What is a derived manifold?

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based on arXiv:1208.4948, arXiv:1206.4207, arXiv:1409.6908, arXiv:1510.07444, and in-progress books available at `people.maths.ox.ac.uk/~joyce/dmanifolds.html`, `people.maths.ox.ac.uk/~joyce/Kuranishi.html`.

These slides available at `people.maths.ox.ac.uk/~joyce/talks.html`.

Plan of talk:

1. What is a moduli space?
2. What is a 'derived' geometric space?
3. What is a derived manifold?
4. Why study derived manifolds and orbifolds?

1. What is a moduli space?

Suppose you want to study some class of geometric objects $\mathcal{O}$, e.g. algebraic vector bundles $E \to X$ over a projective scheme X, or instantons on a compact oriented Riemannian 4-manifold (X, g). Let $\mathcal{M}$ be the set of isomorphism classes $[\mathcal{O}]$ of objects $\mathcal{O}$. Usually $\mathcal{M}$ is infinite, so just knowing $\mathcal{M}$ as a set does not tell us much. We want to put a *geometric structure* on $\mathcal{M}$, which reflects the behaviour of *families* $\{\mathcal{O}_b : b \in B\}$ of objects $\mathcal{O}_b$ over a base space B. Then we call $\mathcal{M}$ with its geometric structure a *moduli space*.

There are two main reasons for doing this. Firstly, classification: if you can give a complete description of $\mathcal{M}$ with its geometric structure, geometers consider you have classified the objects $\mathcal{O}$. Secondly, in problems to do with enumerative invariants, Floer homology, Fukaya categories, etc., 'counting' moduli spaces is used as a tool to define a theory. In these cases it is crucial that $\mathcal{M}$ should have the right kind of geometric structure.

What geometric structures can moduli spaces have?

- If we have a notion of continuous family of objects $\{\mathcal{O}_b : b \in B\}$ over a topological space B, we can put a unique topology on $\mathcal{M}$ such that the map $B \to \mathcal{M}$, $b \mapsto [\mathcal{O}_b]$ is continuous for all such families $\{\mathcal{O}_b : b \in B\}$. So $\mathcal{M}$ is a topological space.
- In algebraic geometry one often has a notion of algebraic family of objects $\{\mathcal{O}_b : b \in B\}$ over a base $\mathbb{K}$-scheme B. Then one can give $\mathcal{M}$ the structure of a $\mathbb{K}$-scheme (or $\mathbb{K}$-stack), such that families $\{\mathcal{O}_b : b \in B\}$ correspond to scheme (or stack) morphisms $B \to \mathcal{M}$.
- In differential geometry one often has a notion of smooth family of objects $\{\mathcal{O}_b : b \in B\}$ over a manifold B. So we would like to put a 'smooth structure' on $\mathcal{M}$, such that smooth families $\{\mathcal{O}_b : b \in B\}$ correspond to 'smooth maps' $B \to \mathcal{M}$.

However, unless we make extra assumptions (genericity of some geometric data), usually $\mathcal{M}$ may not not a manifold, but only some kind of singular smooth space, e.g. a 'C^∞-scheme'.

Enumerative invariants, and virtual cycles of moduli spaces

If X is a compact, oriented manifold of dimension d, it has a fundamental class $[X]$ in the homology group $H_d(X;\mathbb{Q})$.
In some moduli problems used to define enumerative invariants (e.g. Seiberg–Witten theory) one can arrange that moduli spaces $\mathcal{M}$ actually are compact oriented manifolds, and then we use the fundamental classes $[\mathcal{M}]$ to define the invariants.
But in many other cases, the moduli spaces $\mathcal{M}$ may not be manifolds, but to define enumerative invariants we still want to define a 'virtual class' $[\mathcal{M}]_{\mathrm{virt}}$ in $H_d(\mathcal{M};\mathbb{Q})$ which acts like a fundamental class (e.g. in having deformation-invariance properties), although $\mathcal{M}$ may singular, or have dimension $> d$.
In these cases we want $\mathcal{M}$ to have a geometric structure which allows the definition of virtual classes, e.g. schemes with obstruction theory, or Kuranishi spaces. These are 'derived' structures, basically compact, oriented derived manifolds or orbifolds.

2. What is a 'derived' geometric space?

Let $\mathcal{M}$ be a moduli space of geometric objects $\mathcal{O}$, equipped with a (classical) geometric structure. Often one can define a *tangent space* $T_{[\mathcal{O}]}\mathcal{M}$ measuring infinitesimal deformations of $\mathcal{O}$. This tangent space can be computed using deformation theory, and it often turns out to be a cohomology group of some complex, e.g.:

- In the moduli space $\mathcal{M}$ of vector bundles $E \to X$ over a projective scheme X, we have $T_{[E]}\mathcal{M} = \mathrm{Ext}^1(E,E)$. This comes from a complex with cohomology groups $\mathrm{Ext}^k(E,E)$.
- In many differential-geometric moduli spaces, such as moduli of solutions of a nonlinear elliptic p.d.e., $T_{[\mathcal{O}]}\mathcal{M}$ is the kernel of a linear Fredholm map of Banach spaces $L : V \to W$. We think of L as a complex with cohomology $\mathrm{Ker}\, L = T_{[\mathcal{O}]}\mathcal{M}$ in degree 0, and $\mathrm{Coker}\, L$ in degree 1.

Basic Principle of Derived Geometry

To pass from ‘classical’ to ‘derived’, we replace vector spaces or vector bundles with **complexes** *of vector spaces or vector bundles.*

Derived geometric structures are enhanced, souped-up versions of classical geometric structures, containing more information. If we have a derived moduli space $\boldsymbol{\mathcal{M}}$ with classical moduli space $\mathcal{M}$, it should have a ‘tangent bundle’ $\mathbb{T}\boldsymbol{\mathcal{M}}$ which is a complex of vector bundles on $\boldsymbol{\mathcal{M}}$. The classical tangent space $T_{[\mathcal{O}]}\mathcal{M}$ is $H^0(\mathbb{T}\boldsymbol{\mathcal{M}}|_{[\mathcal{O}]})$. For example, if $\boldsymbol{\mathcal{M}}$ is a derived moduli space of vector bundles $E \to X$ then $H^k(\mathbb{T}\boldsymbol{\mathcal{M}}|_{[E]}) = \mathrm{Ext}^{k+1}(E,E)$. So, the derived moduli space $\boldsymbol{\mathcal{M}}$ knows about the full deformation theory $\mathrm{Ext}^*(E,E)$, but the classical space $\mathcal{M}$ only knows about $\mathrm{Ext}^1(E,E)$. One consequence is that in derived geometry there are often exact sequences in $\mathbb{T}\boldsymbol{\mathcal{M}}$, etc., which are not exact at the classical level.

Example: algebras and cdgas, (derived) $\mathbb{K}$-schemes

A classical $\mathbb{K}$-scheme over a field $\mathbb{K}$ is built using a commutative $\mathbb{K}$-algebra A. By our basic principle, to define derived $\mathbb{K}$-schemes, we should replace the $\mathbb{K}$-vector space A (with multiplication) by a complex $A^\bullet$ (with multiplication). This is called a cdga.

Definition

A *commutative differential graded algebra* $A^\bullet$, or *cdga*, over $\mathbb{K}$, is a complex $\cdots \xrightarrow{\mathrm{d}} A^{-2} \xrightarrow{\mathrm{d}} A^{-1} \xrightarrow{\mathrm{d}} A^0 \to 0$ of $\mathbb{K}$-vector spaces in nonpositive degrees, with bilinear maps $\cdot : A^k \times A^l \to A^{k+l}$ such that $a\cdot b=(-1)^{kl}b\cdot a$, and $a\cdot(b\cdot c)=(a\cdot b)\cdot c$, and $\mathrm{d}(a\cdot b)=(\mathrm{d}a)\cdot b+(-1)^k a\cdot \mathrm{d}b$ for all $a\in A^k$, $b\in A^l$, $c\in A^m$, and an identity $1\in A^0$ with $1\cdot a=a$.

Very roughly, a derived $\mathbb{K}$-scheme is built by replacing commutative $\mathbb{K}$-algebras by cdgas in the definition of classical $\mathbb{K}$-scheme. Unfortunately, things are not so simple.

Basic Principle of Derived Geometry

In Derived Geometry, though the local models (e.g. cdgas) may be easy to describe, the way these local models are glued together may be mysterious and very complicated.

Basic Principle of Derived Geometry

In Derived Geometry, objects almost always live in a higher category (e.g. an ∞-category), not an ordinary category.

There is an obvious definition of an ordinary category $\mathbf{cdga}_{\mathbb{K}}$ of cdgas over $\mathbb{K}$. It includes *quasi-isomorphisms* $\alpha^\bullet : A^\bullet \to B^\bullet$ which are not invertible, but $H^*(\alpha^\bullet) : H^*(A^\bullet) \to H^*(B^\bullet)$ is an isomorphism. To define derived schemes we must modify $\mathbf{cdga}_{\mathbb{K}}$ to a mysterious ∞-category $\mathbf{cdga}_{\mathbb{K}}^{\infty}$ in which all quasi-isomorphisms are invertible. We use $\mathbf{cdga}_{\mathbb{K}}^{\infty}$ to glue together local models $\operatorname{Spec} A^\bullet$ using ∞-sheaves to define derived schemes, which are an ∞-category.

Quasi-smooth derived schemes and virtual classes

A derived $\mathbb{K}$-scheme $\boldsymbol{X}$ is called *quasi-smooth* if its tangent bundle $\mathbb{T}\boldsymbol{X}$ is locally equivalent to a complex $E^0 \xrightarrow{\mathrm{d}} E^1$ of vector bundles in degrees 0,1 only. The *virtual dimension* is $\operatorname{vdim} \boldsymbol{X} = \operatorname{rank} E^0 - \operatorname{rank} E^1$. If $\boldsymbol{X}$ is proper (compact and Hausdorff), one can define a virtual cycle $[\boldsymbol{X}]_{\mathrm{virt}}$ in the Chow homology $A_d(X)$ of the classical truncation X of $\boldsymbol{X}$. This works because $\mathbb{T}^*\boldsymbol{X}|_X \to \mathbb{T}^*X$ is a perfect obstruction theory in the sense of Behrend–Fantechi.
For non-quasi-smooth derived schemes, one can show by example that virtual classes cannot exist with the properties we want (e.g. the point $*$ is cobordant to $\emptyset$ through non-quasi-smooth derived schemes, breaking deformation-invariance).

Basic Principle of Derived Geometry

Only quasi-smooth derived spaces $\boldsymbol{X}$ (i.e. those with tangent bundle $\mathbb{T}\boldsymbol{X}$ confined to degrees 0,1) should admit virtual classes.

3. What is a derived manifold?

Derived Algebraic Geometry, the study of derived schemes and stacks, is a major new area of algebraic geometry, with foundations laid by Jacob Lurie and Toën–Vezzosi. By comparison, Derived Differential Geometry, the study of 'derived' smooth manifolds and orbifolds, is very little known.

There are several versions of 'derived manifolds' and 'derived orbifolds' in the literature, in order of increasing simplicity:

- Spivak's ∞-category $\mathbf{DerMan_{Spi}}$ of derived manifolds (2008).
- Borisov–Noël's ∞-category $\mathbf{DerMan_{BN}}$ (2011,2012).
- My *d-manifolds* and *d-orbifolds* (2010–2016), which form strict 2-categories $\mathbf{dMan}, \mathbf{dOrb}$.
- My *μ-Kuranishi spaces*, *m-Kuranishi spaces* and *Kuranishi spaces* (2014), which form a category $\boldsymbol{\mu}\mathbf{Kur}$ and weak 2-categories $\mathbf{mKur}, \mathbf{Kur}$. Here μ-, m-Kuranishi spaces are derived manifolds, and Kuranishi spaces are derived orbifolds.

Relation between these definitions

In fact the Kuranishi space approach is motivated by earlier work by Fukaya, Oh, Ohta and Ono in symplectic geometry (1999,2009–) whose 'Kuranishi spaces' are really a prototype kind of derived orbifold, from before the invention of DAG.

- Borisov–Noel (2011) prove an equivalence of ∞-categories $\mathbf{DerMan_{Spi}} \simeq \mathbf{DerMan_{BN}}$.
- Borisov (2012) gives a 2-functor $\pi_2(\mathbf{DerMan_{BN}}) \to \mathbf{dMan}$ which is nearly an equivalence of 2-categories (e.g. it is a 1-1 correspondence on equivalence classes of objects), where $\pi_2(\mathbf{DerMan_{BN}})$ is the 2-category truncation of $\mathbf{DerMan_{BN}}$.
- I prove (2017) equivalences of 2-categories $\mathbf{dMan} \simeq \mathbf{mKur}$, $\mathbf{dOrb} \simeq \mathbf{Kur}$ and of categories $\mathrm{Ho}(\mathbf{dMan}) \simeq \mathrm{Ho}(\mathbf{mKur}) \simeq \boldsymbol{\mu}\mathbf{Kur}$, where $\mathrm{Ho}(\cdots)$ is the homotopy category.

Thus all these notions of derived manifold are more-or-less equivalent. Kuranishi spaces are easiest.

Two ways to define ordinary manifolds

Definition 3.1

A *manifold* of dimension n is a Hausdorff, second countable topological space X with a sheaf $\mathcal{O}_X$ of $\mathbb{R}$-algebras (or C^∞-rings) locally isomorphic to $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$, where $\mathcal{O}_{\mathbb{R}^n}$ is the sheaf of smooth functions $f : \mathbb{R}^n \to \mathbb{R}$.

Definition 3.2

A *manifold* of dimension n is a Hausdorff, second countable topological space X equipped with an atlas of charts $\{(V_i, \psi_i) : i \in I\}$, where $V_i \subseteq \mathbb{R}^n$ is open, and $\psi_i : V_i \to X$ is a homeomorphism with an open subset $\mathrm{Im}\,\psi_i$ of X for all $i \in I$, and $\psi_j^{-1} \circ \psi_i : \psi_i^{-1}(\mathrm{Im}\,\psi_j) \to \psi_j^{-1}(\mathrm{Im}\,\psi_i)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$ for all $i, j \in I$.

If you define derived manifolds by generalizing Definition 3.1, you get something like d-manifolds; if you generalize Definition 3.2, you get something like (μ- or m-)Kuranishi spaces.

The definition of μ-Kuranishi spaces

I define 2-categories **mKur** of *m-Kuranishi spaces*, the 'manifold' version, and **Kur** of *Kuranishi spaces*, the 'orbifold' version. I also define a simplified ordinary category $\boldsymbol{\mu}$**Kur** of 'μ-Kuranishi spaces', with $\boldsymbol{\mu}$**Kur** $\simeq \mathrm{Ho}($**mKur**$)$. Today I will explain μ-Kuranishi spaces.

Definition 3.3

Let X be a topological space. A *μ-Kuranishi neighbourhood* on X is a quadruple (V, E, s, ψ) such that:

(a) V is a smooth manifold.
(b) $\pi : E \to V$ is a vector bundle over V, the *obstruction bundle*.
(c) $s \in C^\infty(E)$ is a smooth section of E, the *Kuranishi section*.
(d) ψ is a homeomorphism from $s^{-1}(0)$ to an open subset $\mathrm{Im}\,\psi$ in X, where $\mathrm{Im}\,\psi$ is called the *footprint* of (V, E, s, ψ).

If $S \subseteq X$ is open, we call (V, E, s, ψ) a *μ-Kuranishi neighbourhood over S* if $S \subseteq \mathrm{Im}\,\psi \subseteq X$.

Definition 3.4

Let X be a topological space, (V_i,E_i,s_i,ψ_i), (V_j,E_j,s_j,ψ_j) be μ-Kuranishi neighbourhoods on X, and $S\subseteq \mathrm{Im}\,\psi_i\cap\mathrm{Im}\,\psi_j\subseteq X$ be an open set. Consider triples $(V_{ij},\phi_{ij},\hat\phi_{ij})$ satisfying:

(a) V_{ij} is an open neighbourhood of $\psi_i^{-1}(S)$ in V_i.
(b) $\phi_{ij}:V_{ij}\to V_j$ is smooth, with $\psi_i=\psi_j\circ\phi_{ij}$ on $s_i^{-1}(0)\cap V_{ij}$.
(c) $\hat\phi_{ij}:E_i\vert_{V_{ij}}\to\phi_{ij}^*(E_j)$ is a morphism of vector bundles on V_{ij}, with $\hat\phi_{ij}(s_i\vert_{V_{ij}})=\phi_{ij}^*(s_j)+O(s_i^2)$.

Define an equivalence relation $\sim$ on such triples $(V_{ij},\phi_{ij},\hat\phi_{ij})$ by $(V_{ij},\phi_{ij},\hat\phi_{ij})\sim(V'_{ij},\phi'_{ij},\hat\phi'_{ij})$ if there are open $\psi_i^{-1}(S)\subseteq\dot V_{ij}\subseteq V_{ij}\cap V'_{ij}$ and a morphism $\Lambda:E_i\vert_{\dot V_{ij}}\to\phi_{ij}^*(TV_j)\vert_{\dot V_{ij}}$ of vector bundles on $\dot V_{ij}$ satisfying $\phi'_{ij}=\phi_{ij}+\Lambda\circ s_i+O(s_i^2)$ and $\hat\phi'_{ij}=\hat\phi_{ij}+\phi_{ij}^*(\mathrm{d}s_j)\circ\Lambda+O(s_i)$ on $\dot V_{ij}$. We write $[V_{ij},\phi_{ij},\hat\phi_{ij}]$ for the $\sim$-equivalence class of $(V_{ij},\phi_{ij},\hat\phi_{ij})$, and call $[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ a *morphism of μ-Kuranishi neighbourhoods over S*.

Given morphisms $[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$, $[V_{jk},\phi_{jk},\hat\phi_{jk}]:(V_j,E_j,s_j,\psi_j)\to(V_k,E_k,s_k,\psi_k)$ of μ-Kuranishi neighbourhoods over $S\subseteq X$, the *composition* is

$$[V_{jk},\phi_{jk},\hat\phi_{jk}]\circ[V_{ij},\phi_{ij},\hat\phi_{ij}]=\bigl[\phi_{ij}^{-1}(V_{jk}),\phi_{jk}\circ\phi_{ij}\vert_{\cdots},\phi_{ij}^{-1}(\hat\phi_{jk})\circ\hat\phi_{ij}\vert_{\cdots}\bigr]:\\ (V_i,E_i,s_i,\psi_i)\longrightarrow(V_k,E_k,s_k,\psi_k).$$

Then μ-Kuranishi neighbourhoods over $S\subseteq X$ form a category $\mathrm{mKur}_S(X)$. We call $[V_{ij},\phi_{ij},\hat\phi_{ij}]$ a *coordinate change over S* if it is an isomorphism in $\mathrm{mKur}_S(X)$. We have:

Theorem 3.5

A morphism $[V_{ij},\phi_{ij},\hat\phi_{ij}]:(V_i,E_i,s_i,\psi_i)\to(V_j,E_j,s_j,\psi_j)$ is a coordinate change over S if and only if for all $x\in S$ with $v_i=\psi_i^{-1}(x)$ and $v_j=\psi_j^{-1}(x)$, the following sequence is exact:

$$0\longrightarrow T_{v_i}V_i\xrightarrow{\mathrm{d}s_i\vert_{v_i}\oplus T_{v_i}\phi_{ij}}E_i\vert_{v_i}\oplus T_{v_j}V_j\xrightarrow{\hat\phi_{ij}\vert_{v_i}\oplus-\mathrm{d}s_j\vert_{v_j}}E_j\vert_{v_j}\longrightarrow 0.$$

The sheaf property of morphisms

Theorem 3.6

Let (V_i, E_i, s_i, ψ_i), (V_j, E_j, s_j, ψ_j) be μ-Kuranishi neighbourhoods on X. For each open $S \subseteq \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$, write $\mathcal{H}om((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))(S)$ for the set of morphisms $\Phi_{ij} : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ over S, and for all open $T \subseteq S \subseteq \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$ define

$$\rho_{ST} : \mathcal{H}om((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))(S) \longrightarrow \mathcal{H}om((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))(T) \text{ by } \rho_{ST} : \Phi_{ij} \longmapsto \Phi_{ij}|_T.$$

Then $\mathcal{H}om((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$ is a sheaf of sets on $\mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$. Similarly, coordinate changes from (V_i, E_i, s_i, ψ_i) to (V_j, E_j, s_j, ψ_j) are a subsheaf of $\mathcal{H}om((V_i, E_i, s_i, \psi_i), (V_j, E_j, s_j, \psi_j))$.

This is not obvious. It means we can glue (iso)morphisms of μ-Kuranishi neighbourhoods over the sets of an open cover. In the 2-category version for **mKur**, **Kur**, we get a stack instead of a sheaf.

Definition 3.7

Let X be a Hausdorff, second countable topological space, and $n \in \mathbb{Z}$. A *μ-Kuranishi structure $\mathcal{K}$ on X of virtual dimension n* is data $\mathcal{K} = \big(I, (V_i, E_i, s_i, \psi_i)_{i \in I}, \Phi_{ij,\ i,j \in I}\big)$, where:

(a) I is an indexing set.
(b) (V_i, E_i, s_i, ψ_i) is a μ-Kuranishi neighbourhood on X for each $i \in I$, with $\dim V_i - \mathrm{rank}\, E_i = n$.
(c) $\Phi_{ij} = [V_{ij}, \phi_{ij}, \hat{\phi}_{ij}] : (V_i, E_i, s_i, \psi_i) \to (V_j, E_j, s_j, \psi_j)$ is a coordinate change over $S = \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j$ for all $i, j \in I$.
(d) $\bigcup_{i \in I} \mathrm{Im}\,\psi_i = X$.
(e) $\Phi_{ii} = \mathrm{id}_{(V_i, E_i, s_i, \psi_i)}$ for all $i \in I$.
(f) $\Phi_{jk} \circ \Phi_{ij} = \Phi_{ik}$ for all $i, j, k \in I$ over $S = \mathrm{Im}\,\psi_i \cap \mathrm{Im}\,\psi_j \cap \mathrm{Im}\,\psi_k$.

We call $\boldsymbol{X} = (X, \mathcal{K})$ a *μ-Kuranishi space*, of *virtual dimension* $\mathrm{vdim}\,\boldsymbol{X} = n$.

— Compare with the definition of manifolds via an atlas of charts.

Let $\boldsymbol{X}=(X,\mathcal{K})$ with $\mathcal{K}=\bigl(I,(V_i,E_i,s_i,\psi_i)_{i\in I},\Phi_{ii',\,i,i'\in I}\bigr)$ and $\boldsymbol{Y}=(Y,\mathcal{L})$ with $\mathcal{L}=\bigl(J,(W_j,F_j,t_j,\chi_j)_{j\in J},\Psi_{jj',\,j,j'\in J}\bigr)$ be μ-Kuranishi spaces. A *morphism* $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}$ is $\boldsymbol{f}=\bigl(f,\boldsymbol{f}_{ij,\,i\in I,\,j\in J}\bigr)$, where $f:X\to Y$ is a continuous map, and $\boldsymbol{f}_{ij}:(V_i,E_i,s_i,\psi_i)\to(W_j,F_j,t_j,\chi_j)$ is a morphism of μ-Kuranishi neighbourhoods over f for all $i\in I$, $j\in J$, satisfying $\boldsymbol{f}_{i'j}\circ\Phi_{ii'}=\boldsymbol{f}_{ij}$ and $\Psi_{jj'}\circ\boldsymbol{f}_{ij}=\boldsymbol{f}_{ij'}$ on the appropriate open subsets.
If $\boldsymbol{g}:\boldsymbol{Y}\to\boldsymbol{Z}$ is another morphism with $\boldsymbol{Z}=(Z,\mathcal{K})$ with $\mathcal{K}=\bigl(K,(W_k,F_k,t_k,\xi_k)_{k\in K},\Xi_{kk',\,k,k'\in K}\bigr)$, we can define the composition $\boldsymbol{g}\circ\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Z}$, and so make μ-Kuranishi spaces into a category $\boldsymbol{\mu}\mathbf{Kur}$. Now $\boldsymbol{g}\circ\boldsymbol{f}$ must contain $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}:(U_i,D_i,r_i,\phi_i)\to(W_k,F_k,t_k,\xi_k)$ for all $i\in I$, $k\in K$.
For each $j\in J$, we have $\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij}:(U_i,D_i,r_i,\phi_i)\to(W_k,F_k,t_k,\xi_k)$, but it is defined only on an open subset of the domain of $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}$. We can use the sheaf property Theorem 3.6 to show there is a unique $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}$ with $(\boldsymbol{g}\circ\boldsymbol{f})_{ik}|_{\cdots}=\boldsymbol{g}_{jk}\circ\boldsymbol{f}_{ij}$ for all $j\in J$, so $\boldsymbol{g}\circ\boldsymbol{f}$ is well defined. Fukaya–Oh–Ohta–Ono had no sheaf property, and so could not make their Kuranishi spaces into a category.

4. Why study derived manifolds and orbifolds?

Here are some reasons for studying Derived Differential Geometry:

- It is interesting for its own sake. A lot of differential geometry of manifolds (e.g. orientations, submanifolds, transverse fibre products, manifolds with boundary and corners, . . .) extends to derived manifolds in a pretty way.
- Compact oriented derived manifolds and orbifolds have fundamental classes (virtual classes) in homology or bordism. This makes them useful for application in enumerative invariants, Floer homology, Fukaya categories, Symplectic Geometry.
- Many important moduli spaces in Differential Geometry and Algebraic Geometry over $\mathbb{C}$ have the structure of derived manifolds or orbifolds, e.g. moduli of solutions of nonlinear elliptic p.d.e.s on compact manifolds. DDG techniques give new ways to prove moduli spaces are derived manifolds/orbifolds.
- Working with derived manifolds/orbifolds instead of classical manifolds often makes transversality assumptions unnecessary.

Differential geometry of (m-)Kuranishi spaces

Manifolds and orbifolds include into m-Kuranishi spaces and Kuranishi spaces, in a diagram of 2-categories

$$\begin{array}{ccc} \mathbf{Man} & \xrightarrow{\subset} & \mathbf{mKur} \\ \downarrow \subset & & \subset \downarrow \\ \mathbf{Orb} & \xrightarrow{\subset} & \mathbf{Kur}. \end{array}$$

Much of the differential geometry of ordinary manifolds extends nicely to Kuranishi spaces. There are good notions of dimension, orientation, submersions, immersions, embeddings, transversality and fibre products, gluing by equivalences on open covers. There are also good notions of (m-)Kuranishi space with boundary and corners, forming 2-categories $\mathbf{mKur} \subset \mathbf{mKur^b} \subset \mathbf{mKur^c}$ and $\mathbf{Kur} \subset \mathbf{Kur^b} \subset \mathbf{Kur^c}$. Some results are stronger than the classical case. For example, if $\boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Z}$ and $\boldsymbol{h} : \boldsymbol{Y} \to \boldsymbol{Z}$ are 1-morphisms in $\mathbf{Kur}$ with $\boldsymbol{Z}$ a manifold or orbifold then a fibre product $\boldsymbol{X} \times_{\boldsymbol{g},\boldsymbol{Z},\boldsymbol{h}} \boldsymbol{Y}$ exists in $\mathbf{Kur}$, without further transversality conditions.

Virtual classes and virtual chains

If X is a compact, oriented manifold (or orbifold) of dimension k, it has a fundamental class $[X] \in H_k(X;\mathbb{Z})$ (or $[X] \in H_k(X;\mathbb{Q})$). In the same way, if $\boldsymbol{X}$ is a compact, oriented derived manifold (or derived orbifold) of virtual dimension k, it has a *virtual class* $[\boldsymbol{X}]_{\rm virt} \in H_k(X;\mathbb{Z})$ (or $[\boldsymbol{X}]_{\rm virt} \in H_k(X;\mathbb{Q})$). Technically we need to use the Steenrod homology $H_k^{\rm St}(X;\mathbb{Z})$ or Čech homology $\check{H}_k(X;\mathbb{Q})$ here, but they agree with ordinary homology if X is a nice topological space (e.g. a Euclidean Neighbourhood Retract). These virtual classes (also *virtual chains*, for derived manifolds / orbifolds with corners) are very important in applications of DDG. They have deformation-invariance properties under bordism of derived manifolds/orbifolds. They are used in Symplectic Geometry to define Gromov–Witten invariants, and could be used to define other enumerative invariants (Donaldson, Seiberg–Witten, Donaldson–Thomas), Floer theories, and Fukaya categories.

Making moduli spaces into derived manifolds/orbifolds

Many classes of moduli spaces $\mathcal{M}$ in Differential Geometry, and in Algebraic Geometry over $\mathbb{C}$, are known to have the stucture of derived manifolds or derived orbifolds. For example:

Theorem 4.1

Let $\mathcal{V}$ be a Banach manifold, $\mathcal{E} \to \mathcal{V}$ a Banach vector bundle, and $s : \mathcal{V} \to \mathcal{E}$ a smooth Fredholm section, with constant Fredholm index $n \in \mathbb{Z}$. Then there is a canonical derived manifold $\boldsymbol{X}$ with topological space $X = s^{-1}(0)$ and $\mathrm{vdim}\,\boldsymbol{X} = n$.

Nonlinear elliptic equations, when written as maps between suitable Hölder or Sobolev spaces, are the zeroes of Fredholm sections of a Banach vector bundle over a Banach manifold. Thus we have:

Corollary 4.2

Let $\mathcal{M}$ be a moduli space of solutions of a nonlinear elliptic equation on a compact manifold, with fixed topological invariants. Then $\mathcal{M}$ extends to a derived manifold $\boldsymbol{\mathcal{M}}$.

linearization of the elliptic p.d.e. at x, given by the A–S Index Theorem.

Moduli 2-functors

The approaches to moduli spaces in Differential and Algebraic Geometry are very different. In Differential Geometry one constructs the moduli space, as a topological space covered by an atlas of charts. In Algebraic Geometry one writes down a *moduli functor* $F : \mathcal{C} \to \mathrm{Sch}_{\mathbb{K}}$, where objects $O \in \mathcal{C}$ with $F(O) = B$ are families of objects in the moduli problem over a base $\mathbb{K}$-scheme B, and then prove this is equivalent to the functor $\pi : \mathrm{Sch}_{\mathcal{M}} \to \mathrm{Sch}_{\mathbb{K}}$ for some $\mathbb{K}$-scheme $\mathcal{M}$, the *moduli scheme*.

I propose that in Derived Differential Geometry one should write down a *moduli 2-functor* $F : \boldsymbol{\mathcal{C}} \to \mathbf{mKN}$, where $\boldsymbol{\mathcal{C}}$ is a 2-category and $\mathbf{mKN}$ the 2-category of m-Kuranishi neighbourhoods, where objects O in $\boldsymbol{\mathcal{C}}$ with $F(O) = (V, E, s)$ are families of moduli objects over a base m-Kuranishi neighbourhood (V, E, s), and prove this is equivalent (after stackification) to $\pi : \mathbf{mKN}_{\boldsymbol{\mathcal{M}}} \to \mathbf{mKN}$ for some m-Kuranishi space $\boldsymbol{\mathcal{M}}$, with $\mathbf{mKN}_{\boldsymbol{\mathcal{M}}}$ the 2-category of m-Kuranishi neighbourhoods on $\boldsymbol{\mathcal{M}}$.

Some advantages of the moduli 2-functor approach:

- Many current presentations of moduli spaces (e.g. FOOO, HWZ) are long, complicated ad hoc constructions. The effort is mostly in the definition. It is unclear how natural they are. Our definition makes the naturality clear. We have a short definition (the moduli 2-functor), followed by a difficult theorem (the 2-functor is represented by an (m-)Kuranishi space).
- To prove representability we only have to worry about single (m-)Kuranishi neighbourhoods, not double or triple overlaps.
- The definition involves only finite-dimensional families of smooth objects – no Hölder or Sobolev spaces, etc. (though these will be used in the proof of representability). This enables us to sidestep some technical issues in current approaches, e.g. sc-smoothness in polyfolds.
- In our approach, the existence of natural morphisms between moduli spaces (e.g. 'forgetful morphisms' in Symplectic Geometry forgetting a marked point) is essentially trivial.