Homework problems, days 1-4

1. Complex projective space $\mathbb{CP}^n$ is the set of 1-dimensional vector subspaces of $\mathbb{C}^{n+1}$. Points in $\mathbb{CP}^n$ are written $[z_0,\ldots,z_n]$ for $(z_0,\ldots,z_n)$ in $\mathbb{C}^{n+1}\setminus\{0\}$, where $[z_0,\ldots,z_n]=\mathbb{C}\cdot(z_0,\ldots,z_n)\subseteq\mathbb{C}^{n+1}$, and $[\lambda z_0,\ldots,\lambda z_n]=[z_0,\ldots,z_n]$ for $\lambda\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$. Equivalently, $\mathbb{CP}^n=(\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$, where $\mathbb{C}^*$ acts by $\lambda:(z_0,\ldots,z_n)\mapsto(\lambda z_0,\ldots,\lambda z_n)$. It has the quotient topology induced from the surjective projection $\pi:\mathbb{C}^{n+1}\setminus\{0\}\to\mathbb{CP}^n$, $\pi:(z_0,\ldots,z_n)\mapsto[z_0,\ldots,z_n]$.

Write down an explicit atlas on $\mathbb{CP}^n$, compute the transition functions between charts, and prove that it makes $\mathbb{CP}^n$ into a smooth $2n$-dimensional manifold. (Hint: use $n+1$ charts (U_i,ϕ_i) for $i=0,\ldots,n$, such that $\phi_i(U_i)$ is the open subset of $[z_0,\ldots,z_n]\in\mathbb{CP}^n$ with $z_i\neq 0$ — try fixing $z_i=1$.)

2. Let X,Y be manifolds. Show carefully that $X\times Y$ has a unique manifold structure such that if $(U,\phi),(V,\psi)$ are charts on X,Y then $(U\times V,\phi\times\psi)$ is a chart on $X\times Y$, and $\dim(X\times Y)=\dim X+\dim Y$.

3(a) Let X be the sphere $\mathcal{S}^n=\bigl\{(x_0,\ldots,x_n)\in\mathbb{R}^{n+1}:x_0^2+\cdots+x_n^2=1\bigr\}$. Explain why we may identify

$$T_{(x_0,\ldots,x_n)}\mathcal{S}^n\cong\bigl\{(y_0,\ldots,y_n)\in\mathbb{R}^{n+1}:x_0y_0+\cdots+x_ny_n=0\bigr\}.$$

(b) By identifying $\mathbb{R}^{2k+2}\cong\mathbb{C}^{k+1}$, show that any odd-dimensional sphere $\mathcal{S}^{2k+1}$ has a nonvanishing vector field $v\in C^\infty(T\mathcal{S}^{2k+1})$ (i.e. $v\neq 0$ at every point).

For discussion: can the same thing hold for even-dimensional spheres $\mathcal{S}^{2k}$?

4. Let X be a manifold and $v,w\in C^\infty(TX)$. Suppose $(x_1,\ldots,x_n)$ are local coordinates on an open set $U\subseteq X$, so that we may write $v=v_1\frac{\partial}{\partial x_1}+\cdots+v_n\frac{\partial}{\partial x_n}$ and $w=w_1\frac{\partial}{\partial x_1}+\cdots+w_n\frac{\partial}{\partial x_n}$ on U, for $v_i,w_j:U\to\mathbb{R}$ smooth.

Define the Lie bracket $[v,w]\in C^\infty(TX)$ by

$$[v,w]=\sum_{i,j=1}^n\Bigl(v_i\frac{\partial w_j}{\partial x_i}\cdot\frac{\partial}{\partial x_j}-w_j\frac{\partial v_i}{\partial x_j}\cdot\frac{\partial}{\partial x_i}\Bigr)\qquad\text{on }U.\tag{1}$$

Prove that this is independent of choice of local coordinates. That is, if $(y_1,\ldots,y_n)$ is another local coordinate system on $V\subseteq X$, then (1) and its analogue for $(y_1,\ldots,y_n),V$ define the same vector field on $U\cap V$.

5. Take $\alpha\in\Lambda^kV$ where $\dim V=n$ and consider the linear map $A_\alpha:\Lambda^{n-k}V\to\Lambda^nV$ defined by $A_\alpha(\beta)=\alpha\wedge\beta$.

(i) Show that if $\alpha\neq 0$, then $A_\alpha\neq 0$.

(ii) Prove that the map $\alpha\mapsto A_\alpha$ is an isomorphism from Λ^kV to the vector space $\operatorname{Hom}(\Lambda^{n-k}V,\Lambda^nV)$ of linear maps from $\Lambda^{n-k}V$ to Λ^nV. Thus if we choose an isomorphism $\Lambda^nV\cong\mathbb{R}$ we get isomorphisms $\Lambda^kV\cong(\Lambda^{n-k}V)^*$.

6. Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (xy, yz, zx)$. Calculate $F^*(x\,\mathrm{d}y \wedge \mathrm{d}z)$ and $F^*(x\,\mathrm{d}y + y\,\mathrm{d}z)$.

7.(a) Prove explicitly that the de Rham cohomology groups of $\mathbb{R}$ are $H^0(\mathbb{R}) \cong \mathbb{R}$ and $H^1(\mathbb{R}) = 0$.

(b) Similarly, for $\mathcal{S}^1 = \mathbb{R}/\mathbb{Z}$, prove explicitly that $H^0(\mathcal{S}^1) \cong \mathbb{R}$ and $H^1(\mathcal{S}^1) \cong \mathbb{R}$.

Hint: Write $\Omega^0(\mathbb{R}) = \{f(x) \mid f : \mathbb{R} \to \mathbb{R} \text{ is smooth}\}$, and $\Omega^1(\mathbb{R}) = \{g(x)\mathrm{d}x \mid g : \mathbb{R} \to \mathbb{R} \text{ is smooth}\}$, so that $\mathrm{d} : \Omega^0(\mathbb{R}) \to \Omega^1(\mathbb{R})$ maps $f(x) \longmapsto \frac{\mathrm{d}f}{\mathrm{d}x}(x)\mathrm{d}x$. For $\mathcal{S}^1$, do the same, except that f, g are $\mathbb{Z}$-periodic, $f(x) = f(x + n)$ for $n \in \mathbb{Z}$.

8. Show that the product $X \times Y$ of two orientable manifolds is orientable.

9. Is $\mathcal{S}^2 \times \mathbb{RP}^2$ orientable? What about $\mathbb{RP}^2 \times \mathbb{RP}^2$?

10. A *Riemann surface* is defined as a 2-dimensional manifold X with an atlas $\{(U_i, \phi_i) : i \in I\}$ whose transition maps $\phi_j^{-1} \circ \phi_i$ for $i, j \in I$ are maps from an open set $\phi_i^{-1}(\phi_j(U_j))$ of $\mathbb{C} = \mathbb{R}^2$ to another open set $\phi_j^{-1}(\phi_i(U_j))$ which are *holomorphic* and invertible. By considering the Jacobian of $\phi_j^{-1} \circ \phi_i$, show that a Riemann surface is orientable.