Vertex Algebras

Lecture 1 of 8: Borcherds' approach to vertex algebras

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Recommended references:
V. Kac, *Vertex Algebras for Beginners*,
E. Frenkel & D. Ben-Zvi, *Vertex Algebras and Algebraic Curves*.

These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`

Plan of talk:

Introduction

Vertex algebras are complicated algebraic objects. They are difficult to define, and when you have a definition, it is hard, initially, to see the point of it. I hope that the reasons for studying vertex algebras will become clearer in the next few lectures.

The most helpful definitions of vertex algebra are in terms of 'states' and 'fields', and are written in terms of formal power series in a (complex) variable z. Roughly speaking, we have a complex vector space V (usually infinite-dimensional) and a $\mathbb{C}$-linear operator $Y(z) : V \otimes V \to V[[z]][z^{-1}]$ giving a family of 'multiplications' $u \star_z v = Y(z)(u \otimes v) = Y(u,z)v$ depending on (infinitesimally) small $z \in \mathbb{C} \setminus 0$, where $u \star_z v$ may have poles in z at $z = 0$, roughly satisfying $(u \star_{z_1} v) \star_{z_2} w = u \star_{z_1+z_2} (v \star_{z_2} w)$, and other identities.

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We explain the state-field approach next lecture. Today we use the definition of vertex algebra due to Richard Borcherds, who invented them, as it is easy to write down, though difficult to interpret.

Borcherds' vertex algebras (Will see this again later)

Here is the original definition of vertex algebra from Borcherds 1986. Let R be a commutative ring (often $R=\mathbb{C}$). A *vertex algebra* over R is an R-module V equipped with morphisms $D^{(n)}:V\to V$ for $n=0,1,2,\ldots$ with $D^{(0)}=\mathrm{id}_V$ and $v_n:V\to V$ for all $v\in V$ and $n\in\mathbb{Z}$, with v_n R-linear in v, and a distinguished element $\mathbb{1}\in V$ called the *identity* or *vacuum vector*, satisfying:

(i) For all $u,v\in V$ we have $u_n(v)=0$ for $n\gg 0$.

(ii) If $v\in V$ then $\mathbb{1}_{-1}(v)=v$ and $\mathbb{1}_n(v)=0$ for $-1\neq n\in\mathbb{Z}$.

(iii) If $v\in V$ then $v_n(\mathbb{1})=D^{(-n-1)}(v)$ for $n<0$ and $v_n(\mathbb{1})=0$ for $n\geqslant 0$.

(iv) $u_n(v)=\sum_{k\geqslant 0}(-1)^{k+n+1}D^{(k)}(v_{n+k}(u))$ for all $u,v\in V$ and $n\in\mathbb{Z}$, where the sum makes sense by (i), as it has only finitely many nonzero terms.

(v) $$(u_l(v))_m(w)=\sum_{n\geqslant 0}(-1)^n\binom{l}{n}\bigl(u_{l-n}(v_{m+n}(w))-(-1)^l v_{l+m-n}(u_n(w))\bigr)$$

for all $u,v,w\in V$ and $l,m\in\mathbb{Z}$, where the sum makes sense by (i).

Remarks on the definition

- The relation with the state-field definition is that

$$Y(z)(u\otimes v)=Y(u,z)v=\textstyle\sum_{n\in\mathbb{Z}}u_n(v)z^{-n-1}.$$

- Condition (i), that $u_n(v)=0$ for $n\gg 0$, is needed for many formulae to make sense. We do not require $u_n(v)=0$ for $n\ll 0$. So $Y(z)$ maps $V\otimes V\to V[[z]][z^{-1}]$ (more next lecture).
- One can show from the axioms that for $m,n\geqslant 0$

$$D^{(m)}\circ D^{(n)}=\tbinom{m+n}{n}D^{(m+n)}.$$

Hence if R has characteristic 0 (i.e. $\mathbb{Q}\subseteq R$) then by induction $D^{(n)}=\frac{1}{n!}D^n$ for all $n\geqslant 0$, with $D=D^{(1)}$. Thus, when $\mathrm{char}\,R=0$ the operators $D^{(n)}$ simplify to a single operator D, which is called the *translation operator*. By (iii) the $D^{(m)}$ are determined by the operations $(u,v)\mapsto u_n(v)$ and $\mathbb{1}$, so the $D^{(m)}$ are not extra data.
- You can think of the operations $(u,v)\mapsto u_n(v)$ as being like an infinite family of Lie brackets, and (v) as like the Jacobi identity.

What are vertex algebras for? (A first attempt.)

Some reasons to study vertex algebras:

- Vertex algebras come up in *Moonshine:* the mathematics around the Monster (a finite simple group of order $\approx 10^{54}$). One definition of the Monster is as the automorphism group of a certain vertex algebra.

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- Given a vertex algebra, one can define a Lie algebra (see §1.4).

For some classes of interesting infinite-dimensional Lie algebras (e.g. Kac–Moody algebras), perhaps the best way to construct them is to first construct a vertex algebra, and then pass to the Lie algebra. Borcherds invented vertex algebras starting from Lie algebra definitions.

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- Given a vertex algebra, one can define a Lie algebra (see §1.4). For some classes of interesting infinite-dimensional Lie algebras (e.g. Kac–Moody algebras), perhaps the best way to construct them is to first construct a vertex algebra, and then pass to the Lie algebra. Borcherds invented vertex algebras starting from Lie algebra definitions.
- Vertex algebra notation can express some special ∞-dimensional Lie algebras very succinctly, using one, or just a few, vertex operators.
- Vertex algebras are important in Physics: roughly, given a vertex (operator) algebra, one can build a kind of Conformal Field Theory. CFTs are supposed to quantize maps $u : \Sigma \to X$ for Riemann surfaces Σ, with X a classical space-time. The formal complex variable z in $Y(z)$ is roughly a local formal coordinate on Σ.

1.1. Basic definitions and constructions

Definition 1.1 (Borcherds' 1986 definition again)

Let R be a commutative ring. A *vertex algebra* (*VA*) over R is an R-module V equipped with morphisms $D^{(n)}: V \to V$ for $n = 0, 1, 2, \ldots$ with $D^{(0)} = \mathrm{id}_V$ and $v_n : V \to V$ for all $v \in V$ and $n \in \mathbb{Z}$, with v_n R-linear in v, and a distinguished element $\mathbb{1} \in V$ called the *identity* or *vacuum vector*, satisfying:

(i) For all $u, v \in V$ we have $u_n(v) = 0$ for $n \gg 0$.

(ii) If $v \in V$ then $\mathbb{1}_{-1}(v) = v$ and $\mathbb{1}_n(v) = 0$ for $-1 \neq n \in \mathbb{Z}$.

(iii) If $v \in V$ then $v_n(\mathbb{1}) = D^{(-n-1)}(v)$ for $n < 0$ and $v_n(\mathbb{1}) = 0$ for $n \geqslant 0$.

(iv) $u_n(v) = \sum_{k \geqslant 0} (-1)^{k+n+1} D^{(k)}(v_{n+k}(u))$ for all $u, v \in V$ and $n \in \mathbb{Z}$, where the sum makes sense by (i), as it has only finitely many nonzero terms.

(v) $$(u_l(v))_m(w) = \sum_{n \geqslant 0} (-1)^n \binom{l}{n} \bigl(u_{l-n}(v_{m+n}(w)) - (-1)^l v_{l+m-n}(u_n(w))\bigr)$$

for all $u, v, w \in V$ and $l, m \in \mathbb{Z}$, where the sum makes sense by (i).

Vertex superalgebras and graded vertex algebras

Often it is important to take a vertex algebra $V = V_*$ to be graded over $\mathbb{Z}_2$ (i.e. $V = V_{\underline{0}} \oplus V_{\underline{1}}$ is a *super* vector space) or over $\mathbb{Z}$ (i.e. $V = \bigoplus_{n\in\mathbb{Z}} V_n$), with operations graded, and with sign changes (red).

Definition 1.2

Let R be a commutative ring, and V_* be an R-module graded over $\mathbb{Z}_2$ or $\mathbb{Z}$. We call V_* a *vertex superalgebra* (for $\mathbb{Z}_2$ grading) or *graded vertex algebra*) (for $\mathbb{Z}$ grading) if V_* is equipped with R-linear morphisms $D^{(n)} : V_k \to V_{k+2n}$ for $n = 0, 1, 2, \ldots,$ and $v_n : V_b \to V_{a+b-2n-2}$ for all $v \in V_a$ and a, b in $\mathbb{Z}_2$ or $\mathbb{Z}$ and $n \in \mathbb{Z}$, and an *identity* $\mathbb{1} \in V_0$, satisfying Definition 1.1(i)–(iii) and:

(iv)$'$ $u_n(v) = \sum_{k\geqslant 0}(-1)^{ab+k+n+1} D^{(k)}(v_{n+k}(u))$ for all $u \in V_a$, $v \in V_b$, with a, b in $\mathbb{Z}_2$ or $\mathbb{Z}$ and $n \in \mathbb{Z}$.

(v)$'$ $(u_l(v))_m(w) = \sum_{n\geqslant 0}(-1)^n \binom{l}{n} \bigl(u_{l-n}(v_{m+n}(w)) - (-1)^{ab+l} v_{l+m-n}(u_n(w))\bigr)$ for all a, b, c in $\mathbb{Z}_2$ or $\mathbb{Z}$, $u \in V_a$, $v \in V_b$, $w \in V_c$ and $l, m \in \mathbb{Z}$.

If $V_* = \bigoplus_{n\in\mathbb{Z}} V_n$ is a graded vertex algebra then $V_{\underline{0}} \oplus V_{\underline{1}}$ is a vertex superalgebra, where $V_{\underline{0}} = \bigoplus_{k\in\mathbb{Z}} V_{2k}$ and $V_{\underline{1}} = \bigoplus_{k\in\mathbb{Z}} V_{2k+1}$. If $V_* = V_{\underline{0}} \oplus V_{\underline{1}}$ is a vertex superalgebra then $V_{\underline{0}}$ is an ordinary vertex algebra. Definitions and constructions for vertex algebras basically always extend to vertex superalgebras/graded vertex algebras in an obvious way, with sign changes like $(-1)^{ab}$ in (iv)′,(v)′.

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$$\begin{aligned} D^{(n)}(v \oplus w) &= (D^{(n)}v) \oplus (D^{(n)}w), \\ (v \oplus w)_n(v' \oplus w') &= (v_n(v')) \oplus (w_n(w')), \\ \mathbb{1}_{V\oplus W} &= \mathbb{1}_V \oplus \mathbb{1}_W. \end{aligned}$$

This extends to vertex superalgebras and graded vertex algebras.

Tensor products of vertex algebras

If V, W are vertex algebras over R we can make the *tensor product* $V \otimes_R W$ into a vertex algebra, with

$$D^{(n)}(v \otimes w) = \sum_{k=0}^{n} \binom{n}{k} (D^{(k)} v) \otimes (D^{(n-k)} w),$$

$$(v \otimes w)_n(v' \otimes w') = \sum_{k \in \mathbb{Z}} (v_k(v')) \otimes (w_{n-k-1}(w')),$$

$$\mathbb{1}_{V \otimes W} = \mathbb{1}_V \otimes \mathbb{1}_W.$$

This extends to vertex superalgebras and graded vertex algebras. Note that for $(v \otimes w)_n(v' \otimes w')$, the sum is finite as $v_k(v') = 0$ for $k \gg 0$, and $w_{n-k-1}(w') = 0$ for $k \ll 0$.

Morphisms, ideals, quotient vertex algebras

If V, W are vertex algebras over R, a *morphism* $\phi : V \to W$ is an R-module morphism preserving all the structure, i.e. $\phi(\mathbb{1}_V) = \mathbb{1}_W$, $D^{(n)} \circ \phi(v) = \phi \circ D^{(n)}(v)$, $(\phi(v))_n(\phi(v')) = \phi(v_n(v'))$. For vertex superalgebras and graded vertex algebras we also require ϕ to preserve gradings.

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An *ideal* I in a vertex algebra V is an R-submodule $I \subseteq V$ such that if $u \in I$ and $v \in V$ then $D^{(n)}u, u_n(v), v_n(u) \in I$ for all n.

Then the quotient V/I has a unique vertex algebra structure such that $\pi : V \twoheadrightarrow V/I$ is a (surjective) vertex algebra morphism.

(Multiple) intersections of ideals are ideals. If $S \subset V$ is any subset, there is a unique smallest ideal $I_S = \bigcap_{S \subseteq I, \ I \text{ ideal}} I$ containing S, the ideal generated by S, and $V_S = V/I_S$ is a vertex algebra.

A vertex algebra V is *simple* if it has no nonzero ideals.

All this extends to vertex superalgebras and graded vertex algebras.

Representations of vertex algebras

Let V be a vertex algebra over R. A *representation* of V is an R-module W and R-module morphisms $v_n^\rho : W \to W$ for all $v \in V$ and $n \in \mathbb{Z}$, with v_n^ρ R-linear in v, satisfying:

(i) For all $v \in V$ and $w \in W$ we have $v_n^\rho(w) = 0$ for $n \gg 0$.

(ii) If $w \in W$ then $\mathbb{1}_{-1}^\rho(w) = w$ and $\mathbb{1}_n^\rho(w) = 0$ for $-1 \neq n \in \mathbb{Z}$.

(v) $$(u_l(v))_m^\rho(w) = \sum_{n \geqslant 0} (-1)^n \binom{l}{n} \left(u_{l-n}^\rho(v_{m+n}^\rho(w)) - (-1)^l v_{l+m-n}^\rho(u_n^\rho(w)) \right)$$

for all $u, v \in V$, $w \in W$ and $l, m \in \mathbb{Z}$, where the sum exists by (i).

These are the obvious generalizations of Definition 1.1(i),(ii),(v).

Definition 1.1(iii),(iv) do not make sense for representations.

There are obvious notions of morphism, subrepresentation, quotient representation, irreducible representation, etc. V has an obvious representation on itself.

All this extends to vertex superalgebras and graded vertex algebras.

1.2. Vertex operator algebras
The Virasoro algebra (needed for vertex operator algebras)

Let R be a field of characteristic zero. The *Virasoro algebra* Vir is the Lie algebra over R with basis elements L_n, $n \in \mathbb{Z}$ and c (the *central charge*), and Lie bracket

$$[c, L_n] = 0, \ [L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}(m^3-m)\delta_{m,-n}c, \ m, n \in \mathbb{Z}.$$

The factor $\frac{1}{12}$ is a convention, and can be eliminated by replacing c by $12c$. To define the Virasoro algebra Vir_R over a general commutative ring R we omit $\frac{1}{12}$, in case $\frac{1}{12} \notin R$.
The quotient $\mathrm{Vir}/\langle c\rangle$ is called the *Witt algebra*, and may be regarded as the Lie algebra of complex vector fields on the circle $\mathcal{S}^1$ when $R = \mathbb{C}$. The Virasoro algebra is the unique central extension of the Witt algebra. It is very important in Conformal Field Theory and String Theory.

Vertex operator algebras (VOAs)

Vertex operator algebras are vertex algebras with an extra structure. Arguably, they are the most important kind of vertex algebra.

Definition 1.3

Let R be a field of characteristic zero. A *vertex operator algebra* (*VOA*, or *conformal vertex algebra*) over R is a graded vertex algebra $V_* = \bigoplus_{a\in\mathbb{Z}} V_a$ over R as in Definition 1.2, with a distinguished *conformal element* $\omega \in V_4$ and a *central charge* $c_{V_*} \in R$, such that writing $L_n = \omega_{n+1} : V_* \to V_*$, we have

(i) $L_{-1} = D^{(1)} = D$ is the translation operator.

(ii) $L_0|_{V_a} = \frac{1}{2} a \cdot \mathrm{id}_{V_a}$ for $a \in \mathbb{Z}$.

(iii) $[L_m, L_n] = (m-n)L_{m+n} + \frac{1}{12} c_{V_*}(m^3 - m)\delta_{m,-n}\,\mathrm{id}_{V_*}$ for $m, n \in \mathbb{Z}$. That is, the L_n define an action of the *Virasoro algebra* on V_*, with central charge c_{V_*}.

Remarks on vertex operator algebras

- A graded vertex algebra V_* need not admit a conformal element ω, and if ω exists it may not be unique.
- Many authors also impose additional conditions on VOAs, such as $\dim V_n < \infty$ and $V_n = 0$ for $n \ll 0$, but we do not do this.
- The $\mathbb{Z}$-grading on V_* is determined by (ii), so it is not extra data.
- Physicists nearly always care about VOAs, not VAs. VAs/VOAs are connected to CFTs, which quantize maps $u : \Sigma \to X$ for Riemann surfaces Σ, with X a classical space-time. The formal complex variable z in $Y(z)$ is roughly a local formal coordinate on Σ. The VOA structure is something to do with the CFT being independent of choice of coordinates on Σ, it is physically essential. The Virasoro algebra appears as a central extension of a Lie algebra of changes of formal coordinate near a point, I think.
- Many VAs of importance in mathematics are also VOAs.

Why the Virasoro algebra? (Best answer I've got for now.)

Proposition 1.4 (Frenkel and Ben-Zvi Lem. 3.4.5)

Let R be a field of characteristic zero, V_* a graded vertex algebra over R, and $\omega \in V_4$. Then ω is a conformal element for V_*, making V_* into a vertex operator algebra with central charge $c_{V_*} \in R$, if and only if:

(a) $\omega_0 = D$ *is the translation operator.*
(b) $\omega_1|_{V_a} = \frac{1}{2}a \cdot \mathrm{id}_{V_a}$ *for* $a \in \mathbb{Z}$.
(c) $\omega_3(\omega) = \frac{1}{2}c_{V_*} \cdot \mathbb{1}$.

Here we can take (b) as defining the grading on V_*, and (c) as defining the central charge c_{V_*}. So the important condition is (a). Roughly, the proposition says that (a) forces $L_n = \omega_{n+1}$ to satisfy the Virasoro algebra relations. So the Virasoro algebra is already somehow hidden in the structure of vertex algebras, it is not being imposed from the outside in an arbitrary way.

1.3. Commutative vertex algebras

Definition 1.5

A vertex algebra V over R is called *commutative* if $u_n(v) = 0$ for all $u, v \in V$ and $n \geqslant 0$. This implies that $Y(u, z_1)$ and $Y(v, z_2)$ commute as operators on V for all $u, v \in V$.

Given a commutative VA V, define $* : V \times V \to V$ by $u * v = u_{-1}(v)$. This is commutative and associative, and makes V a commutative R-algebra with identity $\mathbb{1}$. The translation operator $D : V \to V$ is a *derivation* of this algebra, i.e. $D(u * v) = (Du) * v + u * (Dv)$. Conversely, if R is a $\mathbb{Q}$-algebra, given a commutative R-algebra $(V, \mathbb{1}, *)$ with a derivation $D : V \to V$, defining $D^{(n)} = \frac{1}{n!} D^n$ and

$$u_n(v) = 0, \quad n \geqslant 0, \quad u_n(v) = \left(\tfrac{1}{(-n-1)!} D^{-n-1}(u)\right) * v, \quad n < 0,$$

gives a commutative vertex algebra over R.

This extends to vertex superalgebras and graded vertex algebras, giving equivalences of categories for R a $\mathbb{Q}$-algebra:

commutative vertex R-algebras	$\Longleftrightarrow$	commutative R-algebras with derivation,
commutative vertex R-superalgebras	$\Longleftrightarrow$	supercommutative R-superalgebras with even derivation,
commutative graded vertex R-algebras	$\Longleftrightarrow$	graded commutative R-algebras with degree 2 derivation.

Commutative VAs are an easy class of examples. Unfortunately:

Proposition 1.6

Let R be a field of characteristic zero and V a non-commutative vertex algebra over R. Then V is infinite-dimensional.

So all non-commutative VAs are complicated, and take quite a lot of work to write down.

1.4. Lie algebras from vertex algebras

Here is an important connection between vertex and Lie algebras:

Proposition 1.7 (Borcherds 1986)

Let V be a vertex algebra over R. Write $D(V) \subset V$ for the R-submodule generated by $D^{(n)}v$ for all $v \in V$ and $n > 0$. If $\mathrm{char}\, R = 0$ *then $D^{(n)} = \frac{1}{n!}D^n$, so $D(V) = \{D(v) : v \in V\}$. Then the quotient R-module $V/D(V)$ has the structure of a* **Lie algebra over** *R, with Lie bracket*

$$\big[u + D(V), v + D(V)\big] = u_0(v) + D(V). \tag{1.1}$$

Similarly, vertex superalgebras V_* over R yield **Lie superalgebras over** *R, and graded vertex superalgebras V_* over R yield* **graded Lie algebras over** *R, but with the shifted grading $\big(V/D(V)\big)_n = V_{n+2}/D(V)_{n+2}$, where $D(V)_{n+2} = D(V_n)$ when* $\mathrm{char}\, R = 0$. *In particular, $\big(V/D(V)\big)_0$ is a Lie algebra over R.*

Proof of Proposition 1.7 (vertex algebra case)

- To prove $[\,,\,]$ in (1.1) is well defined, use the identities in any VA:

$$\begin{aligned}(D^{(m)}(u))_l(v) &= (-1)^m\tbinom{l}{m}\cdot u_{l-m}(v),\\ u_l\bigl(D^{(m)}(v)\bigr) &= \textstyle\sum_{k=0}^m \tbinom{l}{m-k}\cdot D^{(k)}(u_{k+l-m}(v)).\end{aligned}$$

In particular, for $l=0$ and $m>0$ these imply that

$$(D^{(m)}(u))_0(v)=0,\quad u_0\bigl(D^{(m)}(v)\bigr)=D^{(m)}(u_0(v)),$$

so changing u or v by an element of $D(V)$ changes $u_0(v)$ by an element of $D(V)$.

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- To prove $[\,,\,]$ is antisymmetric, by Definition 1.1(iv)

$$u_0(v) = \textstyle\sum_{k\geqslant 0}(-1)^{k+1} D^{(k)}(v_k(u)) = -v_0(u) + \text{terms in } D(V).$$

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- To prove the Jacobi identity for $[\,,\,]$, by Definition 1.1(v)

$$(u_0(v))_0(w) = u_0(v_0(w)) - v_0(u_0(w)).$$

Remarks on vertex algebras and Lie algebras

- Quite often, one can construct graded vertex algebras V_*, such that $\bigl(V/D(V)\bigr)_0$ is an interesting (usually infinite-dimensional) Lie algebra that representation theorists care about. It may be difficult to build $\bigl(V/D(V)\bigr)_0$ without going via the vertex algebra V_*. For example, lattice vertex algebras V_* yield Kac–Moody Lie algebras $\bigl(V/D(V)\bigr)_0$, which generalize finite dim'l semisimple Lie algebras.

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- If V is a vertex algebra and W is a representation of V then the identity from §1.1

$$(u_l(v))^\rho_m(w)=\sum_{n\geqslant 0}(-1)^n\binom{l}{n}\bigl(u^\rho_{l-n}(v^\rho_{m+n}(w))-(-1)^l v^\rho_{l+m-n}(u^\rho_n(w))\bigr)$$

restricts when $l=m=0$ to

$$(u_0(v))^\rho_0(w)=u^\rho_0(v^\rho_0(w))-v^\rho_0(u^\rho_0(w)).$$

Using this we can show that W is a representation of the (perhaps super or graded) Lie algebra $V/D(V)$. So the representation theories of V and $V/D(V)$ are closely linked.

Vertex Algebras

Lecture 2 of 8: Vertex algebras in terms of formal power series

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Summer term 2021

These slides available at
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Plan of talk:

Introduction

Last lecture we defined vertex algebras V following Borcherds, whose structure is determined by an identity $\mathbb{1} \in V$ and operations $(u, v) \mapsto u_n(v)$ for $n \in \mathbb{Z}$. These are usually encoded in one operator

$$Y(z)(u \otimes v) = Y(u, z)v = \sum_{n\in\mathbb{Z}} u_n(v)z^{-n-1},$$

where $Y(z) : V \otimes V \to V[[z]][z^{-1}]$ or $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$ is called the *state-field correspondence*. Today we explain this notation. We need to start with a lot of preliminaries on formal power series spaces like $V[[z]][z^{-1}]$ and operations on them.

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where $Y(z) : V \otimes V \to V[[z]][z^{-1}]$ or $Y : V \to \mathrm{End}(V)[[z, z^{-1}]]$ is called the *state-field correspondence*. Today we explain this notation. We need to start with a lot of preliminaries on formal power series spaces like $V[[z]][z^{-1}]$ and operations on them.

It turns out that $Y(z)$ satisfies many extra identities on top of the original defining identities, and that various subsets of these identities imply all the rest. Therefore there are several equivalent definitions of vertex algebra, depending which identities you choose as primary, and different authors use different definitions.

2.1. Formal power series notation

Throughout let R be a commutative ring (sometimes we want a field of characteristic zero) and V be an R-module (vector space). We define R-modules $V[z], V[[z]], \ldots$ to be the R-modules of formal expressions $v(z) = \sum_{n\in\mathbb{Z}} v_n z^n$ for $v_n \in V$, with vanishing conditions on the v_n as follows:

(i) $V[z]$: $v_n = 0$ if $n < 0$ or $n \gg 0$.
(ii) $V[[z]]$: $v_n = 0$ if $n < 0$.
(iii) $V[z, z^{-1}]$: $v_n = 0$ if $n \ll 0$ or $n \gg 0$.
(iv) $V[[z]][z^{-1}]$ (also written $V((z))$): $v_n = 0$ if $n \ll 0$.
(v) $V[[z, z^{-1}]]$: no vanishing conditions on v_n.

Think of $V[z]$ as polynomials with values in V, $V[[z]]$ as V-valued formal power series, $V[z, z^{-1}]$ as V-valued Laurent polynomials, etc. Note that for $V[[z]], V[[z]][z^{-1}], V[[z, z^{-1}]]$ it does not make sense to give z an actual value in R or $R \setminus 0$, it is just a formal variable.

We can also introduce similar spaces in several variables, such as $V[x,y]$, etc. As a complicated example, consider

$$V[[x,y]][x^{-1},y^{-1},(x-y)^{-1}]=V[[x,y]]\otimes_{R[x,y]}R[x^{\pm 1},y^{\pm 1},(x-y)^{-1}].$$

Elements of this space may be written as $x^a y^b (x-y)^c v(x,y)$ for $v(x,y)\in V[[x,y]]$ and $a,b,c\in\mathbb{Z}$, with the relations for $l,m,n\geqslant 0$

$$x^a y^b (x-y)^c v(x,y) = x^{a-l} y^{b-m} (x-y)^{c-n}\bigl(x^l y^m (x-y)^n v(x,y)\bigr).$$

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$$x^a y^b(x-y)^c v(x,y)=x^{a-l}y^{b-m}(x-y)^{c-n}\bigl(x^l y^m(x-y)^n v(x,y)\bigr).$$

We can multiply by functions, take tensor products, etc., but we must be careful about *doubly infinite* series such as $V[[z,z^{-1}]]$. For example, $R[z], R[z,z^{-1}], R[[z]][z^{-1}]$ are all rings under multiplication $fg(z)=f(z)g(z)$, and if R is a field then $R[[z]][z^{-1}]$ is a field. But $R[[z,z^{-1}]]$ is *not* a ring, as $f(z)g(z)$ is not always defined. For example, $f(z)=\sum_{n\in\mathbb{Z}}z^n$ lies in $R[[z^{\pm 1}]]$, but $f(z)^2$ does not make sense as the coefficient of each z^n is a sum with infinitely many nonzero terms. We want sums to have finitely many nonzero terms, we don't use convergence.

Observe that $R[[x^{\pm 1}, y^{\pm 1}]]$ is a module over the ring $R[x^{\pm 1}, y^{\pm 1}]$, but has zero divisors. For example, $\sum_{n\in\mathbb{Z}} x^n y^{-n} \in R[[x^{\pm 1}, y^{\pm 1}]]$ and $x-y \in R[x^{\pm 1}, y^{\pm 1}]$ with $(x-y)\cdot\bigl(\sum_{n\in\mathbb{Z}} x^n y^{-n}\bigr) = 0$. Because of this there is no unique way to write expressions like $(x-y)^{-1}$ as series in $R[[x^{\pm 1}, y^{\pm 1}]]$. Two obvious guesses would be

$$(x-y)^{-1} \sim \sum_{n\geqslant 0} x^{-n-1} y^n, \; (x-y)^{-1} = -(y-x)^{-1} \sim -\sum_{n\geqslant 0} y^{-n-1} x^n,$$

by expansion using the Binomial Theorem. Define morphisms

$$i_{x,y}, i_{y,x} : V[[x,y]][x^{-1}, y^{-1}, (x-y)^{-1}] \longrightarrow V[[x^{\pm 1}, y^{\pm 1}]]$$

such that $i_{x,y}$ replaces $(x-y)^{-1}$ by $\sum_{n\geqslant 0} x^{-n-1} y^n$, and $i_{y,x}$ replaces $(x-y)^{-1}$ by $-\sum_{n\geqslant 0} y^{-n-1} x^n$. That is, $i_{x,y} v(x,y)$ is the power series expansion of $v(x,y)$ in the region $|x| > |y|$, and $i_{y,x} v(x,y)$ the power series expansion in the region $|y| > |x|$.

The *delta function* $\delta(x-y) \in R[[x^{\pm 1}, y^{\pm 1}]]$ is

$$\delta(x-y) = \sum_{n\in\mathbb{Z}} x^{-n-1} y^n = i_{x,y}(x-y)^{-1} - i_{y,x}(x-y)^{-1}.$$

Then $(x-y)\cdot\delta(x-y) = 0$.

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Then $(x-y)\cdot\delta(x-y) = 0$.
If $v(z) = \sum_{n\in\mathbb{Z}} v_n z^n \in V[[z,z^{-1}]]$, its *residue* is $\mathrm{Res}_z\, v(z) = v_{-1}$.
It satisfies $\mathrm{Res}_z\big(\frac{\mathrm{d}}{\mathrm{d}z}v(z)\big) = 0$ for any $v(z)$. We can also take residues in one of several variables, for example

$$\mathrm{Res}_x : V[[x^{\pm 1}, y^{\pm 1}]] \longrightarrow V[[y^{\pm 1}]]$$

maps $x^{-1}y^n \mapsto y^n$ and $x^m y^n \mapsto 0$ for $m \neq -1$. The delta function has the property that for all $v(x) \in V[[x, x^{-1}]]$ and $k \geqslant 0$

$$\mathrm{Res}_x\Big(v(x)\frac{\partial^k}{\partial x^k}\delta(x-y)\Big) = (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}y^k}v(y) \quad \text{in } V[[y, y^{-1}]].$$

Fields

A *field* $a(z)$ on V is an R-linear morphism $a(z) : V \to V[[z]][z^{-1}]$. It is usual to write $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ where $a_{(n)} \in \mathrm{End}(V)$. But this expression only says that $a(z) \in \mathrm{End}(V)[[z, z^{-1}]]$, or equivalently that $a(z)$ is a morphism $a(z) : V \to V[[z, z^{-1}]]$. Requiring $a(z)$ to map to $V[[z]][z^{-1}]$ (that is, for $a(z)$ to be a field) is equivalent to the condition that for each $v \in V$ there exists N_v with $a_{(n)}(v) = 0$ for all $n > N_v$. But if V is infinite-dimensional, there may be no such $N_v = N$ independent of v for all v (this would mean that $a(z) \in \mathrm{End}(V)[[z]][z^{-1}]$).

2.2. Defining vertex algebras using formal power series

The next definition is equivalent to Definition 1.1:

Definition 2.1

A *vertex algebra* $(V, \mathbb{1}, e^{zD}, Y)$ over a commutative ring R is an R-module V with an *identity element* $\mathbb{1} \in V$, and R-linear operators $e^{zD} : V \to V[[z]]$ written $e^{zD} v = \sum_{n \geqslant 0} D^{(n)}(v)\, z^n$ for $D^{(n)} \in \mathrm{End}(V)$, with $D^{(0)} = \mathrm{id}_V$, and $Y : V \otimes V \to V[[z]][z^{-1}]$ written $Y(z)(u \otimes v) = Y(u,z)v = \sum_{n \in \mathbb{Z}} u_n(v) z^{-n-1}$, satisfying:

(i) $Y(\mathbb{1}, z)v = v$ for all $v \in V$.

(ii) $Y(v, z)\mathbb{1} = e^{zD} v$ for all $v \in V$.

(iii) For all $u, v, w \in V$, in $V[[z_0^{\pm 1}, z_1^{\pm 1}, z_2^{\pm 1}]]$ we have

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2)w = z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2)w - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1)w. \tag{2.1}$$

Equation (2.1) is called the **Jacobi identity**.

Remarks on the definition

- In physics language, V is the space of *states*. Then Y maps states v to fields $Y(v,z)$ on V. We call Y the *state-field correspondence*.
- e^{zD} is determined by (ii), so is not really extra data.
- Definition 1.1(i) says that for $u,v\in V$ we have $u_n(v)=0$ for $n\gg 0$. This is encoded in Y mapping $Y:V\otimes V\to V[[z]][z^{-1}]$, rather than $Y:V\otimes V\to V[[z,z^{-1}]]$ for example.
- Definition 2.1(i),(ii) are equivalent to Definition 1.1(ii),(iii).
- Taking the coefficient of $z_0^{-l-1}z_1^{-1}z_2^{-m-1}$ in (2.1) yields Definition 1.1(v). One can also deduce Definition 1.1(iv) from (2.1).
- Equation (2.1) is an exact identity. Later we meet 'weak' identities which hold only after multiplying by $(x-y)^N$ for $N\gg 0$, for example.
- There are versions of Definition 2.1 for *vertex superalgebras* and *graded vertex algebras*. We take z graded of degree -2, and then $\mathbb{1}\in V_0$ and e^{zD}, $Y(z)$ are grading-preserving. If $u\in V_a$, $v\in V_b$, $w\in V_c$ then the final term in (2.1) has an extra sign $(-1)^{ab}$.

Weak commutativity

For proofs of the next theorems, see e.g. the books by Frenkel–Huang–Lepowsky, Kac, and Lepowksy–Li. We state them for vertex algebras, but analogues hold in the super and graded cases.

Theorem 2.2 (Weak commutativity)

Let $(V, \mathbb{1}, e^{zD}, Y)$ be a vertex algebra over R. Then for all $u, v, w \in V$ there exists $N \geqslant 0$ depending only on u, v such that in $V[[z_1^{\pm 1}, z_2^{\pm 1}]]$ we have

$$(z_1 - z_2)^N \big[Y(u, z_1) \circ Y(v, z_2) w - Y(v, z_2) \circ Y(u, z_1) w \big] = 0. \quad (2.2)$$

That is, $Y(u, z_1)$ and $Y(v, z_2)$ commute in a weak sense. Note that this is nontrivial even if $u = v$.

Equation (2.2) would be a mess to write down in Borcherds' notation. Note that $(z_1 - z_2)^N \cdot \frac{\partial^k}{\partial z_1^k} \delta(z_1 - z_2) = 0$ for $0 \leqslant k < N$, so $[\cdots]$ in (2.2) could (and must) be a sum of terms $\frac{\partial^k}{\partial z_1^k} \delta(z_1 - z_2) \cdot f_k(z_1)$.

Weak associativity

Theorem 2.3 (Weak associativity)

Let $(V, \mathbb{1}, e^{zD}, Y)$ be a vertex algebra over R. Then for all $u, v, w \in V$ there exists $N \geqslant 0$ depending only on u, v such that in $V[[z_1^{\pm 1}, z_2^{\pm 1}]]$ we have the **weak associativity** *property*

$$\begin{aligned}(z_1 + z_2)^N Y(Y(u, z_1)v, z_2)w \\ = (z_1 + z_2)^N i_{z_1,z_2} \circ Y(u, z_1 + z_2) \circ Y(v, z_2)w.\end{aligned} \tag{2.3}$$

If we write $v \star_z w = Y(v, z)w$ this says that modulo weird formal power series issues we have $(u \star_{z_1} v) \star_{z_2} w \approx u \star_{z_1+z_2} (v \star_{z_2} w)$.

Again, (2.3) would be a mess to write down in Borcherds' notation.

Skew symmetry, translation covariance, exponential

Theorem 2.4

Let $(V, \mathbb{1}, e^{zD}, Y)$ be a vertex algebra over R, and $u, v \in V$. Then:

(a) *The* **skew symmetry** *property holds*

$$Y(u,z)v = e^{zD} \circ Y(v,-z)u. \tag{2.4}$$

(b) *The* **translation covariance** *properties hold*

$$e^{z_2 D} \circ Y(u,z_1) \circ e^{-z_2 D}(v) = i_{z_1,z_2} \circ Y(u, z_1+z_2)v, \tag{2.5}$$

$$Y(e^{z_2 D}u, z_1)v = i_{z_1,z_2} \circ Y(u, z_1+z_2)v. \tag{2.6}$$

(c) *The* **exponential** *property holds*

$$e^{(z_1+z_2)D}(v) = e^{z_1 D} \circ e^{z_2 D}(v). \tag{2.7}$$

Equivalent definitions of vertex algebra

Various combinations of (2.1)–(2.7) imply the rest:

Theorem 2.5

The following are equivalent conditions on $(V, \mathbb{1}, e^{zD}, Y)$:
(a) $(V, \mathbb{1}, e^{zD}, Y)$ *is a vertex algebra as in Definition 2.1, in particular, the Jacobi identity* (2.1) *holds. This definition of vertex algebra is used by Frenkel–Lepowksy–Meurmann 1988.*
(b) $(V, \mathbb{1}, e^{zD}, Y)$ *satisfies Definition 2.1(i)-(ii), weak commutativity* (2.2), *and the first translation covariance property* (2.5). *This definition of vertex algebra is used by Kac 1998 and Frenkel–Ben-Zvi 2004.*
(c) $(V, \mathbb{1}, e^{zD}, Y)$ *satisfies Definition 2.1(i)-(ii), weak associativity* (2.3), *and skew symmetry* (2.4). *This definition of vertex algebra appears in Bakalov–Kac 2003.*

2.3. Vertex algebras via meromorphic functions

Theorem 2.6 (Based on Anguelova–Bergvelt, N. Kim, and ...)

Let $(V, \mathbb{1}, e^{zD}, Y)$ *be a vertex algebra over* R. *Then for all* $n \geqslant 1$ *and* $v_1, \ldots, v_{n+1} \in V$ *there exists* $N \geqslant 0$ *such that*

$$\Big[\prod_{1\leqslant i<j\leqslant n}(z_i - z_j)^N\Big]\cdot Y(v_1, z_1)\circ\cdots\circ Y(v_n, z_n)v_{n+1}$$

lies in $V[[z_1, \ldots, z_n]][z_1^{-1}, \ldots, z_n^{-1}]$. *Hence for all* $n \geqslant 0$ *there exist unique* R*-linear maps, where by convention* $V^{\otimes^0} = R$,

$$X_n(z_1, \ldots, z_n) : V^{\otimes^n} \longrightarrow V[[z_1, \ldots, z_n]][(z_i - z_j)^{-1} : i < j], \quad (2.8)$$

such that for all $v_1, \ldots, v_n \in V$ *we have*

$$\begin{aligned} &i_{z_1,z_2,\ldots,z_n}(X_n(z_1, \ldots, z_n)(v_1 \otimes \cdots \otimes v_n)) \\ &\qquad = Y(v_1, z_1)\circ Y(v_2, z_2)\circ\cdots\circ Y(v_n, z_n)\mathbb{1}, \end{aligned} \quad (2.9)$$

where $i_{z_1,z_2,\ldots,z_n}$ *expands* $(z_i - z_j)^{-d}$ *for* $i < j$ *and* $d > 0$ *in nonnegative powers of* z_j, *generalizing* $i_{x,y}$ *in* §2.1.

Theorem 2.6 (... Frenkel–Huang–Lepowsky. Continued.)

Furthermore these X_n satisfy:

(a) *If $1 \leqslant i \leqslant m$, $n \geqslant 0$ and $v_a, w_b \in V$ then*

$$\begin{aligned} &X_m(y_1,\ldots,y_m)(v_1\otimes\cdots\otimes v_{i-1}\otimes \\ &X_n(z_1,\ldots,z_n)(w_1\otimes\cdots\otimes w_n)\otimes v_{i+1}\otimes\cdots\otimes v_m) = \\ &i_{\boldsymbol{y},\boldsymbol{z}}\bigl(X_{m+n-1}(y_1,\ldots,y_{i-1},z_1+y_i,\ldots,z_n+y_i,y_{i+1},\ldots,y_m) \\ &(v_1\otimes\cdots\otimes v_{i-1}\otimes w_1\otimes\cdots\otimes w_n\otimes v_{i+1}\otimes\cdots\otimes v_m)\bigr) \end{aligned} \tag{2.10}$$

in $V[[y_1,\ldots,y_m]][(y_j-y_k)^{-1}:j<k][[z_1,\ldots,z_n]][(z_j-z_k)^{-1}:j<k]$.

(b) *For all $n \geqslant 0$, $\sigma \in S_n$ and $v_1,\ldots,v_n \in V$ we have*

$$\begin{aligned} &X_n(z_1,\ldots,z_n)(v_1\otimes\cdots\otimes v_n) \\ &= X_n(z_{\sigma(1)},\ldots,z_{\sigma(n)})(v_{\sigma(1)}\otimes\cdots\otimes v_{\sigma(n)}). \end{aligned} \tag{2.11}$$

(c) $X_0(a) = a\,\mathbb{1}$ *for all* $a \in R$.

(d) $X_1(z) = e^{zD} : V \to V[[z]]$. *In particular,* $X_1(0) = \mathrm{id}_V$.

(e) $Y(z) = X_2(z,0) : V\otimes V \to V[[z]][z^{-1}]$.

Theorem 2.6 gives us another way to define vertex algebras:

Definition 2.7

A *vertex algebra over* R is a triple $(V, \mathbb{1}, X_*)$ where V is an R-module, $X_* = (X_n)_{n\geqslant 0}$ with each X_n an R-module morphism

$$X_n(z_1,\ldots,z_n) : V^{\otimes^n} \longrightarrow V[[z_1,\ldots,z_n]][(z_i-z_j)^{-1} : i<j], \quad (2.12)$$

satisfying Theorem 2.6(a)–(c). This is equivalent to a vertex algebra $(V, \mathbb{1}, e^{zD}, Y)$ from §2.2 as in Theorem 2.6(d)–(e).

If we work with *graded* vertex algebras $(V_*, \mathbb{1}, X_*)$ then the component of $X_n(z_1,\ldots,z_n)$ in V_k for each $k\in\mathbb{Z}$ lies in $V_k[z_1,\ldots,z_n][(z_i-z_j)^{-1} : i<j]$, that is, it is a *genuine meromorphic function*, not just a formal function, and say over $R=\mathbb{C}$ we could substitute actual complex numbers for $z_1,\ldots,z_n$.

The Borcherds or $(V,\mathbb{1},e^{zD},Y)$ definitions of vertex algebras have the advantage of economy: they put the least information in the definition. The $(V,\mathbb{1},X_*)$ definition requires more data, but has other advantages. For example, weak commutativity for $N\gg 0$

$$(z_1-z_2)^N\bigl[Y(u,z_1)\circ Y(v,z_2)w-Y(v,z_2)\circ Y(u,z_1)w\bigr]=0$$

is replaced by the exact symmetry in $V[[z_1,z_2]][z_1^{-1},z_2^{-1},(z_1-z_2)^{-1}]$

$$X_3(z_1,z_2,0)(u\otimes v\otimes w)=X_3(z_2,z_1,0)(v\otimes u\otimes w). \tag{2.13}$$

That is, $Y(u,z_1)\circ Y(v,z_2)w$ and $Y(v,z_2)\circ Y(u,z_1)w$ are both power series expansions of the same meromorphic (2.13) in the two different regions $|z_1|>|z_2|>0$ and $|z_2|>|z_1|>0$.
In general, the $(V,\mathbb{1},X_*)$ definition tends to replace 'weak' identities by exact identities in meromorphic functions. It also makes clear where the poles are.

2.4. Ways to explain vertex algebras

Borcherds (1997) motivates vertex algebras this way: think of a vertex algebra $(V, \mathbb{1}, e^{zD}, Y)$ as like a commutative ring V with identity $\mathbb{1}$, (partially defined) multiplication $\star$, and an action $\rho : \mathbb{C} \to \mathrm{End}(V)$ of the additive group $\mathbb{C}$ preserving $\mathbb{1}, \star$. Then e^{zD}, Y and X_n in §2.3 are written in terms of $\star, \rho$ by

$$\begin{gathered}\rho(z)v = e^{zD}v = \sum_{n\geqslant 0} z^n D^{(n)}(v), \quad Y(v,z)w = (\rho(z)v)\star w, \\ X_n(z_1,\ldots,z_n)(v_1\otimes\cdots\otimes v_n) = (\rho(z_1)v_1)\star\cdots\star(\rho(z_n)v_n).\end{gathered} \tag{2.14}$$

The catch is that $\star$ is 'singular', so $u \star v$ is not always defined, and in particular $(\rho(z)u)\star v$ might have poles in z.

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$$\begin{gathered}\rho(z)v = e^{zD}v = \sum_{n\geqslant 0} z^n D^{(n)}(v), \quad Y(v,z)w = (\rho(z)v) \star w, \\ X_n(z_1,\ldots,z_n)(v_1 \otimes \cdots \otimes v_n) = (\rho(z_1)v_1) \star \cdots \star (\rho(z_n)v_n).\end{gathered} \tag{2.14}$$

The catch is that $\star$ is 'singular', so $u \star v$ is not always defined, and in particular $(\rho(z)u) \star v$ might have poles in z.
Actually I don't find Borcherds' attempt to make all this rigorous very helpful. But this picture is good for justifying the identities in §2.2–§2.3, modulo weird power series issues. For example:

- Definition 2.1(i) $Y(\mathbb{1},z)v=v$ becomes $(\rho(z)\mathbb{1})\star v=\mathbb{1}\star v=v$.
- If weak commutativity held in the strong sense it would become $Y(u,z_1)\circ Y(v,z_2)w=Y(v,z_2)\circ Y(u,z_1)w$. This corresponds to

$$\bigl(\rho(z_1)(u)\bigr)\star\bigl((\rho(z_2)(v))\star w\bigr)=\bigl(\rho(z_2)(v)\bigr)\star\bigl((\rho(z_1)(u))\star w\bigr),$$

as $\star$ is commutative and associative.
- If weak associativity held in the strong sense it would become $Y(Y(u,z_1)v,z_2)w=Y(u,z_1+z_2)\circ Y(v,z_2)w$. This corresponds to

$$\bigl(\rho(z_2)(\rho(z_1)(u)\star v)\bigr)\star w=\bigl(\rho(z_1+z_2)(u)\bigr)\star\bigl((\rho(z_2)(v))\star w\bigr).$$

- Skew symmetry $Y(u,z)v=e^{zD}\circ Y(v,-z)u$ becomes $(\rho(z)u)\star v=\rho(z)\circ\rho(-z)\circ\bigl(v\star(\rho(z)u)\bigr)=\rho(z)\bigl((\rho(-z)v)\star u\bigr)$.
- Translation covariance $Y(e^{z_2D}u,z_1)v=i_{z_1,z_2}\circ Y(u,z_1+z_2)v$ becomes $\rho(z_1)(\rho(z_2)(u))\star v=(\rho(z_1+z_2)(u))\star v$.

Physical meaning of vertex operator algebras???

I am very confused, so all the following may be lies ...
As I understand it, a VOA $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ roughly corresponds to a kind of QFT in Physics. See Huang 1991 for an attempt to make this precise. Consider the following class of objects $(\Sigma, \boldsymbol{p}, \boldsymbol{x})$: a compact Riemann surface Σ, with $n+1$ marked points $p_0, \ldots, p_n$ where $p_1, \ldots, p_n$ are 'inputs' and p_0 the 'output', and choices of local (formal?) holomorphic coordinates $x_0, \ldots, x_n$ on Σ, with x_j defined near p_j with $x_j|_{p_j} = 0$. To each such triple $(\Sigma, \boldsymbol{p}, \boldsymbol{x})$ we would like to associate a morphism $F_{(\Sigma,\boldsymbol{p},\boldsymbol{x})} : V^{\otimes^n} \to V$ (possibly up to scale?), where $F_{(\Sigma,\boldsymbol{p},\boldsymbol{x})}(v_1 \otimes \cdots \otimes v_n)$ means we insert states $v_1, \ldots, v_n$ from V_* at the marked points $p_1, \ldots, p_n$, and then the physics outputs a state $F_{(\Sigma,\boldsymbol{p},\boldsymbol{x})}(v_1 \otimes \cdots \otimes v_n)$ at p_0.

We would like to make this independent of the choices of centred local coordinates $x_0, \ldots, x_n$ in the following sense: there should be a group G of (germs of centred) coordinate changes with a (projective) representation on V, and acting on $x_0, \ldots, x_n$ by $g_0, \ldots, g_n \in G$ should change $F_{(\Sigma,\boldsymbol{p},\boldsymbol{x})}(v_1 \otimes \cdots \otimes v_n)$ to $F_{(\Sigma,\boldsymbol{p},\boldsymbol{g}\cdot\boldsymbol{x})}(v_1 \otimes \cdots \otimes v_n) = g_0^{-1} \cdot F_{(\Sigma,\boldsymbol{p},\boldsymbol{x})}(g_1 \cdot v_1 \otimes \cdots \otimes g_n \cdot v_n)$. At the Lie algebra level, G corresponds (roughly) to the Witt algebra W, with central extension the Virasoro algebra Vir, so as the VOA V_* is a representation of Vir with central charge c_{V_*}, it is a projective representation of W.

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There should be composition rules for the $F_{(\Sigma,\boldsymbol{p},\boldsymbol{x})}$ as follows: given $(\Sigma, \boldsymbol{p}, \boldsymbol{x})$ and $(\Sigma', \boldsymbol{p}', \boldsymbol{x}')$, suppose the domains of x_k for $k > 0$ and x_0' include closed balls $\overline{B}_r(0)$, $\overline{B}_{1/r}(0)$ for $r > 0$ containing no other x_j, x_j'. We make a new triple $(\Sigma'', \boldsymbol{p}'', \boldsymbol{x}'')$ by gluing $\Sigma \setminus x_k(B_r(0))$ and $\Sigma' \setminus x_0'(B_{1/r}(0))$ along their boundary circle. Then $F_{(\Sigma'',\boldsymbol{p}'',\boldsymbol{x}'')}$ corresponds to substituting $F_{(\Sigma',\boldsymbol{p}',\boldsymbol{x}')}$ into the k^{th} input of $F_{(\Sigma,\boldsymbol{p},\boldsymbol{x})}$.

Then $Y(z): V \otimes V \to V[[z]][z^{-1}]$ should correspond to $F_{(\Sigma,\boldsymbol{p},\boldsymbol{x})}$ for $\Sigma = \mathbb{CP}^1 = \mathbb{C} \amalg \{\infty\}$, $\boldsymbol{p} = (p_0, p_1, p_2) = (\infty, z, 0)$, $x_0(y) = 1/y$, $x_1(y) = y + z$, $x_2(y) = y$, where z varies in $\mathbb{CP}^1$ and we expect singularities at $z = 0, \infty$ when p_1 collides with p_0 or p_2. More generally, $X_n(z_1, \ldots, z_n)$ should correspond to $F_{(\Sigma,\boldsymbol{p},\boldsymbol{x})}$ for $\Sigma = \mathbb{CP}^1$, $\boldsymbol{p} = (\infty, z_1, \ldots, z_n)$, $x_0(y) = 1/y$, $x_k(y) = y + z_k$, $k = 1, \ldots, n$.