Derived Differential Geometry

Lecture 3 of 3: Applications; moduli spaces and virtual cycles

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These slides available at
`http://people.maths.ox.ac.uk/~joyce/`.

Plan of talk:

8. Differential geometry of derived manifolds
Gluing by equivalences

A 1-morphism $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ in **dMan** is an *equivalence* if there exist $\boldsymbol{g}:\boldsymbol{Y}\to\boldsymbol{X}$ and 2-morphisms $\boldsymbol{\eta}:\boldsymbol{g}\circ\boldsymbol{f}\Rightarrow\mathbf{id}_{\boldsymbol{X}}$ and $\boldsymbol{\zeta}:\boldsymbol{f}\circ\boldsymbol{g}\Rightarrow\mathbf{id}_{\boldsymbol{Y}}$.

Theorem 8.1

Let $\boldsymbol{X},\boldsymbol{Y}$ be d-manifolds, $\emptyset\neq\boldsymbol{U}\subseteq\boldsymbol{X}$, $\emptyset\neq\boldsymbol{V}\subseteq\boldsymbol{Y}$ open d-submanifolds, and $\boldsymbol{f}:\boldsymbol{U}\to\boldsymbol{V}$ an equivalence. Suppose the topological space $Z=X\cup_{U=V}Y$ made by gluing X,Y using f is Hausdorff. Then there exists a d-manifold $\boldsymbol{Z}$, unique up to equivalence, open $\hat{\boldsymbol{X}},\hat{\boldsymbol{Y}}\subseteq\boldsymbol{Z}$ with $\boldsymbol{Z}=\hat{\boldsymbol{X}}\cup\hat{\boldsymbol{Y}}$, equivalences $\boldsymbol{g}:\boldsymbol{X}\to\hat{\boldsymbol{X}}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\hat{\boldsymbol{Y}}$, and a 2-morphism $\boldsymbol{\eta}:\boldsymbol{g}|_{\boldsymbol{U}}\Rightarrow\boldsymbol{h}\circ\boldsymbol{f}$.

The theorem generalizes to gluing families of d-manifolds $\boldsymbol{X}_i:i\in I$ by equivalences on double overlaps $\boldsymbol{X}_i\cap\boldsymbol{X}_j$, with (weak) conditions on triple overlaps $\boldsymbol{X}_i\cap\boldsymbol{X}_j\cap\boldsymbol{X}_k$.
(All this holds for m-Kuranishi spaces too, as **dMan** $\simeq$ **mKur**.)

W-transversality and fibre products

Let $g:X\to Z$, $h:Y\to Z$ be smooth maps of manifolds. Then g,h are *transverse* if for all $x\in X$, $y\in Y$ with $g(x)=h(y)=z$ in Z, the map $T_xg\oplus T_yh:T_xX\oplus T_yY\to T_zZ$ is surjective.
Similarly, we call 1-morphisms $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$, $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ in **dMan** *w-transverse* if for all $x\in\boldsymbol{X}$, $y\in\boldsymbol{Y}$ with $\boldsymbol{g}(x)=\boldsymbol{h}(y)=z$ in $\boldsymbol{Z}$, the map $O_x\boldsymbol{g}\oplus O_y\boldsymbol{h}:O_x\boldsymbol{X}\oplus O_y\boldsymbol{Y}\to O_z\boldsymbol{Z}$ is surjective.

Theorem 8.2

Let $\boldsymbol{g}:\boldsymbol{X}\to\boldsymbol{Z}$ and $\boldsymbol{h}:\boldsymbol{Y}\to\boldsymbol{Z}$ be w-transverse 1-morphisms in **dMan**. *Then a fibre product $\boldsymbol{W}=\boldsymbol{X}\times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}}\boldsymbol{Y}$ exists in the 2-category* **dMan**, *with $\mathrm{vdim}\,\boldsymbol{W}=\mathrm{vdim}\,\boldsymbol{X}+\mathrm{vdim}\,\boldsymbol{Y}-\mathrm{vdim}\,\boldsymbol{Z}$.*

If $\boldsymbol{Z}$ is a manifold, $O_z\boldsymbol{Z}=0$ and w-transversality is trivial, giving:

Corollary

All fibre products of the form $\boldsymbol{X}\times_{\boldsymbol{Z}}\boldsymbol{Y}$ with $\boldsymbol{X},\boldsymbol{Y}$ d-manifolds and $\boldsymbol{Z}$ a manifold exist in **dMan**.

W-submersions and submersions

A smooth map of manifolds $f : X \to Y$ is a *submersion* if $T_x f : T_x X \to T_{f(x)} Y$ is surjective for all $x \in X$.

Definition

Let $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ be a 1-morphism of derived manifolds. We call $\boldsymbol{f}$ a *w-submersion* if $O_x \boldsymbol{f} : O_x \boldsymbol{X} \to O_{\boldsymbol{f}(x)} \boldsymbol{Y}$ is surjective for all $x \in \boldsymbol{X}$. We call $\boldsymbol{f}$ a *submersion* if $T_x \boldsymbol{f} : T_x \boldsymbol{X} \to T_{\boldsymbol{f}(x)} \boldsymbol{Y}$ is surjective and $O_x \boldsymbol{f} : O_x \boldsymbol{X} \to O_{\boldsymbol{f}(x)} \boldsymbol{Y}$ is an isomorphism for all $x \in \boldsymbol{X}$.

Theorem 8.3

Suppose $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ is a w-submersion in **dMan**, *and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ is any 1-morphism. Then $\boldsymbol{g}, \boldsymbol{h}$ are w-transverse, so a fibre product $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in* **dMan**.
If $\boldsymbol{g}$ is a submersion and $\boldsymbol{Y}$ is a manifold, then $\boldsymbol{W}$ is a manifold.

Orientations on derived manifolds

Here is one way to define orientations on ordinary manifolds. Let X be a manifold of dimension n. The *canonical bundle* K_X is $\Lambda^n T^* X$. It is a real line bundle over X. An *orientation* o on X is an orientation on the fibres of K_X. That is, o is an equivalence class $[\iota]$ of isomorphisms $\iota : \mathcal{O}_X \to K_X$, where $\mathcal{O}_X = X \times \mathbb{R}$ is the trivial line bundle on X, and two isomorphisms ι, ι' are equivalent if $\iota' = c \cdot \iota$ for $c : X \to (0, \infty)$ a smooth positive function on X. Isomorphisms $\iota : \mathcal{O}_X \to K_X$ are equivalent to non-vanishing n-forms $\omega = \iota(1)$ on X.
The *opposite orientation* is $-o = [-\iota]$.
An *oriented manifold* (X, o) is a manifold X with orientation o.
Usually we just say X is an oriented manifold, and write $-X$ for $(X, -o)$ with the opposite orientation.

There is an analogue of canonical bundles for derived manifolds:

Theorem 8.4

(a) *Every d-manifold or m-Kuranishi space* $\boldsymbol{X}$ *has a* **canonical bundle** $K_{\boldsymbol{X}}$, *a topological real line bundle over the topological space* X, *natural up to canonical isomorphism, with* $K_{\boldsymbol{X}}|_x \cong \Lambda^{\rm top} T^*_x \boldsymbol{X} \otimes \Lambda^{\rm top} O_x \boldsymbol{X}$ *for all* $x \in \boldsymbol{X}$.
(b) *If* $\boldsymbol{f}: \boldsymbol{X} \to \boldsymbol{Y}$ *is an étale 1-morphism (e.g. an equivalence), there is a canonical, functorial isomorphism* $K_{\boldsymbol{f}}: K_{\boldsymbol{X}} \to f^*(K_{\boldsymbol{Y}})$. *If* $\boldsymbol{f}, \boldsymbol{g}: \boldsymbol{X} \to \boldsymbol{Y}$ *are 2-isomorphic then* $K_{\boldsymbol{f}} = K_{\boldsymbol{g}}$.
(c) *If* (V_i, E_i, s_i, ψ_i) *is an m-Kuranishi neighbourhood on* $\boldsymbol{X}$, *there is a canonical isomorphism*

$$\psi_i^*(K_{\boldsymbol{X}}) \cong \left(\Lambda^{\dim V_i} T^* V_i \otimes \Lambda^{\operatorname{rank} E_i} E_i\right)\big|_{s_i^{-1}(0)}.$$

To prove the theorem for m-Kuranishi spaces, we show that the line bundles $\left(\Lambda^{\dim V_i} T^* V_i \otimes \Lambda^{\operatorname{rank} E_i} E_i\right)\big|_{s_i^{-1}(0)}$ on $\operatorname{Im}\psi_i \subseteq X$ can be glued by canonical isomorphisms on overlaps $\operatorname{Im}\psi_i \cap \operatorname{Im}\psi_j$.

Definition

An *orientation* o on a d-manifold or m-Kuranishi space $\boldsymbol{X}$ is an equivalence class $[\iota]$ of isomorphisms $\iota: \mathcal{O}_X \to K_{\boldsymbol{X}}$, where $\mathcal{O}_X$ is the trivial line bundle on X, and two isomorphisms ι, ι' are equivalent if $\iota' = c \cdot \iota$ for $c: X \to (0, \infty)$ continuous.

On a single m-Kuranishi neighbourhood (V_i, E_i, s_i, ψ_i) on $\boldsymbol{X}$, an orientation is equivalent to an orientation (near $s_i^{-1}(0)$) on the total space of E_i. We can do oriented w-transverse fibre products:

Theorem 8.5

Let $\boldsymbol{W} = \boldsymbol{X} \times_{\boldsymbol{g}, \boldsymbol{Z}, \boldsymbol{h}} \boldsymbol{Y}$ *be a w-transverse fibre product in* **dMan**, *as in Theorem 8.2, with projections* $\boldsymbol{e}: \boldsymbol{W} \to \boldsymbol{X}$, $\boldsymbol{f}: \boldsymbol{W} \to \boldsymbol{Y}$. *Then there is a natural isomorphism of line bundles on* W

$$K_{\boldsymbol{W}} \cong e^*(K_{\boldsymbol{X}}) \otimes f^*(K_{\boldsymbol{Y}}) \otimes (g \circ e)^*(K_{\boldsymbol{Z}})^*.$$

Hence orientations on $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}$ *induce an orientation on* $\boldsymbol{W}$.

9. Bordism, virtual classes, and virtual chains

In many important areas of geometry to do with enumerative invariants (e.g. Donaldson and Seiberg–Witten invariants of 4-manifolds, Gromov–Witten invariants of symplectic manifolds, Donaldson–Thomas invariants of Calabi–Yau 3-folds, ...), we form a moduli space $\bar{\mathcal{M}}$ with some geometric structure, and we want to 'count' $\bar{\mathcal{M}}$ to get a number in $\mathbb{Z}$ or $\mathbb{Q}$ (if $\bar{\mathcal{M}}$ has no boundary and dimension 0), or a homology class ('virtual class') $[\bar{\mathcal{M}}]_{\rm virt}$ in some homology theory (if $\bar{\mathcal{M}}$ has no boundary and dimension > 0).
For more complicated theories (Floer homology, Fukaya categories), $\bar{\mathcal{M}}$ has boundary, and then we must define a chain $[\bar{\mathcal{M}}]_{\rm virt}$ in the chain complex (C_*, ∂) of some homology theory (a 'virtual chain'), where ideally we want $\partial[\bar{\mathcal{M}}]_{\rm virt} = [\partial\bar{\mathcal{M}}]_{\rm virt}$.

In general $\bar{\mathcal{M}}$ is not a manifold (or orbifold). However, the point is to treat $\bar{\mathcal{M}}$ *as if* it were a compact, oriented manifold, so that in particular, if $\partial\bar{\mathcal{M}} = \emptyset$ then $\bar{\mathcal{M}}$ has a fundamental class $[\bar{\mathcal{M}}]$ in the homology group $H_{\dim\bar{\mathcal{M}}}(\bar{\mathcal{M}};\mathbb{Z})$.
All of these 'counting invariant' theories over $\mathbb{R}$ or $\mathbb{C}$, in both differential and algebraic geometry, can be understood using derived differential geometry. The point is that the moduli spaces $\bar{\mathcal{M}}$ should be compact, oriented derived manifolds or orbifolds (possibly with corners). Then we show that compact, oriented derived manifolds or orbifolds (with corners) have virtual classes (virtual chains), and these are used to define the invariants.
There is an easy way to define virtual classes for compact, oriented derived manifolds without boundary, using *bordism*, so we explain this first. It does not work as well in the orbifold case, though.

Classical bordism groups

Let Y be a manifold. Define the *bordism group* $B_k(Y)$ to have elements $\sim$-equivalence classes $[X,f]$ of pairs (X,f), where X is a compact oriented k-manifold without boundary and $f:X\rightarrow Y$ is smooth, and $(X,f)\sim(X',f')$ if there exists a compact, oriented $(k+1)$-manifold with boundary W and a smooth map $e:W\rightarrow Y$ with $\partial W\cong X\amalg -X'$ and $e|_{\partial W}\cong f\amalg f'$. It is an abelian group, with addition $[X,f]+[X',f']=[X\amalg X',f\amalg f']$.
If Y is oriented of dimension n, there is a supercommutative, associative *intersection product* $\bullet: B_k(Y)\times B_l(Y)\rightarrow B_{k+l-n}(Y)$ given by $[X,f]\bullet[X',f']=[X\times_{f,Y,f'}X',\pi_Y]$, choosing X,f,X',f' in their bordism classes with $f:X\rightarrow Y$, $f':X'\rightarrow Y$ transverse.
Bordism is a *generalized homology theory*, i.e. it satisfies all the Eilenberg–Steenrod axioms except the Dimension Axiom.
There is a natural morphism $\Pi_{\rm bo}^{\rm hom}: B_k(Y)\rightarrow H_k(Y;\mathbb{Z})$ given by $\Pi_{\rm bo}^{\rm hom}:[X,f]\mapsto f_*([X])$, for $[X]\in H_k(X;\mathbb{Z})$ the fundamental class.

Derived bordism groups

Similarly, define the *derived bordism group* $dB_k(Y)$ to have elements $\approx$-equivalence classes $[\boldsymbol{X},\boldsymbol{f}]$ of pairs $(\boldsymbol{X},\boldsymbol{f})$, where $\boldsymbol{X}$ is a compact oriented d-manifold with $\mathrm{vdim}\,\boldsymbol{X}=k$ and $\boldsymbol{f}:\boldsymbol{X}\rightarrow Y$ is a 1-morphism in **dMan**, and $(\boldsymbol{X},\boldsymbol{f})\approx(\boldsymbol{X}',\boldsymbol{f}')$ if there exists a compact, oriented d-manifold with boundary $\boldsymbol{W}$ with $\mathrm{vdim}\,\boldsymbol{W}=k+1$ and a 1-morphism $\boldsymbol{e}:\boldsymbol{W}\rightarrow\boldsymbol{Y}$ in $\mathbf{dMan^b}$ with $\partial\boldsymbol{W}\simeq\boldsymbol{X}\amalg-\boldsymbol{X}'$ and $\boldsymbol{e}|_{\partial\boldsymbol{W}}\cong\boldsymbol{f}\amalg\boldsymbol{f}'$. It is an abelian group, with $[\boldsymbol{X},\boldsymbol{f}]+[\boldsymbol{X}',\boldsymbol{f}']=[\boldsymbol{X}\amalg\boldsymbol{X},\boldsymbol{f}\amalg\boldsymbol{f}']$.
If Y is oriented of dimension n, there is a supercommutative, associative *intersection product* $\bullet: dB_k(Y)\times dB_l(Y)\rightarrow dB_{k+l-n}(Y)$ given by $[\boldsymbol{X},\boldsymbol{f}]\bullet[\boldsymbol{X}',\boldsymbol{f}']=[\boldsymbol{X}\times_{\boldsymbol{f},Y,\boldsymbol{f}'}\boldsymbol{X}',\boldsymbol{\pi}_Y]$, with no transversality condition on $\boldsymbol{X},\boldsymbol{f},\boldsymbol{X}',\boldsymbol{f}'$.

There is a morphism $\Pi_{\rm bo}^{\rm dbo}:B_k(Y)\rightarrow dB_k(Y)$ mapping $[X,f]\mapsto[X,f]$.

Theorem 9.1 (First version due to David Spivak.)

$\Pi_{\rm bo}^{\rm dbo}:B_k(Y)\rightarrow dB_k(Y)$ is an isomorphism for all k, with $dB_k(Y)=0$ for $k<0$.

This holds as every d-manifold can be perturbed to a manifold.

Composing $(\Pi_{\rm bo}^{\rm dbo})^{-1}$ with $\Pi_{\rm bo}^{\rm hom}:B_k(Y)\rightarrow H_k(Y;\mathbb{Z})$ gives a morphism $\Pi_{\rm dbo}^{\rm hom}:dB_k(Y)\rightarrow H_k(Y;\mathbb{Z})$. We can interpret this as a *virtual class map* for compact, oriented d-manifolds. In particular, this is an easy proof that *the geometric structure on d-manifolds is strong enough to define virtual classes*.

We can also define *orbifold bordism* $B_k^{\rm orb}(Y)$ and *derived orbifold bordism* $dB_k^{\rm orb}(Y)$, replacing (derived) manifolds by (derived) orbifolds. However, the natural morphism $B_k^{\rm orb}(Y)\rightarrow dB_k^{\rm orb}(Y)$ is not an isomorphism, as derived orbifolds cannot always be perturbed to orbifolds.

A virtual class for $\boldsymbol{X}$ in the homology of $\boldsymbol{X}$?

In algebraic geometry, given a moduli space $\bar{\mathcal M}$, it is usual to define the virtual class in the (Chow) homology $H_{{\rm vdim}\,\bar{\mathcal M}}(\bar{\mathcal M};\mathbb{Q})$. But in differential geometry, given $\bar{\mathcal M}$, usually we find a manifold Y with a map $\bar{\mathcal M}\rightarrow Y$, and define the virtual class $[\bar{\mathcal M}]_{\rm virt}$ in the (ordinary) homology $H_{{\rm vdim}\,\bar{\mathcal M}}(Y;\mathbb{Q})$. This is because differential-geometric techniques for defining $[\bar{\mathcal M}]_{\rm virt}$ involve perturbing $\bar{\mathcal M}$, which changes it as a topological space.

Example

Define $f:\mathbb{R}\rightarrow\mathbb{R}$ by $f(x)=e^{-x^{-2}}\sin(\pi/x)$ for $x\neq 0$, and $f(0)=0$. Then f is smooth. Define $\boldsymbol{X}=\mathbb{R}\times_{f,\mathbb{R},0}*$. Then $\boldsymbol{X}$ is a compact, oriented derived manifold without boundary, with ${\rm vdim}\,\boldsymbol{X}=0$. As a topological space we have

$$X=\{1/n:0\neq n\in\mathbb{Z}\}\amalg\{0\}.$$

Then *no virtual class exists* for $\boldsymbol{X}$ in ordinary homology $H_0(X;\mathbb{Z})$.

Virtual classes in Steenrod or Čech homology

Steenrod homology $H_*^{\mathrm{St}}(X;\mathbb{Z})$ (see J. Milnor, *'On the Steenrod homology theory'*, Milnor collected works IV, 2009) is a homology theory of topological spaces. For nice topological spaces X (e.g. manifolds, or finite simplicial complexes) it equals ordinary (e.g. singular) homology $H_*(X;\mathbb{Z})$. It has a useful limiting property:

Theorem 9.2

Let X be a compact subset of a metric space Y, and suppose $W_1, W_2, \ldots$ are open neighbourhoods of X in Y with $\bigcap_{n\geqslant 1} W_n = X$ and $W_1 \supseteq W_2 \supseteq \cdots$. Then $H_k^{\mathrm{St}}(X;\mathbb{Z}) \cong \varprojlim_{n\geqslant 1} H_k^{\mathrm{St}}(W_n;\mathbb{Z})$.

Čech homology $\check{H}_*(X;\mathbb{Q})$ over $\mathbb{Q}$ has the same property. Singular homology does not.

Following an idea due to Dusa McDuff, we can use this to define a virtual class $[\boldsymbol{X}]_{\mathrm{virt}}$ for a compact oriented d-manifold $\boldsymbol{X}$ in $H^{\mathrm{St}}_{\mathrm{vdim}\,\boldsymbol{X}}(X;\mathbb{Z})$ or $\check{H}_*(X;\mathbb{Q})$. We may write $\boldsymbol{X} \simeq \boldsymbol{S}_{V,E,s}$ by Corollary 9.8. This gives a homeomorphism $X \cong s^{-1}(0)$, for $s^{-1}(0)$ a compact subset of V. Choose open neighbourhoods $W_1, W_2, \ldots$ of $s^{-1}(0)$ in V with $\bigcap_{n\geqslant 1} W_n = s^{-1}(0)$ and $W_1 \supseteq W_2 \supseteq \cdots$. The inclusion $\boldsymbol{i}_n : \boldsymbol{X} \hookrightarrow W_n$ defines a d-bordism class $[\boldsymbol{X}, \boldsymbol{i}_n] \in dB_{\mathrm{vdim}\,\boldsymbol{X}}(W_n)$, and hence a homology class $\Pi_{\mathrm{dbo}}^{\mathrm{hom}}([\boldsymbol{X}, \boldsymbol{i}_n])$ in $H_{\mathrm{vdim}\,\boldsymbol{X}}(W_n;\mathbb{Z}) \cong H^{\mathrm{St}}_{\mathrm{vdim}\,\boldsymbol{X}}(W_n;\mathbb{Z})$. These are preserved by the inclusions $W_{n+1} \hookrightarrow W_n$, and so define a class in the inverse limit $\varprojlim_{n\geqslant 1} H_k^{\mathrm{St}}(W_n;\mathbb{Z})$, and thus, by Theorem 9.2, a virtual class $[\boldsymbol{X}]_{\mathrm{virt}}$ in $H^{\mathrm{St}}_{\mathrm{vdim}\,\boldsymbol{X}}(X;\mathbb{Z})$ or $\check{H}_{\mathrm{vdim}\,\boldsymbol{X}}(X;\mathbb{Q})$.

More about virtual classes and virtual chains

- If $\boldsymbol{X}$ is a compact, oriented derived orbifold, we can also define a virtual class $[\boldsymbol{X}]_{\rm virt}$ in $\check{H}_{{\rm vdim}\,\boldsymbol{X}}(X;\mathbb{Q})$, though the process is more complicated.
- If $\boldsymbol{X}$ is a compact oriented derived manifold or orbifold with corners, and Y is a manifold or orbifold, and $\boldsymbol{f}:\boldsymbol{X}\to Y$ is a 1-morphism, then after making some arbitrary choices, one can define a virtual chain $[\boldsymbol{X}]_{\rm virt}$ of $\boldsymbol{X}$ in the chains $C_{{\rm vdim}\,\boldsymbol{X}}(Y;\mathbb{Q})$ of a suitable homology theory of Y. This is important for Floer theories, Fukaya categories, Symplectic Field Theory, and so on. Constructing virtual chains is complicated. Ideally one wants to arrange that $\partial[\boldsymbol{X}]_{\rm virt}=[\partial\boldsymbol{X}]_{\rm virt}$, and several other properties. In arXiv:1509.05672 I define a new homology theory of manifolds, *M-homology* $MH_*(Y;\mathbb{Q})$, which is isomorphic to ordinary homology $H_*(Y;\mathbb{Q})$, but has good chain-level behaviour, and is designed for forming virtual chains of derived manifolds/orbifolds.

10. Derived manifold/orbifold structures on moduli spaces

Theorem 10.1

Let $\mathcal{V}$ be a Banach manifold, $\mathcal{E}\to\mathcal{V}$ a Banach vector bundle, and $s:\mathcal{V}\to\mathcal{E}$ a smooth Fredholm section, with constant Fredholm index $n\in\mathbb{Z}$. Then there is a canonical d-manifold $\boldsymbol{X}$ with topological space $X=s^{-1}(0)$ and ${\rm vdim}\,\boldsymbol{X}=n$.

Nonlinear elliptic equations, when written as maps between suitable Hölder or Sobolev spaces, are the zeroes of Fredholm sections of a Banach vector bundle over a Banach manifold. Thus we have:

Corollary 10.2

Let $\mathcal{M}$ be a moduli space of solutions of a nonlinear elliptic equation on a compact manifold, with fixed topological invariants. Then $\mathcal{M}$ extends to a d-manifold $\boldsymbol{\mathcal{M}}$.

The virtual dimension of $\boldsymbol{\mathcal{M}}$ at $x\in\boldsymbol{\mathcal{M}}$ is the index of the linearization of the elliptic p.d.e. at x, given by the A–S Index Theorem.

Truncation functors from other structures

Theorem 10.3

Suppose X is a Hausdorff, second countable topological space equipped with any of the following geometric structures, each of constant virtual dimension $n \in \mathbb{Z}$:

- (a) *A $\mathbb{C}$-scheme or Deligne–Mumford $\mathbb{C}$-stack with perfect obstruction theory in the sense of Behrend and Fantechi (where X is the underlying complex analytic space).*
- (b) *A quasi-smooth derived $\mathbb{C}$-scheme or D–M $\mathbb{C}$-stack.*
- (c) *An M-polyfold or polyfold Fredholm structure in the sense of Hofer, Wysocki and Zehnder.*
- (d) *A Kuranishi structure in the sense of Fukaya–Oh–Ohta–Ono.*
- (e) *A Kuranishi atlas in the sense of McDuff and Wehrheim.*

Then X may be given the structure of a d-manifold or d-orbifold, natural up to equivalence in **dMan**, **dOrb**, *with $\mathrm{vdim}\,\boldsymbol{X} = n$. We can also allow corners in* (c)–(e), *with $\boldsymbol{X} \in \mathbf{dMan^c}, \mathbf{dOrb^c}$.*

-2-shifted symplectic derived $\mathbb{C}$-schemes

Theorem 10.4 (Borisov–Joyce arXiv:1504.00690)

Suppose $\boldsymbol{X}$ is a derived $\mathbb{C}$-scheme with a -2-shifted symplectic structure $\omega_{\boldsymbol{X}}$ in the sense of Pantev–Toën–Vaquié–Vezzosi arXiv:1111.3209. *Then we can define a d-manifold $\boldsymbol{X}_{\rm dm}$ with the same underlying topological space, and virtual dimension $\mathrm{vdim}_{\mathbb{R}}\,\boldsymbol{X}_{\rm dm} = \frac{1}{2}\,\mathrm{vdim}_{\mathbb{R}}\,\boldsymbol{X}$, i.e. half the expected dimension.*

Note that $\boldsymbol{X}$ is not quasi-smooth, $\mathbb{L}_{\boldsymbol{X}}$ lies in the interval $[-2, 0]$, so this does not follow from Theorem 10.3(b). Also $\boldsymbol{X}_{\rm dm}$ is only canonical up to bordisms fixing the underlying topological space. Derived moduli schemes or stacks of coherent sheaves on a Calabi–Yau m-fold are $(2-m)$-shifted symplectic, so this gives:

Corollary 10.5

Stable moduli schemes of coherent sheaves $\mathcal{M}$ with fixed Chern character on a Calabi–Yau 4-fold can be made into d-manifolds $\boldsymbol{\mathcal{M}}$.

Combining Theorems 10.3 and 10.4 with results from the literature shows that many interesting moduli spaces over $\mathbb{R}$ or $\mathbb{C}$, in both differential and algebraic geometry, have the structure of d-manifolds or d-orbifolds, natural up to equivalence. This includes almost every moduli space used in any enumerative invariant problem over $\mathbb{R}$ or $\mathbb{C}$.

Moduli 2-functors

The approaches to moduli spaces in Differential and Algebraic Geometry are very different. In Differential Geometry one constructs the moduli space, as a topological space covered by an atlas of charts. In Algebraic Geometry one writes down a *moduli functor* $F : \mathcal{C} \to \mathbf{Sch}_{\mathbb{K}}$, where objects $O \in \mathcal{C}$ with $F(O) = B$ are families of objects in the moduli problem over a base $\mathbb{K}$-scheme B, and then prove this is equivalent to the functor $\pi : \mathbf{Sch}_{\mathcal{M}} \to \mathbf{Sch}_{\mathbb{K}}$ for some $\mathbb{K}$-scheme $\mathcal{M}$, the *moduli scheme*.

I propose that in Derived Differential Geometry one should write down a *moduli 2-functor* $F : \boldsymbol{\mathcal{C}} \to \mathbf{GmKN}$, where $\boldsymbol{\mathcal{C}}$ is a 2-category and $\mathbf{GmKN}$ the 2-category of global m-Kuranishi neighbourhoods, where objects O in $\boldsymbol{\mathcal{C}}$ with $F(O) = (V, E, s)$ are families of moduli objects over a base m-Kuranishi neighbourhood (V, E, s), and prove this is equivalent (after stackification) to $\pi : \mathbf{mKN}_{\boldsymbol{\mathcal{M}}} \to \mathbf{GmKN}$ for some m-Kuranishi space $\boldsymbol{\mathcal{M}}$, with $\mathbf{mKN}_{\boldsymbol{\mathcal{M}}}$ the 2-category of m-Kuranishi neighbourhoods on $\boldsymbol{\mathcal{M}}$.

Some advantages of the moduli 2-functor approach:

- Many current presentations of moduli spaces (e.g. FOOO, HWZ) are long, complicated ad hoc constructions. The effort is mostly in the definition. It is unclear how natural they are. Our definition makes the naturality clear. We have a short definition (the moduli 2-functor), followed by a difficult theorem (the 2-functor is represented by an (m-)Kuranishi space).
- To prove representability we only have to worry about single (m-)Kuranishi neighbourhoods, not double or triple overlaps.
- The definition involves only finite-dimensional families of smooth objects – no Hölder or Sobolev spaces, etc. (though these will be used in the proof of representability). This enables us to sidestep some technical issues in current approaches, e.g. sc-smoothness in polyfolds.
- In our approach, the existence of natural morphisms between moduli spaces (e.g. 'forgetful morphisms' in Symplectic Geometry forgetting a marked point) is essentially trivial.