Vertex Algebras

Lecture 3 of 8: Locality, vertex operators, OPEs, and examples

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These slides available at
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Plan of talk:

Introduction

Recall from §2.2 that a vertex algebra $(V, \mathbb{1}, e^{zD}, Y)$ satisfies *weak commutativity*: for all $u, v, w \in V$ there exists $N \geqslant 0$ depending only on u, v such that

$$(z_1 - z_2)^N \big[Y(u, z_1) \circ Y(v, z_2) w - Y(v, z_2) \circ Y(u, z_1) w \big] = 0. \quad (3.1)$$

Here $Y(u, z), Y(v, z)$ are *fields* on V, and (3.1) is a compatibility condition between them. If it holds we say that $Y(u, z), Y(v, z)$ are *mutually local*. This is nontrivial even if $u = v$: we can require a field $a(z)$ such as $Y(v, z)$ to be local with itself, and then we call $a(z)$ a *vertex operator*. Weak commutativity says that the operators $Y(v, z)$ for all $v \in V$ are mutually local vertex operators on V.

Today we explore the approach to vertex algebras which emphasizes local fields, and weak commutativity as the primary property of vertex algebras.

3.1. Local fields

Definition 3.1

Let R be a commutative ring, and V an R-module. We call $a(z)=\sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$ in $\mathrm{End}(V)[[z,z^{-1}]]$ a *distribution* on V. As in §2.1, we call $a(z)$ a *field* on V if it maps $V\to V[[z]][z^{-1}]$. Equivalently, $a(z)$ is a field if for all $v\in V$ there exists N_v with $a_{(n)}(v)=0$ for $n\geqslant N_v$.

Let $a(z),b(z)$ be distributions or fields on V. We call a,b *mutually local* if there exists $N\gg 0$ such that

$$(z-w)^N[a(z),b(w)]=0 \quad \text{in } \mathrm{End}(V)[[z,w,z^{-1},w^{-1}]], \tag{3.2}$$

where $[a(z),b(w)]=a(z)\circ b(w)-b(w)\circ a(z)$. We call a field $a(z)$ a *local field*, or *vertex operator*, if it is mutually local with itself.

Remarks on locality

- The name comes from QFT in physics: we think of $a(z)$ and $b(w)$ as two operators living at points z, w in the Riemann surface $\Sigma = \mathbb{C}$, and locality says that $a(z)$ and $b(w)$ commute if z and w are spacelike separated. That is, if $z \neq w$ then (3.2) becomes $[a(z), b(w)] = 0$.
- We can show that $(z-w)^N[a(z), b(w)] = 0$ if and only if

$$[a(z), b(w)] = \sum_{n=0}^{N-1} c_n(w) \frac{1}{n!} \frac{\partial^n}{\partial w^n} \delta(z-w)$$

for some distributions $c_0(w), \ldots, c_{N-1}(w)$. Note that $\frac{1}{n!}\frac{\partial^n}{\partial w^n}$ is a well defined operator over any R, we don't need $\mathbb{Q} \subseteq R$.
- If instead V_* is a super/graded R-module and a, b are of pure grading we define the supercommutator to be

$$[a(z), b(w)] = a(z) \circ z(w) - (-1)^{\deg a \deg b} b(w) \circ a(z).$$

We should use these to extend this lecture to vertex superalgebras / graded vertex algebras, but for simplicity we do not.

Defining vertex algebras using mutually local fields

Theorem 2.5(b) may now be rewritten:

Theorem 3.2

Let V be an R-module, $\mathbb{1} \in V$ and $e^{zD} : V \to V[[z]]$, $Y : V \to V[[z, z^{-1}]]$ be R-linear maps. Then $(V, \mathbb{1}, e^{zD}, Y)$ is a vertex algebra if for all $u, v \in V$ we have

(i) $Y(\mathbb{1}, z)v = v$.
(ii) $Y(v, z)\mathbb{1} = e^{zD}v$.
(iii) $e^{z_2 D} \circ Y(u, z_1) \circ e^{-z_2 D}(v) = i_{z_1,z_2} \circ Y(u, z_1 + z_2)v$.
(iv) *$Y(u, z)$ and $Y(v, z)$ are mutually local vertex operators on V.*

If we have defined some $(V, \mathbb{1}, e^{zD}, Y)$ and want to show it is a vertex algebra, usually (i)-(iii) are easy, and the difficult thing is to prove (iv). We explain methods for showing fields are mutually local.

Products $a(w)_{(n)}b(w)$, normally ordered products

Definition 3.3

Let V be an R-module, and $a(z), b(z)$ be fields on V. For $n \in \mathbb{Z}$ define a field $a(w)_{(n)}b(w)$ on V by

$$a(w)_{(n)}b(w) = \mathrm{Res}_z\big(a(z)b(w)i_{z,w}(z-w)^n - b(w)a(z)i_{w,z}(z-w)^n\big).$$

For $n \in \mathbb{N}$ this simplifies to

$$a(w)_{(n)}b(w) = \mathrm{Res}_z\big([a(z), b(w)](z-w)^n\big).$$

Writing $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)}z^{-n-1}$, define distributions $a(z)_\pm$ on V by

$$a(z)_+ = \sum_{n<0} a_{(n)}z^{-n-1}, \quad a(z)_- = \sum_{n\geqslant 0} a_{(n)}z^{-n-1}.$$

Define the *normally ordered product* $:a(z)b(w):$ by

$$:a(z)b(w): = a(z)_+b(w) + b(w)a(z)_-.$$

This maps $:a(z)b(w): : V \to V[[z,w]][z^{-1},w^{-1}]$. We may set $z = w$, and $:a(z)b(z):$ is a field, with $:a(z)b(z): = a(z)_{(-1)}b(z)$.

If $(V, \mathbb{1}, e^{zD}, Y)$ is a vertex algebra and $u, v \in V$ then $Y(u,z)$ and $Y(v,z)$ are mutually local fields on V, and for $n \in \mathbb{Z}$ we have

$$Y(u,z)_{(n)} Y(v,z) = Y(u_n(v), z). \tag{3.3}$$

Theorem 3.4 (Kac 1997, §2.3.)

Suppose $a(z), b(z)$ are mutually local fields on V. Then for $N \gg 0$ (the same N as in (3.1)) we have $a(z)_{(n)} b(z) = 0$ for $n \geqslant N$, and

$$[a(z), b(w)] = \textstyle\sum_{n=0}^{N-1} a(w)_{(n)} b(w) \cdot \frac{1}{n!} \frac{\partial^n}{\partial w^n} \delta(z-w), \tag{3.4}$$

$$a(z) \circ b(w) = \textstyle\sum_{n=0}^{N-1} a(w)_{(n)} b(w) \cdot i_{z,w} \frac{1}{(z-w)^{n+1}} + :a(z)b(w): , \tag{3.5}$$

$$b(w) \circ a(z) = \textstyle\sum_{n=0}^{N-1} a(w)_{(n)} b(w) \cdot i_{w,z} \frac{1}{(z-w)^{n+1}} + :a(z)b(w): . \tag{3.6}$$

Operator product expansions

If $a(z), b(z)$ are mutually local fields, equation (3.5) says that

$$a(z)\circ b(w)=\sum_{n=0}^{N-1} a(w)_{(n)}b(w)\cdot i_{z,w}\frac{1}{(z-w)^{n+1}}+:a(z)b(w):\,.$$

Here $:a(z)b(w):$ has no pole when $z=w$. We write the singular part as

$$a(z)\circ b(w)\sim\sum_{n=0}^{N-1} a(w)_{(n)}b(w)\cdot\frac{1}{(z-w)^{n+1}}. \qquad (3.7)$$

This is called an *operator product expansion* (*OPE*), and is important in physics. Theorem 3.4 shows that the r.h.s. of (3.7) is the obstruction to $a(z)$ and $b(w)$ strictly commuting.

Sometimes vertex algebras can be described in an economical way by specifying a small number of generating fields, and the OPEs (3.7) relating them, in a similar way to specifying a Lie algebra by generators and relations.

Theorem 3.5 (Kac 1997, §3.2. Part (a) is 'Dong's Lemma'.)

(a) *Suppose* $a(z), b(z), c(z)$ *are pairwise mutually local fields on* V. *Then* $a(z)_{(n)}b(z)$ *and* $c(z)$ *are mutually local for* $n \in \mathbb{Z}$, *and* $:a(z)b(z):$ *and* $c(z)$ *are mutually local.*
(b) *If* $a(z), b(z)$ *are mutually local then* $\frac{1}{n!}\frac{\partial^n}{\partial z^n}a(z), b(z)$ *are too.*

Theorem 3.5 can be used to construct larger and larger sets of mutually local fields, and so to build vertex algebras.

Theorem 3.6 (Goddard's Uniqueness Thm, Frenkel–Ben-Zvi §3.1.)

Suppose $(V, \mathbb{1}, e^{zD}, Y)$ *is a vertex algebra and* $a(z)$ *is a field on* V *such that* $a(z), Y(v,z)$ *are mutually local for all* $v \in V$, *and there exists* $u \in V$ *with* $a(z)\mathbb{1} = Y(u,z)\mathbb{1} = e^{zD}u$. *Then* $a(z) = Y(u,z)$.

Again, this is helpful for building vertex algebras.

3.2. The Reconstruction Theorem

Theorem 3.7 (Reconstruction Theorem, Kac Th. 4.5, F–B-Z §3.6)

Let R be a field of characteristic zero, and V be an R-vector space. Suppose we are given an element $\mathbb{1} \in V$, a linear map $D : V \to V$, and a countable family $\{a^\alpha(z) : \alpha \in A\}$ of fields on V such that:

(i) $a^\alpha(z)\mathbb{1} \in V[[z]]$ *for all* $\alpha \in A$, *so we set* $a^\alpha = a^\alpha(0)\mathbb{1}$ *in* V.

(ii) $D(\mathbb{1}) = 0$, *and* $[D, a^\alpha(z)] = \frac{\mathrm{d}}{\mathrm{d}z} a^\alpha(z)$ *for all* $\alpha \in A$.

(iii) $a^\alpha(z), a^\beta(z)$ *are mutually local for all* $\alpha, \beta \in A$.

(iv) V *is spanned by the vectors* $a^{\alpha_1}_{(n_1)} \circ \cdots \circ a^{\alpha_m}_{(n_m)}(\mathbb{1})$ *for all* $m \geqslant 0$, $\alpha_1, \ldots, \alpha_m$ *in* A *and* $n_1, \ldots, n_m < 0$.

Then there exists a unique vertex algebra $(V, \mathbb{1}, e^{zD}, Y)$ *with* $a^\alpha(z) = Y(a^\alpha, z)$ *for all* $\alpha \in A$.

Remarks on the Reconstruction Theorem

- This is a generators-and-relations approach to vertex algebras.
- Even though V will be infinite-dimensional, A may be small, even just one or two points. So the main data is a vector space V and a small number of fields $a^\alpha(z)$ on V, which may be a lot less work to write down than the entire structure $(V, \mathbb{1}, e^{zD}, Y)$.
- If (iv) does not hold, replace V by the subspace spanned by all vectors $a^{\alpha_1}_{(n_1)} \circ \cdots \circ a^{\alpha_m}_{(n_m)}(\mathbb{1})$.
- The proof works by defining Y to satisfy

$$Y\bigl(a^{\alpha_1}_{(n_1)} \circ \cdots \circ a^{\alpha_m}_{(n_m)}(\mathbb{1}), z\bigr) = \frac{1}{(-n_1-1)!\cdots(-n_m-1)!} : \frac{\mathrm{d}^{-n_1-1}}{\mathrm{d}z^{-n_1-1}} a^{\alpha_1}(z) \cdots \frac{\mathrm{d}^{-n_m-1}}{\mathrm{d}z^{-n_m-1}} a^{\alpha_m}(z) : ,$$

where $:\cdots:$ extends normal ordering $:a(z)b(z):$ inductively to m operators. Use Theorem 3.5 to deduce these are all mutually local.

3.3. Examples of vertex algebras

Example 3.8 (The Heisenberg vertex algebra, or rank 1 free boson)

Let $R=\mathbb{C}$ and $V=\mathbb{C}[x_1,x_2,\ldots]$ be the space of polynomials in $x_1,x_2,\ldots$. Let $\mathbb{1}\in V$ be the polynomial 1, and $D:V\to V$ act by

$$D\bigl(p(x_1,x_2\ldots)\bigr)=\sum_{n\geqslant 1} nx_{n+1}\frac{\partial}{\partial x_n}p(x_1,x_2\ldots). \tag{3.8}$$

Define a field $a(z)=\sum_{n\in\mathbb{Z}}a_{(n)}z^{-n-1}$ on V, where $a_{(n)}:V\to V$ is a $\mathbb{C}$-linear map, by

$$a_{(n)}:p(x_1,x_2\ldots)\longmapsto\begin{cases} x_{-n}p(x_1,x_2\ldots), & n<0, \\ 0, & n=0, \\ n\frac{\partial}{\partial x_n}p(x_1,x_2\ldots), & n>0. \end{cases} \tag{3.9}$$

The Reconstruction Theorem applies with $\{a^\alpha(z):\alpha\in A\}=\{a(z)\}$ and $a=a(0)\mathbb{1}=x_1$, so there is a unique vertex algebra $(V,\mathbb{1},e^{zD},Y)$ with $a(z)=Y(x_1,z)$.

Example 3.8 (Continued.)

- Can show directly that $(z-w)^2[a(z),a(w)]=0$, so $a(z)$ is local.
- We have $a(z)_{(0)}a(z)=0$ and $a(z)_{(1)}a(z)=\mathrm{id}_V$, so we have the OPE $a(z)a(w)\sim \mathrm{id}_V \frac{1}{(z-w)^2}$.
- We can make V into a *graded* vertex algebra in even degrees by setting $\deg x_n = 2n$.
- V is a *vertex operator algebra* with (nonunique) conformal vector $\omega_s=\frac{1}{2}x_1^2+sx_2$ for any $s\in\mathbb{C}$, and central charge $c_V=1-12s^2$.
- As elements of $\mathrm{End}(V)$, the Fourier coefficients $a_{(n)}$ satisfy

$$\left[a_{(m)},a_{(n)}\right]=m\delta_{m,-n}\,\mathrm{id}_V\,. \tag{3.10}$$

Therefore $\langle a_{(n)}, n\in\mathbb{Z},\ \mathrm{id}_V\rangle_{\mathbb{C}}$ is the *Heisenberg Lie algebra* $\mathfrak{H}$, with centre $\langle a_{(0)},\mathrm{id}_V\rangle_{\mathbb{C}}$. Representation theorists care about $\mathfrak{H}$, and tell us there is a unique irreducible representation of $\mathfrak{H}$ in which $a_{(0)},\mathrm{id}_V$ act by $0,1$, which is V. Note that the infinite-dimensional Lie algebra $\mathfrak{H}$ has been encoded in the single vertex operator $a(z)$.

Example 3.9 (The Virasoro vertex algebra)

As in §1.2, the *Virasoro algebra* Vir is the Lie algebra over $\mathbb{C}$ with basis elements L_n, $n\in\mathbb{Z}$ and c (the *central charge*), and Lie bracket

$$[c,L_n]=0,\;\; [L_m,L_n]=(m-n)L_{m+n}+\tfrac{1}{12}(m^3-m)\delta_{m,-n}c \quad (3.11)$$

for $m,n\in\mathbb{Z}$. Let $\mathfrak{K}=\langle L_n,\; n\geqslant -1, c\rangle_{\mathbb{C}}\subset\mathrm{Vir}$, a Lie subalgebra. For each $\gamma\in\mathbb{C}$ define a representation ρ_γ of $\mathfrak{K}$ on $\mathbb{C}$ by $\rho_\gamma(L_n)=0$ for $n\geqslant -1$ and $\rho_\gamma(c)=\gamma\,\mathrm{id}_{\mathbb{C}}$. Let $V_\gamma=\mathbb{C}\otimes_{\rho_\gamma,U(\mathfrak{K}),\mathrm{inc}}U(\mathrm{Vir})$ be the induced representation of Vir, and let $\mathbb{1}\in V_\gamma$ be the image of the generator $1\in\mathbb{C}$. Regard the L_n as lying in $\mathrm{End}(V_\gamma)$, via the representation of Vir on V_γ. Then $L_n(\mathbb{1})=0$ for $n\geqslant -1$ by definition. Define $D=L_{-1}:V\rightarrow V$. Define a field $T(z)$ on V_γ by $T(z)=\sum_{n\in\mathbb{Z}}L_nz^{-n-2}$ (not the usual power of z). Then $(z-w)^4[T(z),T(w)]=0$, so $T(z)$ is local. Set $\omega=L_{-2}\mathbb{1}$ in V. The Reconstruction Theorem applies with $\{a^\alpha(z):\alpha\in A\}=\{T(z)\}$ and $T(0)\mathbb{1}=\omega$, so there is a unique vertex algebra $(V_\gamma,\mathbb{1},e^{zD},Y)$ with $T(z)=Y(\omega,z)$.

Example 3.9 (Continued)

- $(V_\gamma, \mathbb{1}, e^{zD}, Y)$ is a vertex operator algebra with conformal element ω and central charge γ. This works for each $\gamma \in \mathbb{C}$.
- The field $T(z)$ has OPE

$$T(z) \circ T(w) \sim \frac{\gamma}{2} \frac{\mathrm{id}_{V_\gamma}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\frac{\mathrm{d}T}{\mathrm{d}w}(w)}{z-w}, \qquad (3.12)$$

so by Theorem 3.4 we have

$$[T(z), T(w)] = \qquad (3.13)$$

$$\frac{\gamma}{12} \mathrm{id}_{V_\gamma} \frac{\partial^3}{\partial w^3} \delta(z-w) + 2T(w) \frac{\partial}{\partial w} \delta(z-w) + \frac{\mathrm{d}T}{\mathrm{d}w}(w) \delta(z-w).$$

This is equivalent to the defining relations (3.11) of the Virasoro algebra, with $c = \gamma\, \mathrm{id}_{V_\gamma}$. Note that the single vertex operator $T(z)$ encodes the Virasoro algebra, and its OPE (3.12) the Lie bracket of the Virasoro algebra, without knowing V_γ. Vertex operators and OPEs can characterize interesting Lie algebras very succinctly.

Example 3.10 (Vertex algebras from affine Lie algebras)

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}[t,t^{-1}]$ is an infinite-dimensional Lie algebra: think of it as Laurent polynomials $\gamma(t):\mathbb{C}^*\rightarrow\mathfrak{g}$ with Lie bracket $[\gamma,\delta](t)=[\gamma(t),\delta(t)]$, and restricting to $\mathcal{S}^1\subset\mathbb{C}^*$, this is basically the Lie algebra of the loop group LG. The *affine Lie algebra* $\hat{\mathfrak{g}}$ is a nontrivial central extension of Lie algebras

$$0\longrightarrow\langle c\rangle_{\mathbb{C}}\longrightarrow\hat{\mathfrak{g}}\longrightarrow\mathfrak{g}[t,t^{-1}]\longrightarrow 0,$$

defined using the Killing form $\langle\,,\,\rangle$ of $\mathfrak{g}$. Let the Lie subalgebra $\mathfrak{K}\subset\hat{\mathfrak{g}}$ be the preimage of $\mathfrak{g}[t]$, so $\mathfrak{K}\cong\langle c\rangle_{\mathbb{C}}\oplus\mathfrak{g}[t]$ as Lie algebras. Extend this to a vector space splitting $\hat{\mathfrak{g}}\cong\langle c\rangle_{\mathbb{C}}\oplus\mathfrak{g}[t,t^{-1}]$. For each $k\in\mathbb{C}$ define a representation ρ_k of $\mathfrak{K}$ on $\mathbb{C}$ by $\rho_k(\mathfrak{g}[t])=0$ and $\rho_\gamma(c)=k\,\mathrm{id}_{\mathbb{C}}$. We call k the 'level'. Let $V_k^{\hat{\mathfrak{g}}}=\mathbb{C}\otimes_{\rho_k,U(\mathfrak{K}),\mathrm{inc}}U(\hat{\mathfrak{g}})$ and $\sigma:\hat{\mathfrak{g}}\rightarrow\mathrm{End}(V_k^{\hat{\mathfrak{g}}})$ be the induced representation of $\hat{\mathfrak{g}}$, and let $\mathbb{1}\in V_k^{\hat{\mathfrak{g}}}$ be the image of $1\in\mathbb{C}$.

Example 3.10 (Continued)

For each $\gamma \in \mathfrak{g}$, define a field $a^\gamma(z)$ on $V_k^{\hat{\mathfrak{g}}}$ by

$$a^\gamma(z) = \sum_{n \in \mathbb{Z}} \sigma(\gamma t^n) z^{-n-1}. \tag{3.14}$$

The $a^\gamma(z)$ for $\gamma \in \mathfrak{g}$ are mutually local, with OPE

$$a^\gamma(z) a^\delta(w) \sim k \,\mathrm{id}_{V_k^{\hat{\mathfrak{g}}}} \frac{\langle \gamma, \delta \rangle}{(z-w)^2} + a^{[\gamma,\delta]}(w) \frac{1}{z-w}. \tag{3.15}$$

so by Theorem 3.4 we have

$$[a^\gamma(z), a^\delta(w)] = k \langle \gamma, \delta \rangle \,\mathrm{id}_{V_k^{\hat{\mathfrak{g}}}} \tfrac{\partial}{\partial w} \delta(z-w) + a^{[\gamma,\delta]}(w) \delta(z-w). \tag{3.16}$$

For D I won't give, the Reconstruction Theorem applies with $\{a^\alpha(z) : \alpha \in A\} = \{a^{\gamma_i}(z) : i = 1, \ldots, n\}$ for $\gamma_1, \ldots, \gamma_n$ a basis of $\mathfrak{g}$, giving a vertex algebra $(V_k^{\hat{\mathfrak{g}}}, \mathbb{1}, e^{zD}, Y)$. If k is not a certain critical value $k = -h^\vee$, this is a vertex operator algebra. The OPEs (3.15) for $\gamma, \delta \in \{\gamma_1, \ldots, \gamma_n\}$ encode the Lie algebra $\hat{\mathfrak{g}}$ by (3.16).

Example 3.11 (Lattice vertex algebras)

Let (Λ,χ) be an even lattice. That is, $\Lambda\cong\mathbb{Z}^d$ is a free abelian group of rank d and $\chi:\Lambda\times\Lambda\to\mathbb{Z}$ is biadditive and symmetric with $\chi(\lambda,\lambda)\in 2\mathbb{Z}$ for all $\lambda\in\Lambda$, and write $\Lambda_{\mathbb{C}}=\Lambda\otimes_{\mathbb{Z}}\mathbb{C}$. Define

$$V_\Lambda=\mathbb{C}[\Lambda]\otimes_{\mathbb{C}}\mathrm{Sym}(t\Lambda_{\mathbb{C}}[t]). \tag{3.17}$$

Here $\mathbb{C}[\Lambda]$ is the group algebra of Λ, a $\mathbb{C}$-vector space with basis e^λ for $\lambda\in\Lambda$, and $\mathrm{Sym}(W)=\bigoplus_{n\geqslant 0}S^nW$ the symmetric algebra. So V_Λ is a $\mathbb{C}$-algebra spanned over $\mathbb{C}$ by elements of the form

$$e^{\lambda_0}\otimes(t^{a_1}\lambda_1)\otimes\cdots\otimes(t^{a_n}\lambda_n) \tag{3.18}$$

for $\lambda_0,\ldots,\lambda_n\in\Lambda$ and $a_1,\ldots,a_n>0$, and generated by elements $e^\lambda\otimes 1$ and $e^0\otimes(t^a\lambda)$ for $0\neq\lambda\in\Lambda$ and $a>0$. We make V_Λ graded by giving (3.18) degree $\chi(\lambda_0,\lambda_0)+2a_1+\cdots+2a_n$ (note this is even). Define $\mathbb{1}=e^0\otimes 1$, where $1\in S^0(t\Lambda_{\mathbb{C}}[t])=\mathbb{C}$. Choose signs $\epsilon_{\lambda,\mu}=\pm 1$ for $\lambda,\mu\in\Lambda$ satisfying $\epsilon_{\lambda,0}=\epsilon_{0,\lambda}=1$ and

$$\epsilon_{\lambda,\mu}\cdot\epsilon_{\mu,\lambda}=(-1)^{\chi(\lambda,\mu)+\chi(\lambda,\lambda)\chi(\mu,\mu)},\ \epsilon_{\lambda,\mu}\cdot\epsilon_{\lambda+\mu,\nu}=\epsilon_{\lambda,\mu+\nu}\cdot\epsilon_{\mu,\nu}.$$

Example 3.11 (Continued)

Define $D : V_\Lambda \to V_\Lambda$ to be the $\mathbb{C}$-algebra derivation with

$$D(e^\lambda \otimes 1) = e^\lambda \otimes (t\lambda), \quad D(e^0 \otimes (t^a\lambda)) = a\, e^0 \otimes (t^{a+1}\lambda).$$

For each $\mu \in \Lambda$ and $n \in \mathbb{Z}$, define $\mu_n : V_\Lambda \to V_\Lambda$ by

(i) If $n > 0$ then $\mu_n : V_\Lambda \to V_\Lambda$ is the derivation of V_Λ determined by

$$\mu_n(e^\lambda \otimes 1) = 0, \quad \mu_n(e^0 \otimes (t^a\lambda)) = n\delta_{an}\chi(\mu,\lambda)\, e^0 \otimes 1.$$

(ii) $\mu_0(e^\lambda \otimes p) = \chi(\mu,\lambda)\, e^\lambda \otimes p$ for any λ, p.

(iii) If $n < 0$ then μ_n is multiplication by $e^0 \otimes (t^{-n}\mu)$.

Define $\mu(z) = \sum_{n\in\mathbb{Z}} \mu_n z^{-n-1}$. Define $\tilde\mu(z) : V_\Lambda \to V_\Lambda[[z]][z^{-1}]$ by

$$\tilde\mu(z)(e^\lambda \otimes p) = \epsilon_{\lambda,\mu} z^{\chi(\lambda,\mu)}(e^\mu \otimes 1)\cdot \exp\Bigl[-\sum_{n<0} \tfrac{1}{n} z^{-n}\mu_n\Bigr] \circ \exp\Bigl[-\sum_{n>0} \tfrac{1}{n} z^{-n}\mu_n\Bigr](e^\lambda \otimes p).$$

Then $\mu(z), \tilde\mu(z)$ are fields on V_Λ for $\mu \in \Lambda$. The Reconstruction Theorem applies with $\{a^\alpha(z) : \alpha \in A\} = \{\mu_1(z),\ldots,\mu_d(z),\tilde\mu_1(z),\ldots,\tilde\mu_d(z)\}$ for $\mu_1,\ldots,\mu_d$ a basis of Λ, giving a vertex algebra $(V_\Lambda, \mathbb{1}, e^{zD}, Y)$, a *lattice vertex algebra*.

Example 3.11 (Continued)

- If χ is nondegenerate then V_Λ is a vertex operator algebra with central change $c_{V_\Lambda} = d$ and conformal vector

$$\omega = \tfrac{1}{2} \sum_{i,j=1}^{d} A_{ij} e^0 \otimes (t\mu_i) \otimes (t\mu_j),$$

where $(A_{ij})_{i,j=1}^d$ is the inverse matrix to $(\chi(\mu_i, \mu_j))_{i,j=1}^d$.

- For non-even lattices the construction generalizes to vertex superalgebras / non-even graded vertex algebras.
- The subspace $e^0 \otimes \mathrm{Sym}(-) \subset V_\Lambda$ is a vertex subalgebra of V_Λ. If χ is nondegenerate then by choosing an orthonormal basis of $(\Lambda_{\mathbb{C}}, \chi_{\mathbb{C}})$ we may identify $e^0 \otimes \mathrm{Sym}(-)$ with the tensor product of d copies of the Heisenberg vertex algebra in Example 3.8.
- If Λ is the lattice of root vectors of a simple Lie algebra $\mathfrak{g}$ then V_Λ is a simple vertex algebra (has no nontrivial ideals) and is the unique simple quotient of $V_1^{\hat{\mathfrak{g}}}$ in Example 3.10 for at level $k = 1$.
- The Monster vertex algebra is related to V_Λ for Λ the rank 24 Leech lattice.

Vertex Algebras

Lecture 4 of 8: Representation theory of vertex algebras

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References for this lecture: Frenkel and Ben-Zvi, 2nd ed. (2004), §5,
Y. Zhu, *Modular invariance of characters of vertex operator algebras*, J. A.M.S. 9 (1996), 237–302.
M. Miyamoto, Duke Math. J. 122 (2004), 51–91. math.QA/0209101.

These slides available at
`http://people.maths.ox.ac.uk/~joyce/`

Plan of talk:

4.1. Basic definitions on representations of VAs and VOAs

Let $\mathbb{K}$ be a field of characteristic zero, e.g. $\mathbb{K}=\mathbb{C}$. Recall from §1.1:

Definition 4.1 (Representations of VAs, in the style of Borcherds.)

Let V be a vertex algebra over $\mathbb{K}$. A *representation* of V is a $\mathbb{K}$-vector space W and linear maps $v_n^\rho : W \to W$ for all $v\in V$ and $n\in\mathbb{Z}$, with v_n^ρ linear in v, satisfying:

(i) For all $v\in V$ and $w\in W$ we have $v_n^\rho(w)=0$ for $n\gg 0$.

(ii) If $w\in W$ then $\mathbb{1}_{-1}^\rho(w)=w$ and $\mathbb{1}_n^\rho(w)=0$ for $-1\neq n\in\mathbb{Z}$.

(v) $$(u_l(v))_m^\rho(w)=\sum_{n\geqslant 0}(-1)^n\binom{l}{n}\bigl(u_{l-n}^\rho(v_{m+n}^\rho(w))-(-1)^l v_{l+m-n}^\rho(u_n^\rho(w))\bigr)$$

for all $u,v\in V$, $w\in W$ and $l,m\in\mathbb{Z}$, where the sum exists by (i).

These are the obvious generalizations of Definition 1.1(i),(ii),(v).

V has an obvious representation on itself.

All this extends to vertex superalgebras and graded vertex algebras V_*, when we take $W=W_*$ to be graded over $\mathbb{Z}_2$ or $\mathbb{Z}$.

Here is an equivalent definition in the language of states and fields:

Definition 4.2

Let $(V, \mathbb{1}, e^{zD}, Y)$ be a vertex algebra over $\mathbb{K}$. A *representation* (W, Y^ρ) of $(V, \mathbb{1}, e^{zD}, Y)$, or *V-module*, is a $\mathbb{K}$-vector space W and a linear map $Y^\rho : V \otimes W \to W[[z]][z^{-1}]$ (hence a map $V \to \mathrm{End}(W)[[z, z^{-1}]]$) satisfying:

(i) $Y^\rho(\mathbb{1}, z) = \mathrm{id}_W$.

(ii) For all $u, v \in V$ and $w \in W$, in $W[[z_0^{\pm 1}, z_1^{\pm 1}, z_2^{\pm 1}]]$ we have

$$z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y^\rho(Y(u, z_0)v, z_2)w = z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y^\rho(u, z_1) Y^\rho(v, z_2)w - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y^\rho(v, z_2) Y^\rho(u, z_1)w. \tag{4.1}$$

In Physics, V-modules are called *primary fields*.

If $u, v \in V$ and $w \in W$ then as for Theorems 2.2–2.4 there exist $N \gg 0$ such that

$$(z_1 - z_2)^N \bigl[Y^\rho(u, z_1) \circ Y^\rho(v, z_2) w - Y^\rho(v, z_2) \circ Y^\rho(u, z_1) w \bigr] = 0, \quad (4.2)$$

$$\begin{aligned} &(z_1 + z_2)^N Y^\rho(Y(u, z_1)v, z_2) w \\ &\quad = (z_1 + z_2)^N i_{z_1, z_2} \circ Y^\rho(u, z_1 + z_2) \circ Y^\rho(v, z_2) w. \end{aligned} \quad (4.3)$$

$$Y^\rho(e^{z_2 D} u, z_1) w = i_{z_1, z_2} \circ Y^\rho(u, z_1 + z_2) w. \quad (4.4)$$

As for Theorem 2.5, there are equivalent definitions of V-module in which we replace (4.1) by (4.2) or (4.3) (as in Frenkel–Ben-Zvi).

Finite-dimensionality and boundedness assumptions

To make progress on representations of VAs or VOAs, it is usual to make simplifying assumptions on $(V, \mathbb{1}, e^{zD}, Y)$ and (W, Y^{ρ}):

- We usually take V_*, W_* to be graded over $\mathbb{Z}$ (this is automatic if V is a VOA). Although V_*, W_* will be infinite-dimensional, we can assume that $\dim_{\mathbb{K}} V_n, \dim_{\mathbb{K}} W_n < \infty$ for all $n \in \mathbb{Z}$.
- We can also assume that $V_{\mathrm{odd}} = W_{\mathrm{odd}} = 0$, and that $V_n, W_n = 0$ for $n \ll 0$, or (stronger) that $V_n = 0$ for $n < 0$.
- The grading of V_* is fixed, but we can shift the grading of W_* by $W_n \mapsto W_{n+c}$ without changing anything important. So if $W_n = 0$ for $n \ll 0$ we can shift gradings so that $W_n = 0$ for $n < 0$ and $W_0 \neq 0$.
- For (W_*, Y^{ρ}) with $\dim_{\mathbb{K}} W_n < \infty$, $W_n = 0$ for $n < 0$ and n odd, and $W_0 \neq 0$, the *character* is $\mathrm{ch}\, W_* = \sum_{n \geqslant 0} \dim W_{2n} q^n$. If assuming $V_{\mathrm{odd}} = W_{\mathrm{odd}} = 0$, people tend to re-grade $V_{2n} \mapsto V_n$, but I won't.
- Authors often incorporate these assumptions (e.g. $\dim V_n < \infty$, $V_n = 0$ for $n < 0$) into their *definitions* of VAs, VOAs, representations.
- I will say V_*, W_* are *well behaved* if such assumptions hold.

Sub- and quotient representations, simples

Definition 4.3

Let $(V, \mathbb{1}, e^{zD}, Y)$ be a vertex algebra over $\mathbb{K}$, and (W, Y^ρ) a representation of V. A *subrepresentation* is a vector subspace $W' \subset W$ such that Y^ρ maps $V \otimes W' \to W'[[z]][z^{-1}]$. Then W' is a representation of V, and so is $W'' = W/W'$.

For example, if $w \in W$ we can consider the subrepresentation $W' = \langle w \rangle \subset W$ generated by w, spanned by all elements $(v_1)^\rho_{n_1} \circ (v_2)^\rho_{n_2} \circ \cdots \circ (v_k)^\rho_{n_k}(w)$ for $v_1, \ldots, v_k \in V$, $n_1, \ldots, n_k \in \mathbb{Z}$.

We call W *irreducible*, or *simple*, if $W \neq 0$ and the only subrepresentations $W' \subset W$ are 0 and W. Then W is generated by any $0 \neq w \in W$.

For well behaved V_*, W_*, it is reasonable to expect any representation W_* to be built from finitely many simple representations by extensions. Thus, to classify V_*-representations, it is enough to classify simple representations.

Representations of vertex operator algebras

For vertex operator algebras we can add a compatibility condition with the conformal vector ω:

Definition 4.4

Let $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ be a vertex operator algebra over $\mathbb{K}$, and (W_*, Y^ρ) a (graded) representation of $(V_*, \mathbb{1}, e^{zD}, Y)$. Recall that by definition of VOA we have $L_{-1} = \omega_0 = D$ and $L_0|_{V_a} = \omega_1|_{V_a} = \frac{1}{2}a\,\mathrm{id}_{V_a}$ for $a \in \mathbb{Z}$.
We call (W_*, Y^ρ) a *conformal representation* if $\omega_1^\rho|_{W_a} = (\frac{1}{2}a + h)\,\mathrm{id}_{W_a}$ for some $h \in \mathbb{K}$. If $W_n = 0$ for $n < 0$ and $W_0 \neq 0$, we call h the *highest weight*.
Alternatively, we can take W_* to be graded over $\mathbb{K}$ not $\mathbb{Z}$, relabel $W_a \mapsto W_{a+2h}$, and require that $\omega_1^\rho|_{W_a} = \frac{1}{2}a\,\mathrm{id}_{W_a}$ for $a \in \mathbb{K}$. This is better for taking direct sums of representations with different h.

4.2. Rational VOAs and Zhu's Theorem

Rational VOAs have particularly nice representation theory:

Definition 4.5

Let $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ be a VOA over $\mathbb{K}$ with $V_n = 0$ for $n < 0$. Consider only conformal representations (W_*, Y^ρ) with $W_n = 0$ for $n \ll 0$. We call $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ *rational* if:

- (i) There are only finitely many isomorphism classes of simple V_*-modules W_*, up to shifts $W_n \mapsto W_{n+c}$.
- (ii) Every simple V_*-module W_* has $\dim W_n < \infty$ for $n \in \mathbb{Z}$.
- (iii) Every V_*-module W_* is a direct sum of simple V_*-modules.

Actually (iii) implies (i),(ii) (Dong–Li–Mason).
We call $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ *holomorphic* if it is rational with only one simple V_*-module, which is V_* itself.

Rational VOAs are a bit like finite groups: we would like to understand them and classify their simple representations.

Examples of rational VOAs

Example 4.6

(a) (Dong.) Let Λ be an even positive definite lattice. Then the lattice VOA V_Λ over $\mathbb{C}$ from Example 3.11 is rational. The simple modules of V_Λ are in 1-1 correspondence with $\Lambda^\vee/\Lambda$.
(b) (Frenkel–Zhu.) Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and k be a positive integer. Example 3.10 constructs a VOA $(V_k^{\hat{\mathfrak{g}}}, \mathbb{1}, e^{zD}, Y)$. It turns out that $V_k^{\hat{\mathfrak{g}}}$ has a maximal proper ideal $I \subset V_k^{\hat{\mathfrak{g}}}$ whose quotient $L_k^{\hat{\mathfrak{g}}} = V_k^{\hat{\mathfrak{g}}}/I$ is a simple VOA. Then $L_k^{\hat{\mathfrak{g}}}$ is a rational VOA whose (simple) representations correspond to (simple) representations of $\hat{\mathfrak{g}}$ of level k.
(c) The Monster vertex operator algebra V_*^{Mon} is rational, and in fact holomorphic.

Zhu's cofiniteness condition

Definition 4.7 (Zhu 1996.)

Let $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ be a VOA over $\mathbb{K}$ with $V_n = 0$ for $n < 0$ or n odd. We say that V_* satisfies *Zhu's cofiniteness condition* if

(a) Write $C_2(V_*)$ for the vector subspace of V_* spanned by $u_{-2}(v)$ for $u, v \in V_*$. Then $\dim V_*/C_2(V_*) < \infty$.
(b) Let $L_n = \omega_{n+1} : V_* \to V_*$ be the Virasoro action on V_*. Then V_* is spanned by vectors of the form $L_{n_1} \circ \cdots \circ L_{n_k}(v)$ for $n_i < 0$, where $v \in V_*$ satisfies $L_n(v) = 0$ for all $n > 0$.

This holds in Example 4.6(a)–(c).

Zhu's Theorem

Theorem 4.8 (Zhu 1996, extended by Miyamoto 2004.)

Let $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ be a VOA over $\mathbb{C}$ with central charge $c \in \mathbb{C}$ satisfying Zhu's cofiniteness condition. Then $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ is rational. Let $W_*^1, \ldots, W_*^N$ be the simple V_*-representations up to isomorphism, where $W_0^i \neq 0$ and $W_n^i = 0$ for $n < 0$ or n odd, and let W_*^i have highest weight $h_i \in \mathbb{C}$. Consider the functions

$$f_i(q) = q^{h_i - c/24}\,\mathrm{ch}(W_*^i) = \textstyle\sum_{n \geqslant 0} q^{h_i - c/24 + n} \dim W_{2n}. \tag{4.5}$$

Here $\mathrm{ch}(W_*^i)$ converges on $\{q \in \mathbb{C} : |q| < 1\}$. Thus changing variables to τ with $q = e^{2\pi i \tau}$, $f_i(\tau)$ is a holomorphic function on the upper half-plane $\mathbb{H} = \{\tau \in \mathbb{C} : \mathrm{Im}\,\tau > 0\}$. Then $c, h_i \in \mathbb{Q}$, and $f_1(\tau), \ldots, f_N(\tau)$ are linearly independent, and their span $\langle f_i(\tau) : i = 1, \ldots, N\rangle_{\mathbb{C}}$ is invariant under the action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{H}$, so the $f_i(\tau)$ are **vector-valued modular forms**, *a generalization of modular forms.*

Remarks on Zhu's Theorem

• Characters of representations of infinite-dimensional Lie algebras (e.g. affine Lie algebras, Virasoro) are often modular forms.

• A heuristic explanation for Theorem 4.8 is as follows: to each VOA V_*, Frenkel–Ben-Zvi associate a $\mathscr{D}$-module on the moduli space of Riemann surfaces, and so in particular on the moduli space $\mathbb{H}/\operatorname{SL}(2,\mathbb{Z})$ of elliptic curves. For rational VOAs this $\mathscr{D}$-module is expected to be a vector bundle $E \to \mathbb{H}/\operatorname{SL}(2,\mathbb{Z})$ with flat connection, and the W_*^i should induce a basis of constant sections of E on the universal cover $\mathbb{H}$ of $\mathbb{H}/\operatorname{SL}(2,\mathbb{Z})$ near $\tau = i\infty$. The $\operatorname{SL}(2,\mathbb{Z})$ action on $\langle f_i(\tau) : i = 1, \ldots, N\rangle_{\mathbb{C}}$ comes from the monodromy action of $\operatorname{SL}(2,\mathbb{Z}) = \pi_1(\mathbb{H}/\operatorname{SL}(2,\mathbb{Z}))$.

• I hope to return to the modular forms aspect later in term, but it involves too much background to explain now.

Today I will just explain a more elementary part of the proof, relating V_*-modules to representations of an algebra $A(V_*)$, the *Zhu algebra*.

4.3. The Zhu algebra and simple representations

Definition 4.9

Let $\mathbb{K}$ be a field of characteristic zero, and $(V_*, \mathbb{1}, e^{zD}, Y)$ be a graded vertex algebra over $\mathbb{K}$ with $V_{\mathrm{odd}} = 0$. Define bilinear operations $*, \circ : V_* \times V_* \to V_*$ by, for $u \in V_{2a}$ and $v \in V_{2b}$,

$$u * v = \mathrm{Res}_z \left(\frac{(1+z)^a}{z} Y(u,z)v \right) = \sum_{n=0}^{\infty} \binom{a}{n} u_{n-1}(v), \qquad (4.6)$$

$$u \circ v = \mathrm{Res}_z \left(\frac{(1+z)^a}{z^2} Y(u,z)v \right) = \sum_{n=0}^{\infty} \binom{a}{n} u_{n-2}(v). \qquad (4.7)$$

These are well-defined as $u_n(v) = 0$ for $n \gg 0$. Note that $*, \circ$ are not grading-preserving. Write $O(V_*)$ for the vector subspace of V_* spanned by elements $u \circ v$ for all $u, v \in V_*$. Define $A(V_*) = V_*/O(V_*)$ to be the quotient vector space.

Theorem 4.10 (Zhu 1996.)

In Definition 4.9, $O(V_*)$ is a two-sided ideal for $*$, so $*$ descends to a bilinear multiplication $* : A(V_*) \times A(V_*) \to A(V_*)$. Furthermore:

(a) *The product $*$ on $A(V_*)$ is associative, and makes $A(V_*)$ into a $\mathbb{K}$-algebra with identity $\mathbb{1} + O(V_*)$, the* **Zhu algebra**.

(b) *If $(V_*, \mathbb{1}, e^{zD}, Y)$ is a vertex operator algebra with conformal element ω then $\omega + O(V_*)$ lies in the centre of $A(V_*)$.*

(c) *As $A(V_*)$ is an associative algebra, it is also a Lie algebra, with Lie bracket $[\alpha, \beta] = \alpha * \beta - \beta * \alpha$. In §1.4 we defined a Lie bracket on $V_2/D(V_0)$. We have $D(V_0) \subset A(V_*)$, and the natural map $V_2/D(V_0) \to A(V_*)$ is a Lie algebra morphism. This induces an algebra morphism $U(V_2/D(V_0)) \to A(V_*)$, where $U(V_2/D(V_0))$ is the universal enveloping algebra.*

Theorem 4.10 (Continued.)

(d) *Suppose* (W_*, Y^ρ) *is a representation of* $(V_*, \mathbb{1}, e^{zD}, Y)$ *with* $W_a = 0$ *for* $a < 0$. *Then* W_0 *is a left module over* $A(V_*)$, *with action for* $v \in V_{2a}$ *and* $w \in W_0$ *given by*

$$(v + O(V_*)) \cdot w = v^\rho_{a-1}(w). \tag{4.8}$$

(e) *Let* W_0 *be a left module over* $A(V_*)$. *Then* W_0 *extends to a representation* (W_*, Y^ρ) *of* $(V_*, \mathbb{1}, e^{zD}, Y)$ *with* $W_a = 0$ *for* $a < 0$, *such that the* $A(V_*)$*-module structure on* W_0 *in* (d) *is the given one, and there are no nonzero* $(V_*, \mathbb{1}, e^{zD}, Y)$*-subrepresentations* $\tilde{W}_* \subset W_*$ *with* $\tilde{W}_0 = 0$.

(f) *Parts* (d),(e) *induce a* 1-1 *correspondence between isomorphism classes of nonzero simple* $A(V_*)$*-representations* W_0, *and isomorphism classes of simple representations* (W_*, Y^ρ) *of* $(V_*, \mathbb{1}, e^{zD}, Y)$ *with* $W_a = 0$ *for* $a < 0$ *and* $W_0 \neq 0$.

Partial proof of Theorem 4.10

For $u,v\in V_*$, $w\in W_*$ and $a,b\in\mathbb{Z}$, the Jacobi identity implies that

$$\sum_{n=0}^\infty \binom{a}{n}(u_{n-1}(v))^\rho_{a+b-n-1}(w)=\sum_{n=0}^\infty \bigl(u^\rho_{a-1-n}(v^\rho_{b-1+n}(w)) + v^\rho_{b-2-n}(u^\rho_{a+n}(w))\bigr). \tag{4.9}$$

Apply this with $u\in V_{2a}$, $v\in V_{2b}$ and $w\in W_0$. By (4.6) and (4.8), the l.h.s. is $(u\star v)\cdot w$. The first term on the r.h.s. when $n=0$ is $u\cdot(v\cdot w)$. Also $v^\rho_{b-1+n}(w)\in W_{-2n}$ and $u^\rho_{a+n}(w)\in W_{-2n-2}$. Thus the first term on the r.h.s. is zero for $n>0$, and the second term for $n\geqslant 0$, as $W_{<0}=0$. Hence (4.9) becomes

$$(u\star v)\cdot w=u\cdot(v\cdot w). \tag{4.10}$$

This proves (d), assuming (a). It also motivates the definition of the product $\star$ in (4.6). We have a map $\cdot: V_*\rightarrow \mathrm{End}(W_0)$ taking $\star$ to the (associative) composition in $\mathrm{End}(W_0)$. So there should be some subspace $O(V_*)\subset V_*$ with $\star$ associative on $V_*/O(V_*)$.

Remarks on Zhu algebras and V_*-representations

- It is crucial that $(V_*, \mathbb{1}, e^{zD}, Y)$ is a *graded* vertex algebra, with $V_{\rm odd} = 0$, since $v \in V_{2a}$ acts on $w \in W_0$ by $v \cdot w = v^\rho_{a-1}(w)$, which would not make sense without these conditions.
- Theorem 4.10 reduces understanding simple V_*-representations to simple $A_*(V)$-representations. If V_* is an (infinite-dimensional) *rational* vertex algebra, one can prove that $A(V_*)$ is a finite-dimensional semisimple $\mathbb{K}$-algebra – a much simpler object. The simple V_*-representations W_* also have $\dim W_0 < \infty$. So we reduce to ordinary algebra in finite dimensions.
- Rational VOAs are important in Physics.

Example: the Heisenberg VOA

Recall the Heisenberg VOA $V_*^{\mathrm{Heis}} = \mathbb{C}[x_1, x_2, \ldots]$ from Example 3.8. This is graded with $\deg x_n = 2n$, so $V_{\mathrm{odd}}^{\mathrm{Heis}} = 0$, and with conformal vector $\omega_0 = \frac{1}{2}x_1^2$ has central charge $c = 1$. It is a simple representation over itself, with highest weight $h = 0$.

The character of V_* is $\mathrm{ch}\, V_* = \sum_{n\geqslant 0} \dim V_{2n} q^n$. Writing $V_*^{\mathrm{Heis}} = \bigotimes_{n\geqslant 1} \mathbb{C}[x_n]$, where $\mathrm{ch}\, \mathbb{C}[x_n] = \sum_{k\geqslant 0} q^{nk} = (1-q^n)^{-1}$, we see that $\mathrm{ch}\, V_* = \prod_{n\geqslant 1}(1-q^n)^{-1}$.

Thus the function $f_i(q)$ in (4.5) corresponding to V_*^{Heis} is

$$f_i(q) = q^{h-c/24}\,\mathrm{ch}(V_*) = q^{-1/24}\prod_{n\geqslant 1}(1-q^n)^{-1} = \eta(q)^{-1}, \quad (4.11)$$

where $\eta(q)$ is *Dedekind's η-function*, a modular form of weight $\frac{1}{2}$.

The Heisenberg VOA is not rational.

Example: lattice VOAs

Let (Λ,χ) be an even positive definite lattice of rank d. Example 3.11 defines the lattice VOA V_*^Λ, with central charge d, which is rational. As a graded vector space we have $V_*^\Lambda\cong\mathbb{C}[\Lambda]\otimes\bigotimes^d V_*^{\mathrm{Heis}}$, where $\mathbb{C}[\Lambda]$ has character the *lattice theta function* $\Theta_\Lambda(q)$

$$\mathrm{ch}\,\mathbb{C}[\Lambda]=\Theta_\Lambda(q)=\sum_{\lambda\in\Lambda}q^{\frac{1}{2}\chi(\lambda,\lambda)}.$$

Hence $\mathrm{ch}\,V_*^\Lambda=\Theta_\Lambda(q)\cdot\prod_{n\geqslant 1}(1-q^n)^{-d}$, and the function $f_i(q)$ in (4.5) corresponding to V_*^Λ is

$$f_i(q)=\Theta_\Lambda(q)\cdot\eta(q)^{-d}. \tag{4.12}$$

As (Λ,χ) is integral there is a natural morphism $\Lambda\hookrightarrow\Lambda^\vee$. We call (Λ,χ) *unimodular* if this is an isomorphism. Then it is known that $\Theta_\Lambda(q)$ is a modular form of weight $d/2$, so $f_i(q)$ has weight 0.