Extensions of Donaldson–Thomas theory of Calabi–Yau 3-folds

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Plan of talk:

1. Categorification using perverse sheaves
2. Motives of d-critical loci
3. D–T style invariants for Calabi-Yau 4-folds
4. More about perverse sheaves
5. 'Fukaya categories' of complex symplectic manifolds
6. Cohomological Hall Algebras
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1. Categorification using perverse sheaves

Theorem (Brav, Bussi, Dupont, Joyce, Szendrői arXiv:1211.3259)

Let (X, s) be an algebraic d-critical locus over $\mathbb{K}$, with an orientation $K_{X,s}^{1/2}$. Then we can construct a canonical perverse sheaf $P_{X,s}^\bullet$ on X, such that if (X, s) is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, then $P_{X,s}^\bullet$ is locally modelled on the perverse sheaf of vanishing cycles $\mathcal{PV}_{U,f}^\bullet$ of (U, f).
Similarly, we can construct a natural $\mathscr{D}$-module $D_{X,s}^\bullet$ on X, and when $\mathbb{K} = \mathbb{C}$ a natural mixed Hodge module $M_{X,s}^\bullet$ on X.

Sketch of the proof of the theorem

Roughly, we prove the theorem by taking a Zariski open cover $\{R_i : i \in I\}$ of X with $R_i \cong \mathrm{Crit}(f_i : U_i \to \mathbb{A}^1)$, and showing that $\mathcal{PV}_{U_i,f_i}^\bullet$ and $\mathcal{PV}_{U_j,f_j}^\bullet$ are canonically isomorphic on $R_i \cap R_j$, so we can glue the $\mathcal{PV}_{U_i,f_i}^\bullet$ to get a global perverse sheaf $P_{X,s}^\bullet$ on X.
In fact things are more complicated: the (local) isomorphisms $\mathcal{PV}_{U_i,f_i}^\bullet \cong \mathcal{PV}_{U_j,f_j}^\bullet$ are only canonical *up to sign*. To make them canonical, we use the orientation $K_{X,s}^{1/2}$ to define natural principal $\mathbb{Z}_2$-bundles Q_i on R_i, such that $\mathcal{PV}_{U_i,f_i}^\bullet \otimes_{\mathbb{Z}_2} Q_i \cong \mathcal{PV}_{U_j,f_j}^\bullet \otimes_{\mathbb{Z}_2} Q_j$ is canonical, and then we glue the $\mathcal{PV}_{U_i,f_i}^\bullet \otimes_{\mathbb{Z}_2} Q_i$ to get $P_{X,s}^\bullet$.

The first corollary in §2 of the first talk implies:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a classical moduli $\mathbb{K}$-scheme of coherent sheaves, or complexes of coherent sheaves, on Y, with (symmetric) obstruction theory $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. **orientation data**, *K–S). Then we have a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a finite-dimensional graded vector space. The pointwise Euler characteristic $\chi(P^\bullet_{\mathcal{M},s})$ is the *Behrend function* $\nu_{\mathcal{M}}$ of $\mathcal{M}$. Thus

$$\textstyle\sum_{i\in\mathbb{Z}}(-1)^i\dim\mathbb{H}^i(P^\bullet_{\mathcal{M},s})=\chi(\mathcal{M},\nu_{\mathcal{M}}).$$

Now by Behrend 2005, the Donaldson–Thomas invariant of $\mathcal{M}$ is $DT(\mathcal{M})=\chi(\mathcal{M},\nu_{\mathcal{M}})$. So, $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a graded vector space with dimension $DT(\mathcal{M})$, that is, a *categorification* of $DT(\mathcal{M})$.

Categorifying Lagrangian intersections

The second corollary in §2 implies:

Corollary

Let (S,ω) be a classical smooth symplectic $\mathbb{K}$-scheme of dimension $2n$, and $L,M\subseteq S$ be smooth algebraic Lagrangians, with square roots $K_L^{1/2},K_M^{1/2}$ of their canonical bundles. Then we have a natural perverse sheaf $P^\bullet_{L,M}$ on $X=L\cap M$.

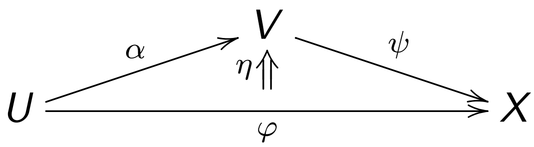
This is related to Behrend and Fantechi 2009. We think of the hypercohomology $\mathbb{H}^*(P^\bullet_{L,M})$ as being morally related to the *Lagrangian Floer cohomology* $HF^*(L,M)$ by

$$\mathbb{H}^i(P^\bullet_{L,M})\approx HF^{i+n}(L,M).$$

We are working on defining 'Fukaya categories' for algebraic/complex symplectic manifolds using these ideas.

Extension to Artin stacks

Let (X, s) be a d-critical stack, with an orientation $K_{X,s}^{1/2}$. Then for any smooth $\varphi : U \to X$ with U a scheme, $\bigl(U, s(U, \varphi)\bigr)$ is an oriented d-critical locus, so as above, BBDJS constructs a perverse sheaf $P^\bullet_{U,\varphi}$ on U. Given a diagram



with U, V schemes and φ, ψ smooth, we can construct a natural isomorphism $P^\bullet_{\alpha,\eta} : \alpha^*(P^\bullet_{V,\psi})[\dim\varphi - \dim\psi] \to P^\bullet_{U,\varphi}$.
All this data $P^\bullet_{U,\varphi}, P^\bullet_{\alpha,\eta}$ is equivalent to a perverse sheaf on X.

Thus we prove:

Theorem (Ben-Bassat, Brav, Bussi, Joyce)

Let (X, s) be a d-critical stack, with an orientation $K_{X,s}^{1/2}$. Then we can construct a canonical perverse sheaf $P^\bullet_{X,s}$ on X.

Corollary

Suppose Y is a Calabi–Yau 3-fold and $\mathcal{M}$ a classical moduli stack of coherent sheaves F on Y, or of complexes $F^\bullet$ in $D^b \operatorname{coh}(Y)$ with $\operatorname{Ext}^{<0}(F^\bullet, F^\bullet) = 0$, with (symmetric) obstruction theory $\phi : \mathcal{E}^\bullet \to \mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$. Then we construct a natural perverse sheaf $P^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.

The *hypercohomology* $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$ is a categorification of the Donaldson–Thomas theory of Y.

2. Motives of d-critical loci

By similar arguments to those used to construct the perverse sheaves $P^\bullet_{X,s}$ in §1, we prove:

Theorem (Bussi, Joyce and Meinhardt arXiv:1305.6428)

Let (X,s) be a finite type algebraic d-critical locus over $\mathbb{K}$, with an orientation $K^{1/2}_{X,s}$. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat{\mu}$-equivariant motives $\bar{\mathcal{M}}^{\hat{\mu}}_X$ on X, such that if (X,s) is locally modelled on $\mathrm{Crit}(f:U\to\mathbb{A}^1)$, then $MF_{X,s}$ is locally modelled on $\mathbb{L}^{-\dim U/2}([X]-MF^{\mathrm{mot}}_{U,f})$, where $MF^{\mathrm{mot}}_{U,f}$ is the **motivic Milnor fibre** *or* **motivic nearby cycle** *of f.*

Relation to motivic D–T invariants

The first corollary in §2 implies:

Corollary

Let Y be a Calabi–Yau 3-fold over $\mathbb{K}$ and $\mathcal{M}$ a finite type classical moduli $\mathbb{K}$-scheme of (complexes of) coherent sheaves on Y, with (symmetric) obstruction theory $\phi:\mathcal{E}^\bullet\to\mathbb{L}_{\mathcal{M}}$. Suppose we are given a square root $\det(\mathcal{E}^\bullet)^{1/2}$ for $\det(\mathcal{E}^\bullet)$ (i.e. **orientation data**, *K–S). Then we have a natural motive $MF^\bullet_{\mathcal{M},s}$ on $\mathcal{M}$.*

This motive $MF^\bullet_{\mathcal{M},s}$ is essentially the motivic Donaldson–Thomas invariant of $\mathcal{M}$ defined (partially conjecturally) by Kontsevich and Soibelman 2008. K–S work with motivic Milnor fibres of formal power series at each point of $\mathcal{M}$. Our results show the formal power series can be taken to be a regular function, and clarify how the motivic Milnor fibres vary in families over $\mathcal{M}$.

Extension to Artin stacks

We can also generalize BJM to d-critical stacks:

Theorem (Ben-Bassat, Brav, Bussi, Joyce)

Let (X, s) be an oriented d-critical stack, of finite type and locally a global quotient. Then we can construct a natural motive $MF_{X,s}$ in a certain ring of $\hat{\mu}$-equivariant motives $\bar{\mathcal{M}}_X^{\mathrm{st},\hat{\mu}}$ on X, such that if $\varphi : U \to X$ is smooth and U is a scheme then

$$\varphi^*(MF_{X,s}) = \mathbb{L}^{\dim\varphi/2} \odot MF_{U,s(U,\varphi)},$$

where $MF_{U,s(U,\varphi)}$ for the scheme case is as in BJM above.

For CY3 moduli stacks, these $MF_{X,s}$ are basically Kontsevich–Soibelman motivic Donaldson–Thomas invariants.

3. D–T style invariants for Calabi-Yau 4-folds

If Y is a Calabi–Yau 3-fold (say over $\mathbb{C}$), then the *Donaldson-Thomas invariants* $DT^\alpha(\tau)$ in $\mathbb{Z}$ or $\mathbb{Q}$ 'count' τ-(semi)stable coherent sheaves on Y with Chern character $\alpha \in H^{\mathrm{even}}(Y, \mathbb{Q})$, for τ a (say Gieseker) stability condition. The $DT^\alpha(\tau)$ are unchanged under continuous deformations of Y, and transform by a wall-crossing formula under change of stability condition τ.

We have τ-(semi)stable moduli schemes $\mathcal{M}_{\mathrm{st}}^\alpha(\tau) \subseteq \mathcal{M}_{\mathrm{ss}}^\alpha(\tau)$, where $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau)$ is proper, and $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ has a symmetric obstruction theory.

The easy case (Thomas 1998) is when $\mathcal{M}_{\mathrm{ss}}^\alpha(\tau) = \mathcal{M}_{\mathrm{st}}^\alpha(\tau)$. Then $DT^\alpha(\tau) \in \mathbb{Z}$ is the *virtual cycle* (which has dimension zero) of the proper scheme with obstruction theory $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$.

Note that the derived moduli scheme $\boldsymbol{\mathcal{M}}_{\mathrm{st}}^\alpha(\tau)$ is -1-shifted symplectic by PTVV, and $\mathcal{M}_{\mathrm{st}}^\alpha(\tau)$ is a d-critical locus by BBJ.

Holomorphic Donaldson invariants?

In joint work with Dennis Borisov (in progress), I am developing a similar story for Calabi–Yau 4-folds. We want to define invariants 'counting' τ-(semi)stable coherent sheaves on Calabi–Yau 4-folds. If CY3 Donaldson–Thomas invariants are 'holomorphic Casson invariants', as in Thomas 1998, these should be thought of as 'holomorphic Donaldson invariants'.
The idea for doing this goes back to Donaldson–Thomas 1998, using gauge theory: one wants to 'count' moduli spaces of Spin(7)-instantons on a Calabi–Yau 4-fold (or more generally a Spin(7)-manifold). However, it has not gone very far, as compactifying such higher-dimensional gauge-theoretic moduli spaces in a nice way is (much) too difficult.

Virtual cycles using algebraic geometry?

Rather than using gauge theory, we stay within algebraic geometry, so we get compactness of moduli spaces more-or-less for free. So, suppose Y is a Calabi–Yau 4-fold, and $\alpha \in H^{\rm even}(Y,\mathbb{Q})$ such that $\mathcal{M}_{\rm ss}^\alpha(\tau)=\mathcal{M}_{\rm st}^\alpha(\tau)$ (the easy case). Then $\mathcal{M}_{\rm st}^\alpha(\tau)$ is proper, and the corresponding derived moduli scheme $\boldsymbol{\mathcal{M}}_{\rm st}^\alpha(\tau)$ is -2-shifted symplectic by PTVV. It need not have virtual dimension zero. Our task is to define a virtual cycle for $\boldsymbol{\mathcal{M}}_{\rm st}^\alpha(\tau)$, or more generally for any proper -2-shifted symplectic derived scheme $(\mathbf{X},\omega)$.
There is a natural obstruction theory $\phi:\mathcal{E}^\bullet\rightarrow\mathbb{L}_{\mathcal{M}}$ on $\mathcal{M}_{\rm st}^\alpha(\tau)$, but $\mathcal{E}^\bullet$ is perfect in $[-2,0]$ not $[-1,0]$, so the usual Behrend–Fantechi virtual cycles do not work.

Virtual cycles using d-manifolds

Here is the first part of what we want to prove:

"Theorem" (Borisov–Joyce, in progress, proof nearly finished)

Let $(\mathbf{X}, \omega)$ be a -2-shifted symplectic derived scheme over $\mathbb{C}$. Then one can construct a **d-manifold** *(derived smooth manifold) $X_{\rm dm}$ which has the same underlying topological space X as $(\mathbf{X}, \omega)$, with the complex analytic topology.*
The construction involves arbitrary choices, but $X_{\rm dm}$ is unique up to bordisms which fix the topological space X.
The (real) virtual dimension of $X_{\rm dm}$ is

$$\mathrm{vdim}_{\mathbb{R}} X_{\rm dm} = \mathrm{vdim}_{\mathbb{C}} \mathbf{X} = \tfrac{1}{2} \mathrm{vdim}_{\mathbb{R}} \mathbf{X},$$

which is half what one would have expected.

D-manifolds and bordisms

I haven't time to explain d-manifolds – see my webpage `people.maths.ox.ac.uk/~joyce/dmanifolds.html`.
For now, note that any (compact) d-manifold $\mathbf{X}$ can be perturbed to a (compact) ordinary manifold $\tilde{X}$, which is unique up to bordism. Thus, if $(\mathbf{X}, \omega)$ is a proper -2-shifted symplectic derived $\mathbb{C}$-scheme, the "Theorem" will give us a bordism class of (unoriented) compact manifolds $\tilde{X}$, which is basically a virtual cycle over $\mathbb{Z}_2$. To lift this to a virtual cycle over $\mathbb{Z}$, we need to include *orientations* of $(\mathbf{X}, \omega)$ and $X_{\rm dm}$.
Recall that if $(\mathbf{X}, \omega)$ is a -1-shifted symplectic derived scheme (the CY3 case), an *orientation* of $(\mathbf{X}, \omega)$ is a square root line bundle $\det(\mathbb{L}_{\mathbf{X}})^{1/2}$. These were introduced by Kontsevich and Soibelman, and are essential for motivic and categorified D–T theory.

Orientations of -2-shifted symplectic derived schemes

Here is the CY4 analogue:

Definition

Let $(\mathbf{X},\omega)$ be a -2-shifted symplectic derived scheme. There is a natural isomorphism $\iota:\det(\mathbb{L}_{\mathbf{X}})^{\otimes^2}\longrightarrow\mathcal{O}_{\mathbf{X}}$. An *orientation* of $(\mathbf{X},\omega)$ is an isomorphism $\alpha:\det(\mathbb{L}_{\mathbf{X}})\longrightarrow\mathcal{O}_{\mathbf{X}}$ with $\alpha\otimes\alpha=\iota$.

"Lemma"

In the "Theorem", there is a natural 1-1 *correspondence between orientations on* $(\mathbf{X},\omega)$ *and orientations on the d-manifold* $X_{\rm dm}$.

"Corollary"

Let $(\mathbf{X},\omega)$ *be a proper, oriented* -2*-shifted symplectic derived* $\mathbb{C}$*-scheme. Then we construct a bordism class* $[X_{\rm dm}]$ *of compact oriented manifolds. We consider this a* **virtual cycle** *for* $(\mathbf{X},\omega)$.

Sketch proof of "Theorem"

Let $(\mathbf{X},\omega)$ be a -2-shifted symplectic derived $\mathbb{C}$-scheme. Then the BBJ 'Darboux Theorem' gives local models for $(\mathbf{X},\omega)$ in the Zariski topology. In the -2-shifted case, the local models reduce to the following data:

- A smooth $\mathbb{C}$-scheme U
- A vector bundle $E\rightarrow U$
- A section $s\in H^0(E)$
- A nondegenerate quadratic form Q on E with $Q(s,s)=0$.

The underlying topological space of $\mathbf{X}$ is $\{x\in U:s(x)=0\}$. The virtual dimension of $\mathbf{X}$ is $\mathrm{vdim}_{\mathbb{C}}\mathbf{X}=2\dim_{\mathbb{C}}U-\mathrm{rank}_{\mathbb{C}}E$. The cotangent complex $\mathbb{L}_{\mathbf{X}}|_X$ of $\mathbf{X}$ is (roughly)

$$\bigl[\underset{-2}{TU}\xrightarrow{Q\circ\mathrm{d}s}\underset{-1}{E^*}\xrightarrow{\mathrm{d}s}\underset{0}{T^*U}\bigr].$$

The local model for $X_{\rm dm}$

Here is how to build the d-manifold $X_{\rm dm}$ locally: regard $E\to U$ as a real vector bundle over the real manifold U. Choose a splitting $E=E_+\oplus E_-$, where $Q|_{E_+}$ is real and positive definite, and $E_-=iE_+$ so that $Q|_{E_-}$ is real and negative definite. Write $s=s_+\oplus s_-$ with $s_\pm\in C^\infty(E_\pm)$. Then $X_{\rm dm}$ is the derived fibre product $U\times_{0,E_+,s_+}U$. It has virtual dimension

$$\dim_{\mathbb R}U-\mathrm{rank}_{\mathbb R}E_+=2\dim_{\mathbb C}U-\mathrm{rank}_{\mathbb C}E=\mathrm{vdim}_{\mathbb C}\mathbf{X}.$$

Observe that $Q(s,s)=0$ implies that $|s_+|^2=|s_-|^2$, where norms $|\,.\,|$ on E_+,E_- are defined using $\pm\,\mathrm{Re}\,Q$. Hence as sets we have

$$\{x\in U:s(x)=0\}=\{x\in U:s_+(x)=0\}\subseteq U.$$

This is why $\mathbf{X}$ and $X_{\rm dm}$ have the same topological space X.

The difficult bit is to show we can choose compatible splittings $E=E_+\oplus E_-$ on an open cover of $\mathbf{X}$, and glue the local models to make a global d-manifold $X_{\rm dm}$.

4. More about perverse sheaves

Let $(\mathbf{X},\omega_{\mathbf{X}})$ be a -1-shifted symplectic derived scheme, and $\mathbf{i}:\mathbf{L}\to\mathbf{X}$ a Lagrangian, in the sense of PTVV.

Choose an orientation $K_{X,s}^{1/2}$ for $(\mathbf{X},\omega_{\mathbf{X}})$. There is then a notion of relative orientation for $\mathbf{i}:\mathbf{L}\to\mathbf{X}$, choose one of these.

We get a perverse sheaf $P^\bullet_{\mathbf{X},\omega_{\mathbf{X}}}$ on $\mathbf{X}$, by BBDJS in §1.

Conjecture

There is a natural morphism in $D^b_c(\mathbf{L})$

$$\mu_{\mathbf{L}}:\mathbb{Q}_{\mathbf{L}}[\mathrm{vdim}\,\mathbf{L}]\longrightarrow\mathbf{i}^!(P^\bullet_{\mathbf{X},\omega_{\mathbf{X}}}),\qquad(1)$$

with given local models in 'Darboux form' presentations for $\mathbf{X},\mathbf{L}$.

This Conjecture has important consequences.

I already know local models for $\boldsymbol{i}:\boldsymbol{L}\to\boldsymbol{X}$ and $\mu_{\boldsymbol{L}}$ in (1). What makes the Conjecture difficult is that local models are not enough: $\mu_{\boldsymbol{L}}$ is a morphism of complexes, not of (perverse) sheaves, and such morphisms do not glue like sheaves. For instance, one could imagine $\mu_{\boldsymbol{L}}$ to be globally nonzero, but zero on the sets of an open cover of $\boldsymbol{L}$.

So to construct $\mu_{\boldsymbol{L}}$, we have to do a gluing problem in an ∞-category, probably using hypercovers. I have a sketch of one way to do this (over $\mathbb{C}$). It is not easy.

Maybe gluing local models naïvely is not the best approach for this problem, need some more advanced Lurie-esque technology?

Relation to Calabi–Yau 4-fold sheaf counting

Let $(\boldsymbol{X},\omega)$ be an oriented -2-shifted symplectic derived scheme over $\mathbb{C}$, e.g. a Calabi–Yau 4-fold derived moduli scheme. Regard the point $*$ as an oriented -1-shifted symplectic derived scheme. Its perverse sheaf is the constant sheaf $\mathbb{Q}_*$. Then $\boldsymbol{\pi}:\boldsymbol{X}\to *$ is Lagrangian in $(*,\omega)$, so the Conjecture gives a morphism

$$\mu_{\boldsymbol{X}}:\mathbb{Q}_{\boldsymbol{X}}[\mathrm{vdim}\,\boldsymbol{X}]\longrightarrow \boldsymbol{\pi}^!(\mathbb{Q}_*)=\mathcal{D}_{\boldsymbol{X}}(\mathbb{Q}_{\boldsymbol{X}}).$$

Taking hypercohomology induces a linear map

$$H^{\mathrm{vdim}\,\boldsymbol{X}}_{\mathrm{cs}}(X,\mathbb{Q})\longrightarrow\mathbb{Q}.$$

If $\boldsymbol{X}$ is compact, this should be contraction with a class $[\boldsymbol{X}]\in H_{\mathrm{vdim}\,\boldsymbol{X}}(X,\mathbb{Q})$. I expect this to be the virtual cycle from §3. Note that this also works over other fields.

5. 'Fukaya categories' of complex symplectic manifolds

Let (S,ω) be a complex symplectic manifold, with $\dim_{\mathbb{C}} S = 2n$, and $L, M \subset S$ be complex Lagrangians (not supposed compact or closed). The intersection $L \cap M$, as a complex analytic space, has a d-critical structure s (Vittoria Bussi, work in progress). Given square roots of canonical bundles $K_L^{1/2}, K_M^{1/2}$, we get an orientation on $(L \cap M, s)$, and so a perverse sheaf $P^\bullet_{L,M}$ on $L \cap M$.
I claim that we should think of the shifted hypercohomology $\mathbb{H}^{*-n}(P^\bullet_{L,M})$ as a substitute for the Lagrangian Floer cohomology $HF^*(L,M)$ in symplectic geometry. But $HF^*(L,M)$ is the morphisms in the derived Fukaya category $D^b\mathrm{Fuk}(S,\omega)$.

Problem

Given a complex symplectic manifold (S,ω), build a 'Fukaya category' with objects $(L, K_L^{1/2})$ for L a complex Lagrangian, and graded morphisms $\mathbb{H}^{*-n}(P^\bullet_{L,M})$.
Extend to **derived** *Lagrangians L in (S,ω).*
Work out the 'right' way to form a 'derived Fukaya category' for (S,ω) out of this, as a (Calabi–Yau?) triangulated category.
Show that (derived) Lagrangian correspondences induce functors between these derived Fukaya categories.

Question

Can we include complex coisotropic submanifolds as objects? Maybe using $\mathscr{D}$-modules?

The Conjecture in §4 is what we need to define composition of morphisms in this 'Fukaya category', as follows. If L, M, N are Lagrangians in (S, ω), then $M \cap L, N \cap M, L \cap N$ are -1-shifted symplectic / d-critical loci, and $L \cap M \cap N$ is Lagrangian in the product $(M \cap L) \times (N \cap M) \times (L \cap N)$ (Ben-Bassat arXiv:1309.0596). Applying the Conjecture to $L \cap M \cap N$ and rearranging gives a morphism of constructible complexes

$$\mu_{L,M,N} : P^\bullet_{L,M} \overset{L}{\otimes} P^\bullet_{M,N}[n] \longrightarrow P^\bullet_{L,N}.$$

Taking hypercohomology gives the multiplication $\mathrm{Hom}^*(L, M) \times \mathrm{Hom}^*(M, N) \to \mathrm{Hom}^*(L, N)$.

6. Cohomological Hall Algebras

Let Y be a Calabi–Yau 3-fold, and $\mathcal{M}$ the moduli stack of coherent sheaves (or suitable complexes) on Y. Then BBBJ makes $\mathcal{M}$ into a d-critical stack $(\mathcal{M}, s)$. Suppose we have 'orientation data' for Y, i.e. an orientation $K^{1/2}_{\mathcal{M},s}$, with compatibility condition on exact sequences.

Then we have a perverse sheaf $P^\bullet_{\mathcal{M},s}$, with hypercohomology $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$.

We would like to define an associative multiplication on $\mathbb{H}^*(P^\bullet_{\mathcal{M},s})$, making it into a *Cohomological Hall Algebra*, Kontsevich–Soibelman style (arXiv:1006.2706).

Let $\mathcal{E}xact$ be the stack of short exact sequences $0 \to F_1 \to F_2 \to F_3 \to 0$ in $\mathrm{coh}(Y)$ (or distinguished triangles in $D^b \mathrm{coh}(Y)$), with projections $\pi_1, \pi_2, \pi_3 : \mathcal{E}xact \to \mathcal{M}$.

"Theorem" (Oren Ben-Bassat, work in progress.)

$\pi_1 \times \pi_2 \times \pi_3 : \mathcal{E}xact \to \mathcal{M} \times \bar{\mathcal{M}} \times \mathcal{M}$ is Lagrangian in -1-shifted symplectic, where the central term $\bar{\mathcal{M}}$ has the opposite sign -1-shifted symplectic structure.

Then apply the stack version of the Conjecture in §4 to get COHA multiplication, in a similar way to the Fukaya category case.

7. Gluing matrix factorization categories

Suppose $f : U \to \mathbb{A}^1$ is a regular function on a smooth scheme U. The *matrix factorization category* $\mathrm{MF}(U, f)$ is a $\mathbb{Z}_2$-periodic triangulated category. It depends only on U, f in a neighbourhood of $\mathrm{Crit}(f)$, and we can think of it as a sheaf of triangulated categories on $\mathrm{Crit}(f)$. By BBJ, -1-shifted symplectic derived schemes $(X, \omega_{\mathbf{X}})$ are locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$.

Problem

Given a -1-shifted symplectic derived scheme $(X, \omega_{\mathbf{X}})$ with extra data (orientation and 'spin structure'?), construct a sheaf of $\mathbb{Z}_2$-periodic triangulated categories $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ on $\mathbf{X}$, such that if $(\mathbf{X}, \omega_{\mathbf{X}})$ is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, then $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ is locally modelled on $\mathrm{MF}(U, f)$.

Although d-critical loci (X, s) are also locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$, I do not expect the analogue for d-critical loci to work; $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ will encode derived data in $(\mathbf{X}, \omega_{\mathbf{X}})$ which is forgotten by the d-critical locus (X, s).
I expect that a Lagrangian $\mathbf{i} : \mathbf{L} \to \mathbf{X}$ (plus extra data) should define an object (global section of sheaf of objects) in $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$, with nice properties.
It is conceivable that one could actually *define* $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$ as a derived category of Lagrangians $\mathbf{i} : \mathbf{L} \to \mathbf{X}$ in $(\mathbf{X}, \omega_{\mathbf{X}})$.

Kapustin–Rozansky 2-categories for complex symplectic manifolds

Given a complex symplectic manifold (S, ω), Kapustin and Rozansky conjecture the existence of an interesting 2-category, with objects complex Lagrangians L with $K_L^{1/2}$, such that $\mathrm{Hom}(L, M)$ is a $\mathbb{Z}_2$-periodic triangulated category (or sheaf of such on $L \cap M$), and if $L \cap M$ is locally modelled on $\mathrm{Crit}(f : U \to \mathbb{A}^1)$ then $\mathrm{Hom}(L, M)$ is locally modelled on $\mathrm{MF}(U, f)$.
A lot of this K–R Conjecture would follow by combining §5, Fukaya categories and §7, Gluing matrix factorization categories above.
Seeing what the rest of the K–R Conjecture requires should tell us some interesting properties to expect of $\mathrm{MF}_{\mathbf{X},\omega_{\mathbf{X}}}$.