Riemannian holonomy groups and calibrated geometry

Dominic Joyce, Oxford

Lecture 13. Calibrated geometry

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6. Calibrated geometry

6.1 Minimal submanifolds

Let (M, g) be a compact Riemannian manifold, and N a compact submanifold of M. Using g, define the *volume* Vol(N) of N by integration. We call N *minimal* if Vol(N) is stationary under small variations of N. Let $\nu \to N$ be the normal bundle of N in M, so that $TM|_N = TN \oplus \nu$.

The *second fundamental form* $B \in C^\infty(S^2T^*N\otimes\nu)$ satisfies $B\cdot\bigl(v\vert_N\otimes w\vert_N\bigr)=\pi_\nu\bigl(\nabla_v w\vert_N\bigr)$ when $v,w\in C^\infty(TM)$ with $v\vert_N, w\vert_N\in C^\infty(TN)$.

The *mean curvature vector* $\kappa\in C^\infty(\nu)$ is the trace of B.

By the Euler–Lagrange method, a submanifold N is minimal if and only if $\kappa\equiv 0$. We use this to *define* minimal submanifolds in the noncompact case.

To find a compact minimal k-fold N in M with $[N] = \alpha$ in $H_k(M, \mathbb{Z})$, choose a *minimizing sequence* $(N_i)_{i=1}^{\infty}$ of compact k-folds N_i with $[N_i] = \alpha$, such that Vol(N_i) approaches the infimum of volumes with homology class α as $i \to \infty$.

If the set of k-folds N with Vol(N) $\leqslant C$ were *compact*, we could then choose a subsequence $(N_{i_j})_{j=1}^{\infty}$ converging to a minimal limit N.

The set of submanifolds N with $\mathrm{Vol}(N) \leqslant C$ is not compact. But the set of *rectifiable currents* N with $\mathrm{Vol}(N) \leqslant C$ is. Therefore, every $\alpha \in H_k(M, \mathbb{Z})$ is represented by a minimal rectifiable current.

Rectifiable currents are generalizations of submanifolds, and have singularities. They are studied in *Geometric Measure Theory*.

Some important questions: how close are minimal rectifiable currents to being submanifolds? How bad are their singularities? What are the singularities like?

A k-dimensional minimal rectifiable current is an embedded submanifold except on a singular set of Hausdorff dimension at most $k-2$.

6.2 Calibrations

Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane* V on M is an oriented vector subspace V of some tangent space T_xM to M with $\dim V = k$. Each has a *volume form* vol_V defined using g.

A *calibration* on M is a closed k-form φ with $\varphi|_V \leqslant \mathrm{vol}_V$ for every oriented tangent k-plane V on M.

Let N be an oriented k-fold in M with $\dim N = k$. We call N *calibrated* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

If N is compact then $\mathrm{vol}(N) \geqslant [\varphi] \cdot [N]$, and if N is compact and calibrated then $\mathrm{vol}(N) = [\varphi] \cdot [N]$, where $[\varphi] \in H^k(M, \mathbb{R})$ and $[N] \in H_k(M, \mathbb{Z})$.

Thus calibrated submanifolds are volume-minimizing in their homology class, and are *minimal submanifolds*.

Let (M, g) be Riemannian with calibration φ, and let $\iota : N \to M$ be an immersion. For N to be calibrated is a first-order p.d.e. on ι, but for N to be minimal is a second-order p.d.e. on ι.

Thus, the calibrated equations are often easier to solve than the minimal equations, and are a good way of finding examples of minimal submanifolds.

6.3 Calibrations on $\mathbb{R}^n$

Let $(\mathbb{R}^n, g)$ be Euclidean, and φ be a constant k-form on $\mathbb{R}^n$ with $\varphi|_V \leqslant \mathrm{vol}_V$ for all oriented k-planes V in $\mathbb{R}^n$.

Let $\mathcal{F}_\varphi$ be the set of oriented k-planes V in $\mathbb{R}^n$ with $\varphi|_V = \mathrm{vol}_V$. Then an oriented k-fold N in $\mathbb{R}^n$ is a φ-submanifold iff $T_x N \in \mathcal{F}_\varphi$ for all $x \in N$.

For φ to be interesting, $\mathcal{F}_\varphi$ must be fairly large, or there will be few φ-submanifolds.

Write $\varphi \preceq \varphi'$ if $\mathcal{F}_\varphi \subseteq \mathcal{F}_{\varphi'}$. Call a calibration φ *maximal* if it is maximal with respect to this partial order.

Maximal calibrations φ are the most interesting, as $\mathcal{F}_\varphi$ is as big as possible. They usually have quite large symmetry groups G, and $\mathcal{F}_\varphi$ may be a G-orbit. Often G is also a Riemannian holonomy group.

6.4 Calibrations and special holonomy metrics

Let $G \subset \mathrm{O}(n)$ be the holonomy group of a Riemannian metric. Then G acts on $\Lambda^k(\mathbb{R}^n)^*$. Suppose $\varphi_0 \in \Lambda^k(\mathbb{R}^n)^*$ is nonzero and G-invariant. Rescale φ_0 so that $\varphi_0|_V \leqslant \mathrm{vol}_V$ for all oriented k-planes $V \subset \mathbb{R}^n$, and $\varphi_0|_U = \mathrm{vol}_U$ for some U. Then $U \in \mathcal{F}_{\varphi_0}$, so by G-invariance $\mathcal{F}_{\varphi_0}$ contains the G-orbit of U. Usually $\mathcal{F}_{\varphi_0}$ is 'fairly big'.

Let (M,g) be have holonomy G. Then there is constant k-form φ on M corresponding to the G-invariant k-form φ_0. It is a *calibration* on M.

At each $x \in M$ the family of oriented tangent k-planes V with $\varphi|_V = \mathrm{vol}_V$ is $\mathcal{F}_{\varphi_0}$, which is 'fairly big'. So we expect many φ-submanifolds N in M. Thus manifolds with special holonomy often have interesting calibrations.

Here are some examples.

• The group $\mathrm{U}(m) \subset \mathrm{O}(2m)$ preserves a 2-form ω_0 on $\mathbb{R}^{2m}$. If (M, g) has holonomy $\mathrm{U}(m)$ then g is *Kähler*, with complex structure J, and the 2-form ω on M associated to ω_0 is the *Kähler form* of (g, J). Now $\omega^k/k!$ is a calibration for $1 \leqslant k \leqslant m$, with calibrated submanifolds the *complex* k*-submanifolds* of (M, J).

• The group SU(m) $\subset$ O($2m$) preserves a complex m-form Ω_0 on $\mathbb{R}^{2m}$. A manifold (M, g) with holonomy SU(m) is a *Calabi–Yau* m*-fold*, with *complex volume form* Ω corresponding to Ω_0.
The real part Re Ω is a calibration on M, and its calibrated submanifolds are called *special Lagrangian submanifolds*.

• The group $G_2 \subset \mathrm{O}(7)$ preserves a 3-form φ_0 and a 4-form $*\varphi_0$ on $\mathbb{R}^7$. A manifold (M, g) with holonomy G_2 carries a constant 3-form φ and 4-form $*\varphi$, which are both calibrations. Their calibrated submanifolds are called *associative 3-folds* and *coassociative 4-folds*.

• The group Spin(7) $\subset$ O(8) preserves a 4-form Ω_0 on $\mathbb{R}^8$. A manifold (M, g) with holonomy Spin(7) carries a constant 4-form Ω, which is a calibration. Its calibrated submanifolds are called *Cayley 4-folds*.

For each calibration φ on an n-manifold (M,g) with special holonomy constructed this way, there is a constant calibration φ_0 on $\mathbb{R}^n$. Locally, φ-submanifolds in M look like φ_0-submanifolds in $\mathbb{R}^n$.
In particular, *singularities* of φ-submanifolds in M are locally modelled on singularities of φ_0-submanifolds in $\mathbb{R}^n$.