Vertex Algebras

Lecture 7 of 8: Vertex algebra bundles on curves, and chiral algebras

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References for this lecture: Frenkel and Ben-Zvi, 2nd ed. (2004), §6, §19.
A. Beilinson and V. Drinfeld, *Chiral algebras*, A.M.S. 2004.
N. Rozenblyum, *Introduction to chiral algebras*, unpublished notes.

These slides available at
`http://people.maths.ox.ac.uk/∼joyce/`

Plan of talk:

Introduction

The last two lectures will be about two approaches to vertex algebras (mostly VOAs) by Beilinson and Drinfeld, motivated by Physics: *chiral algebras* and *factorization algebras*. The goal is that to an even VOA $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ with $V_k = 0$ for $k \ll 0$ we will associate geometric structures on X for every algebraic curve X:

- a *chiral algebra on* X: a $\mathscr{D}$-module $\mathscr{V}_X \to X$ — essentially an infinite-dimensional vector bundle over X with fibre V_*, and a flat connection ∇ — with a $\mathscr{D}$-module morphism $Y_X : j_* j^*(\mathscr{V}_X \boxtimes \mathscr{V}_X) \to \Delta_!(\mathscr{V}_X)$ on $X \times X$, where $\Delta : X \to X \times X$ is the diagonal and $j : X \times X \setminus \Delta(X) \hookrightarrow X \times X$ is the inclusion.
- a *factorization algebra on* X: a quasicoherent sheaf $\mathcal{F} \to \mathrm{Ran}(X)$ on the *Ran space* $\mathrm{Ran}(X)$, the prestack of all finite subsets $\{x_1, \ldots, x_n\} \subset X$, satisfying the factorization property $\mathcal{F}|_{S \amalg T} \cong \mathcal{F}|_S \otimes \mathcal{F}|_T$ in canonical isomorphisms for disjoint finite $S, T \subset X$, where $\mathcal{F}|_{\{x_1,\ldots,x_n\}} \cong (V_*)^{\otimes^n}$.

The basic idea here is that for the state-field correspondence $Y(z): V\otimes V\to V[[z]][z^{-1}]$, we should consider $\mathrm{Spec}\,\mathbb{C}[[z]]$ as the formal disc $\mathbb{D}$ and $\mathrm{Spec}\,\mathbb{C}[[z]][z^{-1}]$ as the punctured formal disc $\mathbb{D}'=\mathbb{D}\setminus\{0\}$. If X is an algebraic curve and $x\in X$ then the formal completion X_x of X at x has a non-canonical isomorphism $X_x\cong\mathbb{D}$, natural up to the action of $\mathrm{Aut}(\mathbb{D})$, so $X'_x=X_x\setminus\{x\}\cong\mathbb{D}\setminus\{0\}$. The Lie algebra $\mathfrak{aut}(\mathbb{D})$ is (nearly) a Lie subalgebra of the Virasoro algebra Vir which acts on V_*, and $V_n=0$ for $n\ll 0$ implies that we can exponentiate this to an action of $\mathrm{Aut}(\mathbb{D})$ on V_*. There is a principal $\mathrm{Aut}(\mathbb{D})$-bundle $P_X\to X$ with fibre at x the isomorphisms $X_x\cong\mathbb{D}$, so we can define an infinite-dimensional vector bundle $\mathscr{V}_X\to X$ by $\mathscr{V}_X=(V_*\times P_X)/\mathrm{Aut}(\mathbb{D})$ with fibre V_*.
Then we can ask: how can we translate the VOA structure on V_* to geometric operations on the vector bundle $\mathscr{V}_X\to X$?
Note that this is all physically motivated: VOAs should correspond to CFTs in String Theory, which quantize maps $X\to S$ from a Riemann surface X to a space-time S.

7.1. Vertex algebra bundles on curves
Another definition of the Virasoro algebra

We defined the *Virasoro algebra* Vir to be the $\mathbb{C}$-Lie algebra with basis elements L_n, $n \in \mathbb{Z}$ and c (the *central charge*), and Lie bracket

$$[c, L_n] = 0,\ [L_m, L_n] = (m-n)L_{m+n} + \tfrac{1}{12}(m^3 - m)\delta_{m,-n}c,\ \ m, n \in \mathbb{Z}.$$

It may be written as a universal central extension

$$0 \longrightarrow \langle c\rangle_{\mathbb{C}} \longrightarrow \begin{array}{c}\mathrm{Vir} = \\ \langle c\rangle_{\mathbb{C}} \oplus \mathrm{Der}\,\mathbb{C}[z, z^{-1}]\end{array} \xrightarrow{\pi} \mathrm{Der}\,\mathbb{C}[z, z^{-1}] \longrightarrow 0,$$

where $\mathrm{Der}\,\mathbb{C}[z, z^{-1}] = \mathbb{C}[z, z^{-1}]\frac{\mathrm{d}}{\mathrm{d}z}$ is the Lie algebra of derivations of the algebra $\mathbb{C}[z, z^{-1}]$, and $\pi(L_n) = -z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z}$, and Vir has Lie bracket

$$\bigl[p(z)\tfrac{\mathrm{d}}{\mathrm{d}z}, q(z)\tfrac{\mathrm{d}}{\mathrm{d}z}\bigr] = (pq' - p'q)\tfrac{\mathrm{d}}{\mathrm{d}z} - \tfrac{1}{12}\,\mathrm{Res}_z(pq''') \cdot c.$$

Observe that $\mathrm{Der}\,\mathbb{C}[z] = \mathbb{C}[z]\frac{\mathrm{d}}{\mathrm{d}z} = \langle L_n : n \geqslant -1\rangle$ and $\mathrm{Der}_0\,\mathbb{C}[z] = z\mathbb{C}[z]\frac{\mathrm{d}}{\mathrm{d}z} = \langle L_n : n \geqslant 0\rangle$ are Lie subalgebras.

We may extend this to a completion $\overline{\mathrm{Vir}}$ of Vir

$$0 \longrightarrow \langle c\rangle_{\mathbb{C}} \longrightarrow \overline{\mathrm{Vir}} \xrightarrow{\pi} \mathrm{Der}\,\mathbb{C}[[z]][z^{-1}] \longrightarrow 0,$$

given by the same formula, with $p(z), q(z) \in \mathbb{C}[[z]][z^{-1}]$, and $\mathrm{Der}\,\mathbb{C}[[z]]$, $\mathrm{Der}_0\,\mathbb{C}[[z]]$ are Lie subalgebras of $\overline{\mathrm{Vir}}$. Here $\mathrm{Der}_0\,\mathbb{C}[[z]] = \mathfrak{aut}(\mathbb{C}[[z]]) = \mathfrak{aut}(\mathbb{D})$ is the Lie algebra of automorphisms of $\mathbb{D} = \mathrm{Spec}\,\mathbb{C}[[z]]$.
Now let $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ be an even VOA with $V_k = 0$ for $k \ll 0$. Then Vir acts on V_*. Also elements of $\overline{\mathrm{Vir}}$ are of the form $\gamma c + \sum_{n\in\mathbb{Z}} \lambda_n L_n$ with $\lambda_n = 0$ for $n \ll 0$, and if $v \in V_*$ then $L_n(v) = 0$ for $n \gg 0$ as $V_k = 0$ for $k \ll 0$, so the action of Vir on V_* extends to an action of $\overline{\mathrm{Vir}}$, which restricts to an action of $\mathrm{Der}_0\,\mathbb{C}[[z]] = \mathfrak{aut}(\mathbb{D})$. One can show that this exponentiates to an action of the infinite-dimensional Lie group $\mathrm{Aut}(\mathbb{D})$ on V_* (this needs that L_0 has eigenvalues in $\mathbb{Z}$). It preserves the filtered subspaces $V_{\leqslant m} \subset V_*$ for each $m \in \mathbb{Z}$.

Now let X be an algebraic curve. Define a space P_X to have points (x,ϕ) for $x\in X$ and $\phi:X_x\rightarrow\mathbb{D}$ an isomorphism, where X_x is the formal completion of X at x, so ϕ is a formal coordinate at x. Then $\mathrm{Aut}(\mathbb{D})$ acts on P_X by $\gamma:(x,\phi)\mapsto(x,\gamma\circ\phi)$, making $P_X\rightarrow X$ into a principal $\mathrm{Aut}(\mathbb{D})$-bundle. Define $\mathscr{V}_X=(V_*\times P_X)/\mathrm{Aut}(\mathbb{D})$ to be the associated infinite-dimensional vector bundle with fibre V_*. It has a canonical filtration $\cdots\subset\mathscr{V}_{\leqslant m}\subset\mathscr{V}_{\leqslant m+1}\subset\cdots\subset\mathscr{V}_X$ modelled on $\cdots\subset V_{\leqslant m}\subset V_{\leqslant m+1}\subset\cdots\subset V_*$, where $\mathscr{V}_{\leqslant m}$ is a finite rank vector bundle if $\dim V_{\leqslant m}<\infty$.

$\mathscr{V}_X$ (or better, the sheaf of algebraic sections of $\mathscr{V}_X$) is a *quasicoherent sheaf* on X, and the $\mathscr{V}_{\leqslant m}$ are coherent sheaves (algebraic vector bundles) if $\dim V_{\leqslant m}<\infty$.

Actually all this and the following results work not just for VOAs $(V_*,\mathbb{1},e^{zD},Y,\omega)$ with $V_k=0$ for $k\ll 0$, but for the larger class of *quasi-conformal vertex algebras* (see Frenkel–Ben-Zvi §6.3), which have a locally nilpotent $\mathrm{Der}\,\mathbb{C}[z]$-action rather than a Vir-action.

Proposition 7.1 (Frenkel and Ben-Zvi §6.6.)

There is a natural connection ∇ on $\mathscr{V}_X \to X$, which is automatically flat as X is a curve. It satisfies $\nabla \mathscr{V}_{\leqslant m} \subset \mathscr{V}_{\leqslant m+1} \otimes K_X$ for $m \in \mathbb{Z}$.

Sketch proof. We built $\mathscr{V}_X \to X$ using the action of $\mathrm{Der}_0\,\mathbb{C}[z] = z\mathbb{C}[z]\frac{\mathrm{d}}{\mathrm{d}z} = \langle L_n : n \geqslant 0\rangle$ on $\mathcal{V}_*$, which preserves the subspaces $V_{\leqslant m} \subset V_*$. But $\mathrm{Der}\,\mathbb{C}[z] = \mathbb{C}[z]\frac{\mathrm{d}}{\mathrm{d}z} = \langle L_n : n \geqslant -1\rangle$ also acts on V_*, where $\frac{\mathrm{d}}{\mathrm{d}z} = -L_{-1}$ corresponds to infinitesimal translation in $\mathbb{D}$, which does not exponentiate to an actual automorphism of $\mathbb{D}$, and $\frac{\mathrm{d}}{\mathrm{d}z}$ maps $V_{\leqslant m} \to V_{\leqslant m+1}$.
If $z : U \to \mathbb{A}^1$ is any local algebraic coordinate on open $U \subset X$, then z induces an isomorphism $X_x \cong \mathbb{D}$ for $x \in U$, so that $P_X|_U \cong U \times \mathrm{Aut}(\mathbb{D})$, and $\mathscr{V}_X|_U \cong U \times V_*$. Then ∇ acts by $\frac{\mathrm{d}}{\mathrm{d}z} + L_{-1}$ in this trivialization. This is independent of coordinate z. □

As $\mathscr{V}_X$ is a quasicoherent sheaf with connection ∇, it is a *$\mathscr{D}$-module* on X (explained later).

As $\mathbb{1} \in V_*$ is invariant under Vir, it induces a section $\mathbb{1}_X \in \Gamma(\mathscr{V}_X)$ with $\nabla \mathbb{1}_X = 0$. Here is how to interpret the state-field correspondence $Y(z)$:

Proposition 7.2 (Frenkel and Ben-Zvi §6.5.)

$Y(z) : V_* \otimes V_* \to V_*[[z]][z^{-1}]$ induces for each $x \in X$ a morphism of quasicoherent sheaves on $X'_x = X_x \setminus \{x\}$, where $\mathscr{V}_x = \mathscr{V}_X|_x$

$$Y'_x : \mathscr{V}_X|_{X'_x} \otimes_{\mathbb{C}} \mathscr{V}_x \longrightarrow \mathscr{V}_x \otimes_{\mathbb{C}} \mathcal{O}_{X'_x}. \tag{7.1}$$

Alternatively we may regard this as a meromorphic morphism on X_x

$$Y_x : \mathscr{V}_X|_{X_x} \otimes_{\mathbb{C}} \mathscr{V}_x \dashrightarrow \mathscr{V}_x \otimes_{\mathbb{C}} \mathcal{O}_{X_x}, \tag{7.2}$$

which is allowed to have poles at $x \in X_x$.

Here $X'_x \cong \mathbb{D}' = \mathrm{Spec}\,\mathbb{C}[[z]][z^{-1}]$, but the important thing is that (7.1)–(7.2) are independent of choice of formal coordinate on X_x. Roughly, (7.1) says we have morphisms $Y'_x(w) : \mathscr{V}_w \otimes \mathscr{V}_x \longrightarrow \mathscr{V}_x$ for $w \in X'_x$, that is, w 'infinitesimally close to x in X', or $(w, x) \in X \times X$ 'infinitesimally close to the diagonal $\Delta(X)$'.

Conformal blocks

So far we have considered only the local geometry of X, in an infinitesimal neighbourhood of a point $x\in X$. But we can also do global geometry on X, e.g. by considering meromorphic sections of $\mathscr{V}_X$ with poles at prescribed points $x_1,\ldots,x_n$.

Definition 7.3 (Frenkel and Ben–Zvi §9.)

Suppose $(V_*,\mathbb{1},e^{zD},Y,\omega)$ is an even VOA with $V_k=0$ for $k\ll 0$, and X is a smooth projective curve, and $x\in X$. Let $\mathscr{V}_X$ and Y'_x be as above. We say that $\varphi\in\mathscr{V}_x^\vee$ is a *conformal block* for V_* (in the simplest case) if for all $v\in\mathscr{V}_x$, the section $Y_{\varphi,v}$ of $\mathscr{V}_X^\vee|_{X'_x}$, $Y_{\varphi,v}(w)=\varphi\circ Y'_x(w\otimes v)$, extends to a regular section of $\mathscr{V}_X^\vee|_{X\setminus\{x\}}$. Conformal blocks form a vector subspace of $\mathscr{V}_x^\vee\cong V_*^\vee$.

Conformal blocks are a central concept in Conformal Field Theory. One can also consider conformal blocks with multiple points $x_1,\ldots,x_n$ and V_*-modules $M_1,\ldots,M_n$. Conformal blocks are used in a kind of 'Fourier decomposition' of correlation functions in CFT.

7.2. Chiral algebras
Introduction to $\mathscr{D}$-modules

Let X be a smooth $\mathbb{C}$-scheme, e.g. a curve, and $E \to X$ an algebraic vector bundle. An (*algebraic*) *connection* ∇ on E is a sheaf morphism $\nabla : TX \otimes_{\mathbb{C}_X} E \to E$ satisfying the Leibniz rule $\nabla(v \otimes fe) = f\nabla(v \otimes e) + (v \cdot \mathrm{d}f)e$ for all local sections $v \in \Gamma(TX)$, $e \in \Gamma(E)$, $f \in \Gamma(\mathcal{O}_X)$. Here $\mathbb{C}_X$ is the sheaf of locally constant functions $f : X \to \mathbb{C}$. Write $\nabla_v e = \nabla v \otimes e$.

We say ∇ is *flat* if $\nabla_v \nabla_w e - \nabla_w \nabla_v e = \nabla_{[v,w]} e$ for all local sections $v, w \in \Gamma(TX)$, $e \in \Gamma(E)$. This is automatic if X is a curve.

These definitions also make sense if E is a coherent sheaf or quasicoherent sheaf on X, and if ∇ is a flat connection on E then (E, ∇) is a (left) $\mathscr{D}_X$*-module* or just (left) $\mathscr{D}$*-module*.

An alternative way to define $\mathscr{D}$-modules is to define a sheaf of $\mathbb{C}$-algebras $\mathscr{D}_X$ on X to be the subsheaf of $\mathrm{End}_{\mathbb{C}_X}(\mathcal{O}_X)$ (that is, $\mathbb{C}_X$-linear morphisms $\mathcal{O}_X \to \mathcal{O}_X$, not $\mathcal{O}_X$-linear morphisms $\mathcal{O}_X \to \mathcal{O}_X$) generated by $\mathcal{O}_X$ acting by multiplication and TX acting by Lie bracket. Then a $\mathscr{D}_X$-module is a sheaf of $\mathscr{D}_X$-modules on X, quasicoherent as an $\mathcal{O}_X$-module. As $\mathscr{D}_X$ is non-commutative we must distinguish *left $\mathscr{D}_X$-modules* and *right $\mathscr{D}_X$-modules*. If $\mathcal{E}$ is a left $\mathscr{D}$-module then $\mathcal{E} \otimes_{\mathcal{O}_X} K_X$ is a right $\mathscr{D}$-module, and vice versa.

The categories $\mathcal{D}_X\text{-mod}^l, \mathcal{D}_X\text{-mod}^r$ of left and right $\mathscr{D}_X$-modules are abelian and have many nice properties, like $\mathrm{coh}(X)$ and $\mathrm{qcoh}(X)$. There is an equivalence $\mathcal{D}_X\text{-mod}^l \to \mathcal{D}_X\text{-mod}^r$, $\mathcal{E} \mapsto \mathcal{E} \otimes_{\mathcal{O}_X} K_X$.

If $f : X \to Y$ is a morphism satisfying suitable conditions then we have two kinds of pushforwards $f_*, f_! : \mathcal{D}_X\text{-mod}^r \to \mathcal{D}_Y\text{-mod}^r$, and two kinds of pullbacks $f^*, f^! : \mathcal{D}_Y\text{-mod}^r \to \mathcal{D}_X\text{-mod}^r$, satisfying the Grothendieck six-functor formalism.

From now on '$\mathscr{D}$-module' means 'right $\mathscr{D}$-module'.

Two operations on $\mathscr{D}$-modules will be important to us:

- Let $\mathcal{E}$ be a $\mathscr{D}$-module on a curve X. Then the external tensor product $\mathcal{E}\boxtimes\mathcal{E}$ is a $\mathscr{D}$-module on $X\times X$. Let $j: X\times X\setminus\Delta(X)\hookrightarrow X\times X$ be the inclusion, where $\Delta: X\to X\times X$, $\Delta: x\mapsto(x,x)$. Then $j_*\circ j^*(\mathcal{E}\boxtimes\mathcal{E})$ is another $\mathscr{D}$-module on $X\times X$.

Local sections of $j_*\circ j^*(\mathcal{E}\boxtimes\mathcal{E})$ are meromorphic local sections of $\mathcal{E}\boxtimes\mathcal{E}$ on $X\times X$ which are regular on $X\times X\setminus\Delta(X)$ but are allowed to have arbitrary poles on $\Delta(X)\subset X\times X$. (Needs $\dim X=1$.)

- Let $\mathcal{E}$ be a $\mathscr{D}$-module on a curve X. Then $\Delta_!(\mathcal{E})$ is a $\mathscr{D}$-module on $X\times X$ supported on the diagonal $\Delta(X)$.

We can consider $\mathscr{D}$-module morphisms $\mu: j_*\circ j^*(\mathcal{E}\boxtimes\mathcal{E})\to\Delta_!(\mathcal{E})$. Morally $j_*\circ j^*(\mathcal{E}\boxtimes\mathcal{E})$ lives on $X\times X\setminus\Delta(X)$, and $\Delta_!(\mathcal{E})$ lives on $\Delta(X)$, so you might expect such μ to be trivial, but they are not. Roughly, such μ induce morphisms $\mu(w,x):\mathcal{E}_w\otimes\mathcal{E}_x\longrightarrow\mathcal{E}_x$ for $w\in X'_x$, that is, w 'infinitesimally close to x in X', as for $Y'_x(w)$ in Proposition 7.2.

Chiral algebras

Definition 7.4 (Beilinson–Drinfeld, notation abused a bit.)

Let X be an algebraic curve. A *chiral algebra* on X is a $\mathscr{D}$-module $\mathcal{A}$ on X with $\mathscr{D}$-module morphisms $\mu : j_* \circ j^*(\mathcal{A} \boxtimes \mathcal{A}) \to \Delta_!(\mathcal{A})$, where $j : X \times X \setminus \Delta(X) \hookrightarrow X \times X$ is the inclusion and $\Delta : X \to X \times X$ is the diagonal map, and $\mathbb{1} : K_X \to \mathcal{A}$, satisfying

(i) **Antisymmetry:** $\sigma_* \circ \mu \circ \sigma_* = -\mu$, where $\sigma(x_1, x_2) = (x_2, x_1)$.

(ii) **Jacobi identity:** define $\Delta_{***} : X \times X \to X \times X \times X$ to map

$$\Delta_{(12)3} : (x_1, x_2) \mapsto (x_1, x_1, x_2), \quad \Delta_{2(13)} : (x_1, x_2) \mapsto (x_2, x_1, x_2),$$
$$\Delta_{1(23)} : (x_1, x_2) \mapsto (x_1, x_2, x_2),$$

Define $k : X^3 \setminus \text{all diagonals} \hookrightarrow X^3$ to be the inclusion, and $\Delta_3 : X \hookrightarrow X^3$ the diagonal. Let $\mu_{(12)3}$ be the composition

$$k_* \circ k^*(\mathcal{A} \boxtimes \mathcal{A} \boxtimes \mathcal{A}) \xrightarrow{\mu_{12}\boxtimes \mathrm{id}_3} (\Delta_{(12)3})_!(\mathcal{A} \boxtimes \mathcal{A}) \xrightarrow{(\Delta_{(12)3})_!(\mu)} (\Delta_3)_!(\mathcal{A})$$

on $X \times X \times X$, and similarly for $\mu_{2(13)}, \mu_{1(23)}$. Then

$$\mu_{(12)3} = \mu_{2(13)} + \mu_{1(23)}.$$

Definition 7.3 (Continued.)

(iii) **Unit:** the following diagram commutes on $X \times X$

$$\begin{array}{ccc} j_* \circ j^*(K_X \boxtimes \mathcal{A}) & \xrightarrow[j_*\circ j^*(\mathbb{1}\boxtimes\mathrm{id}_{\mathcal{A}})]{} & j_* \circ j^*(\mathcal{A} \boxtimes \mathcal{A}) \\ & \searrow^{\text{canonical map}} & \downarrow \mu \\ & & \Delta_!(\mathcal{A}). \end{array}$$

Note that (i),(ii) basically say that a chiral algebra is a Lie algebra in $\mathscr{D}$-modules on X (technically, a Lie algebra object for a certain 'pseudotensor category structure' on $\mathcal{D}_X\text{-mod}^r$).

Theorem 7.4 (Frenkel and Ben–Zvi §19.3.)

Let $(V_*, \mathbb{1}, e^{zD}, Y, \omega)$ be an even VOA with $V_k = 0$ for $k \ll 0$ (or more generally, a 'quasi-conformal vertex algebra'). Then for each smooth curve X, the $\mathscr{D}_X$-module $\mathscr{V}_X$ on X has the structure of a chiral algebra on X.

Sketch proof. We have already explained the $\mathscr{D}_X$-module $\mathscr{V}_X$ and section $\mathbb{1}_X \in \Gamma(\mathscr{V}_X)$. As in Proposition 7.2, for each $x \in X$ we use $Y(z)$ to define a morphism

$$Y'_x : \mathscr{V}_X|_{X'_x} \otimes_{\mathbb{C}} \mathscr{V}_x \longrightarrow \mathscr{V}_x \otimes_{\mathbb{C}} \mathcal{O}_{X'_x}.$$

We build a morphism $\mu_X : j_* \circ j^*(\mathscr{V}_X \boxtimes \mathscr{V}_X) \to \Delta_!(\mathscr{V}_X)$ such that $\mu_X|_{X'_x \times \{x\}} = Y'_x$ for each $x \in X$, and show that $(\mathscr{V}_X, \mu_X, \mathbb{1}_X)$ is a chiral algebra on X. □

Theorem 7.5 (Frenkel and Ben–Zvi §19.3.)

A quasi-conformal vertex algebra is equivalent to the data of a chiral algebra $\mathscr{V}_X$ on every smooth curve X, together with compatible, functorial isomorphisms $\phi^*(\mathscr{V}_Y) \cong \mathscr{V}_X$ for all étale maps $\phi : X \to Y$ of curves. That is, quasi-conformal vertex algebras are equivalent to **universal chiral algebras***.*

Note however that there are examples of chiral algebras on a curve X which do not come from the universal curve, but are special to X. One can also show that:

- Quasi-conformal vertex algebras are also equivalent to chiral algebras on $\mathbb{D}$ equivariant under $\mathrm{Aut}(\mathbb{D})$.
- Ordinary graded vertex algebras (not VOAs) are equivalent to chiral algebras on $\mathbb{A}^1$ which are translation-equivariant on $\mathbb{A}^1$.

7.3. Vertex Lie algebras and Lie*-algebras

A vertex Lie algebra is like 'half a vertex algebra':

Definition 7.6 (Borcherds style definition.)

Let R be a commutative ring. A *vertex Lie algebra* over R is an R-module V equipped with morphisms $D^{(n)}: V \to V$ for $n = 0, 1, 2, \ldots$ with $D^{(0)} = \mathrm{id}_V$ and $v_n : V \to V$ for all $v \in V$ and $n \in \mathbb{N}$, with v_n R-linear in v, satisfying:

(i) For all $u, v \in V$ we have $u_n(v) = 0$ for $n \gg 0$.

(ii) If $u, v \in V$ then $(D^{(k)}(u))_n(v) = (-1)^k \binom{n}{k} u_{n-k}(v)$ for $0 \leqslant k \leqslant n$, and $(D^{(k)}(u))_n(v) = 0$ for $0 \leqslant n < k$.

(iii) $u_n(v) = \sum_{k \geqslant 0} (-1)^{k+n+1} D^{(k)}(v_{n+k}(u))$ for all $u, v \in V$ and $n \in \mathbb{N}$, where the sum makes sense by (i).

(iv) $$(u_l(v))_m(w) = \sum_{n \geqslant 0} (-1)^n \binom{l}{n} \bigl(u_{l-n}(v_{m+n}(w)) - (-1)^l v_{l+m-n}(u_n(w)) \bigr)$$

for all $u, v, w \in V$ and $l, m \in \mathbb{N}$, where the sum makes sense by (i).

We have operations $u_n(v)$ for $n \in \mathbb{N}$ only, and no identity. Here (ii),(iii) are consequences of the usual vertex algebra axioms.

We may rewrite the definition in terms of a morphism

$$Y(z) : V \otimes V \longrightarrow z^{-1} V[z^{-1}] \cong V[[z]][z^{-1}]/V[[z]],$$

mapping $Y(z) : u \otimes v \mapsto \sum_{n \geqslant 0} z^{-n-1} u_n(v)$. We think of a vertex Lie algebra as remembering only the poles of an ordinary vertex algebra (and hence the OPEs). Any vertex algebra gives a vertex Lie algebra by forgetting $\mathbb{1}$ and the operations $u_n(v)$ for $n < 0$.

There is a VOA version of vertex Lie algebras, called a *conformal vertex Lie algebra*. It includes the data of a conformal vector $\omega \in V$ such that $L_n = \omega_{n+1}$ for $n \geqslant -1$ satisfy the relations of the Lie subalgebra $\langle L_n, n \geqslant -1 \rangle_{\mathbb{C}}$ of the Virasoro algebra.

One can prove (Primc 1999) that the forgetful functor F : (vertex algebras) $\to$ (vertex Lie algebras) has a left adjoint U : (vertex Lie algebras) $\to$ (vertex algebras), sending a vertex Lie algebra to its *universal enveloping vertex algebra*. This is a lot like the universal enveloping algebra of a Lie algebra.

There is a chiral algebra version of vertex Lie algebras:

Definition 7.7 (Beilinson–Drinfeld.)

Let X be an algebraic curve. A *Lie* algebra* on X is a $\mathscr{D}$-module $\mathcal{A}$ on X with a $\mathscr{D}$-module morphism $\mu:\mathcal{A}\boxtimes\mathcal{A}\to\Delta_!(\mathcal{A})$, where $\Delta:X\to X\times X$ is the diagonal map, satisfying

(i) **Antisymmetry:** $\sigma_*\circ\mu\circ\sigma_*=-\mu$, where $\sigma(x_1,x_2)=(x_2,x_1)$.

(ii) **Jacobi identity:** define $\Delta_{***}:X\times X\to X\times X\times X$ to map

$$\Delta_{(12)3}:(x_1,x_2)\mapsto(x_1,x_1,x_2),\quad \Delta_{2(13)}:(x_1,x_2)\mapsto(x_2,x_1,x_2),$$
$$\Delta_{1(23)}:(x_1,x_2)\mapsto(x_1,x_2,x_2),$$

Write $\Delta_3:X\hookrightarrow X^3$ for the diagonal. Let $\mu_{(12)3}$ be the composition

$$\mathcal{A}\boxtimes\mathcal{A}\boxtimes\mathcal{A}\xrightarrow{\mu_{12}\boxtimes\mathrm{id}_3}(\Delta_{(12)3})_!(\mathcal{A}\boxtimes\mathcal{A})\xrightarrow{(\Delta_{(12)3})_!(\mu)}(\Delta_3)_!(\mathcal{A})$$

on $X\times X\times X$, and similarly for $\mu_{2(13)},\mu_{1(23)}$. Then

$$\mu_{(12)3}=\mu_{2(13)}+\mu_{1(23)}.$$

This is as for chiral algebras, but without $j_*\circ j^*$ and $\mathbb{1}$.

Note that Lie* algebras are simpler than chiral algebras: they are more-or-less the naïve notion of Lie algebras in $\mathscr{D}$-modules on X.

Theorem 7.8 (Frenkel and Ben–Zvi §19.4.)

Let V_* be a conformal vertex Lie algebra. Then for each smooth curve X, the $\mathscr{D}_X$-module $\mathscr{V}_X$ on X has the structure of a Lie* algebra on X.

If $(\mathcal{A},\mu,\mathbb{1})$ is a chiral algebra on X then the composition

$$\mathcal{A}\boxtimes\mathcal{A}\xrightarrow{\text{adjunction}} j_*\circ j^*(\mathcal{A}\boxtimes\mathcal{A})\xrightarrow{\quad\mu\quad}\Delta_!(\mathcal{A})$$

makes $\mathcal{A}$ into a Lie* algebra. This is the analogue of the forgetful functor F : (vertex algebras) $\rightarrow$ (vertex Lie algebras). There is a left adjoint functor called the *chiral envelope* U : (Lie* algebras on X) $\rightarrow$ (chiral algebras on X).

Vertex Algebras

Lecture 8 of 8: Factorization algebras and geometric Langlands

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References for this lecture: Frenkel and Ben-Zvi, 2nd ed. (2004), §20.
A. Beilinson and V. Drinfeld, *Chiral algebras*, A.M.S. 2004.

These slides available at
http://people.maths.ox.ac.uk/∼joyce/

Plan of talk:

Introduction

Finally we discuss *factorization algebras*, Beilinson and Drinfeld's second way of generalizing vertex operator algebras (the first being chiral algebras, as in §7). Given a curve X (or topological space, or scheme), we define the *Ran space* $\mathrm{Ran}(X)$ to be the set of all finite subsets $\{x_1,\ldots,x_n\}\subset X$, made into a geometric space. A *factorization algebra* on X is roughly a quasicoherent sheaf $\mathcal{F}\to\mathrm{Ran}(X)$ with functorial isomorphisms $\mathcal{F}|_{I\amalg J}\cong\mathcal{F}|_I\otimes\mathcal{F}|_J$ for all disjoint finite subsets $I,J\subset X$, with an identity section $\mathbb{1}\in H^0(\mathcal{F})$. Quasi-conformal vertex algebras are equivalent to universal factorization algebras.

An interesting feature of this framework is that there is a non-linear, space-level version, a *factorization space* $\mathcal{G}_X\to\mathrm{Ran}(X)$, and factorization algebras may be obtained from factorization spaces by applying some kind of cohomology theory. These 'factorization' ideas have important applications in the geometric Langlands programme.

8.1. Ran spaces

Let X be a topological space. The *Ran space* $\mathrm{Ran}(X)$ is the set of nonempty finite subsets $\{x_1,\ldots,x_n\}$ of X, made into a topological space with the strongest topology (most open sets) such that for $n \geqslant 1$ the maps $X^n \to \mathrm{Ran}(X)$, $(x_1,\ldots,x_n) \mapsto \{x_1,\ldots,x_n\}$ are continuous.

Theorem 8.1 (Beilinson–Drinfeld 2004.)

If X is connected then $\mathrm{Ran}(X)$ is weakly contractible.

Another way to define $\mathrm{Ran}(X)$: let $m \geqslant n \geqslant 1$ and $f:\{1,\ldots,m\} \twoheadrightarrow \{1,\ldots,n\}$ be surjective. Define the f-diagonal $\Delta_f : X^n \to X^m$ by $\Delta_f : (x_1,\ldots,x_n) \mapsto (x_{f(1)},\ldots,x_{f(m)})$. Then $\mathrm{Ran}(X)$ is the colimit in **Top** of all the spaces X^n, $n \geqslant 1$ and f-diagonals $\Delta_f : X^n \to X^m$. That is, $\mathrm{Ran}(X)$ is the universal topological space with maps $\Pi_n : X^n \to \mathrm{Ran}(X)$ for $n \geqslant 1$ satisfying $\Pi_m \circ \Delta_f = \Delta_n$ for all $f:\{1,\ldots,m\} \twoheadrightarrow \{1,\ldots,n\}$.

Now suppose we want an analogue of $\mathrm{Ran}(X)$ in algebraic geometry, for X a $\mathbb{C}$-scheme say. The obvious thing to do is to pick a suitable (higher) category of algebro-geometric spaces, and define $\mathrm{Ran}(X)$ to be the colimit of spaces X^n and f-diagonals $\Delta_f : X^n \to X^m$ in this category. Unfortunately, the colimit doesn't exist in schemes, or ind-schemes (though it nearly does), or Artin stacks. We have to take the colimit in *prestacks*, which are basically functors (commutative $\mathbb{C}$-algebras) $\to$ (groupoids).

Now prestacks are a pretty horrible kind of space, so we shouldn't expect to be able to say much about $\mathrm{Ran}(X)$ in general. But one thing we can understand reasonably well is *sheaves on* $\mathrm{Ran}(X)$, as these are characterized by their pullbacks to X^n. For example, a quasicoherent sheaf (or $\mathscr{D}$-module) $\mathcal{E}$ on $\mathrm{Ran}(X)$ is equivalent to quasicoherent sheaves $\mathcal{E}_n = \Pi_n^*(\mathcal{E})$ on X^n for all $n \geqslant 1$, together with isomorphisms $\mathcal{E}_f : \Delta_f^*(\mathcal{E}_m) \to \mathcal{E}_n$ for all $f : \{1, \ldots, m\} \twoheadrightarrow \{1, \ldots, n\}$ satisfying $\mathcal{E}_{g\circ f} = \mathcal{E}_g \circ \Delta_g^*(\mathcal{E}_f)$ for all $f : \{1, \ldots, m\} \twoheadrightarrow \{1, \ldots, n\}$ and $g : \{1, \ldots, n\} \twoheadrightarrow \{1, \ldots, p\}$.

Bundles on curves and affine Grassmannians

Last lecture I told you that although chiral algebras can be defined on any scheme X, they are most interesting on smooth curves X, because on a curve X we can consider meromorphic functions on $X \times X$ with poles on the diagonal $\Delta(X) \subset X \times X$. The same applies here: although $\mathrm{Ran}(X)$ makes sense for any scheme X, the most interesting applications (I believe) are to curves X.

To show you why, suppose G is an algebraic $\mathbb{C}$-group, and X a smooth projective curve, and write Bun_G for the moduli space of principal G-bundles $P \to X$, which is an Artin stack. If $G = \mathrm{GL}(n, \mathbb{C})$ then Bun_G is the moduli stack of rank n vector bundles $E \to X$. Algebraic geometers care a lot about Bun_G, and it is central in the geometric Langlands programme. I will explain a method for studying Bun_G using $\mathrm{Ran}(X)$.

Bundles on curves and affine Grassmannians

Define the 'adelic Grassmannian' $\mathrm{Gr}_G^{\mathrm{ad}}$ to be the moduli stack of pairs (P,ϕ), a prestack where $P\to X$ is a principal G-bundle and $\phi:P\dashrightarrow X\times G$ is a *rational trivialization*, that is, we have an isomorphism $\phi|_{X\setminus\{x_1,\ldots,x_n\}}:P|_{X\setminus\{x_1,\ldots,x_n\}}\to(X\setminus\{x_1,\ldots,x_n\})\times G$ for some finite subset $\{x_1,\ldots,x_n\}\subset X$, but ϕ may have poles at $x_1,\ldots,x_n$. There is a projection $\Pi_{\mathrm{Bun}_G}:\mathrm{Gr}_G^{\mathrm{ad}}\to\mathrm{Bun}_G$ mapping $(P,\phi)\mapsto P$. For fixed P, two rational trivializations ϕ,ϕ' satisfy $\phi'=\psi\circ\phi$ for a unique rational map $\psi:X\dashrightarrow G$. Thus $\Pi_{\mathrm{Bun}_G}:\mathrm{Gr}_G^{\mathrm{ad}}\to\mathrm{Bun}_G$ is a principal $\mathrm{Map}(X,G)^{\mathrm{rat}}$-bundle, where $\mathrm{Map}(X,G)^{\mathrm{rat}}$ is an infinite-dimensional group, and an ind-scheme.

Theorem 8.2 (Gaitsgory 2012.)

$\mathrm{Map}(X,G)^{\mathrm{rat}}$ *is homologically contractible.*

Thus $\Pi_{\mathrm{Bun}_G}:\mathrm{Gr}_G^{\mathrm{ad}}\to\mathrm{Bun}_G$ is basically a homotopy equivalence. So, for example, the global sections of a sheaf or $\mathscr{D}$-module $\mathcal{E}$ on Bun_G should be the same as the global sections of its pullback to $\mathrm{Gr}_G^{\mathrm{ad}}$.

Now define a morphism $\Pi_{\mathrm{Ran}(X)}:\mathrm{Gr}_G^{\mathrm{ad}}\to\mathrm{Ran}(X)$ to map $(P,\phi)\mapsto\mathrm{Sing}(\phi)$, the set of poles of ϕ. The fibre of $\Pi_{\mathrm{Ran}(X)}$ over $\{x_1,\ldots,x_n\}$ is the ind-scheme

$$\prod_{i=1}^n\frac{\mathrm{Map}(X'_{x_i},G)\setminus\mathrm{Map}(X_{x_i},G)}{\mathrm{Map}(X_{x_i},G)},\tag{8.1}$$

where $X_{x_i}\cong\mathrm{Spec}\,\mathbb{C}[[z]]$ is the formal completion of X at x_i, and $X'_{x_i}=X_{x_i}\setminus\{x_i\}\cong\mathrm{Spec}\,\mathbb{C}[[z]][z^{-1}]$. To see why (8.1) holds, given (P,ϕ) mapping to $\{x_1,\ldots,x_n\}$, choose a local trivialization $\chi_i:P|_{X_{x_i}}\xrightarrow{\cong}X_{x_i}\times G$. This identifies $\phi_i|_{X'_{x_i}}$ with an element of $\mathrm{Map}(X'_{x_i},G)\setminus\mathrm{Map}(X_{x_i},G)$. Two choices of χ_i differ by the action of $\mathrm{Map}(X_{x_i},G)$. Thus, given a sheaf or $\mathscr{D}$-module $\mathcal{E}$ on Bun_G, we have

$$\begin{aligned}H^0(\mathrm{Bun}_G,\mathcal{E})&\cong H^0(\mathrm{Gr}_G^{\mathrm{ad}},\Pi_{\mathrm{Bun}_G}^*(\mathcal{E}))\\&\cong H^0(\mathrm{Ran}(X),(\Pi_{\mathrm{Ran}(X)})_*\circ\Pi_{\mathrm{Bun}_G}^*(\mathcal{E})).\end{aligned}\tag{8.2}$$

Thus we reduce computations on Bun_G to computations on $\mathrm{Ran}(X)$.

As $\mathrm{Map}(X'_{x_i},G)/\mathrm{Map}(X_{x_i},G)$ is generally infinite-dimensional, deleting the point $\mathrm{Map}(X_{x_i},G)/\mathrm{Map}(X_{x_i},G)$ doesn't change $\mathrm{Map}(X'_{x_i},G)/\mathrm{Map}(X_{x_i},G)$ up to homotopy equivalence.

The *Beilinson–Drinfeld Grassmannian* $\mathrm{Gr}_G^{\mathrm{BD}}$ is the moduli space of $(\{x_1,\ldots,x_n\},P,\phi)$, where $\{x_1,\ldots,x_n\}\in\mathrm{Ran}(X)$, $P\to X$ is a principal G-bundle, and $\phi|_{X\setminus\{x_1,\ldots,x_n\}}:P|_{X\setminus\{x_1,\ldots,x_n\}}\to (X\setminus\{x_1,\ldots,x_n\})\times G$ is an isomorphism. There is a projection $\Pi_{\mathrm{Ran}(X)}:\mathrm{Gr}_G^{\mathrm{BD}}\to\mathrm{Ran}(X)$ mapping $(\{x_1,\ldots,x_n\},P,\phi)\mapsto\{x_1,\ldots,x_n\}$, with fibre $\prod_{i=1}^n\mathrm{Map}(X'_{x_i},G)/\mathrm{Map}(X_{x_i},G)$. There is an open inclusion $\mathrm{Gr}_G^{\mathrm{ad}}\hookrightarrow\mathrm{Gr}_G^{\mathrm{BD}}$, a homotopy equivalence.

Thus we have a diagram

$$\begin{array}{ccc}
\mathrm{Gr}_G^{\mathrm{ad}} & \xrightarrow[\mathrm{inc}]{\simeq} & \mathrm{Gr}_G^{\mathrm{BD}} \\
{\scriptstyle\simeq}\big\downarrow{\scriptstyle\Pi_{\mathrm{Bun}_G}} & \searrow^{\Pi_{\mathrm{Ran}(X)}} & \big\downarrow{\scriptstyle\Pi_{\mathrm{Ran}(X)}} \\
\mathrm{Bun}_G & & \mathrm{Ran}(X),
\end{array}$$

with morphisms '$\simeq$' (homological) homotopy equivalences.

If we choose a local formal coordinate z_i near x_i then we get a particular isomorphism $X_{x_i} \cong \operatorname{Spec} \mathbb{C}[[z]] =: \mathbb{D}$, which identifies

$$\frac{\mathrm{Map}(X'_{x_i}, G)}{\mathrm{Map}(X_{x_i}, G)} \cong \frac{\mathrm{Map}(\mathbb{D}', G)}{\mathrm{Map}(\mathbb{D}, G)}.$$

Here $\mathrm{Gr}_G^{\mathrm{aff}} := \mathrm{Map}(\mathbb{D}', G)/\mathrm{Map}(\mathbb{D}, G)$ is the *affine Grassmannian* of G, a formally smooth ind-scheme. The fibre of $\Pi_{\mathrm{Ran}(X)} : \mathrm{Gr}_G^{\mathrm{BD}} \to \mathrm{Ran}(X)$ over $\{x_1, \ldots, x_n\}$ is $(\mathrm{Gr}_G^{\mathrm{aff}})^n$. The *Hecke category* $\mathcal{H}_G := \mathcal{D}\text{-mod}(\mathrm{Gr}_G^{\mathrm{aff}})$ is the category of $\mathscr{D}$-modules on $\mathrm{Gr}_G^{\mathrm{aff}}$. The *geometric Satake correspondence*, for G reductive, is an equivalence of monoidal categories

$$\mathcal{H}_G \simeq \mathrm{Rep}\, {}^L G,$$

where ${}^L G$ is the Langlands dual group of G. This is a kind of mirror symmetry for algebraic groups.

8.2. Factorization algebras

Definition 8.3 (Beilinson–Drinfeld 2004.)

Let X be an algebraic curve. Write $\mathrm{Ran}(X)_{\mathrm{disj}} \subset \mathrm{Ran}(X) \times \mathrm{Ran}(X)$ for the open subset of $(I, J) \in \mathrm{Ran}(X) \times \mathrm{Ran}(X)$ with I, J disjoint, and let $\Phi : \mathrm{Ran}(X)_{\mathrm{disj}} \to \mathrm{Ran}(X)$ map $(I, J) \mapsto I \amalg J$. A *factorization algebra* on X consists of:

(i) A quasicoherent sheaf $\mathcal{F} \to \mathrm{Ran}(X)$.

(ii) An isomorphism $\Psi : (\mathcal{F} \boxtimes \mathcal{F})|_{\mathrm{Ran}(X)_{\mathrm{disj}}} \to \Phi^*(\mathcal{F})$, functorial under unions of disjoint triples $I, J, K \in \mathrm{Ran}(X)$.

(iii) A morphism $\mathbb{1} : \mathcal{O}_{\mathrm{Ran}(X)} \to \mathcal{F}$ (i.e. section $\mathbb{1} \in H^0(\mathcal{F})$) called the *unit*, such that for every local section f of $\Pi_1^*(\mathcal{F}) \to X$, the local section $\Psi(\mathbb{1} \boxtimes f)$ of $\Pi_2^*(\mathcal{F})$ over $(X \times X) \setminus \Delta(X)$ extends over $\Delta(X)$, and restricts on $\Delta(X) \cong X$ to f.

Here $\mathcal{F}_X = \Pi_1^*(\mathcal{F})$ is a quasicoherent sheaf on X. Part (ii) implies that $\mathcal{F}|_{\{x_1,\ldots,x_n\}} \cong \mathcal{F}_X|_{x_1} \otimes \cdots \otimes \mathcal{F}_X|_{x_n}$.

We can think of a factorization algebra as consisting of a quasicoherent sheaf $\mathcal{F}_X \to X$ with section $\mathbb{1}_X \in H^0(\mathcal{F}_X)$, together with data on how to glue the sheaves $\mathcal{F}_X \to X$, $(\mathcal{F}_X \boxtimes \mathcal{F}_X)|_{(X\times X)\setminus\Delta(X)} \to (X\times X)\setminus\Delta(X),\ldots,$ $\mathcal{F}_X^{\boxtimes^n}|_{X^n\setminus\{\text{all diagonals}\}} \to X^n\setminus\{\text{all diagonals}\},\ldots$ on the strata $(X^n\setminus\{\text{all diagonals}\})/S_n$, $n\geqslant 1$, of $\mathrm{Ran}(X)$. This gluing information is provided magically by working with sheaves on $\mathrm{Ran}(X)$. It is essentially the same data as the $\mathscr{D}$-module morphism $\mu : j_*\circ j^*(\mathcal{F}\boxtimes\mathcal{F}) \to \Delta_!(\mathcal{F})$ in a chiral algebra.

It is a surprising fact that any factorization algebra (defined as a quasicoherent sheaf $\mathcal{F}\to\mathrm{Ran}(X)$ with extra data) has a unique left $\mathscr{D}$-module structure (i.e. a flat connection ∇ on $\mathcal{F}\to\mathrm{Ran}(X)$) compatible with the factorization structure. Here is how to build ∇ on $\mathcal{F}_X\to X$. A connection on $\mathcal{F}_X$ is equivalent to an isomorphism $(\mathcal{O}_X\boxtimes\mathcal{F}_X)|_{\Delta(X)^{(1)}} \cong (\mathcal{F}_X\boxtimes\mathcal{O}_X)|_{\Delta(X)^{(1)}}$ on the first-order neighbourhood $\Delta(X)^{(1)}$ of $\Delta(X)$ in $X\times X$ which restricts to $\mathrm{id}_{\mathcal{F}_X}$ on $\Delta(X)$. Using $\mathbb{1}$ we get isomorphisms which induce ∇

$$(\mathcal{O}_X\boxtimes\mathcal{F}_X)|_{\Delta(X)^{(1)}} \cong \Pi_2^*(\mathcal{F}_X)|_{\Delta(X)^{(1)}} \cong (\mathcal{F}_X\boxtimes\mathcal{O}_X)|_{\Delta(X)^{(1)}}.$$

Factorization algebras and chiral algebras

Theorem 8.4 (Beilinson–Drinfeld 2004.)

Let X be an algebraic curve. There is an equivalence of categories

$$\{\textit{factorization algebras on } X\} \longrightarrow \{\textit{chiral algebras on } X\},$$

which maps a factorization algebra $(\mathcal{F}, \Psi, \mathbb{1})$ to the right $\mathscr{D}$-module $(\mathcal{F}_X, \nabla) \otimes K_X$ on X obtained from the left $\mathscr{D}$-module $\mathcal{F}_X = \Pi_1^*(\mathcal{F})$ with canonical connection ∇, with identity $\Pi_1^*(\mathbb{1}) \otimes \mathrm{id}_{K_X} : K_X \to \mathcal{F}_X \otimes K_X$, and with chiral morphism μ constructed from $\Pi_2^*(\Psi)$.

Combined with Theorem 7.5, this implies

Corollary 8.5 (Frenkel and Ben–Zvi §20.2.)

A quasi-conformal vertex algebra is equivalent to the data of a factorization algebra $\mathcal{F}_X \to \mathrm{Ran}(X)$ for every smooth curve X, together with compatible, functorial isomorphisms $\phi^*(\mathcal{F}_Y) \cong \mathcal{F}_X$ for all étale maps $\phi : X \to Y$ of curves. That is, quasi-conformal vertex algebras are equivalent to **universal factorization algebras**.

Write $\mathrm{Ran}(X)_{\leqslant n}$ for the closed subspace of $I \in \mathrm{Ran}(X)$ with $|I| \leqslant n$. Then $\mathrm{Ran}(X)_{\leqslant 1} \cong X$. Given a factorization algebra $\mathcal{F} \to \mathrm{Ran}(X)$, the corresponding chiral algebra $(\mathcal{A}, \mu, \mathbb{1}_{\mathcal{A}})$ is determined as an $\mathcal{O}_X$-module $\mathcal{A} \in \mathcal{O}_X$-mod by $\mathcal{A} = \mathcal{F}|_{\mathrm{Ran}(X)_{\leqslant 1}} \otimes K_X$, with $\mathbb{1}_{\mathcal{A}} = \mathbb{1}|_{\mathrm{Ran}(X)_{\leqslant 1}} \otimes \mathrm{id}_{K_X}$. The remaining structures ∇, μ are determined by $\mathcal{F}|_{\mathrm{Ran}(X)_{\leqslant 2}}$, and the relations on these structures are determined by $\mathcal{F}|_{\mathrm{Ran}(X)_{\leqslant 3}}$. So we could actually write the theory just in terms of $\mathrm{Ran}(X)_{\leqslant 3}$, which is finite-dimensional. But then we would miss the consequences of $\mathrm{Ran}(X)$ being weakly contractible, for instance.

A topic which uses the full geometry of $\mathrm{Ran}(X)$, not just $\mathrm{Ran}(X)_{\leqslant 3}$, is Beilinson–Drinfeld's *chiral homology groups*

$$H^i_{\mathrm{dR}}(\mathrm{Ran}(X), \mathcal{F}),$$

the de Rham cohomology groups of $\mathcal{F} \to \mathrm{Ran}(X)$ as a left $\mathscr{D}$-module. Here $H^0_{\mathrm{dR}}(\mathrm{Ran}(X), \mathcal{F})$ is roughly the dual of the space of *conformal blocks*.

8.3. Factorization spaces

Definition 8.6 (Beilinson–Drinfeld 2004.)

Let X be an algebraic curve. A *factorization space* $\mathcal{G}$ over X is a morphism of prestacks $\Pi : \mathcal{G} \to \mathrm{Ran}(X)$ which is a formallly smooth ind-scheme over $\mathrm{Ran}(X)$ (i.e. the fibres are formally smooth ind-schemes), with an isomorphism

$$\Psi : (\mathcal{G} \times \mathcal{G}) \times_{\Pi\times\Pi,\mathrm{Ran}(X)\times\mathrm{Ran}(X),\mathrm{inc}} \mathrm{Ran}(X)_{\mathrm{disj}} \longrightarrow \mathcal{G} \times_{\Pi,\mathrm{Ran}(X),\Phi} \mathrm{Ran}(X)_{\mathrm{disj}}. \tag{8.3}$$

Writing $\mathcal{G}_I = \Pi^{-1}(I)$ for $I \in \mathrm{Ran}(X)$, as a formally smooth ind-scheme, (8.3) gives isomorphisms $\mathcal{G}_I \times \mathcal{G}_J \to \mathcal{G}_{I\amalg J}$ for all disjoint $I, J \in \mathrm{Ran}(X)$. We require Ψ to be functorial under unions of disjoint triples $I, J, K \in \mathrm{Ran}(X)$. A *unit* for $\mathcal{G}$ is a section $\mathbb{1} : X \to \mathcal{G}$ of $\Pi : \mathcal{G} \to \mathrm{Ran}(X)$, which is compatible with factorization and restriction to diagonals.

This is a non-linear, space-level analogue of factorization algebras.

The prototypical example of a factorization space is the Beilinson–Drinfeld Grassmannian $\Pi_{\mathrm{Ran}(X)}:\mathrm{Gr}_G^{\mathrm{BD}}\to\mathrm{Ran}(X)$ in §8.1. We can pass from factorization spaces to factorization algebras by 'linearization', passing to some kind of cohomology: given a suitable cohomology functor $\mathbb{H}(-)$, from a factorization space $\Pi:\mathcal{G}\to\mathrm{Ran}(X)$ we associate a factorization algebra $\mathcal{F}\to\mathrm{Ran}(X)$ with $\mathcal{F}(U)=\mathbb{H}(\Pi^{-1}(U))$. One way to do this is to take $\mathcal{F}=\Pi_*\circ\mathbb{1}_!(K_{\mathrm{Ran}(X)})$, where $K_{\mathrm{Ran}(X)}$ is the unit right $\mathscr{D}$-module on $\mathrm{Ran}(X)$, and $\mathbb{1}_!$ the $\mathscr{D}$-module pushforward along $\mathbb{1}:\mathrm{Ran}(X)\to\mathcal{G}_X$, and Π_* the $\mathcal{O}$-module pushforward along $\Pi:\mathcal{G}_X\to\mathrm{Ran}(X)$.

The geometric Langlands correspondence

Let X be an algebraic curve and G a reductive algebraic group, with Langlands dual group ${}^L G$. The *geometric Langlands correspondence* is a conjectural equivalence of categories

$$D(\mathscr{D}\text{-mod}(\mathrm{Bun}_G)) \simeq D(\mathrm{qcoh}(\mathrm{LocSys}_{{}^L G})) \tag{8.4}$$

between the derived categories of $\mathscr{D}$-modules on Bun_G, the moduli stack of principal G-bundles on X, and of quasicoherent sheaves on $\mathrm{LocSys}_{{}^L G}$, the moduli stack of ${}^L G$-local systems on X. This is supposed to identify skyscraper sheaves on the right with 'Hecke eigensheaves' on the left.

Actually in this form the conjecture is false, and needs to be refined as in Arinkin–Gaitsgory 2012.

A programme of Beilinson–Drinfeld, starting with an ${}^L G$-local system $E \to X$ (and hence a skyscraper sheaf $\mathcal{O}_E$ on $\mathrm{LocSys}_{{}^L G}$), explains how to use factorization spaces over $\mathrm{Ran}(X)$ to construct a Hecke eigensheaf in $\mathscr{D}$-mod(Bun_G) with eigenvalue E. This reduces geometric Langlands to a question in factorization algebras.