Riemannian holonomy groups and calibrated geometry

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1 Introduction to holonomy groups

Sections 1.1, 1.2 and 1.4 are included as background material for those unfamiliar with the subject, and to establish notation. They will be covered only briefly in the lecture, as I assume most people will know the material already.

1.1 Tensors and forms

Let M be a smooth n-dimensional manifold, with tangent bundle TM and cotangent bundle T^*M. Then TM and T^*M are *vector bundles* over M. If E is a vector bundle over M, we use the notation $C^\infty(E)$ for the vector space of smooth sections of E. Elements of $C^\infty(TM)$ are called *vector fields*, and elements of $C^\infty(T^*M)$ are called 1-*forms*. By taking tensor products of the vector bundles TM and T^*M we obtain the bundles of *tensors* on M. A *tensor* T on M is a smooth section of a bundle $\bigotimes^k TM \otimes \bigotimes^l T^*M$ for some $k, l \in \mathbb{N}$.

It is convenient to write tensors using the *index notation*. Let U be an open set in M, and $(x^1, \dots, x^n)$ coordinates on U. Then at each point $x \in U$, $\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}$ are a basis for T_xU. Hence, any vector field v on U may be uniquely written $v = \sum_{a=1}^n v^a \frac{\partial}{\partial x^a}$ for some smooth functions $v^1, \dots, v^n : U \to \mathbb{R}$. We denote v by v^a, which is understood to mean the collection of n functions $v^1, \dots, v^n$, so that a runs from 1 to n.

Similarly, at each $x \in U$, $\mathrm{d}x^1, \dots, \mathrm{d}x^n$ are a basis for T_x^*U. Hence, any 1-form α on U may be uniquely written $\alpha = \sum_{b=1}^n \alpha_b \mathrm{d}x^b$ for some smooth functions $\alpha_1, \dots, \alpha_n : U \to \mathbb{R}$. We denote α by α_b, where b runs from 1 to n. In the same way, a general tensor T in $C^\infty(\bigotimes^k TM \otimes \bigotimes^l T^*M)$ is written $T^{a_1 \dots a_k}_{b_1 \dots b_l}$, where

$$T = \sum_{\substack{1 \leqslant a_i \leqslant n,\ 1 \leqslant i \leqslant k \\ 1 \leqslant b_j \leqslant n,\ 1 \leqslant j \leqslant l}} T^{a_1 \dots a_k}_{b_1 \dots b_l} \frac{\partial}{\partial x^{a_1}} \otimes \cdots \frac{\partial}{\partial x^{a_k}} \otimes \mathrm{d}x^{b_1} \otimes \cdots \otimes \mathrm{d}x^{b_l}.$$

The k^{th} exterior power of the cotangent bundle T^*M is written $\Lambda^k T^*M$. Smooth sections of $\Lambda^k T^*M$ are called *k-forms*, and the vector space of k-forms is written $C^\infty(\Lambda^k T^*M)$. They are examples of tensors. In the index notation they are written $T_{b_1\ldots b_k}$, and are antisymmetric in the indices $b_1,\ldots,b_k$. The *exterior product* $\wedge$ and the *exterior derivative* d are important natural operations on forms. If α is a k-form and β an l-form then $\alpha\wedge\beta$ is a $(k+l)$-form and $\mathrm{d}\alpha$ a $k+1$-form, which are given in index notation by

$$(\alpha\wedge\beta)_{a_1\ldots a_{k+l}} = \alpha_{[a_1\ldots a_k}\beta_{a_{k+1}\ldots a_{k+l}]} \quad \text{and} \quad (\mathrm{d}\alpha)_{a_1\ldots a_{k+1}} = T_{[a_1\ldots a_{k+1}]},$$

where $T_{a_1\ldots a_{k+1}} = \frac{\partial}{\partial x^{a_1}}\alpha_{a_2\ldots a_{k+1}}$ and $[\cdots]$ denotes antisymmetrization over the enclosed group of indices.

Let v,w be vector fields on M. The *Lie bracket* $[v,w]$ of v and w is another vector field on M, given in index notation by

$$[v,w]^a = v^b\frac{\partial w^a}{\partial x^b} - w^b\frac{\partial v^a}{\partial x^b}. \tag{1}$$

Here we have used the *Einstein summation convention*, that is, the repeated index b on the right hand side is summed from 1 to n. The important thing about this definition is that it is independent of choice of coordinates $(x^1,\ldots,x^n)$.

1.2 Connections on vector bundles and curvature

Let M be a manifold, and $E\rightarrow M$ a vector bundle. A *connection* ∇^E on E is a linear map $\nabla^E : C^\infty(E)\rightarrow C^\infty(E\otimes T^*M)$ satisfying the condition

$$\nabla^E(\alpha\, e) = \alpha\nabla^E e + e\otimes \mathrm{d}\alpha,$$

whenever $e\in C^\infty(E)$ is a smooth section of E and α is a smooth function on M.

If ∇^E is such a connection, $e\in C^\infty(E)$, and $v\in C^\infty(TM)$ is a vector field, then we write $\nabla^E_v e = v\cdot\nabla^E e\in C^\infty(E)$, where '$\cdot$' contracts together the TM and T^*M factors in v and $\nabla^E e$. Then if $v\in C^\infty(TM)$ and $e\in C^\infty(E)$ and α,β are smooth functions on M, we have

$$\nabla^E_{\alpha v}(\beta e) = \alpha\beta\nabla^E_v e + \alpha(v\cdot\beta)e.$$

Here $v\cdot\beta$ is the Lie derivative of β by v. It is a smooth function on M, and could also be written $v\cdot \mathrm{d}\beta$.

There exists a unique, smooth section $R(\nabla^E)\in C^\infty\big(\mathrm{End}(E)\otimes\Lambda^2 T^*M\big)$ called the *curvature* of ∇^E, that satisfies the equation

$$R(\nabla^E)\cdot(e\otimes v\wedge w) = \nabla^E_v\nabla^E_w e - \nabla^E_w\nabla^E_v e - \nabla^E_{[v,w]}e \tag{2}$$

for all $v,w\in C^\infty(TM)$ and $e\in C^\infty(E)$, where $[v,w]$ is the Lie bracket of v,w.

Here is one way to understand the curvature of ∇^E. Define $v_i=\partial/\partial x^i$ for $i=1,\ldots,n$. Then v_i is a vector field on U, and $[v_i,v_j]=0$. Let e be a smooth section of E. Then we may interpret $\nabla^E_{v_i} e$ as a kind of *partial derivative* $\partial e/\partial x^i$ of e. Equation (2) then implies that

$$R(\nabla^E)\cdot(e\otimes v_i\wedge v_j)=\frac{\partial^2 e}{\partial x^i\partial x^j}-\frac{\partial^2 e}{\partial x^j\partial x^i}. \tag{3}$$

Thus, *the curvature $R(\nabla^E)$ measures how much partial derivatives in E fail to commute.*

Now let ∇ be a connection on the tangent bundle TM of M, rather than a general vector bundle E. Then there is a unique tensor $T = T^a_{bc}$ in $C^\infty\big(TM\otimes\Lambda^2T^*M\big)$ called the *torsion* of ∇, satisfying

$$T\cdot(v\wedge w)=\nabla_v w-\nabla_w v-[v,w]\quad\text{for all } v,w\in C^\infty(TM).$$

A connection ∇ with zero torsion is called *torsion-free*. Torsion-free connections have various useful properties, so we usually restrict attention to torsion-free connections on TM.

A connection ∇ on TM extends naturally to connections on all the bundles of tensors $\bigotimes^k TM\otimes\bigotimes^l T^*M$ for $k,l\in\mathbb{N}$, which we will also write ∇. That is, we can use ∇ to differentiate not just vector fields, but any tensor on M.

1.3 Parallel transport and holonomy groups

Let M be a manifold, $E\to M$ a vector bundle over M, and ∇^E a connection on E. Let $\gamma:[0,1]\to M$ be a smooth curve in M. Then the pull-back $\gamma^*(E)$ of E to $[0,1]$ is a vector bundle over $[0,1]$ with fibre $E_{\gamma(t)}$ over $t\in[0,1]$, where E_x is the fibre of E over $x\in M$. The connection ∇^E pulls back under γ to give a connection on $\gamma^*(E)$ over $[0,1]$.

Definition 1.1 Let M be a manifold, E a vector bundle over M, and ∇^E a connection on E. Suppose $\gamma:[0,1]\to M$ is (piecewise) smooth, with $\gamma(0)=x$ and $\gamma(1)=y$, where $x,y\in M$. Then for each $e\in E_x$, there exists a unique smooth section s of $\gamma^*(E)$ satisfying $\nabla^E_{\dot\gamma(t)}s(t)=0$ for $t\in[0,1]$, with $s(0)=e$. Define $P_\gamma(e)=s(1)$. Then $P_\gamma:E_x\to E_y$ is a well-defined linear map, called the *parallel transport map*.

We use parallel transport to define the *holonomy group* of ∇^E.

Definition 1.2 Let M be a manifold, E a vector bundle over M, and ∇^E a connection on E. Fix a point $x\in M$. We say that γ is a *loop based at* x if $\gamma:[0,1]\to M$ is a piecewise-smooth path with $\gamma(0)=\gamma(1)=x$. The parallel transport map $P_\gamma:E_x\to E_x$ is an invertible linear map, so that P_γ lies in $\mathrm{GL}(E_x)$, the group of invertible linear transformations of E_x. Define the *holonomy group* $\mathrm{Hol}_x(\nabla^E)$ of ∇^E based at x to be

$$\mathrm{Hol}_x(\nabla^E)=\big\{P_\gamma:\gamma\text{ is a loop based at }x\big\}\subset\mathrm{GL}(E_x). \tag{4}$$

The holonomy group has the following important properties.

- It is a *Lie subgroup* of $\mathrm{GL}(E_x)$. To show that $\mathrm{Hol}_x(\nabla^E)$ is a subgroup of $\mathrm{GL}(E_x)$, let γ, δ be loops based at x, and define loops $\gamma\delta$ and γ^{-1} by

$$\gamma\delta(t) = \begin{cases} \delta(2t) & t \in [0, \frac{1}{2}] \\ \gamma(2t-1) & t \in [\frac{1}{2}, 1] \end{cases} \quad \text{and} \quad \gamma^{-1}(t) = \gamma(1-t) \quad \text{for } t \in [0,1].$$

  Then $P_{\gamma\delta} = P_\gamma \circ P_\delta$ and $P_{\gamma^{-1}} = P_\gamma^{-1}$, so $\mathrm{Hol}_x(\nabla^E)$ is closed under products and inverses.

- It is *independent of basepoint* $x \in M$, in the following sense. Let $x, y \in M$, and let $\gamma : [0,1] \to M$ be a smooth path from x to y. Then $P_\gamma : E_x \to E_y$, and $\mathrm{Hol}_x(\nabla^E)$ and $\mathrm{Hol}_y(\nabla^E)$ satisfy $\mathrm{Hol}_y(\nabla^E) = P_\gamma \mathrm{Hol}_x(\nabla^E) P_\gamma^{-1}$.

  Suppose E has fibre $\mathbb{R}^k$, so that $\mathrm{GL}(E_x) \cong \mathrm{GL}(k, \mathbb{R})$. Then we may regard $\mathrm{Hol}_x(\nabla^E)$ as a subgroup of $\mathrm{GL}(k, \mathbb{R})$ defined up to conjugation, and it is then independent of basepoint x.

- If M is simply-connected, then $\mathrm{Hol}_x(\nabla^E)$ is connected. To see this, note that any loop γ based at x can be continuously shrunk to the constant loop at x. The corresponding family of parallel transports is a continuous path in $\mathrm{Hol}_x(\nabla^E)$ joining P_γ to the identity.

The holonomy group of a connection is closely related to its curvature. Here is one such relationship. As $\mathrm{Hol}_x(\nabla^E)$ is a Lie subgroup of $\mathrm{GL}(E_x)$, it has a *Lie algebra* $\mathfrak{hol}_x(\nabla^E)$, which is a Lie subalgebra of $\mathrm{End}(E_x)$. It can be shown that the curvature $R(\nabla^E)_x$ at x lies in the linear subspace $\mathfrak{hol}_x(\nabla^E) \otimes \Lambda^2 T_x^* M$ of $\mathrm{End}(E_x) \otimes \Lambda^2 T_x^* M$. Thus, *the holonomy group of a connection places a linear restriction upon its curvature.*

Now let ∇ be a connection on TM. Then from §1.2, ∇ extends to connections on all the tensor bundles $\bigotimes^k TM \otimes \bigotimes^l T^*M$. We call a tensor S on M *constant* if $\nabla S = 0$. The constant tensors on M are determined by the holonomy group $\mathrm{Hol}(\nabla)$.

Theorem 1.3 *Let M be a manifold, and ∇ a connection on TM. Fix $x \in M$, and let $H = \mathrm{Hol}_x(\nabla)$. Then H acts naturally on the tensor powers $\bigotimes^k T_x M \otimes \bigotimes^l T_x^* M$.*

Suppose $S \in C^\infty\bigl(\bigotimes^k TM \otimes \bigotimes^l T^*M\bigr)$ is a constant tensor. Then $S|_x$ is fixed by the action of H on $\bigotimes^k T_x M \otimes \bigotimes^l T_x^* M$. Conversely, if $S|_x \in \bigotimes^k T_x M \otimes \bigotimes^l T_x^* M$ is fixed by H, it extends to a unique constant tensor $S \in C^\infty\bigl(\bigotimes^k TM \otimes \bigotimes^l T^*M\bigr)$.

The main idea in the proof is that if S is a constant tensor and $\gamma : [0,1] \to M$ is a path from x to y, then $P_\gamma(S|_x) = S|_y$. Thus, constant tensors are invariant under parallel transport.

1.4 Riemannian metrics and the Levi-Civita connection

Let g be a *Riemannian metric* on M. We refer to the pair (M, g) as a *Riemannian manifold.* Here g is a tensor in $C^\infty(S^2T^*M)$, so that $g = g_{ab}$ in index notation with $g_{ab} = g_{ba}$. There exists a unique, torsion-free connection ∇ on TM with $\nabla g = 0$, called the *Levi-Civita connection*, which satisfies

$$\begin{aligned}2g(\nabla_u v, w) = &u \cdot g(v, w) + v \cdot g(u, w) - w \cdot g(u, v)\\ &+ g([u, v], w) - g([v, w], u) - g([u, w], v)\end{aligned}$$

for all $u, v, w \in C^\infty(TM)$. This result is known as the *fundamental theorem of Riemannian geometry.*

The curvature $R(\nabla)$ of the Levi-Civita connection is a tensor $R^a{}_{bcd}$ on M. Define $R_{abcd} = g_{ae}R^e{}_{bcd}$. We shall refer to both $R^a{}_{bcd}$ and R_{abcd} as the *Riemann curvature* of g. The following theorem gives a number of symmetries of R_{abcd}. Equations (6) and (7) are known as the *first* and *second Bianchi identities*, respectively.

Theorem 1.4 *Let (M, g) be a Riemannian manifold, ∇ the Levi-Civita connection of g, and R_{abcd} the Riemann curvature of g. Then R_{abcd} and $\nabla_e R_{abcd}$ satisfy the equations*

$$R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab}, \tag{5}$$

$$R_{abcd} + R_{adbc} + R_{acdb} = 0, \tag{6}$$

and

$$\nabla_e R_{abcd} + \nabla_c R_{abde} + \nabla_d R_{abec} = 0. \tag{7}$$

Let (M, g) be a Riemannian manifold, with Riemann curvature $R^a{}_{bcd}$. The *Ricci curvature* of g is $R_{ab} = R^c{}_{acb}$. It is a component of the full Riemann curvature, and satisfies $R_{ab} = R_{ba}$. We say that g is *Einstein* if $R_{ab} = \lambda g_{ab}$ for some constant $\lambda \in \mathbb{R}$, and *Ricci-flat* if $R_{ab} = 0$. Einstein and Ricci-flat metrics are of great importance in mathematics and physics.

1.5 Riemannian holonomy groups

Let (M, g) be a Riemannian manifold. We define the *holonomy group* $\mathrm{Hol}_x(g)$ of g to be the holonomy group $\mathrm{Hol}_x(\nabla)$ of the Levi-Civita connection ∇ of g, as in §1.3. Holonomy groups of Riemannian metrics, or *Riemannian holonomy groups*, have stronger properties than holonomy groups of connections on arbitrary vector bundles. We shall explore some of these.

Firstly, note that g is a *constant tensor* as $\nabla g = 0$, so g is invariant under $\mathrm{Hol}(g)$ by Theorem 1.3. That is, $\mathrm{Hol}_x(g)$ lies in the subgroup of $\mathrm{GL}(T_xM)$ which preserves $g|_x$. This subgroup is isomorphic to $\mathrm{O}(n)$. Thus, $\mathrm{Hol}_x(g)$ may be regarded as a *subgroup of* $\mathrm{O}(n)$ *defined up to conjugation*, and it is then independent of $x \in M$, so we will often write it as $\mathrm{Hol}(g)$, dropping the basepoint x.

Secondly, the holonomy group $\mathrm{Hol}(g)$ constrains the Riemann curvature of g, in the following way. The Lie algebra $\mathfrak{hol}_x(\nabla)$ of $\mathrm{Hol}_x(\nabla)$ is a vector subspace of $T_xM \otimes T_x^*M$. From §1.3, we have $R^a{}_{bcd}|_x \in \mathfrak{hol}_x(\nabla) \otimes \Lambda^2 T_x^*M$.

Use the metric g to identify $T_xM \otimes T_x^*M$ and $\otimes^2 T_x^*M$, by equating $T^a{}_b$ with $T_{ab} = g_{ac}T^c{}_b$. This identifies $\mathfrak{hol}_x(\nabla)$ with a vector subspace of $\otimes^2 T_x^*M$ that we will write as $\mathfrak{hol}_x(g)$. Then $\mathfrak{hol}_x(g)$ lies in $\Lambda^2 T_x^*M$, and $R_{abcd}|_x \in \mathfrak{hol}_x(g) \otimes \Lambda^2 T_x^*M$. Applying the symmetries (5) of R_{abcd}, we have:

Theorem 1.5 *Let (M, g) be a Riemannian manifold with Riemann curvature R_{abcd}. Then R_{abcd} lies in the vector subspace $S^2\mathfrak{hol}_x(g)$ in $\Lambda^2 T_x^*M \otimes \Lambda^2 T_x^*M$ at each $x \in M$.*

Combining this theorem with the Bianchi identities, (6) and (7), gives strong restrictions on the curvature tensor R_{abcd} of a Riemannian metric g with a prescribed holonomy group $\mathrm{Hol}(g)$. These restrictions are the basis of the classification of Riemannian holonomy groups, which will be explained in Lecture 2.

Reading

D.D. Joyce, *Compact Manifolds with Special Holonomy*, OUP, Oxford, 2000, Chapters 2 and 3.

S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, volume 1, Wiley, New York, 1963, Chapters I–IV.

2 Berger's classification of holonomy groups

2.1 Reducible Riemannian manifolds

Let (P, g) and (Q, h) be Riemannian manifolds with positive dimension, and $P \times Q$ the product manifold. Then at each (p, q) in $P \times Q$ we have $T_{(p,q)}(P \times Q) \cong T_p P \oplus T_q Q$. Define the *product metric* $g \times h$ on $P \times Q$ by $g \times h|_{(p,q)} = g|_p + h|_q$ for all $p \in P$ and $q \in Q$. We call $(P \times Q, g \times h)$ a *Riemannian product*.

A Riemannian manifold (M, g') is said to be *(locally) reducible* if every point has an open neighbourhood isometric to a Riemannian product $(P \times Q, g \times h)$, and *irreducible* if it is not locally reducible. It is easy to show that the holonomy of a product metric $g \times h$ is the product of the holonomies of g and h.

Proposition 2.1 *Let (P, g) and (Q, h) be Riemannian manifolds. Then* $\mathrm{Hol}(g \times h) = \mathrm{Hol}(g) \times \mathrm{Hol}(h)$.

Here is a kind of converse to this.

Theorem 2.2 *Let M be an n-manifold, and g an irreducible Riemannian metric on M. Then the representation of* $\mathrm{Hol}(g)$ *on* $\mathbb{R}^n$ *is irreducible.*

To prove the theorem, suppose $\mathrm{Hol}(g)$ acts reducibly on $\mathbb{R}^n$, so that $\mathbb{R}^n$ is the direct sum of representations $\mathbb{R}^k$, $\mathbb{R}^l$ of $\mathrm{Hol}(g)$ with $k, l > 0$. Using parallel transport, one can define a splitting $TM = E \oplus F$, where E, F are vector subbundles with fibres $\mathbb{R}^k, \mathbb{R}^l$. These vector subbundles are *integrable*, so locally $M \cong P \times Q$ with $E = TP$ and $F = TQ$. One can then show that the metric on M is the product of metrics on P and Q, so that g is locally reducible.

2.2 Riemannian symmetric spaces

Next we discuss Riemannian symmetric spaces.

Definition 2.3 A Riemannian manifold (M, g) is said to be a *Riemannian symmetric space* if for every point $p \in M$ there exists an isometry $s_p : M \to M$ that is an involution (that is, s_p^2 is the identity), such that p is an isolated fixed point of s_p.

Examples include $\mathbb{R}^n$, spheres $\mathcal{S}^n$, projective spaces $\mathbb{CP}^m$ with the Fubini–Study metric, and so on. Symmetric spaces have a transitive group of isometries.

Proposition 2.4 *Suppose (M, g) is a connected, simply-connected Riemannian symmetric space. Then g is complete. Let G be the group of isometries of (M, g) generated by elements of the form $s_q \circ s_r$ for $q, r \in M$. Then G is a connected Lie group acting transitively on M. Choose $p \in M$, and let H be the subgroup of G fixing p. Then H is a closed, connected Lie subgroup of G, and M is the homogeneous space G/H.*

Because of this, symmetric spaces can be classified completely using the theory of Lie groups. This was done in 1925 by Élie Cartan. From Cartan's classification one can quickly deduce the list of holonomy groups of symmetric spaces.

A Riemannian manifold (M, g) is called *locally symmetric* if every point has an open neighbourhood isometric to an open set in a symmetric space, and *nonsymmetric* if it is not locally symmetric. It is a surprising fact that Riemannian manifolds are locally symmetric if and only if they have *constant curvature.*

Theorem 2.5 *Let (M, g) be a Riemannian manifold, with Levi-Civita connection ∇ and Riemann curvature R. Then (M, g) is locally symmetric if and only if $\nabla R = 0$.*

2.3 Berger's classification

In 1955, Berger proved the following result.

Theorem 2.6 (Berger) *Suppose M is a simply-connected manifold of dimension n, and that g is a Riemannian metric on M, that is irreducible and nonsymmetric. Then exactly one of the following seven cases holds.*

(i) $\mathrm{Hol}(g) = \mathrm{SO}(n)$,

(ii) $n = 2m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{U}(m)$ *in* $\mathrm{SO}(2m)$,

(iii) $n = 2m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{SU}(m)$ *in* $\mathrm{SO}(2m)$,

(iv) $n = 4m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)$ *in* $\mathrm{SO}(4m)$,

(v) $n = 4m$ *with* $m \geqslant 2$, *and* $\mathrm{Hol}(g) = \mathrm{Sp}(m)\,\mathrm{Sp}(1)$ *in* $\mathrm{SO}(4m)$,

(vi) $n = 7$ *and* $\mathrm{Hol}(g) = G_2$ *in* $\mathrm{SO}(7)$, *or*

(vii) $n = 8$ *and* $\mathrm{Hol}(g) = \mathrm{Spin}(7)$ *in* $\mathrm{SO}(8)$.

Notice the three simplifying assumptions on M and g: that M is simply-connected, and g is irreducible and nonsymmetric. Each condition has consequences for the holonomy group $\mathrm{Hol}(g)$.

- As M is simply-connected, $\mathrm{Hol}(g)$ is connected, from §1.3.
- As g is irreducible, $\mathrm{Hol}(g)$ acts irreducibly on $\mathbb{R}^n$ by Theorem 2.2.
- As g is nonsymmetric, $\nabla R \not\equiv 0$ by Theorem 2.5.

The point of the third condition is that there are some holonomy groups H which can *only* occur for metrics g with $\nabla R = 0$, and these holonomy groups are excluded from the theorem.

One can remove the three assumptions, at the cost of making the list of holonomy groups much longer. To allow g to be symmetric, we must include the holonomy groups of Riemannian symmetric spaces, which are known from

Cartan's classification. To allow g to be reducible, we must include all products of holonomy groups already on the list. To allow M not simply-connected, we must include non-connected Lie groups groups whose identity components are already on the list.

Berger proved that the groups on his list were the only possibilities, but he did not show whether the groups actually do occur as holonomy groups. It is now known (but this took another thirty years to find out) that all of the groups on Berger's list do occur as the holonomy groups of irreducible, nonsymmetric metrics.

2.4 A sketch of the proof of Berger's Theorem

Let (M, g) be a Riemannian n-manifold with M simply-connected and g irreducible and nonsymmetric, and let $H = \mathrm{Hol}(g)$. Then it is known that H is a closed, connected Lie subgroup of $\mathrm{SO}(n)$. The classification of such subgroups follows from the classification of Lie groups. Berger's method was to take the list of all closed, connected Lie subgroups H of $\mathrm{SO}(n)$, and apply two tests to each possibility to find out if it could be a holonomy group. The only groups H which passed both tests are those in the Theorem 2.6.

Berger's tests are algebraic and involve the curvature tensor. Suppose R_{abcd} is the Riemann curvature of a metric g with $\mathrm{Hol}(g) = H$, and let $\mathfrak{h}$ be the Lie algebra of H. Then Theorem 1.4 shows that $R_{abcd} \in S^2\mathfrak{h}$, and the first Bianchi identity (6) applies.

If $\mathfrak{h}$ has large codimension in $\mathfrak{so}(n)$, then the vector space $\mathfrak{R}^H$ of elements of $S^2\mathfrak{h}$ satisfying (6) will be small, or even zero. But the *Ambrose–Singer Holonomy Theorem* shows that $\mathfrak{R}^H$ must be big enough to generate $\mathfrak{h}$, in a certain sense. For many of the candidate groups H this does not hold, and so H cannot be a holonomy group. This is the first test.

Now $\nabla_e R_{abcd}$ lies in $(\mathbb{R}^n)^* \otimes \mathfrak{R}^H$, and also satisfies the second Bianchi identity (7). Frequently these requirements imply that $\nabla R = 0$, so that g is locally symmetric. Therefore we may exclude such H, and this is Berger's second test.

2.5 The groups on Berger's list

Here are some brief remarks about each group on Berger's list.

(i) $\mathrm{SO}(n)$ is the holonomy group of generic Riemannian metrics.

(ii) Riemannian metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$ are called *Kähler metrics.* Kähler metrics are a natural class of metrics on complex manifolds, and generic Kähler metrics on a given complex manifold have holonomy $\mathrm{U}(m)$.

(iii) Metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$ are called *Calabi–Yau metrics.* Since $\mathrm{SU}(m)$ is a subgroup of $\mathrm{U}(m)$, all Calabi–Yau metrics are Kähler. If g is Kähler and M is simply-connected, then $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$ if and only if g is Ricci-flat. Thus Calabi–Yau metrics are locally the same as Ricci-flat Kähler metrics.

(iv) Metrics g with $\mathrm{Hol}(g) \subseteq \mathrm{Sp}(m)$ are called *hyperkähler metrics*. As $\mathrm{Sp}(m) \subseteq \mathrm{SU}(2m) \subset \mathrm{U}(2m)$, hyperkähler metrics are Ricci-flat and Kähler.

(v) Metrics g with holonomy group $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$ for $m \geqslant 2$ are called *quaternionic Kähler metrics*. (Note that quaternionic Kähler metrics are not in fact Kähler.) They are Einstein, but not Ricci-flat.

(vi) and (vii) The holonomy groups G_2 and $\mathrm{Spin}(7)$ are called the *exceptional holonomy groups*.

The groups can be understood in terms of the four *division algebras*: the *real numbers* $\mathbb{R}$, the *complex numbers* $\mathbb{C}$, the *quaternions* $\mathbb{H}$, and the *octonions* or *Cayley numbers* $\mathbb{O}$.

- $\mathrm{SO}(n)$ is a group of automorphisms of $\mathbb{R}^n$.
- $\mathrm{U}(m)$ and $\mathrm{SU}(m)$ are groups of automorphisms of $\mathbb{C}^m$
- $\mathrm{Sp}(m)$ and $\mathrm{Sp}(m)\,\mathrm{Sp}(1)$ are automorphism groups of $\mathbb{H}^m$.
- G_2 is the automorphism group of $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$. $\mathrm{Spin}(7)$ is a group of automorphisms of $\mathbb{O} \cong \mathbb{R}^8$.

Here are three ways in which we can gather together the holonomy groups on Berger's list into subsets with common features.

- The *Kähler holonomy groups* are $\mathrm{U}(m)$, $\mathrm{SU}(m)$ and $\mathrm{Sp}(m)$. Any Riemannian manifold with one of these holonomy groups is a Kähler manifold, and thus a complex manifold.
- The *Ricci-flat holonomy groups* are $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2 and $\mathrm{Spin}(7)$. Any metric with one of these holonomy groups is Ricci-flat. This follows from the effect of holonomy on curvature discussed in §1.5 and §2.4: if H is one of these holonomy groups and R_{abcd} any curvature tensor lying in $S^2\mathfrak{h}$ and satisfying (6), then R_{abcd} has zero Ricci component.
- The *exceptional holonomy groups* are G_2 and $\mathrm{Spin}(7)$. They are the exceptional cases in Berger's classification, and they are rather different from the other holonomy groups.

Reading

On Berger's classification:

D.D. Joyce, *Compact Manifolds with Special Holonomy*, OUP, Oxford, 2000, Chapter 3.

S.M. Salamon, *Riemannian geometry and holonomy groups*, Longman, Harlow, 1989, Chapter 10.

On symmetric spaces:

S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, vol. 2, Wiley, 1963, Chapter XI.

Exercises

1. Let M be a manifold and u, v, w be vector fields on M. The *Jacobi identity* for the Lie bracket of vector fields is

$$[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0.$$

Prove the Jacobi identity in coordinates $(x^1,\ldots,x^n)$ on a coordinate patch U. Use the coordinate expression (1) for the Lie bracket of vector fields.

2. In §1.3 we explained that if M is a manifold, $E \to M$ a vector bundle and ∇^E a connection, then $\mathrm{Hol}(\nabla^E)$ is connected when M is simply-connected. If M is not simply-connected, what is the relationship between the fundamental group $\pi_1(M)$ and $\mathrm{Hol}(\nabla^E)$?

3. Work out your own proof of Theorem 1.3.

4. Work out your own proof of Proposition 2.1 and (rather harder) Theorem 2.2.

5. Suppose that (M, g) is a simply-connected Ricci-flat Kähler manifold of complex dimension 4. What are the possibilities for $\mathrm{Hol}(g)$?

[You may use the fact that the only simply-connected Ricci-flat symmetric spaces are $\mathbb{R}^n$, $n \in \mathbb{N}$.]

3 Kähler geometry and the Kähler holonomy groups

3.1 Complex manifolds

We begin by defining *complex manifolds* M. The usual definition of complex manifolds involves an atlas of complex coordinate patches covering M, whose transition functions are holomorphic. However, for our purposes we need a more differential geometric definition, involving a tensor J on M called a *complex structure*.

Let M be a real manifold of dimension $2m$. An *almost complex structure* J on M is a tensor J_a^b on M satisfying $J_a^b J_b^c = -\delta_a^c$. For each vector field v on M define Jv by $(Jv)^b = J_a^b v^a$. Then $J^2 = -1$, so J gives each tangent space T_pM the structure of a *complex vector space*.

We can associate a tensor $N = N_{bc}^a$ to J, called the *Nijenhuis tensor*, which satisfies

$$N_{bc}^a v^b w^c = \big([v,w] + J\big([Jv,w] + [v,Jw]\big) - [Jv,Jw]\big)^a$$

for all vector fields v, w on M, where $[\,,\,]$ is the Lie bracket of vector fields. The almost complex structure J is called a *complex structure* if $N \equiv 0$. A *complex manifold* (M, J) is a manifold M with a complex structure J.

Here is why this is equivalent to the usual definition. A smooth function $f : M \to \mathbb{C}$ is called *holomorphic* if $J_a^b (\mathrm{d}f)_b \equiv i(\mathrm{d}f)_a$ on M. These are called the *Cauchy–Riemann equations*. It turns out that the Nijenhuis tensor N is the obstruction to the existence of holomorphic functions. If $N \equiv 0$ there are many holomorphic functions locally, enough to form a set of holomorphic coordinates around every point.

3.2 Kähler manifolds

Let (M, J) be a complex manifold, and let g be a Riemannian metric on M. We call g a *Hermitian metric* if $g(v, w) = g(Jv, Jw)$ for all vector fields v, w on M, or $g_{ab} = J_a^c J_b^d g_{cd}$ in index notation. When g is Hermitian, define the *Hermitian form* ω of g by $\omega(v, w) = g(Jv, w)$ for all vector fields v, w on M, or $\omega_{ac} = J_a^b g_{bc}$ in index notation. Then ω is a (1,1)-form, and we may reconstruct g from ω by $g(v, w) = \omega(v, Jw)$.

A Hermitian metric g on a complex manifold (M, J) is called *Kähler* if one of the following three equivalent conditions holds:

(i) $\mathrm{d}\omega = 0$,

(ii) $\nabla J = 0$, or

(iii) $\nabla\omega = 0$,

where ∇ is the Levi-Civita connection of g. We then call (M, J, g) a *Kähler manifold*. Kähler metrics are a natural and important class of metrics on complex manifolds.

By parts (ii) and (iii), if g is Kähler then J and ω are *constant tensors* on M. Thus by Theorem 1.3, the holonomy group $\mathrm{Hol}(g)$ must preserve a complex structure J_0 and 2-form ω_0 on $\mathbb{R}^{2m}$. The subgroup of $\mathrm{O}(2m)$ preserving J_0 and ω_0 is $\mathrm{U}(m)$, so $\mathrm{Hol}(g) \subseteq \mathrm{U}(m)$. So we prove:

Proposition 3.1 *A metric g on a $2m$-manifold M is Kähler with respect to some complex structure J on M if and only if* $\mathrm{Hol}(g) \subseteq \mathrm{U}(m) \subset \mathrm{O}(2m)$.

3.3 Kähler potentials

Let (M, J) be a complex manifold. We have seen that to each Kähler metric g on M there is associated a closed real (1,1)-form ω, called the Kähler form. Conversely, if ω is a closed real (1,1)-form on M, then ω is the Kähler form of a Kähler metric if and only if ω is *positive*, that is, $\omega(v, Jv) > 0$ for all nonzero vectors v.

Now there is an easy way to manufacture closed real (1,1)-forms, using the ∂ and $\bar{\partial}$ operators on M. If $\phi : M \to \mathbb{R}$ is smooth, then $i\partial\bar{\partial}\phi$ is a closed real (1,1)-form, and every closed real (1,1)-form may be locally written in this way. Therefore, every Kähler metric g on M may be described locally by a function $\phi : M \to \mathbb{R}$ called a *Kähler potential*, such that the Kähler form ω satisfies $\omega = i\partial\bar{\partial}\phi$.

However, in general one cannot write $\omega = i\partial\bar{\partial}\phi$ globally on M, because $i\partial\bar{\partial}\phi$ is *exact*, but ω is usually not exact (never, if M is compact). Thus we are led to consider the *de Rham cohomology class* $[\omega]$ of ω in $H^2(M, \mathbb{R})$. We call $[\omega]$ the *Kähler class* of g. If two Kähler metrics g, g' on M lie in the same Kähler class, then they differ by a Kähler potential.

Proposition 3.2 *Let (M, J) be a compact complex manifold, and let g, g' be Kähler metrics on M with Kähler forms ω, ω'. Suppose that $[\omega] = [\omega'] \in H^2(M, \mathbb{R})$. Then there exists a smooth, real function ϕ on M such that $\omega' = \omega + i\partial\bar{\partial}\phi$. This function ϕ is unique up to the addition of a constant.*

Note also that if ω is the Kähler form of a fixed Kähler metric g and ϕ is sufficiently small in $\mathbb{C}^2$, then $\omega' = \omega + i\partial\bar{\partial}\phi$ is the Kähler form of another Kähler metric g' on M, in the same Kähler class as g. This implies that if there exists one Kähler metric on M, then there exists an infinite-dimensional family — Kähler metrics are very abundant.

3.4 Ricci curvature and the Ricci form

Let (M, J, g) be a Kähler manifold, with Ricci curvature R_{ab}. Define the *Ricci form* ρ by $\rho_{ac} = J_a^b R_{bc}$. Then it turns out that $\rho_{ac} = -\rho_{ca}$, so that ρ is a 2-form. Furthermore, it is a remarkable fact that ρ is a *closed, real* $(1,1)$*-form.* Note also that the Ricci curvature can be recovered from ρ by the formula $R_{ab} = \rho_{ac} J_b^c$.

To explain this, we will give an explicit expression for the Ricci form. Let $(z_1, \ldots, z_m)$ be holomorphic coordinates on an open set U in M. Define a smooth function $f : U \to (0,\infty)$ by

$$\omega^m = f \cdot \frac{(-1)^{m(m-1)/2} i^m m!}{2^m} \cdot \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_1 \wedge \cdots \wedge \mathrm{d}\bar{z}_m. \qquad (8)$$

Here the constant factor ensures that f is positive, and gives $f \equiv 1$ when ω is the standard Hermitian form on $\mathbb{C}^m$. Then it can be shown that

$$\rho = -i\partial\bar{\partial}(\log f) \quad \text{on } U, \qquad (9)$$

so that ρ is indeed a closed real (1,1)-form.

Using some algebraic geometry, we can interpret this. The *canonical bundle* $K_M = \Lambda^{(m,0)}T^*M$ is a holomorphic line bundle over M. The Kähler metric g on M induces a metric on K_M, and the combination of metric and holomorphic structure induces a connection ∇^K on K_M. The curvature of this connection is a closed 2-form with values in the Lie algebra $\mathfrak{u}(1)$, and identifying $\mathfrak{u}(1) \cong \mathbb{R}$ we get a closed 2-form, which is the Ricci form.

Thus the Ricci form ρ may be understood as the curvature 2-form of a connection ∇^K on the canonical bundle K_M. So by characteristic class theory we may identify the de Rham cohomology class $[\rho]$ of ρ in $H^2(M,\mathbb{R})$: it satisfies

$$[\rho] = 2\pi\, c_1(K_M) = 2\pi\, c_1(M), \qquad (10)$$

where $c_1(M)$ is the first Chern class of M in $H^2(M,\mathbb{Z})$. It is a topological invariant depending on the homotopy class of the (almost) complex structure J.

3.5 Calabi–Yau manifolds

Now suppose (M,J,g) is a *Ricci-flat* Kähler manifold. Then the Ricci form $\rho \equiv 0$, so $c_1(M) = 0$ by (10). But ρ is the curvature of a connection ∇^K on K_M, so ∇^K is *flat*. It follows that K_M locally admits constant sections. Suppose also that M is simply-connected. Then K_M has a family of global constant sections, isomorphic to $\mathbb{C}$.

Thus, if (M,J,g) is a simply-connected Ricci-flat Kähler manifold, then there is a family of nonzero constant $(m,0)$-forms Ω on M. So by Theorem 1.3, the holonomy group $\mathrm{Hol}(g)$ preserves a nonzero $(m,0)$-form Ω_0 on $\mathbb{R}^{2m} \equiv \mathbb{C}^m$. It follows that $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$, since $\mathrm{SU}(m)$ is the subgroup of $\mathrm{U}(m)$ preserving $\mathrm{d}z_1 \wedge \cdots \mathrm{d}z_m$. We have proved:

Proposition 3.3 *Let (M,J,g) be a simply-connected Kähler manifold. Then* $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$ *if and only if g is Ricci-flat.*

There are several different definitions of Calabi–Yau manifolds in use. We define a *Calabi–Yau manifold* to be a compact Kähler manifold (M,J,g) of dimension $m \geqslant 2$, with $\mathrm{Hol}(g) = \mathrm{SU}(m)$. Calabi–Yau manifolds are Ricci-flat with $c_1(M) = 0$, and have finite fundamental group. Each Calabi–Yau manifold (M,J,g) admits a holomorphic volume form Ω, which is a nonzero constant $(m,0)$-form on M. By rescaling Ω we can choose it to satisfy

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}, \qquad (11)$$

and Ω is then unique up to phase. The constant factor in (11) is chosen to make $\operatorname{Re}\Omega$ a *calibration*.

3.6 Hyperkähler manifolds

The *quaternions* are the associative algebra $\mathbb{H} = \langle 1, i_1, i_2, i_3 \rangle \cong \mathbb{R}^4$, with multiplication given by

$$i_1 i_2 = -i_2 i_1 = i_3, \quad i_2 i_3 = -i_3 i_2 = i_1, \quad i_3 i_1 = -i_1 i_3 = i_2, \quad i_1^2 = i_2^2 = i_3^2 = -1.$$

The holonomy group $\mathrm{Sp}(m)$ is the group of $m \times m$ matrices A over $\mathbb{H}$ satisfying $A\bar{A}^T = I$, where $x \mapsto \bar{x}$ is the conjugation on $\mathbb{H}$ defined by $\bar{x} = x_0 - x_1 i_1 - x_2 i_2 - x_3 i_3$ when $x = x_0 + x_1 i_1 + x_2 i_2 + x_3 i_3$.

Now $\mathrm{Sp}(m)$ acts on $\mathbb{H}^m = \mathbb{R}^{4m}$, regarded as column matrices of quaternions, preserving the Euclidean metric and also three complex structures J_1, J_2, J_3, induced by right multiplication of $\mathbb{H}^m$ by i_1, i_2, i_3. In fact, if $a_1, a_2, a_3 \in \mathbb{R}$ with $a_1^2 + a_2^2 + a_3^2 = 1$ then $a_1 J_1 + a_2 J_2 + a_3 J_3$ is also a complex structure on $\mathbb{R}^{4m}$ preserved by $\mathrm{Sp}(m)$.

Thus, if (M, g) is a Riemannian $4m$-manifold and g has holonomy $\mathrm{Sp}(m)$, then there exists an $\mathcal{S}^2$ family of constant complex structures $a_1 J_1 + a_2 J_2 + a_3 J_3$ on M for $a_1^2 + a_2^2 + a_3^2 = 1$, by Theorem 1.3. Now g is Kähler with respect to each complex structure $a_1 J_1 + a_2 J_2 + a_3 J_3$ from §3.2, so g is Kähler in many different ways. Therefore metrics with holonomy $\mathrm{Sp}(m)$ are called *hyperkähler*.

The three complex structures J_1, J_2, J_3 each have their own Kähler form $\omega_1, \omega_2, \omega_3$, with $\nabla \omega_k = 0$. The complex 2-form $\omega_2 + i\omega_3$ is a *holomorphic* $(2,0)$-*form* with respect to the complex structure J_1, which is a *complex symplectic structure* on the complex manifold (M, J_1). Thus, one way to interpret the geometric structure on a hyperkähler manifold is as a complex symplectic manifold $(M, J, \omega_2 + i\omega_3)$ together with a compatible Kähler metric g.

As $\mathrm{Sp}(m) \subseteq \mathrm{SU}(2m)$, hyperkähler metrics are automatically Ricci-flat, from §3.5. In complex dimension 2 we have $\mathrm{Sp}(1) = \mathrm{SU}(2)$, so (compact) hyperkähler 4-manifolds and Calabi–Yau 2-folds coincide. But in higher dimensions the inclusion $\mathrm{Sp}(m) \subset \mathrm{SU}(2m)$ is proper. Many examples of noncompact hyperkähler manifolds are known, but rather fewer examples of compact hyperkähler manifolds.

Reading

D. Huybrechts, *Compact hyperkähler manifolds: basic results*, Inventiones mathematicae 135 (1999), 63–113. alg-geom/9705025.

D.D. Joyce, *Compact Manifolds with Special Holonomy*, OUP, 2000, Chapters 4, 6 and 7.

S.M. Salamon, *Riemannian geometry and holonomy groups*, Longman, 1989, Chapters 3, 4 and 8.

4 The Calabi Conjecture and constructions of Calabi–Yau manifolds

4.1 The Calabi Conjecture

Let (M, J) be a compact, complex manifold, and g a Kähler metric on M, with Ricci form ρ. From §3.4, ρ is a closed real (1,1)-form and $[\rho] = 2\pi\, c_1(M) \in H^2(M, \mathbb{R})$. The Calabi Conjecture specifies which closed (1,1)-forms can be the Ricci forms of a Kähler metric on M.

The Calabi Conjecture *Let (M, J) be a compact, complex manifold, and g a Kähler metric on M, with Kähler form ω. Suppose that ρ' is a real, closed $(1, 1)$-form on M with $[\rho'] = 2\pi\, c_1(M)$. Then there exists a unique Kähler metric g' on M with Kähler form ω', such that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$, and the Ricci form of g' is ρ'.*

Note that $[\omega'] = [\omega]$ says that g and g' are in the *same Kähler class*. The conjecture was posed by Calabi in 1954, and was eventually proved by Yau in 1976. Its importance to us is that when $c_1(M) = 0$ we can take $\rho' \equiv 0$, and then g' is Ricci-flat. Thus, assuming the Calabi Conjecture we prove:

Corollary 4.1 *Let (M, J) be a compact complex manifold with $c_1(M) = 0$ in $H^2(M, \mathbb{R})$. Then every Kähler class on M contains a unique Ricci-flat Kähler metric g.*

If in addition M is simply-connected, then Proposition 3.3 implies that these Ricci-flat Kähler metrics g have $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$. When $\mathrm{Hol}(g) = \mathrm{SU}(m)$, by definition (M, J, g) is a *Calabi–Yau manifold*. So Yau's proof of the Calabi Conjecture gives a way to find examples of Calabi–Yau manifolds, which is how Calabi–Yau manifolds got their name.

All we have to do is to find examples of complex manifolds (M, J) satisfying some simple topological conditions, and then the Calabi Conjecture guarantees the existence of a family of metrics g on M such that (M, J, g) is a Calabi–Yau manifold. However, note that we know *almost nothing* about g except that it exists; we cannot write it down explicitly in coordinates, for instance. In fact, *no* explicit examples of Calabi–Yau metrics on compact manifolds are known at all.

4.2 Sketch of the proof of the Calabi Conjecture

The Calabi Conjecture is proved by rewriting it as a second-order nonlinear elliptic p.d.e. upon a real function ϕ on M, and then showing that this p.d.e. has a unique solution. We first explain how to rewrite the Calabi Conjecture as a p.d.e.

Let (M, J) be a compact, complex manifold, and let g, g' be two Kähler metrics on M with Kähler forms ω, ω' and Ricci forms ρ, ρ'. Suppose g, g' are in the same Kähler class, so that $[\omega'] = [\omega] \in H^2(M, \mathbb{R})$. Define a smooth function $f : M \to \mathbb{R}$ by $(\omega')^m = \mathrm{e}^f \omega^m$. Then from equations (8) and (9) of §3.4, we find that $\rho' = \rho - i\partial\bar\partial f$. Furthermore, as $[\omega'] = [\omega]$ in $H^2(M,\mathbb{R})$, we have $[\omega']^m = [\omega]^m$ in $H^{2m}(M,\mathbb{R})$, and thus $\int_M \mathrm{e}^f \omega^m = \int_M \omega^m$.

Now suppose that we are given the real, closed (1,1)-form ρ' with $[\rho'] = 2\pi\, c_1(M)$, and want to construct a metric g' with ρ' as its Ricci form. Since $[\rho] = [\rho'] = 2\pi\, c_1(M)$, $\rho - \rho'$ is an *exact* real (1,1)-form, and so by the $\partial\bar\partial$-Lemma there exists a smooth function $f : M \to \mathbb{R}$ with $\rho - \rho' = i\partial\bar\partial f$. This f is unique up to addition of a constant, but the constant is fixed by requiring that $\int_M \mathrm{e}^f \omega^m = \int_M \omega^m$. Thus we have proved:

Proposition 4.2 *Let (M, J) be a compact complex manifold, g a Kähler metric on M with Kähler form ω and Ricci form ρ, and ρ' a real, closed $(1,1)$-form on M with $[\rho'] = 2\pi\, c_1(M)$. Then there is a unique smooth function $f : M \to \mathbb{R}$ such that*

$$\rho' = \rho - i\partial\bar\partial f \quad \text{and} \quad \int_M \mathrm{e}^f \omega^m = \int_M \omega^m, \tag{12}$$

and a Kähler metric g on M with Kähler form ω' satisfying $[\omega'] = [\omega]$ in $H^2(M,\mathbb{R})$ has Ricci form ρ' if and only if $(\omega')^m = \mathrm{e}^f \omega^m$.

Thus we have transformed the Calabi Conjecture from seeking a metric g' with *prescribed Ricci curvature* ρ' to seeking a metric g' with *prescribed volume form* $(\omega')^m$. This is an important simplification, because the Ricci curvature depends on the second derivatives of g', but the volume form depends only on g' and not on its derivatives.

Now by Proposition 3.2, as $[\omega'] = [\omega]$ we may write $\omega' = \omega + i\partial\bar\partial\phi$ for ϕ a smooth real function on M, unique up to addition of a constant. We can fix the constant by requiring that $\int_M \phi\, \mathrm{d}V_g = 0$. So, from Proposition 4.2 we deduce that the Calabi Conjecture is equivalent to:

The Calabi Conjecture (second version) *Let (M, J) be a compact, complex manifold, and g a Kähler metric on M, with Kähler form ω. Let f be a smooth real function on M satisfying $\int_M \mathrm{e}^f \omega^m = \int_M \omega^m$. Then there exists a unique smooth real function ϕ such that*

(i) *$\omega + i\partial\bar\partial\phi$ is a positive $(1,1)$-form, that is, it is the Kähler form of some Kähler metric g',*

(ii) *$\int_M \phi\, \mathrm{d}V_g = 0$, and*

(iii) *$(\omega + i\partial\bar\partial\phi)^m = \mathrm{e}^f \omega^m$ on M.*

This reduces the Calabi Conjecture to a problem in analysis, that of showing that the nonlinear p.d.e. $(\omega + i\partial\bar\partial\phi)^m = \mathrm{e}^f \omega^m$ has a solution ϕ for every suitable function f. To prove this second version of the Calabi Conjecture, Yau used the *continuity method.*

For each $t \in [0,1]$, define $f_t = tf + c_t$, where c_t is the unique real constant such that $\mathrm{e}^{c_t} \int_M \mathrm{e}^{f_t} \omega^m = \int_M \omega^m$. Then f_t depends smoothly on t, with $f_0 \equiv 0$ and $f_1 \equiv f$. Define S to be the set of $t \in [0,1]$ such that there exists a smooth real function ϕ on M satisfying parts (i) and (ii) above, and also (iii)′ $(\omega + i\partial\bar\partial\phi)^m = e^{f_t}\omega^m$ on M.

The idea of the continuity method is to show that S is both *open* and *closed* in $[0,1]$. Thus, S is a connected subset of $[0,1]$, so $S = \emptyset$ or $S = [0,1]$. But $0 \in S$, since as $f_0 \equiv 0$ parts (i), (ii) and (iii)′ are satisfied by $\phi \equiv 0$. Thus $S = [0,1]$. In particular, (i), (ii) and (iii)′ admit a solution ϕ when $t = 1$. As $f_1 \equiv f$, this ϕ satisfies (iii), and the Calabi Conjecture is proved.

Showing that S is open is fairly easy, and was done by Calabi. It depends on the fact that (iii) is an *elliptic* p.d.e. — basically, the operator $\phi \mapsto (\omega + i\partial\bar\partial\phi)^m$ is rather like a nonlinear Laplacian — and uses only standard facts about elliptic operators.

However, showing that S is closed is much more difficult. One must prove that S contains its limit points. That is, if $(t_n)_{n=1}^\infty$ is a sequence in S converging to $t \in [0,1]$ then there exists a sequence $(\phi_n)_{n=1}^\infty$ satisfying (i), (ii) and $(\omega + i\partial\bar\partial\phi_n)^m = e^{f_{t_n}}\omega^m$ for $n = 1, 2, \ldots$, and we need to show that $\phi_n \to \phi$ as $n \to \infty$ for some smooth real function ϕ satisfying (i), (ii) and (iii)′, so that $t \in S$.

The thing you have to worry about is that the sequence $(\phi_n)_{n=1}^\infty$ might converge to some horrible non-smooth function, or might not converge at all. To prove this doesn't happen you need *a priori estimates* on the ϕ_n and all their derivatives. In effect, you need upper bounds on $|\nabla^k\phi_n|$ for all n and k, bounds which are allowed to depend on M, J, g, k and f_{t_n}, but not on n or ϕ_n. These a priori estimates were difficult to find, because the nonlinearities in ϕ of $(\omega + i\partial\bar\partial\phi)^m = e^f\omega^m$ are of a particularly nasty kind, and this is why it took so long to prove the Calabi Conjecture.

4.3 Calabi–Yau 2-folds and $K3$ surfaces

Recall from §3.5–§3.6 that Calabi–Yau manifolds of complex dimension m have holonomy $\mathrm{SU}(m)$ for $m \geqslant 2$, and hyperkähler manifolds of complex dimension $2k$ have holonomy $\mathrm{Sp}(k)$ for $k \geqslant 1$. In complex dimension 2 these coincide, as $\mathrm{SU}(2) = \mathrm{Sp}(1)$. Because of this, Calabi–Yau 2-folds have special features which are not present in Calabi–Yau m-folds for $m \geqslant 3$.

Calabi–Yau 2-folds are very well understood, through Kodaira's classification of compact complex surfaces. A $K3$ *surface* is defined to be a compact, complex surface (X, J) with $h^{1,0}(X) = 0$ and trivial canonical bundle. All Calabi–Yau 2-folds are $K3$ surfaces, and conversely, every $K3$ surface (X, J) admits a family of Kähler metrics g making it into a Calabi–Yau 2-fold. All $K3$ surfaces (X, J) are diffeomorphic, sharing the same smooth 4-manifold X, which is simply-connected, with Betti numbers $b^2 = 22$, $b^2_+ = 3$, and $b^2_- = 19$.

The moduli space $\mathscr{M}_{K3}$ of $K3$ surfaces is a connected 20-dimensional singular complex manifold, which can be described very precisely via the 'Torelli Theorems'. Some $K3$ surfaces are *algebraic*, that is, they can be embedded as complex submanifolds in $\mathbb{CP}^N$ for some N, and some are not. The set of algebraic $K3$ surfaces is a countable, dense union of 19-dimensional subvarieties in $\mathscr{M}_{K3}$. Each $K3$ surface (X, J) admits a real 20-dimensional family of Calabi–Yau metrics g, so the family of Calabi–Yau 2-folds (X, J, g) is a nonsingular

60-dimensional real manifold.

4.4 General properties of Calabi–Yau m-folds for $m \geqslant 3$

Using general facts about Ricci-flat manifolds (the *Cheeger–Gromoll Theorem*) one can show that every Calabi–Yau m-fold (M, J, g) has finite fundamental group. Also, using the 'Bochner argument' one can show that any closed $(p, 0)$-form ξ on M is constant under the Levi-Civita connection ∇ of g.

However, the set of constant tensors on M is determined by the holonomy group $\mathrm{Hol}(g)$ of g, which is $\mathrm{SU}(m)$ by definition. It is easy to show that the vector space of closed $(p, 0)$-forms on M is $\mathbb{C}$ if $p = 0, m$ and 0 otherwise. But the vector space of closed $(p, 0)$ forms is the *Dolbeault cohomology group* $H^{p,0}(M)$, whose dimension is the *Hodge number* $h^{p,0}$ of M. Thus we prove:

Proposition 4.3 *Let (M, J, g) be a Calabi–Yau m-fold with Hodge numbers $h^{p,q}$. Then M has finite fundamental group, $h^{0,0} = h^{m,0} = 1$ and $h^{p,0} = 0$ for $p \neq 0, m$.*

For $m \geqslant 3$ this gives $h^{2,0}(M) = 0$, and this has important consequences for the complex manifold (M, J). It can be shown that a complex line bundle L over a compact Kähler manifold (M, J, g) admits a holomorphic structure if and only if $c_1(L)$ lies in $H^{1,1}(M) \subseteq H^2(M, \mathbb{C})$. But $H^2(M, \mathbb{C}) = H^{2,0}(M) \oplus H^{1,1}(M) \oplus H^{0,2}(M)$, and $H^{2,0}(M) = H^{0,2}(M) = 0$ as $h^{2,0}(M) = 0$. Thus $H^{1,1}(M) = H^2(M, \mathbb{C})$, and so *every* complex line bundle L over M admits a holomorphic structure.

Thus, Calabi–Yau m-folds for $m \geqslant 3$ are richly endowed with holomorphic line bundles. Using the *Kodaira Embedding Theorem* one can show that some of these holomorphic line bundles admit many holomorphic sections. By taking a line bundle with enough holomorphic sections (a *very ample* line bundle) we can construct an embedding of M in $\mathbb{CP}^N$ as a complex submanifold. So we prove:

Theorem 4.4 *Let (M, J, g) be a Calabi–Yau manifold of dimension $m \geqslant 3$. Then M is projective. That is, (M, J) is isomorphic as a complex manifold to a complex submanifold of $\mathbb{CP}^N$, and is an algebraic variety.*

This shows that Calabi–Yau manifolds (or at least, the complex manifolds underlying them) can be studied using *complex algebraic geometry*.

4.5 Constructions of Calabi–Yau m-folds

The easiest way to find examples of Calabi–Yau m-folds for $m \geqslant 3$ is to choose a method of generating a large number of complex algebraic varieties, and then check the topological conditions to see which of them are Calabi–Yau. Here are some ways of doing this.

- **Hypersurfaces in $\mathbb{CP}^{m+1}$**. Suppose that X is a smooth degree d hypersurface in $\mathbb{CP}^{m+1}$. When is X a Calabi–Yau manifold? Well, using the *adjunction formula* one can show that the canonical bundle of X is given by $K_X = L^{d-m-2}|_X$, where $L \to \mathbb{CP}^{m+1}$ is the hyperplane line bundle on $\mathbb{CP}^{m+1}$.

  Therefore K_X is trivial if and only if $d = m + 2$. It is not difficult to show that any smooth hypersurface of degree $m+2$ in $\mathbb{CP}^{m+1}$ is a Calabi–Yau m-fold. All such hypersurfaces are diffeomorphic, for fixed m. For instance, the *Fermat quintic*

$$\left\{[z_0, \ldots, z_4] \in \mathbb{CP}^4 : z_0^5 + \cdots + z_4^5 = 0\right\}$$

  is a Calabi–Yau 3-fold, with Betti numbers $b^0 = 1$, $b^1 = 0$, $b^2 = 1$ and $b^3 = 204$.

- **Complete intersections in $\mathbb{CP}^{m+k}$**. In the same way, suppose X is a *complete intersection* of transverse hypersurfaces $H_1, \ldots, H_k$ in $\mathbb{CP}^{m+k}$ of degrees $d_1, \ldots, d_k$, with each $d_j \geqslant 2$. It can be shown that X is Calabi–Yau m-fold if and only if $d_1 + \cdots + d_k = m + k + 1$. This yields a finite number of topological types in each dimension m.

- **Hypersurfaces in toric varieties**. A *toric variety* is a complex m-manifold X with a holomorphic action of $(\mathbb{C}^*)^m$ which is transitive and free upon a dense open set in X. Toric varieties can be constructed and studied using only a finite amount of combinatorial data.

  The conditions for a smooth hypersurface in a compact toric variety to be a Calabi–Yau m-fold can be calculated using this combinatorial data. Using a computer, one can generate a large (but finite) number of Calabi–Yau m-folds, at least when $m = 3$, and calculate their topological invariants such as Hodge numbers. This has been done by Candelas, and other authors.

- **Resolution of singularities**. Suppose you have some way of producing examples of *singular* Calabi–Yau m-folds Y. Often it is possible to find a *resolution* X of Y with holomorphic map $\pi : X \to Y$, such that X is a nonsingular Calabi–Yau m-fold. Basically, each singular point in Y is replaced by a finite union of complex submanifolds in X.

  Resolutions which preserve the Calabi–Yau property are called *crepant resolutions*, and are well understood when $m = 3$. For certain classes of singularities, such as singularities of Calabi–Yau 3-orbifolds, a crepant resolution always exists.

  This technique can be applied in a number of ways. For instance, you can start with a nonsingular Calabi–Yau m-fold X, deform it till you get a singular Calabi–Yau m-fold Y, and then resolve the singularities of Y to get a second nonsingular Calabi–Yau m-fold X' with different topology to X.

Another method is to start with a nonsingular Calabi–Yau m-fold X, divide by the action of a finite group G preserving the Calabi–Yau structure to get a singular Calabi–Yau manifold (orbifold) $Y = X/G$, and then resolve the singularities of Y to get a second nonsingular Calabi–Yau m-fold X' with different topology to X.

Reading

For the whole lecture:

D.D. Joyce, *Compact manifolds with special holonomy*, OUP, 2000, Chapters 5 and 6, and §7.3.

Here are some references on the solution to the Calabi Conjecture. The Yau paper is the original proof that everyone cites, but it's pretty hard going unless you know a lot of analysis; I'd recommend my book, Chapter 5, or Aubin instead.

T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer–Verlag, 1998.

S.-T. Yau. *On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equations. I*, Comm. pure appl. math. 31 (1978), 339–411.

Here are some (somewhat random) references which may help if you want to follow up constructions of Calabi–Yau manifolds. Candelas et al. describe the construction by computer of Calabi–Yau 3-folds as hypersurfaces in toric 4-folds, and give a nice graph of the resulting Hodge numbers displaying 'Mirror Symmetry'. Fulton is an introduction to toric geometry to help you understand the toric hypersurface construction. Hübsch is the only book on Calabi–Yau manifolds I know, though now somewhat out of date.

P. Candelas, X.C. de La Ossa and S. Katz, *Mirror symmetry for Calabi–Yau hypersurfaces in weighted* $\mathbb{P}_4$ *and extensions of Landau–Ginsburg theory*, Nuclear Physics B450 (1995), 267–290.

W. Fulton, *Introduction to Toric Varieties*, Princeton University Press, 1993.

T. Hübsch, *Calabi–Yau manifolds, a bestiary for physicists*, World Scientific, Singapore, 1992.

Exercises

1. Let U be a simply-connected subset of $\mathbb{C}^m$ with coordinates $(z_1, \ldots, z_m)$, and g a Ricci-flat Kähler metric on U with Kähler form ω. Use equations (8) and (9) to show that there exists a holomorphic $(m, 0)$-form Ω on U satisfying

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}.$$

Hint: Write $\Omega = F\,\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m$ for some holomorphic function F. Use the fact that if f is a real function on a simply-connected subset U of $\mathbb{C}^m$ and $\partial\bar{\partial} f \equiv 0$, then f is the real part of a holomorphic function on U.

2. Let $\mathbb{C}^2$ have complex coordinates (z_1, z_2), and define $u = |z_1|^2 + |z_2|^2$. Let $f : [0, \infty) \to \mathbb{R}$ be a smooth function, and define a closed real (1,1)-form ω on $\mathbb{C}^2$ by $\omega = i\partial\bar{\partial} f(u)$.

(a) Calculate the conditions on f for ω to be the Kähler form of a Kähler metric g on $\mathbb{C}^2$.

(You can define g by $g(v, w) = \omega(v, Jw)$, and need to ensure that g is positive definite).

(b) Supposing g is a metric, calculate the conditions on f for g to be Ricci-flat. You should get an o.d.e. on f. If you can, solve this o.d.e., and write down the corresponding Kähler metrics in coordinates.

3. The most well-known examples of Calabi–Yau 3-folds are *quintics* X in $\mathbb{CP}^4$, defined by

$$X = \{[z_0, \ldots, z_4] \in \mathbb{CP}^4 : p(z_0, \ldots, z_4) = 0\},$$

where $p(z_0, \ldots, z_4)$ is a homogeneous quintic polynomial in its arguments. Every nonsingular quintic has Hodge numbers $h^{1,1} = h^{2,2} = 1$ and $h^{2,1} = h^{1,2} = 101$.

(i) Calculate the dimension of the vector space of homogeneous quintic polynomials $p(z_0, \ldots, z_4)$. Hence find the dimension of the moduli space of nonsingular quintics in $\mathbb{CP}^4$. (A generic quintic is nonsingular).

(ii) Identify the group of complex automorphisms of $\mathbb{CP}^4$ and calculate its dimension.

(iii) Hence calculate the dimension of the moduli space of quintics in $\mathbb{CP}^4$ up to automorphisms of $\mathbb{CP}^4$.

It is a general fact that if (X, J, g) is a Calabi–Yau 3-fold, then the moduli space of complex deformations of (X, J) has dimension $h^{2,1}(X)$, and each nearby deformation is a Calabi–Yau 3-fold. In this case, $h^{2,1}(X) = 101$, and this should be your answer to (iii). That is, deformations of quintics in $\mathbb{CP}^4$ are also quintics in $\mathbb{CP}^4$.

4. One can also construct Calabi–Yau 3-folds as the complete intersection of two cubics in $\mathbb{CP}^5$,

$$X = \big\{[z_0, \ldots, z_5] \in \mathbb{CP}^5 : p(z_0, \ldots, z_5) = q(z_0, \ldots, z_5) = 0\big\},$$

where p, q are linearly independent homogeneous cubic polynomials. Using the method of question 3, calculate the dimension of the moduli space of such complete intersections up to automorphisms of $\mathbb{CP}^5$, and hence predict $h^{2,1}(X)$.

5 Introduction to calibrated geometry

The theory of *calibrated geometry* was invented by Harvey and Lawson. It concerns *calibrated submanifolds*, a special kind of *minimal submanifold* of a Riemannian manifold M, which are defined using a closed form on M called a *calibration*. It is closely connected with the theory of Riemannian holonomy groups because Riemannian manifolds with special holonomy usually come equipped with one or more natural calibrations.

5.1 Minimal submanifolds

Let (M, g) be an n-dimensional Riemannian manifold, and N a compact k-dimensional submanifold of M. Regard N as an immersed submanifold (N, ι), with immersion $\iota : N \to M$. Using the metric g we can define the *volume* $\mathrm{Vol}(N)$ of N, by integration over N. We call N a *minimal submanifold* if its volume is stationary under small variations of the immersion $\iota : N \to M$. When $k = 1$, a curve in M is minimal if and only if it is a *geodesic*.

Let $\nu \to N$ be the normal bundle of N in M, so that $TM|_N = TN \oplus \nu$ is an orthogonal direct sum. The *second fundamental form* is a section B of $S^2T^*N \otimes \nu$ such that whenever v, w are vector fields on M with $v|_N, w|_N$ sections of TN over N, then $B \cdot \bigl(v|_N \otimes w|_N\bigr) = \pi_\nu\bigl(\nabla_v w|_N\bigr)$, where '$\cdot$' contracts S^2T^*N with $TN \otimes TN$, ∇ is the Levi-Civita connection of g, and π_ν is the projection to ν in the splitting $TM|_N = TN \oplus \nu$.

The *mean curvature vector* κ of N is the trace of the second fundamental form B taken using the metric g on N. It is a section of the normal bundle ν. It can be shown by the Euler–Lagrange method that a submanifold N is minimal if and only if its mean curvature vector κ is zero. Note that this is a local condition. Therefore we can also define noncompact submanifolds N in M to be minimal if they have zero mean curvature. This makes sense even when N has infinite volume.

If $\iota : N \to M$ is a immersed submanifold, then the mean curvature κ of N depends on ι and its first and second derivatives, so the condition that N be minimal is a *second-order* equation on ι. Note that minimal submanifolds may not have minimal area, even amongst nearby homologous submanifolds. For instance, the equator in $\mathcal{S}^2$ is minimal, but does not minimize length amongst lines of latitude.

The following argument is important in the study of minimal submanifolds. Let (M, g) be a compact Riemannian manifold, and α a nonzero homology class in $H_k(M, \mathbb{Z})$. We would like to find a compact, minimal immersed, k-dimensional submanifold N in M with homology class $[N] = \alpha$. To do this, we choose a minimizing sequence $(N_i)_{i=1}^\infty$ of compact submanifolds N_i with $[N_i] = \alpha$, such that $\mathrm{Vol}(N_i)$ approaches the infimum of volumes of submanifolds with homology class α as $i \to \infty$.

Pretend for the moment that the set of all closed k-dimensional submanifolds N with $\mathrm{Vol}(N) \leqslant C$ is a *compact* topological space. Then there exists a subsequence $(N_{i_j})_{j=1}^\infty$ which converges to some submanifold N, which is the minimal submanifold we want. In fact this does not work, because the set of submanifolds N does not have the compactness properties we need.

However, if we work instead with *rectifiable currents*, which are a measure-theoretic generalization of submanifolds, one can show that every integral homology class α in $H_k(M,\mathbb{Z})$ is represented by a minimal rectifiable current. One should think of rectifiable currents as a class of singular submanifolds, obtained by completing the set of nonsingular submanifolds with respect to some norm. They are studied in the subject of *Geometric Measure Theory*.

The question remains: how close are these minimal rectifiable currents to being submanifolds? For example, it is known that a k-dimensional minimal rectifiable current in a Riemannian n-manifold is an embedded submanifold except on a singular set of Hausdorff dimension at most $k-2$. When $k=2$ or $k=n-1$ one can go further. In general, it is important to understand the possible singularities of such singular minimal submanifolds.

5.2 Calibrations and calibrated submanifolds

Let (M,g) be a Riemannian manifold. An *oriented tangent k-plane V on M* is a vector subspace V of some tangent space T_xM to M with $\dim V = k$, equipped with an orientation. If V is an oriented tangent k-plane on M then $g|_V$ is a Euclidean metric on V, so combining $g|_V$ with the orientation on V gives a natural *volume form* vol_V on V, which is a k-form on V.

Now let φ be a closed k-form on M. We say that φ is a *calibration* on M if for every oriented k-plane V on M we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha\cdot\mathrm{vol}_V$ for some $\alpha\in\mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha\leqslant 1$. Let N be an oriented submanifold of M with dimension k. Then each tangent space T_xN for $x\in N$ is an oriented tangent k-plane. We say that N is a *calibrated submanifold* or *φ-submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x\in N$.

All calibrated submanifolds are automatically *minimal submanifolds*. We prove this in the compact case, but it is true for noncompact submanifolds as well.

Proposition 5.1 *Let (M,g) be a Riemannian manifold, φ a calibration on M, and N a compact φ-submanifold in M. Then N is volume-minimizing in its homology class.*

Proof. Let $\dim N = k$, and let $[N]\in H_k(M,\mathbb{R})$ and $[\varphi]\in H^k(M,\mathbb{R})$ be the homology and cohomology classes of N and φ. Then

$$[\varphi]\cdot[N] = \int_{x\in N}\varphi\big|_{T_xN} = \int_{x\in N}\mathrm{vol}_{T_xN} = \mathrm{Vol}(N),$$

since $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for each $x\in N$, as N is a calibrated submanifold. If N' is any other compact k-submanifold of M with $[N'] = [N]$ in $H_k(M,\mathbb{R})$, then

$$[\varphi]\cdot[N] = [\varphi]\cdot[N'] = \int_{x\in N'}\varphi\big|_{T_xN'} \leqslant \int_{x\in N'}\mathrm{vol}_{T_xN'} = \mathrm{Vol}(N'),$$

since $\varphi|_{T_xN'} \leqslant \mathrm{vol}_{T_xN'}$ because φ is a calibration. The last two equations give $\mathrm{Vol}(N) \leqslant \mathrm{Vol}(N')$. Thus N is volume-minimizing in its homology class. □

Now let (M, g) be a Riemannian manifold with a calibration φ, and let $\iota : N \to M$ be an immersed submanifold. Whether N is a φ-submanifold depends upon the tangent spaces of N. That is, it depends on ι and its first derivative. So, to be calibrated with respect to φ is a *first-order* equation on ι. But if N is calibrated then N is minimal, and we saw in §5.1 that to be minimal is a *second-order* equation on ι.

One moral is that the calibrated equations, being first-order, are often easier to solve than the minimal submanifold equations, which are second-order. So calibrated geometry is a fertile source of examples of minimal submanifolds.

5.3 Calibrated submanifolds of $\mathbb{R}^n$

One simple class of calibrations is to take (M, g) to be $\mathbb{R}^n$ with the Euclidean metric, and φ to be a constant k-form on $\mathbb{R}^n$, such that $\varphi|_V \leqslant \mathrm{vol}_V$ for every oriented k-dimensional vector subspace $V \subseteq \mathbb{R}^n$. Each such φ defines a class of minimal k-submanifolds in $\mathbb{R}^n$. However, this class may be very small, or even empty. For instance, $\varphi = 0$ is a calibration on $\mathbb{R}^n$, but has no calibrated submanifolds.

For each constant calibration k-form φ on $\mathbb{R}^n$, define

$$\mathcal{F}_\varphi = \bigl\{ V : V \text{ an oriented } k\text{-dimensional vector subspace of } \mathbb{R}^n,\ \varphi|_V = \mathrm{vol}_V \bigr\}.$$

Then an oriented submanifold N of $\mathbb{R}^n$ is a φ-submanifold if and only if each tangent space T_xN lies in $\mathcal{F}_\varphi$. To be interesting, a calibration φ should define a fairly abundant class of calibrated submanifolds, and this will only happen if $\mathcal{F}_\varphi$ is reasonably large.

Define a *partial order* $\preceq$ on the set of constant calibration k-forms φ on $\mathbb{R}^n$ by $\varphi \preceq \varphi'$ if $\mathcal{F}_\varphi \subseteq \mathcal{F}_{\varphi'}$. A calibration φ is *maximal* if it is maximal with respect to this partial order. A maximal calibration φ is one in which $\mathcal{F}_\varphi$ is as large as possible.

It is an interesting problem to determine the maximal calibrations φ on $\mathbb{R}^n$. The symmetry group $G \subset \mathrm{O}(n)$ of a maximal calibration is usually quite large. This is because if $V \in \mathcal{F}_\varphi$ and $\gamma \in G$ then $\gamma \cdot G \in \mathcal{F}_\varphi$, that is, G acts on $\mathcal{F}_\varphi$. So if G is big we expect $\mathcal{F}_\varphi$ to be big too. Also, symmetry groups of maximal calibrations are often possible holonomy groups of Riemannian metrics.

5.4 Calibrated submanifolds of manifolds with special holonomy

Next we explain the connection with Riemannian holonomy. Let $G \subset \mathrm{O}(n)$ be a possible holonomy group of a Riemannian metric. In particular, we can take G to be one of the holonomy groups $\mathrm{U}(m)$, $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2 or $\mathrm{Spin}(7)$ from Berger's classification. Then G acts on the k-forms $\Lambda^k(\mathbb{R}^n)^*$ on $\mathbb{R}^n$, so we can look for G-invariant k-forms on $\mathbb{R}^n$.

Suppose φ_0 is a nonzero, G-invariant k-form on $\mathbb{R}^n$. By rescaling φ_0 we can arrange that for each oriented k-plane $U \subset \mathbb{R}^n$ we have $\varphi_0|_U \leqslant \mathrm{vol}_U$, and that $\varphi_0|_U = \mathrm{vol}_U$ for at least one such U. Thus $\mathcal{F}_{\varphi_0}$ is nonempty. Since φ_0 is G-invariant, if $U \in \mathcal{F}_{\varphi_0}$ then $\gamma \cdot U \in \mathcal{F}_{\varphi_0}$ for all $\gamma \in G$. Generally this means that $\mathcal{F}_{\varphi_0}$ is 'reasonably large'.

Let M be a manifold of dimension n, and g a metric on M with Levi-Civita connection ∇ and holonomy group G. Then by Theorem 1.3 there is a k-form φ on M with $\nabla\varphi = 0$, corresponding to φ_0. Hence $\mathrm{d}\varphi = 0$, and φ is closed. Also, the condition $\varphi_0|_U \leqslant \mathrm{vol}_U$ for all oriented k-planes U in $\mathbb{R}^n$ implies that $\varphi|_V \leqslant \mathrm{vol}_V$ for all oriented tangent k-planes in M. Thus φ is a *calibration* on M.

At each point $x \in M$ the family of oriented tangent k-planes V with $\varphi|_V = \mathrm{vol}_V$ is isomorphic to $\mathcal{F}_{\varphi_0}$, which is 'reasonably large'. This suggests that locally there should exist many φ-submanifolds N in M, so the calibrated geometry of φ on (M, g) is nontrivial.

This gives us a general method for finding interesting calibrations on manifolds with reduced holonomy. Here are the most important examples of this, which we will look at in detail next lecture.

- Let $G = \mathrm{U}(m) \subset \mathrm{O}(2m)$. Then G preserves a 2-form ω_0 on $\mathbb{R}^{2m}$. If g is a metric on M with holonomy $\mathrm{U}(m)$ then g is *Kähler* with complex structure J, and the 2-form ω on M associated to ω_0 is the *Kähler form* of g.

  One can show that ω is a calibration on (M, g), and the calibrated submanifolds are exactly the *holomorphic curves* in (M, J). More generally $\omega^k/k!$ is a calibration on M for $1 \leqslant k \leqslant m$, and the corresponding calibrated submanifolds are the complex k-dimensional submanifolds of (M, J).

- Let $G = \mathrm{SU}(m) \subset \mathrm{O}(2m)$. Riemannian manifolds (M, g) with holonomy $\mathrm{SU}(m)$ are called *Calabi–Yau manifolds*. A Calabi–Yau manifold comes equipped with a complex m-form Ω called a *holomorphic volume form*. The real part $\mathrm{Re}\,\Omega$ is a calibration on M, and the corresponding calibrated submanifolds are called *special Lagrangian submanifolds*.

- The group $G_2 \subset \mathrm{O}(7)$ preserves a 3-form φ_0 and a 4-form $*\varphi_0$ on $\mathbb{R}^7$. Thus a Riemannian 7-manifold (M, g) with holonomy G_2 comes with a 3-form φ and 4-form $*\varphi$, which are both calibrations. The corresponding calibrated submanifolds are called *associative 3-folds* and *coassociative 4-folds*.

- The group $\mathrm{Spin}(7) \subset \mathrm{O}(8)$ preserves a 4-form Ω_0 on $\mathbb{R}^8$. Thus a Riemannian 8-manifold (M, g) with holonomy $\mathrm{Spin}(7)$ has a 4-form Ω, which is a calibration. We call Ω-submanifolds *Cayley 4-folds*.

It is an important general principle that to each calibration φ on an n-manifold (M, g) with special holonomy we construct in this way, there corresponds a constant calibration φ_0 on $\mathbb{R}^n$. Locally, φ-submanifolds in M will look very like φ_0-submanifolds in $\mathbb{R}^n$, and have many of the same properties. Thus, to understand the calibrated submanifolds in a manifold with special holonomy, it is often a good idea to start by studying the corresponding calibrated submanifolds of $\mathbb{R}^n$.

In particular, singularities of φ-submanifolds in M will be locally modelled on singularities of φ_0-submanifolds in $\mathbb{R}^n$. (Formally, the *tangent cone* at a singular point of a φ-submanifold in M is a conical φ_0-submanifold in $\mathbb{R}^n$.) So by studying singular φ_0-submanifolds in $\mathbb{R}^n$, we may understand the singular behaviour of φ-submanifolds in M.

Reading

Here are some references on calibrated geometry.

R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Mathematica 148 (1982), 47–157, sections I and II.

R. Harvey, *Spinors and calibrations*, Academic Press, San Diego, 1990.

D.D. Joyce, *Compact Manifolds with Special Holonomy*, OUP, 2000, §3.7.

And here is some background reading on minimal submanifolds and geometric measure theory.

H.B. Lawson, *Lectures on Minimal Submanifolds, volume 1*, Publish or Perish, 1980.

F. Morgan, *Geometric Measure Theory, a Beginner's Guide*, Academic Press, San Diego, 1995.

6 Calibrated submanifolds in $\mathbb{R}^n$

6.1 Special Lagrangian submanifolds in $\mathbb{C}^m$

Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$.

Definition 6.1 Let $\mathbb{C}^m \cong \mathbb{R}^{2m}$ have complex coordinates $(z_1, \ldots, z_m)$ and complex structure I, and define a metric g, Kähler form ω and a complex volume form Ω on $\mathbb{C}^m$ by

$$\begin{aligned} g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \\ \text{and} \quad \Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \end{aligned} \tag{13}$$

Then $\operatorname{Re}\Omega$ and $\operatorname{Im}\Omega$ are real m-forms on $\mathbb{C}^m$. Let L be an oriented real submanifold of $\mathbb{C}^m$ of real dimension m. We call L a *special Lagrangian submanifold* or *special Lagrangian m-fold* of $\mathbb{C}^m$ if L is calibrated with respect to $\operatorname{Re}\Omega$, in the sense of §5.2.

In fact there is a more general definition involving a *phase* $\mathrm{e}^{i\theta}$: if $\theta \in [0, 2\pi)$, we say that L is *special Lagrangian with phase* $\mathrm{e}^{i\theta}$ if it is calibrated with respect to $\cos\theta \operatorname{Re}\Omega + \sin\theta \operatorname{Im}\Omega$. But we will not use this.

We shall identify the family $\mathcal{F}$ of tangent m-planes in $\mathbb{C}^m$ calibrated with respect to $\operatorname{Re}\Omega$. The subgroup of $\mathrm{GL}(2m, \mathbb{R})$ preserving g, ω and Ω is the Lie group $\mathrm{SU}(m)$ of complex unitary matrices with determinant 1. Define a real vector subspace U in $\mathbb{C}^m$ to be

$$U = \big\{(x_1, \ldots, x_m) : x_j \in \mathbb{R}\big\} \subset \mathbb{C}^m, \tag{14}$$

and let U have the usual orientation. Then U is calibrated w.r.t. $\operatorname{Re}\Omega$.

Furthermore, any oriented real vector subspace V in $\mathbb{C}^m$ calibrated w.r.t. $\operatorname{Re}\Omega$ is of the form $V = \gamma \cdot U$ for some $\gamma \in \mathrm{SU}(m)$. Therefore $\mathrm{SU}(m)$ acts transitively on $\mathcal{F}$. The stabilizer subgroup of U in $\mathrm{SU}(m)$ is the subset of matrices in $\mathrm{SU}(m)$ with real entries, which is $\mathrm{SO}(m)$. Thus $\mathcal{F} \cong \mathrm{SU}(m)/\mathrm{SO}(m)$, and we prove:

Proposition 6.2 *The family $\mathcal{F}$ of oriented real m-dimensional vector subspaces V in $\mathbb{C}^m$ with $\operatorname{Re}\Omega|_V = \mathrm{vol}_V$ is isomorphic to $\mathrm{SU}(m)/\mathrm{SO}(m)$, and has dimension $\frac{1}{2}(m^2 + m - 2)$.*

The dimension follows because $\dim \mathrm{SU}(m) = m^2 - 1$ and $\dim \mathrm{SO}(m) = \frac{1}{2}m(m-1)$. It is easy to see that $\omega|_U = \operatorname{Im}\Omega|_U = 0$. As $\mathrm{SU}(m)$ preserves ω and $\operatorname{Im}\Omega$ and acts transitively on $\mathcal{F}$, it follows that $\omega|_V = \operatorname{Im}\Omega|_V = 0$ for any $V \in \mathcal{F}$. Conversely, if V is a real m-dimensional vector subspace of $\mathbb{C}^m$ and $\omega|_V = \operatorname{Im}\Omega|_V = 0$, then V lies in $\mathcal{F}$, with some orientation. This implies an alternative characterization of special Lagrangian submanifolds:

Proposition 6.3 *Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into a special Lagrangian submanifold of $\mathbb{C}^m$ if and only if $\omega|_L \equiv 0$ and $\operatorname{Im}\Omega|_L \equiv 0$.*

Note that an m-dimensional submanifold L in $\mathbb{C}^m$ is called *Lagrangian* if $\omega|_L \equiv 0$. (This is a term from symplectic geometry, and ω is a symplectic structure.) Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $\operatorname{Im}\Omega|_L \equiv 0$, which is how they get their name.

6.2 Special Lagrangian 2-folds in $\mathbb{C}^2$ and the quaternions

The smallest interesting dimension, $m = 2$, is a special case. Let $\mathbb{C}^2$ have complex coordinates (z_1, z_2), complex structure I, and metric g, Kähler form ω and holomorphic 2-form Ω defined in (13). Define real coordinates (x_0, x_1, x_2, x_3) on $\mathbb{C}^2 \cong \mathbb{R}^4$ by $z_0 = x_0 + ix_1$, $z_1 = x_2 + ix_3$. Then

$$g = \mathrm{d}x_0^2 + \cdots + \mathrm{d}x_3^2, \qquad \omega = \mathrm{d}x_0 \wedge \mathrm{d}x_1 + \mathrm{d}x_2 \wedge \mathrm{d}x_3,$$
$$\operatorname{Re}\Omega = \mathrm{d}x_0 \wedge \mathrm{d}x_2 - \mathrm{d}x_1 \wedge \mathrm{d}x_3 \quad \text{and} \quad \operatorname{Im}\Omega = \mathrm{d}x_0 \wedge \mathrm{d}x_3 + \mathrm{d}x_1 \wedge \mathrm{d}x_2.$$

Now define a *different* set of complex coordinates (w_1, w_2) on $\mathbb{C}^2 = \mathbb{R}^4$ by $w_1 = x_0 + ix_2$, $w_2 = -x_1 + ix_3$. Then $\omega + i\operatorname{Im}\Omega = \mathrm{d}w_1 \wedge \mathrm{d}w_2$.

But by Proposition 6.3, a real 2-submanifold $L \subset \mathbb{R}^4$ is special Lagrangian if and only if $\omega|_L \equiv \operatorname{Im}\Omega|_L \equiv 0$. Thus, L is special Lagrangian if and only if $(\mathrm{d}w_1 \wedge \mathrm{d}w_2)|_L \equiv 0$. But this holds if and only if L is a *holomorphic curve* with respect to the complex coordinates (w_1, w_2).

Here is another way to say this. There are *two different* complex structures I and J involved in this problem, associated to the two different complex coordinate systems (z_1, z_2) and (w_1, w_2) on $\mathbb{R}^4$. In the coordinates $(x_0, \ldots, x_3)$, I and J are given by

$$I\bigl(\tfrac{\partial}{\partial x_0}\bigr) = \tfrac{\partial}{\partial x_1}, \; I\bigl(\tfrac{\partial}{\partial x_1}\bigr) = -\tfrac{\partial}{\partial x_0}, \; I\bigl(\tfrac{\partial}{\partial x_2}\bigr) = \tfrac{\partial}{\partial x_3}, \quad I\bigl(\tfrac{\partial}{\partial x_3}\bigr) = -\tfrac{\partial}{\partial x_2},$$
$$J\bigl(\tfrac{\partial}{\partial x_0}\bigr) = \tfrac{\partial}{\partial x_2}, \; J\bigl(\tfrac{\partial}{\partial x_1}\bigr) = -\tfrac{\partial}{\partial x_3}, \; J\bigl(\tfrac{\partial}{\partial x_2}\bigr) = -\tfrac{\partial}{\partial x_0}, \; J\bigl(\tfrac{\partial}{\partial x_3}\bigr) = \tfrac{\partial}{\partial x_1}.$$

The usual complex structure on $\mathbb{C}^2$ is I, but a 2-fold L in $\mathbb{C}^2$ is special Lagrangian if and only if it is holomorphic w.r.t. the alternative complex structure J. This means that special Lagrangian 2-folds are already very well understood, so we generally focus our attention on dimensions $m \geqslant 3$.

We can explain all this in terms of the *quaternions* $\mathbb{H}$, which were mentioned briefly in §3.6. The complex structures I, J anticommute, so that $IJ = -JI$, and $K = IJ$ is also a complex structure on $\mathbb{R}^4$, and $\langle 1, I, J, K\rangle$ is an algebra of automorphisms of $\mathbb{R}^4$ isomorphic to $\mathbb{H}$.

6.3 Special Lagrangian submanifolds in $\mathbb{C}^m$ as graphs

In symplectic geometry, there is a well-known way of manufacturing *Lagrangian* submanifolds of $\mathbb{R}^{2m} \cong \mathbb{C}^m$, which works as follows. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function, and define

$$\Gamma_f = \bigl\{\bigl(x_1 + i\tfrac{\partial f}{\partial x_1}(x_1, \ldots, x_m), \ldots, x_m + i\tfrac{\partial f}{\partial x_m}(x_1, \ldots, x_m)\bigr) : x_1, \ldots, x_m \in \mathbb{R}\bigr\}. \tag{15}$$

Then Γ_f is a smooth real m-dimensional submanifold of $\mathbb{C}^m$, with $\omega|_{\Gamma_f} \equiv 0$. Identifying $\mathbb{C}^m \cong \mathbb{R}^{2m} \cong \mathbb{R}^m \times (\mathbb{R}^m)^*$, we may regard Γ_f as the graph of the 1-form $\mathrm{d}f$ on $\mathbb{R}^m$, so that Γ_f is the *graph of a closed* 1*-form*. Locally, but not globally, every Lagrangian submanifold arises from this construction.

Now by Proposition 6.3, a special Lagrangian m-fold in $\mathbb{C}^m$ is a Lagrangian m-fold L satisfying the additional condition that $\mathrm{Im}\,\Omega|_L \equiv 0$. We shall find the condition for Γ_f to be a special Lagrangian m-fold. Define the *Hessian* $\mathrm{Hess}\, f$ of f to be the $m \times m$ matrix $\bigl(\frac{\partial^2 f}{\partial x_i \partial x_j}\bigr)_{i,j=1}^m$ of real functions on $\mathbb{R}^m$. Then it is easy to show that $\mathrm{Im}\,\Omega|_{\Gamma_f} \equiv 0$ if and only if

$$\mathrm{Im}\,\det_{\mathbb{C}}\bigl(I + i\,\mathrm{Hess}\, f\bigr) \equiv 0 \quad \text{on } \mathbb{C}^m. \tag{16}$$

This is a *nonlinear second-order elliptic partial differential equation* upon the function $f : \mathbb{R}^m \to \mathbb{R}$.

6.4 Local discussion of deformations of special Lagrangian submanifolds

Suppose L_0 is a special Lagrangian submanifold in $\mathbb{C}^m$ (or, more generally, in some Calabi–Yau m-fold). What can we say about the family of *special Lagrangian deformations* of L_0, that is, the set of special Lagrangian m-folds L that are 'close to L_0' in a suitable sense? Essentially, deformation theory is one way of thinking about the question 'how many special Lagrangian submanifolds are there in $\mathbb{C}^m$'?

Locally (that is, in small enough open sets), every special Lagrangian m-fold looks quite like $\mathbb{R}^m$ in $\mathbb{C}^m$. Therefore deformations of special Lagrangian m-folds should look like special Lagrangian deformations of $\mathbb{R}^m$ in $\mathbb{C}^m$. So, we would like to know what special Lagrangian m-folds L in $\mathbb{C}^m$ close to $\mathbb{R}^m$ look like.

Now $\mathbb{R}^m$ is the graph Γ_f of the function $f \equiv 0$. Thus, a graph Γ_f will be close to $\mathbb{R}^m$ if the function f and its derivatives are small. But then $\mathrm{Hess}\, f$ is small, so we can approximate equation (16) by its *linearization*. For

$$\mathrm{Im}\,\det_{\mathbb{C}}\bigl(I + i\,\mathrm{Hess}\, f\bigr) = \mathrm{Tr}\,\mathrm{Hess}\, f + \text{higher order terms}.$$

Thus, when the second derivatives of f are small, equation (16) reduces approximately to $\mathrm{Tr}\,\mathrm{Hess}\, f \equiv 0$. But $\mathrm{Tr}\,\mathrm{Hess}\, f = \frac{\partial^2 f}{(\partial x_1)^2} + \cdots + \frac{\partial^2 f}{(\partial x_m)^2} = \Delta f$, where Δ is the *Laplacian* on $\mathbb{R}^m$.

Hence, the small special Lagrangian deformations of $\mathbb{R}^m$ in $\mathbb{C}^m$ are approximately parametrized by small *harmonic functions* on $\mathbb{R}^m$. Actually, because adding a constant to f has no effect on Γ_f, this parametrization is degenerate. We can get round this by parametrizing instead by $\mathrm{d}f$, which is a closed and coclosed 1-form. This justifies the following:

Principle. *Small special Lagrangian deformations of a special Lagrangian m-fold L are approximately parametrized by closed and coclosed* 1*-forms α on L.*

This is the idea behind McLean's Theorem, which we will discuss next lecture, Theorem 7.1.

We have seen using (16) that the deformation problem for special Lagrangian m-folds can be written as an *elliptic equation*. In particular, there are the same number of equations as functions, so the problem is overdetermined nor underdetermined. Therefore we do not expect special Lagrangian m-folds to be very few and very rigid (as would be the case if (16) were overdetermined), nor to be very abundant and very flabby (as would be the case if (16) were underdetermined).

If we think about Proposition 6.2 for a while, this may seem surprising. For the set $\mathcal{F}$ of special Lagrangian m-planes in $\mathbb{C}^m$ has dimension $\frac{1}{2}(m^2+m-2)$, but the set of all real m-planes in $\mathbb{C}^m$ has dimension m^2. So the special Lagrangian m-planes have codimension $\frac{1}{2}(m^2-m+2)$ in the set of all m-planes.

This means that the condition for a real m-submanifold L in $\mathbb{C}^m$ to be special Lagrangian is $\frac{1}{2}(m^2-m+2)$ real equations on each tangent space of L. However, the freedom to vary L is the sections of its normal bundle in $\mathbb{C}^m$, which is m real functions. When $m \geqslant 3$, there are more equations than functions, so we would expect the deformation problem to be *overdetermined*.

The explanation is that because ω is a *closed* 2-form, submanifolds L with $\omega|_L \equiv 0$ are much more abundant than would otherwise be the case. So the closure of ω is a kind of integrability condition necessary for the existence of many special Lagrangian submanifolds, just as the integrability of an almost complex structure is a necessary condition for the existence of many complex submanifolds in a complex submanifold.

6.5 Associative and coassociative submanifolds of $\mathbb{R}^7$

Let $(x_1, \dots, x_7)$ be coordinates on $\mathbb{R}^7$. Write $\mathrm{d}\mathbf{x}_{ij\dots l}$ for the exterior form $\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \cdots \wedge \mathrm{d}x_l$ on $\mathbb{R}^7$. Define a 3-form φ on $\mathbb{R}^7$ by

$$\varphi = \mathrm{d}\mathbf{x}_{123} + \mathrm{d}\mathbf{x}_{145} + \mathrm{d}\mathbf{x}_{167} + \mathrm{d}\mathbf{x}_{246} - \mathrm{d}\mathbf{x}_{257} - \mathrm{d}\mathbf{x}_{347} - \mathrm{d}\mathbf{x}_{356}. \quad (17)$$

The subgroup of $\mathrm{GL}(7,\mathbb{R})$ preserving φ is the *exceptional Lie group* G_2, one of the exceptional cases G_2 and Spin(7) in Berger's classification of Riemannian holonomy groups in §2.3. It is compact, connected, simply-connected, semisimple and 14-dimensional, and it also fixes the 4-form

$$*\varphi = \mathrm{d}\mathbf{x}_{4567} + \mathrm{d}\mathbf{x}_{2367} + \mathrm{d}\mathbf{x}_{2345} + \mathrm{d}\mathbf{x}_{1357} - \mathrm{d}\mathbf{x}_{1346} - \mathrm{d}\mathbf{x}_{1256} - \mathrm{d}\mathbf{x}_{1247}, \quad (18)$$

the Euclidean metric $g = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_7^2$, and the orientation on $\mathbb{R}^7$. Note that φ and $*\varphi$ are related by the *Hodge star*.

Now φ and $*\varphi$ are both calibrations on $\mathbb{R}^7$. We define an *associative 3-fold* in $\mathbb{R}^7$ to be a 3-dimensional submanifold of $\mathbb{R}^7$ calibrated with respect to φ, and a *coassociative 4-fold* in $\mathbb{R}^7$ to be a 4-dimensional submanifold of $\mathbb{R}^7$ calibrated with respect to $*\varphi$.

Define an *associative 3-plane* to be an oriented 3-dimensional vector subspace V of $\mathbb{R}^7$ with $\varphi|_V = \mathrm{vol}_V$, and a *coassociative 4-plane* to be an oriented 4-

dimensional vector subspace V of $\mathbb{R}^7$ with $*\varphi|_V = \mathrm{vol}_V$. By analogy with Proposition 6.2, we can prove:

Proposition 6.4 *The family $\mathcal{F}^3$ of associative 3-planes in $\mathbb{R}^7$ and the family $\mathcal{F}^4$ of coassociative 4-planes in $\mathbb{R}^7$ are both isomorphic to $G_2/SO(4)$, with dimension 8.*

There is also the following analogue of Proposition 6.3 for coassociative 4-folds:

Proposition 6.5 *Let L be a real 4-dimensional submanifold of $\mathbb{R}^7$. Then L admits an orientation making it into a coassociative 4-fold of $\mathbb{R}^7$ if and only if $\varphi|_L \equiv 0$.*

The set of all 3-planes in $\mathbb{R}^7$ has dimension 12, and the set of associative 3-planes in $\mathbb{R}^7$ has dimension 8 by Proposition 6.4. Thus the associative 3-planes are of codimension 4 in the set of all 3-planes. This means that the condition for a 3-fold L in $\mathbb{R}^7$ to be associative is 4 equations on each tangent space. The freedom to vary L is the sections of its normal bundle in $\mathbb{R}^7$, which is 4 real functions.

Thus, the deformation problem for associative 3-folds involves 4 equations on 4 functions, so it is a *determined* problem. In fact, the relevant equation is *elliptic*, essentially the Dirac equation on L. This implies that the deformation theory of associative 3-folds is quite well-behaved.

For coassociative 4-folds, the deformation problem has 4 equations on 3 real functions, which is apparently overdetermined. But because φ is closed, we can rewrite the problem as an elliptic equation, as we did for special Lagrangian m-folds. So, the closure of φ can be seen as an integrability condition for the existence of many coassociative 4-folds.

6.6 Cayley 4-folds in $\mathbb{R}^8$

Let $\mathbb{R}^8$ have coordinates $(x_1, \ldots, x_8)$. Write $\mathrm{d}\mathbf{x}_{ijkl}$ for the 4-form $\mathrm{d}x_i \wedge \mathrm{d}x_j \wedge \mathrm{d}x_k \wedge \mathrm{d}x_l$ on $\mathbb{R}^8$. Define a 4-form Ω on $\mathbb{R}^8$ by

$$\begin{aligned}\Omega =&\mathrm{d}\mathbf{x}_{1234} + \mathrm{d}\mathbf{x}_{1256} + \mathrm{d}\mathbf{x}_{1278} + \mathrm{d}\mathbf{x}_{1357} - \mathrm{d}\mathbf{x}_{1368} - \mathrm{d}\mathbf{x}_{1458} - \mathrm{d}\mathbf{x}_{1467} \\ &-\mathrm{d}\mathbf{x}_{2358} - \mathrm{d}\mathbf{x}_{2367} - \mathrm{d}\mathbf{x}_{2457} + \mathrm{d}\mathbf{x}_{2468} + \mathrm{d}\mathbf{x}_{3456} + \mathrm{d}\mathbf{x}_{3478} + \mathrm{d}\mathbf{x}_{5678}.\end{aligned} \tag{19}$$

Then Ω is a calibration on $\mathbb{R}^8$. Submanifolds of $\mathbb{R}^8$ calibrated with respect to Ω are called *Cayley 4-folds*.

The subgroup of $\mathrm{GL}(8, \mathbb{R})$ preserving Ω is the *exceptional holonomy group* $\mathrm{Spin}(7)$. It is a compact, 21-dimensional Lie group, isomorphic to the double cover of $\mathrm{SO}(7)$. It also preserves the orientation on $\mathbb{R}^8$ and the Euclidean metric $g = \mathrm{d}x_1^2 + \cdots + \mathrm{d}x_8^2$ on $\mathbb{R}^8$.

Define a *Cayley 4-plane* to be an oriented 4-dimensional vector subspace V of $\mathbb{R}^8$ with $\Omega|_V = \mathrm{vol}_V$. Then one can prove the following analogue of Propositions 6.2 and 6.4.

Proposition 6.6 *The family $\mathcal{F}$ of Cayley 4-planes in $\mathbb{R}^8$ is isomorphic to* $\mathrm{Spin}(7)/K$*, where* $K \cong \big(\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\big)/\mathbb{Z}_2$ *is a Lie subgroup of* $\mathrm{Spin}(7)$*. The dimension of $\mathcal{F}$ is* 12*.*

The set of all 4-planes in $\mathbb{R}^8$ has dimension 16, so that the Cayley 4-planes have codimension 4. Thus, the deformation problem for a Cayley 4-fold L may be written as 4 real equations on 4 real functions, a determined problem. In fact this is an elliptic equation, essentially the positive Dirac equation upon L. Therefore the deformation theory of Cayley 4-folds is quite well-behaved.

Reading

R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Mathematica 148 (1982), 47–157, sections III and IV.

For the associative, coassociative and Cayley material only:

D.D. Joyce, *Compact Manifolds with Special Holonomy*, OUP, 2000, §§10.1, 10.5, 10.8.

We haven't had time to look at examples of special Lagrangian m-folds in $\mathbb{C}^m$. You can find examples in Harvey and Lawson, and also:

D.D. Joyce, *Special Lagrangian m-folds in $\mathbb{C}^m$ with symmetries*, Duke Mathematical Journal 115 (2002), 1–51. math.DG/0008021.

and other papers in the same series — see the references in this paper.

Exercises

1. The metric g and Kähler form ω on $\mathbb{C}^m$ are given by

$$g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2 \quad \text{and} \quad \omega = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m).$$

Show that a tangent 2-plane in $\mathbb{C}^m$ is calibrated w.r.t. ω if and only if it is a complex line in $\mathbb{C}^m$. (Harder) generalize to tangent $2k$-planes and $\frac{1}{k!}\,\omega^k$.

2. The group of automorphisms of $\mathbb{C}^m$ preserving g, ω and Ω is $\mathrm{SU}(m) \ltimes \mathbb{C}^m$, where $\mathbb{C}^m$ acts by translations. Let G be a Lie subgroup of $\mathrm{SU}(m) \ltimes \mathbb{C}^m$, let $\mathfrak{g}$ be its Lie algebra, and let $\phi : \mathfrak{g} \rightarrow \mathrm{Vect}(\mathbb{C}^m)$ be the natural map associating an element of $\mathfrak{g}$ to the corresponding vector field on $\mathbb{C}^m$.

A *moment map* for the action of G on $\mathbb{C}^m$ is a smooth map $\mu : \mathbb{C}^m \rightarrow \mathfrak{g}^*$, such that $\phi(x) \cdot \omega = x \cdot \mathrm{d}\mu$ for all $x \in \mathfrak{g}$, and $\mu : \mathbb{C}^m \rightarrow \mathfrak{g}^*$ is equivariant with respect to the G-action on $\mathbb{C}^m$ and the coadjoint G-action on $\mathfrak{g}^*$. Moment maps always exist if G is compact or semisimple, and are unique up to the addition of a constant in the centre $Z(\mathfrak{g}^*)$ of $\mathfrak{g}^*$, that is, the G-invariant subspace of $\mathfrak{g}^*$.

Suppose L is a (special) Lagrangian m-fold in $\mathbb{C}^m$ invariant under a Lie subgroup G in $\mathrm{SU}(m) \ltimes \mathbb{C}^m$, with moment map μ. Show that $\mu \equiv c$ on L for some $c \in Z(\mathfrak{g}^*)$.

3. Define a smooth map $f : \mathbb{C}^3 \rightarrow \mathbb{R}^3$ by

$$f(z_1, z_2, z_3) = \bigl(|z_1|^2 - |z_3|^2, |z_2|^2 - |z_3|^2, \mathrm{Im}(z_1 z_2 z_3)\bigr).$$

For each $a, b, c \in \mathbb{R}^3$, define $N_{a,b,c} = f^{-1}(a, b, c)$. Then $N_{a,b,c}$ is a real 3-dimensional submanifold of $\mathbb{C}^3$, which may be singular.

(i) At $\mathbf{z} = (z_1, z_2, z_3) \in \mathbb{C}^3$, determine $\mathrm{d}f|_{\mathbf{z}} : \mathbb{C}^3 \rightarrow \mathbb{R}^3$. Find the conditions on $\mathbf{z}$ for $\mathrm{d}f|_{\mathbf{z}}$ to be surjective.

Now $N_{a,b,c}$ is nonsingular at $\mathbf{z} \in N_{a,b,c}$ if and only if $\mathrm{d}f|_{\mathbf{z}}$ is surjective. Hence determine which of the $N_{a,b,c}$ are singular, and find their singular points.

(ii) If $\mathbf{z}$ is a nonsingular point of $N_{a,b,c}$, then $T_{\mathbf{z}} N_{a,b,c} = \mathrm{Ker}\, \mathrm{d}f|_{\mathbf{z}}$. Determine $\mathrm{Ker}\, \mathrm{d}f|_{\mathbf{z}}$ in this case, and show that it is a special Lagrangian 3-plane in $\mathbb{C}^3$.

Hence prove that $N_{a,b,c}$ is a special Lagrangian 3-fold wherever it is nonsingular, and that $f : \mathbb{C}^3 \rightarrow \mathbb{R}^3$ is a *special Lagrangian fibration.*

(iii) Observe that $N_{a,b,c}$ is invariant under the Lie group $G = \mathrm{U}(1)^2$, acting by

$$(\mathrm{e}^{i\theta_1}, \mathrm{e}^{i\theta_2}) : (z_1, z_2, z_3) \mapsto (\mathrm{e}^{i\theta_1} z_1, \mathrm{e}^{i\theta_2} z_2, \mathrm{e}^{-i\theta_1 - i\theta_2} z_3).$$

How is the form of f related to the ideas of question 2? How might G-invariance have been used to construct the fibration f?

(iv) Describe the topology of $N_{a,b,c}$, distinguishing different cases according to the singularities.

7 7. Compact calibrated submanifolds in manifolds of special holonomy

Let (M, J, g) be a Calabi–Yau manifold of complex dimension m with Kähler form ω, and choose a holomorphic volume form Ω on M, normalized to satisfy (11). We will regard Ω as part of the Calabi–Yau structure, so that the Calabi–Yau manifold is (M, J, g, Ω). Then $\mathrm{Re}\,\Omega$ is a *calibration* on M. An oriented real m-dimensional submanifold N in M is called a *special Lagrangian submanifold* if it is calibrated with respect to $\mathrm{Re}\,\Omega$.

In this lecture we shall discuss *compact* special Lagrangian submanifolds in Calabi–Yau manifolds, and also compact calibrated submanifolds in manifolds with exceptional holonomy. Here are three important questions which motivate work in this area.

- **Question 1.** Let N be a compact special Lagrangian m-fold in a Calabi–Yau m-fold (M, J, g, Ω). Let $\mathscr{M}_N$ be the moduli space of *special Lagrangian deformations* of N, that is, the connected component of the set of special Lagrangian m-folds containing N. What can we say about $\mathscr{M}_N$? For instance, is it a smooth manifold, and of what dimension?

- **Question 2.** Let $\{(M, J_t, g_t, \Omega_t) : t \in (-\epsilon, \epsilon)\}$ be a smooth 1-parameter family of Calabi–Yau m-folds. Suppose N_0 is a special Lagrangian m-fold of (M, J_0, g_0, Ω_0). Under what conditions can we extend N_0 to a smooth family of special Lagrangian m-folds N_t in (M, J_t, g_t, Ω_t) for $t \in (-\epsilon, \epsilon)$?

- **Question 3.** In general the moduli space $\mathscr{M}_N$ in Question 1 will be noncompact. Can we enlarge $\mathscr{M}_N$ to a compact space $\overline{\mathscr{M}}_N$ by adding a 'boundary' consisting of *singular* special Lagrangian m-folds? If so, what is the nature of the singularities that develop?

Briefly, these questions concern the *deformations* of special Lagrangian m-folds, *obstructions* to their existence, and their *singularities* respectively. The local answers to Questions 1 and 2 are fairly well understood, and we shall discuss them in this lecture. Question 3 is an active area of research at the moment, and we will touch on it next lecture.

7.1 Deformations of compact special Lagrangian submanifolds

The deformation theory of special Lagrangian submanifolds was studied by McLean, who proved the following result.

Theorem 7.1 *Let (M, J, g, Ω) be a Calabi–Yau m-fold, and N a compact special Lagrangian m-fold in M. Then the moduli space $\mathscr{M}_N$ of special Lagrangian deformations of N is a smooth manifold of dimension $b^1(N)$, the first Betti number of N.*

Sketch proof. Suppose for simplicity that N is an embedded submanifold. There is a natural orthogonal decomposition $TM|_N = TN \oplus \nu$, where $\nu \to N$ is the *normal bundle* of N in M. As N is Lagrangian, the complex structure $J : TM \to TM$ gives an isomorphism $J : \nu \to TN$. But the metric g gives an isomorphism $TN \cong T^*N$. Composing these two gives an isomorphism $\nu \cong T^*N$.

Let T be a small *tubular neighbourhood* of N in M. Then we can identify T with a neighbourhood of the zero section in ν. Using the isomorphism $\nu \cong T^*N$, we have an identification between T and a neighbourhood of the zero section in T^*N. This can be chosen to identify the Kähler form ω on T with the natural symplectic structure on T^*N. Let $\pi : T \to N$ be the obvious projection.

Under this identification, submanifolds N' in $T \subset M$ which are C^1 close to N are identified with the graphs of small smooth sections α of T^*N. That is, submanifolds N' of M close to N are identified with 1-*forms* α on N. We need to know: which 1-forms α are identified with *special Lagrangian* submanifolds N'?

Well, N' is special Lagrangian if $\omega|_{N'} \equiv \operatorname{Im}\Omega|_{N'} \equiv 0$. Now $\pi|_{N'} : N' \to N$ is a diffeomorphism, so we can push $\omega|_{N'}$ and $\operatorname{Im}\Omega|_{N'}$ down to N, and regard them as functions of α. Calculation shows that

$$\pi_*\bigl(\omega|_{N'}\bigr) = \mathrm{d}\alpha \quad \text{and} \quad \pi_*\bigl(\operatorname{Im}\Omega|_{N'}\bigr) = F(\alpha, \nabla\alpha),$$

where F is a nonlinear function of its arguments. Thus, the moduli space $\mathscr{M}_N$ is locally isomorphic to the set of small 1-forms α on N such that $\mathrm{d}\alpha \equiv 0$ and $F(\alpha, \nabla\alpha) \equiv 0$.

Now it turns out that F satisfies $F(\alpha, \nabla\alpha) \approx \mathrm{d}(*\alpha)$ when α is small. Therefore $\mathcal{M}_N$ is locally approximately isomorphic to the vector space of 1-forms α with $\mathrm{d}\alpha = \mathrm{d}(*\alpha) = 0$. But by Hodge theory, this is isomorphic to the de Rham cohomology group $H^1(N, \mathbb{R})$, and is a manifold with dimension $b^1(N)$.

To carry out this last step rigorously requires some technical machinery: one must work with certain *Banach spaces* of sections of T^*N, $\Lambda^2 T^*N$ and $\Lambda^m T^*N$, use *elliptic regularity results* to prove that the map $\alpha \mapsto \bigl(\mathrm{d}\alpha, F(\alpha, \nabla\alpha)\bigr)$ has *closed image* in these Banach spaces, and then use the *Implicit Function Theorem for Banach spaces* to show that the kernel of the map is what we expect. □

7.2 Natural coordinates on the moduli space $\mathscr{M}_N$

Suppose N is a compact special Lagrangian m-fold in a Calabi–Yau m-fold (M, J, g, Ω). Theorem 7.1 shows that the moduli space $\mathscr{M}_N$ has dimension $b^1(N)$. By Poincaré duality $b^1(N) = b^{m-1}(N)$. Thus $\mathscr{M}_N$ has the same dimension as the de Rham cohomology groups $H^1(M, \mathbb{R})$ and $H^{m-1}(M, \mathbb{R})$. We shall construct natural local identifications Φ between $\mathscr{M}_N$ and $H^1(N, \mathbb{R})$, and Ψ between $\mathscr{M}_N$ and $H^{m-1}(N, \mathbb{R})$. These induce two natural *affine structures* on $\mathscr{M}_N$, and can be thought of as two *natural coordinate systems* on $\mathscr{M}_N$.

Here is how to define Φ and Ψ. Let U be a connected and simply-connected open neighbourhood of N in $\mathscr{M}_N$. We will construct smooth maps $\Phi : U \to$

$H^1(N,\mathbb{R})$ and $\Psi : U \to H^{m-1}(N,\mathbb{R})$ with $\Phi(N) = \Psi(N) = 0$, which are local diffeomorphisms.

Let $N' \in U$. Then as U is connected, there exists a smooth path $\gamma : [0,1] \to U$ with $\gamma(0) = N$ and $\gamma(1) = N'$, and as U is simply-connected, γ is unique up to isotopy. Now γ parametrizes a family of submanifolds of M diffeomorphic to N, which we can lift to a smooth map $\Gamma : N \times [0,1] \to M$ with $\Gamma(N \times \{t\}) = \gamma(t)$.

Consider the 2-form $\Gamma^*(\omega)$ on $N \times [0,1]$. As each fibre $\gamma(t)$ is Lagrangian, we have $\Gamma^*(\omega)|_{N\times\{t\}} \equiv 0$ for each $t \in [0,1]$. Therefore we may write $\Gamma^*(\omega) = \alpha_t \wedge \mathrm{d}t$, where α_t is a closed 1-form on N for $t \in [0,1]$. Define $\Phi(N') = \left[\int_0^1 \alpha_t \,\mathrm{d}t\right] \in H^1(N,\mathbb{R})$. That is, we integrate the 1-forms α_t with respect to t to get a closed 1-form $\int_0^1 \alpha_t \,\mathrm{d}t$, and then take its cohomology class.

Similarly, write $\Gamma^*(\operatorname{Im}\Omega) = \beta_t \wedge \mathrm{d}t$, where β_t is a closed $(m-1)$-form on N for $t \in [0,1]$, and define $\Psi(N') = \left[\int_0^1 \beta_t \,\mathrm{d}t\right] \in H^{m-1}(N,\mathbb{R})$. Then Φ and Ψ are independent of choices made in the construction (exercise). We need to restrict to a simply-connected subset U of $\mathscr{M}_N$ so that γ is unique up to isotopy. Alternatively, one can define Φ and Ψ on the universal cover $\widetilde{\mathscr{M}}_N$ of $\mathscr{M}_N$.

7.3 Obstructions to the existence of compact special Lagrangian submanifolds

Let N be a compact real m-submanifold in a Calabi–Yau m-fold (M,J,g,Ω). Then N is special Lagrangian if $\omega|_N \equiv \operatorname{Im}\Omega|_N = 0$. Thus, a necessary condition for N to be special Lagrangian is that $[\omega|_N] = 0$ in $H^2(N,\mathbb{R})$, and $[\operatorname{Im}\Omega|_N] = 0$ in $H^m(N,\mathbb{R})$.

Regard N as an immersed submanifold, with immersion $\iota : N \to M$. Then $[\omega|_N]$ and $[\operatorname{Im}\Omega|_N]$ are unchanged under continuous variations of the immersion ι. Thus, $[\omega|_N] = [\operatorname{Im}\Omega|_N] = 0$ is a necessary condition not just for N to be special Lagrangian, but also for any isotopic submanifold N' in M to be special Lagrangian. This proves:

Lemma 7.2 *Let (M,J,g,Ω) be a Calabi–Yau m-fold, and N a compact real m-submanifold in M. Then a necessary condition for N to be isotopic to a special Lagrangian submanifold N' in M is that $[\omega|_N] = 0$ in $H^2(N,\mathbb{R})$ and $[\operatorname{Im}\Omega|_N] = 0$ in $H^m(N,\mathbb{R})$.*

This gives a simple, necessary topological condition for an isotopy class of m-submanifolds of a Calabi–Yau m-fold to contain a special Lagrangian submanifold.

Next we address Question 2 above. Let $\{(M,J_t,g_t,\Omega_t) : t \in (-\epsilon,\epsilon)\}$ be a smooth 1-parameter family of Calabi–Yau m-folds. Suppose N_0 is a special Lagrangian m-fold of (M,J_0,g_0,Ω_0). When can we extend N_0 to a smooth family of special Lagrangian m-folds N_t in (M,J_t,g_t,Ω_t) for $t \in (-\epsilon,\epsilon)$?

By Lemma 7.2, a necessary condition is that $[\omega_t|_{N_0}] = [\operatorname{Im}\Omega_t|_{N_0}] = 0$ for all t. Our next result shows that locally, this is also a *sufficient* condition.

Theorem 7.3 *Let $\{(M, J_t, g_t, \Omega_t) : t \in (-\epsilon, \epsilon)\}$ be a smooth 1-parameter family of Calabi–Yau m-folds. Let N_0 be a compact special Lagrangian m-fold of (M, J_0, g_0, Ω_0), and suppose that $[\omega_t|_{N_0}] = 0$ in $H^2(N_0, \mathbb{R})$ and $[\operatorname{Im}\Omega_t|_{N_0}] = 0$ in $H^m(N_0, \mathbb{R})$ for all $t \in (-\epsilon, \epsilon)$. Then N_0 extends to a smooth 1-parameter family $\{N_t : t \in (-\delta, \delta)\}$, where $0 < \delta \leqslant \epsilon$ and N_t is a compact special Lagrangian m-fold of (M, J_t, g_t, Ω_t).*

This can be proved using similar techniques to Theorem 7.1, though McLean did not prove it.

Remark. Recall from §6.1 that we can generalize the definition of special Lagrangian m-folds as follows. For $\theta \in [0, 2\pi)$, define a *special Lagrangian m-fold with phase* $e^{i\theta}$ to be calibrated with respect to $\cos\theta \operatorname{Re}\Omega + \sin\theta \operatorname{Im}\Omega$. This is in many ways a better and more useful definition.

Then if N is a compact special Lagrangian m-fold with phase $e^{i\theta}$, it is easy to show that $[\Omega] \cdot [N] = \operatorname{Vol}(N)e^{i\theta}$, where $[\Omega] \in H^m(M, \mathbb{C})$ and $[N] \in H_m(M, \mathbb{Z})$. Therefore, *the homology class $[N] \in H_m(M, \mathbb{Z})$ determines the phase $e^{i\theta}$ and volume $\operatorname{Vol}(N)$ of N uniquely.*

Because of this, if we adopt the 'phase $e^{i\theta}$' definition of special Lagrangian submanifolds, then $[\operatorname{Im}\Omega|_N] = 0$ is no longer a necessary condition for N to be isotopic to a special Lagrangian submanifold. In Theorem 7.3 we can drop the condition $[\operatorname{Im}\Omega_t|_{N_0}] = 0$, and instead introduce a family of phases $e^{i\theta_t}$ determined by $[\Omega_t] \cdot [N_0] = \operatorname{Vol}(N_t)e^{i\theta_t}$.

Also, if the image of $H_2(N, \mathbb{Z})$ in $H_2(M, \mathbb{R})$ is zero, then the condition $[\omega|_N] = 0$ holds automatically. So the obstructions $[\omega_t|_{N_0}] = [\operatorname{Im}\Omega_t|_{N_0}] = 0$ in Theorem 7.3 are actually fairly mild restrictions, and special Lagrangian m-folds should be thought of as pretty stable under small deformations of the Calabi–Yau structure.

7.4 Coassociative 4-folds in 7-manifolds with holonomy G_2

Let (M, g) be a Riemannian 7-manifold with holonomy G_2. Then as in §6.5 there is a natural 3-form φ and a natural 4-form $*\varphi$ on M, of the form (17) and (18) at each point. They are calibrations, and the corresponding calibrated submanifolds are called *associative 3-folds* and *coassociative 4-folds* respectively.

By Proposition 6.5, a 4-fold N in M is coassociative if and only if $\varphi|_N \equiv 0$. Thus, coassociative 4-folds may be defined by the vanishing of a closed form, in the same way as special Lagrangian m-folds are. This gives coassociative 4-folds similar properties to special Lagrangian m-folds. Here is the analogue of Theorem 7.1, proved by McLean.

Theorem 7.4 *Let (M, g) be a 7-manifold with holonomy G_2 and 3-form φ, and N a compact coassociative 4-fold in M. Then the moduli space of coassociative 4-folds isotopic to N in M is a smooth manifold of dimension $b^2_+(N)$.*

Briefly, the theorem holds because the normal bundle ν of N in M is naturally isomorphic to the bundle $\Lambda^2_+ N$ of *self-dual 2-forms* on N. Nearby submanifolds N' correspond to small sections α of $\Lambda^2_+ N$, and to leading order N' is coassociative if and only α is closed. So the tangent space $T_N\mathscr{M}_N$ at N to the moduli space $\mathscr{M}_N$ of coassociative 4-folds is the vector space of closed self-dual 2-forms on N, which has dimension $b^2_+(N)$.

There are also analogues for coassociative 4-folds of §7.2, Lemma 7.2 and Theorem 7.3, which we will not give.

7.5 Associative 3-folds and Cayley 4-folds

There are two other classes of calibrated submanifolds in Riemannian manifolds with exceptional holonomy: *associative* 3*-folds* in 7-manifolds with holonomy G_2, and *Cayley* 4*-folds* in 8-manifolds with holonomy Spin(7). These cannot be defined in terms of the vanishing of closed forms, and this gives their deformation and obstruction theory a different character. Here is how the theories work.

Let N be a compact associative 3-fold or Cayley 4-fold in a 7- or 8-manifold M. Then there are vector bundles $E, F \to N$ with $E \cong \nu$, the normal bundle of N in M, and a first-order elliptic operator $D_N : C^\infty(E) \to C^\infty(F)$ on N. The *kernel* $\operatorname{Ker} D_N$ is the set of *infinitesimal deformations* of N as an associative 3-fold or Cayley 4-fold. The *cokernel* $\operatorname{Coker} D_N$ is the *obstruction space* for these deformations.

Both are finite-dimensional vector spaces, and

$$\dim \operatorname{Ker} D_N - \dim \operatorname{Coker} D_N = \operatorname{ind}(D_N),$$

the *index* of D_N. It is a topological invariant, given in terms of characteristic classes by the *Atiyah–Singer Index Theorem*. In the associative case we have $E \cong F$, and D_N is anti-self-adjoint, so that $\operatorname{Ker}(D_N) \cong \operatorname{Coker}(D_N)$ and $\operatorname{ind}(D_N) = 0$ automatically. In the Cayley case we have

$$\operatorname{ind}(D_N) = \tau(N) - \tfrac{1}{2}\chi(N) - \tfrac{1}{2}[N]\cdot[N],$$

where $\tau(N)$ is the signature, $\chi(N)$ the Euler characteristic and $[N]\cdot[N]$ the self-intersection of N.

In a *generic* situation we expect $\operatorname{Coker} D_N = 0$, and then deformations of N will be unobstructed, so that the moduli space $\mathscr{M}_N$ of associative or Cayley deformations of N will locally be a smooth manifold of dimension $\operatorname{ind}(D_N)$. However, in nongeneric situations the obstruction space may be nonzero, and then we cannot predict the dimension of the moduli space.

This general structure is found in the deformation theory of many other important mathematical objects — for instance, pseudo-holomorphic curves in almost complex manifolds, and instantons and Seiberg–Witten solutions on 4-manifolds. In each case, we can only predict the dimension of the moduli space under a *genericity assumption* which forces the obstructions to vanish.

However, special Lagrangian and coassociative submanifolds do not follow this pattern. Instead, there are *no obstructions*, and the dimension of the moduli space is *always* given by a topological formula. This should be regarded as a minor mathematical miracle.

Reading

McLean is the primary reference for Theorems 7.1 and 7.4. Hitchin is helpful on the ideas of §7.2 above. My book covers the exceptional holonomy material above, and also gives examples of compact associative, coassociative and Cayley submanifolds in compact 7- and 8-manifolds with exceptional holonomy.

R.C. McLean, *Deformations of calibrated submanifolds*, Communications in Analysis and Geometry 6 (1998), 705–747.

N.J. Hitchin, *The moduli space of Special Lagrangian submanifolds*, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze 25 (1997), 503–515. dg-ga/9711002.

D.D. Joyce, *Compact Manifolds with Special Holonomy*, OUP, 2000, §§10.8, 12.6, 14.3.

8 The SYZ Conjecture and special Lagrangian singularities

8.1 String Theory and Mirror Symmetry

String Theory is a branch of high-energy theoretical physics in which particles are modelled not as points but as 1-dimensional objects – 'strings' – propagating in some background space-time M. String theorists aim to construct a *quantum theory* of the string's motion. The process of quantization is extremely complicated, and fraught with mathematical difficulties that are as yet still poorly understood.

The most popular version of String Theory requires the universe to be 10 dimensional for this quantization process to work. Therefore, String Theorists suppose that the space we live in looks locally like $M = \mathbb{R}^4 \times X$, where $\mathbb{R}^4$ is Minkowski space, and X is a compact Riemannian 6-manifold with radius of order 10^{-33}cm, the Planck length. Since the Planck length is so small, space then appears to macroscopic observers to be 4-dimensional.

Because of supersymmetry, X has to be a *Calabi–Yau 3-fold*. Therefore String Theorists are very interested in Calabi–Yau 3-folds. They believe that each Calabi–Yau 3-fold X has a quantization, which is a *Super Conformal Field Theory* (SCFT), a complicated mathematical object. Invariants of X such as the Dolbeault groups $H^{p,q}(X)$ and the number of holomorphic curves in X translate to properties of the SCFT.

However, two entirely different Calabi–Yau 3-folds X and $\hat{X}$ may have the *same* SCFT. In this case, there are powerful relationships between the invariants of X and of $\hat{X}$ that translate to properties of the SCFT. This is the idea behind *Mirror Symmetry* of Calabi–Yau 3-folds.

It turns out that there is a very simple automorphism of the structure of a SCFT — changing the sign of a U(1)-action — which does *not* correspond to a classical automorphism of Calabi–Yau 3-folds. We say that X and $\hat{X}$ are *mirror* Calabi–Yau 3-folds if their SCFT's are related by this automorphism. Then one can argue using String Theory that

$$H^{1,1}(X) \cong H^{2,1}(\hat{X}) \quad \text{and} \quad H^{2,1}(X) \cong H^{1,1}(\hat{X}).$$

Effectively, the mirror transform exchanges even- and odd-dimensional cohomology. This is a very surprising result!

More involved String Theory arguments show that, in effect, the Mirror Transform exchanges things related to the complex structure of X to things related to the symplectic structure of $\hat{X}$, and vice versa. Also, a generating function for the number of holomorphic rational curves in X is exchanged with a simple invariant to do with variation of complex structure on $\hat{X}$, and so on.

Because the quantization process is poorly understood and not at all rigorous — it involves non-convergent path-integrals over horrible infinite-dimensional spaces — String Theory generates only conjectures about Mirror Symmetry, not proofs. However, many of these conjectures have been verified in particular cases.

8.2 Mathematical interpretations: Kontsevich and the SYZ Conjecture

In the beginning (the 1980's), Mirror Symmetry seemed mathematically completely mysterious. But there are now two competing conjectural theories, due to Kontsevich and Strominger–Yau–Zaslow, which explain Mirror Symmetry in a fairly mathematical way. Probably both are true, at some level.

The first proposal was due to Kontsevich in 1994. This says that for mirror Calabi–Yau 3-folds X and $\hat{X}$, the derived category $D^b(X)$ of coherent sheaves on X is equivalent to the derived category $D^b(\mathrm{Fuk}(\hat{X}))$ of the Fukaya category of $\hat{X}$, and vice versa. Basically, $D^b(X)$ has to do with X as a complex manifold, and $D^b(\mathrm{Fuk}(\hat{X}))$ with $\hat{X}$ as a symplectic manifold, and its Lagrangian submanifolds. We shall not discuss this here; the algebra of derived categories is gruesome.

The second proposal, due to Strominger, Yau and Zaslow in 1996, is known as the *SYZ Conjecture*. Here is an attempt to state it.

The SYZ Conjecture. *Suppose X and $\hat{X}$ are mirror Calabi–Yau 3-folds. Then (under some additional conditions) there should exist a compact topological 3-manifold B and surjective, continuous maps $f : X \to B$ and $\hat{f} : \hat{X} \to B$, such that*

(i) *There exists a dense open set $B_0 \subset B$, such that for each $b \in B_0$, the fibres $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are nonsingular special Lagrangian 3-tori T^3 in X and $\hat{X}$. Furthermore, $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are in some sense dual to one another.*

(ii) *For each $b \in \Delta = B \setminus B_0$, the fibres $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are expected to be singular special Lagrangian 3-folds in X and $\hat{X}$.*

We call f and $\hat{f}$ *special Lagrangian fibrations*, and the set of singular fibres Δ is called the *discriminant*. In part (i), the nonsingular fibres of f and $\hat{f}$ are supposed to be *dual tori*. What does this mean?

On the topological level, we can define duality between two tori $T, \hat{T}$ to be a choice of isomorphism $H^1(T, \mathbb{Z}) \cong H_1(\hat{T}, \mathbb{Z})$. We can also define duality between tori equipped with flat Riemannian metrics. Write $T = V/\Lambda$, where V is a Euclidean vector space and Λ a *lattice* in V. Then the dual torus $\hat{T}$ is defined to be V^*/Λ^*, where V^* is the dual vector space and Λ^* the dual lattice. However, there is no notion of duality between non-flat metrics on dual tori.

Strominger, Yau and Zaslow argue only that their conjecture holds when $X, \hat{X}$ are close to the 'large complex structure limit'. In this case, the diameters of the fibres $f^{-1}(b), \hat{f}^{-1}(b)$ are expected to be small compared to the diameter of the base space B, and away from singularities of $f, \hat{f}$, the metrics on the nonsingular fibres are expected to be approximately flat.

So, part (i) of the SYZ Conjecture says that for $b \in B \setminus B_0$, $f^{-1}(b)$ is approximately a flat Riemannian 3-torus, and $\hat{f}^{-1}(b)$ is approximately the dual flat Riemannian torus. Really, the SYZ Conjecture makes most sense as a statement about the limiting behaviour of *families* of mirror Calabi–Yau 3-folds X_t, $\hat{X}_t$ which approach the 'large complex structure limit' as $t \to 0$.

8.3 Consequences of the SYZ Conjecture: symplectic topological approach

The most successful approach to the SYZ Conjecture so far could be described as *symplectic topological*. Its principal exponents are Mark Gross and Wei-Dong Ruan. In this approach, we mostly forget about complex structures, and treat $X, \hat{X}$ just as *symplectic manifolds*. We mostly forget about the 'special' condition, and treat $f, \hat{f}$ just as *Lagrangian fibrations*. We also impose the condition that B is a *smooth* 3-manifold and $f : X \to B$ and $\hat{f} : \hat{X} \to B$ are *smooth maps*. (It is not clear that $f, \hat{f}$ can in fact be smooth at every point, though).

Under these simplifying assumptions, Gross, Ruan and others have built up a beautiful, detailed picture of how dual SYZ fibrations work at the global topological level, in particular for examples such as the quintic and its mirror, and for Calabi–Yau 3-folds constructed as hypersurfaces in toric 4-folds, using combinatorial data.

8.4 Local geometric approach, and special Lagrangian singularities

There is also another approach to the SYZ Conjecture, which is not yet well understood, and is the author's area of research. We could describe it as a *local geometric* approach. In it we try to take the special Lagrangian condition seriously from the outset, and our focus is on the local behaviour of special Lagrangian submanifolds, and especially their singularities, rather than on global topological questions. Also, we are interested in what fibrations of *generic* Calabi–Yau 3-folds might look like.

The basic premise is that in a *generic* Calabi–Yau 3-fold X, compact special Lagrangian 3-folds N are singular at only finitely many points. (In contrast, N can be singular along a real curve in a nongeneric X.) Near each singular point, X looks locally like $\mathbb{C}^3$, and N looks locally like a singular special Lagrangian 3-fold L in $\mathbb{C}^3$.

Therefore, to understand singularities of compact special Lagrangian 3-folds in Calabi–Yau 3-folds, we begin by studying singularities of special Lagrangian 3-folds L in $\mathbb{C}^3$. In general L is noncompact, but satisfies suitable asymptotic boundary conditions at infinity. Using various techniques one can construct and classify many examples of singular special Lagrangian 3-folds L in $\mathbb{C}^3$, and these serve as *local models* for singularities of special Lagrangian 3-folds in Calabi–Yau 3-folds.

Once we have a good understanding of the local behaviour of singularities of special Lagrangian 3-folds, we integrate it into the global topological picture using the ideas of §7.1–§7.3, in particular Theorem 7.1, and the 3-manifold topology of N. This approach has little to say about the global topology of the Calabi–Yau 3-fold X.

One of the first-fruits of this approach has been the understanding that for *generic* Calabi–Yau 3-folds X, special Lagrangian fibrations $f : X \to B$ will not be smooth maps, but only piecewise smooth. Furthermore, their behaviour at the singular set is rather different to the smooth Lagrangian fibrations discussed in §7.3.

For smooth special Lagrangian fibrations $f : X \to B$, the discriminant Δ is of codimension 2 in B, and the typical singular fibre is singular along an $\mathcal{S}^1$. But in a generic special Lagrangian fibration $f : X \to B$ the discriminant Δ is of codimension 1 in B, and the typical singular fibre is singular at finitely many points.

One can also show that if $X, \hat{X}$ are a mirror pair of generic Calabi–Yau 3-folds and $f : X \to B$ and $\hat{f} : \hat{X} \to B$ are dual special Lagrangian fibrations, then in general the discriminants Δ of f and $\hat{\Delta}$ of $\hat{f}$ cannot coincide in B, because they have different topological properties in the neighbourhood of a certain kind of codimension 3 singular fibre.

This contradicts part (ii) of the SYZ Conjecture above, as we have stated it. In the author's view, these calculations support the idea that the SYZ Conjecture in its present form should be viewed primarily as a limiting statement, about what happens at the 'large complex structure limit', rather than as simply being about pairs of Calabi–Yau 3-folds. But new ways to understand and state the SYZ Conjecture may well emerge in future.

Reading

Kontsevich's Mirror Symmetry proposal is contained in

M. Kontsevich, *Homological Algebra of Mirror Symmetry*, in Proc. Int. Cong. Math. Zürich, 1994. alg-geom/9411018.

The original SYZ Conjecture paper is:

A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Physics B479 (1996), 243–259. hep-th/9606040.

It is written in String Theory language, and not easy for mathematicians to follow. But most of the papers below have short introductions to it.
For the symplectic topological approach to the SYZ Conjecture, see for instance:

M. Gross, *Special Lagrangian fibrations I: Topology*, alg-geom/9710006.

M. Gross, *Special Lagrangian fibrations II: Geometry*, math.AG/9809072, 1998.

M. Gross, *Topological mirror symmetry*, math.AG/9909015, 1999.

For the local geometric approach to the SYZ Conjecture, see

D.D. Joyce, *Singularities of special Lagrangian fibrations and the SYZ Conjecture*, math.DG/0011179, 2000.

Exercises

1. Show that the maps Φ, Ψ between special Lagrangian moduli space $\mathscr{M}_N$ and $H^1(N,\mathbb{R})$, $H^{m-1}(N,\mathbb{R})$ defined in §7.2 are well-defined and independent of choices.

Prove also that Φ and Ψ are *local diffeomorphisms*, that is, that $\mathrm{d}\Phi|_{N'}$ and $\mathrm{d}\Psi|_{N'}$ are isomorphisms between $T_{N'}\mathscr{M}_N$ and $H^1(N,\mathbb{R})$, $H^{m-1}(N,\mathbb{R})$ for each $N' \in U$.

2. Putting together the maps Φ, Ψ of question 1 gives a map $\Phi \times \Psi : U \to H^1(N,\mathbb{R}) \times H^{m-1}(N,\mathbb{R})$. Now $H^1(N,\mathbb{R})$ and $H^{m-1}(N,\mathbb{R})$ are dual by Poincaré duality, so $H^1(N,\mathbb{R}) \times H^{m-1}(N,\mathbb{R})$ has a natural *symplectic structure*. Show that the image of U is a *Lagrangian submanifold* in $H^1(N,\mathbb{R}) \times H^{m-1}(N,\mathbb{R})$.

Hint: From the proof of McLean's theorem in §7.1, the tangent space $T_N\mathscr{M}_N$ is isomorphic to the vector space of 1-forms α with $\mathrm{d}\alpha = \mathrm{d}(*\alpha) = 0$. Then $\mathrm{d}\Phi|_N : T_N\mathscr{M} \to H^1(M,\mathbb{R})$ takes $\alpha \mapsto [\alpha]$, and $\mathrm{d}\Psi|_N : T_N\mathscr{M} \to H^{m-1}(M,\mathbb{R})$ takes $\alpha \mapsto [*\alpha]$. Use the fact that for 1-forms α, β on an oriented Riemannian manifold we have $\alpha \wedge (*\beta) = \beta \wedge (*\alpha)$.

3. In my paper math.DG/0011179 I define a map $f : \mathbb{C}^3 \to \mathbb{R}^3$ by $f(z_1, z_2, z_3) = (a, \mathrm{Re}\, c, \mathrm{Im}\, c)$, where

$$a = |z_1|^2 - |z_2|^2$$

$$\text{and} \quad c = \begin{cases} z_3 - \bar{z}_1\bar{z}_2/|z_1|, & a = 0 \text{ and } z_1, z_2 \neq 0, \\ z_3, & a = z_1 = z_2 = 0, \\ z_3 - \bar{z}_1\bar{z}_2/|z_1|, & a > 0, \\ z_3 - \bar{z}_1\bar{z}_2/|z_2|, & a < 0. \end{cases}$$

It is a conjectural local model for the most generic kind of singularity of a special Lagrangian fibration of a Calabi–Yau 3-fold.

- **(a)** Show that f is continuous, surjective, and piecewise smooth.
- **(b)** Show that $f^{-1}(a, b, c)$ is a (possibly singular) special Lagrangian 3-fold for all $(a, b, c) \in \mathbb{R}^3$.
- **(c)** Identify the singular fibres and describe their singularities. Describe the topology of the singular and the nonsingular fibres.

The idea of a 'special Lagrangian fibration' $f : X \to B$ is in some ways a rather unnatural one. One of the problems is that the map f doesn't satisfy a particularly nice equation, locally; the level sets of f do, but the 'coordinates' on B are determined globally rather than locally. To understand the problems with special Lagrangian fibrations, try the following (rather difficult) exercise.

4. Let X be a Calabi–Yau 3-fold, N a compact special Lagrangian 3-fold in X diffeomorphic to T^3, $\mathscr{M}_N$ the family of special Lagrangian deformations of N, and $\overline{\mathscr{M}}_N$ be $\mathscr{M}_N$ together with the singular special Lagrangian 3-folds occurring as limits of elements of $\mathscr{M}_N$.

In good cases, SYZ hope that $\overline{\mathscr{M}}_N$ is the family of level sets of a special Lagrangian fibration $f : X \to B$, where B is homeomorphic to $\overline{\mathscr{M}}_N$. How many different ways can you think of for this not to happen? (There are at least two mechanisms not involving singular fibres, and others which do).