Dominic Joyce **Kähler Geometry** **Nairobi 2019**

Miniprojects

Project 1. Let (X, J, g) be a Kähler manifold, and G a Lie group which acts on X by diffeomorphisms fixing J and g, with Lie algebra $\mathfrak{g}$. In the *Kähler quotient construction*, one defines a moment map $\mu : X \to \mathfrak{g}^*$, and then for $c \in Z(\mathfrak{g}^*)$, under good conditions $X//G := \mu^{-1}(c)/G$ is a new Kähler manifold, of real dimension $\dim X - 2\dim G$.
Explain all this. You could illustrate your discussion in terms of the quotient of $\mathbb{C}^4$ by U(1) acting by

$$u : (z_1, z_2, z_3, z_4) \longmapsto (uz_1, uz_2, u^{-1}z_3, u^{-1}z_4).$$

There are two different nonsingular cases and one singular case, according to the value of the moment map in $Z(\mathfrak{u}(1)^*)$; make sure you cover them all. Can you describe the quotients explicitly?
Other topics you could look at (pick one):

- The *hyperkähler quotient construction* for hyperkähler manifolds (holonomy $\mathrm{Sp}(m)$).
- *Geometric Invariant Theory*, and quotients in algebraic geometry. This is related to the question: in what sense can we regard the Kähler quotient $X//G$, as a complex manifold, as a complex quotient $X/G^{\mathbb{C}}$ by the complexification of G?
- Marsden–Weinstein symplectic reduction for symplectic manifolds.

Useful references:
D.D. Joyce, *Riemannian holonomy groups and calibrated geometry*, Oxford Graduate Texts in Mathematics 12, Oxford University Press, 2007, §10.6.4.
F.C. Kirwan, *Cohomology of quotients in algebraic and symplectic geometry*, Princeton University Press, 1985.
D. Mumford, J. Fogarty and F. Kirwan, *Geometric Invariant Theory*, 3rd edition, Springer, 1994.
A. Cannas da Silva, *Lectures on Symplectic Geometry*, Lecture Notes in Mathematics 1764, Springer, 2001.

Project 2. Write about what is known about the moduli space of $K3$ surfaces (Calabi–Yau 2-folds), both as complex surfaces, and as hyperkähler 4-manifolds. Include Torelli Theorems, etc., and discuss why some $K3$ surfaces are projective and some are not.

Useful references:
D.D. Joyce, *Riemannian holonomy groups and calibrated geometry*, Oxford Graduate Texts in Mathematics 12, Oxford University Press, 2007, §10.3.

A. Beauville et al. *Géometrie des surfaces K3: modules et périodes*, Astérisque 126 (1985).
W. Barth, C. Peters and A. van de Ven, *Compact complex surfaces*, Springer, 1984, §VIII.
A. Besse, *Einstein manifolds*, Springer, 1987, p.365-368.

Project 3. Outline the famous String Theory calculation of:

> P. Candelas and de la Ossa, X.C. and P.S. Green and L. Parkes, *A pair of Calabi–Yau manifolds as an exactly soluble superconformal theory*, Nuclear Physics B359 (1991), 21–74,

which computes (conjectural) numbers of holomorphic curves on the quintic in $\mathbb{CP}^4$ (Gromov–Witten invariants), in terms of the variation of Hodge structure of a 'mirror' Calabi–Yau 3-fold.
Say something about the mathematics that is needed to make some part of this calculation mathematically rigorous, for instance, how Gromov–Witten invariants are defined, or what variation of Hodge structure is.

Useful references:
D.A. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*, Mathematical Surveys and Monographs 68, A.M.S., 1999.
C. Voisin, *Mirror Symmetry*, A.M.S./S.M.F., 1999.

Project 4. *Sheaves* are a fundamental idea in modern Algebraic Geometry, for instance, a *scheme* is a special kind of topological space with a sheaf of rings. Discuss sheaves of abelian groups on a topological space and their cohomology. As examples you could consider:

- The constant sheaf $\underline{\mathbb{R}}$ with fibre $\mathbb{R}$ on a (sufficiently nice) topological space X, whose sheaf cohomology $H^k(X;\underline{\mathbb{R}})$ is the Čech cohomology $H^k_{\text{Čech}}(X;\mathbb{R})$ of X;
- The sheaves of de Rham k-forms on a manifold X;
- The sheaf of holomorphic functions on a complex manifold (X, J); and
- The sheaf of holomorphic sections of a holomorphic vector bundle E on a complex manifold (X, J).

Outline the proof that Čech cohomology $H^k_{\text{Čech}}(X;\mathbb{R})$ of a smooth manifold X is isomorphic to de Rham cohomology $H^k_{\text{dR}}(X;\mathbb{R})$. The same method is used to show that Dolbeault-style cohomology of a holomorphic vector bundle $H^q(E)$ on (X, J), as defined in the lectures, is isomorphic to the Čech cohomology of the sheaf of holomorphic sections of E.

Useful references:
P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, 1994, §0.3.
R. Hartshorne, *Algebraic Geometry*, Springer, 1997, §II.1, §III.
B. Iversen, Birger (1986), *Cohomology of sheaves*, Springer, 1986.

Project 5. Find out and write about *moduli spaces* in Algebraic Geometry. (You will need some background on schemes.)
Some topics you could consider:

- moduli of curves (Riemann surfaces);
- moduli of (algebraic) vector bundles (or more generally coherent sheaves), and Grothendieck's Quot scheme;
- Geometric Invariant Theory, and quotients in algebraic geometry.

Useful references:
F. Kirwan et al., *European women in mathematics: workshop on moduli spaces in mathematics and physics, Oxford, July 1998*, Hindawi, 2001, in particular, survey articles by F. Kirwan and R. Miro-Roig.
J. Harris and I. Morrison, *Moduli of curves*, Springer, 1998.
D. Gieseker, *Geometric invariant theory and applications to moduli problems*, pages 45-73 in Springer Lecture Notes in Math. 996, Springer, 1983.
D. Mumford, J. Fogarty and F. Kirwan, *Geometric invariant theory*, 3rd edition, Springer, 1994.