = t_0(\mathbf{X})$. Thus we deduce:

Corollary

Let $(\mathbf{X}, \omega_{\mathbf{X}})$ be a -1-shifted symplectic derived stack. Then the classical Artin stack $X = t_0(\mathbf{X})$ locally admits smooth atlases $\varphi : U \to X$ with $U = \mathrm{Crit}(f : S \to \mathbb{A}^1)$, for S a smooth scheme and f a regular function.

Calabi–Yau 3-fold moduli stacks

If Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a moduli stack of coherent sheaves F on Y, or complexes $F^\bullet$ in $D^b\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, then by PTVV the corresponding derived moduli stack $\boldsymbol{\mathcal{M}}$ with $t_0(\boldsymbol{\mathcal{M}}) = \mathcal{M}$ has a -1-shifted symplectic structure $\omega_{\boldsymbol{\mathcal{M}}}$. So the previous corollary gives:

Corollary

Suppose Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b\mathrm{coh}(Y)$ with $\mathrm{Ext}^{<0}(F^\bullet, F^\bullet) = 0$. Then $\mathcal{M}$ locally admits smooth atlases $\varphi : U \to X$ with $U = \mathrm{Crit}(f : S \to \mathbb{A}^1)$, for S a smooth scheme.

A holomorphic version of this was proved by Joyce and Song using gauge theory, and is important in Donaldson–Thomas theory. Bussi (work in progress) uses this to give a new algebraic proof of the 'Behrend function identities' in Donaldson–Thomas theory.