Orientations on moduli spaces of coherent sheaves on Calabi–Yau 4-folds. I

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Work in progress. Everything joint with Markus Upmeier.

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1. Introduction

Let X be a compact Calabi–Yau 4-fold, and $\boldsymbol{\mathcal{M}}$ the derived moduli stack of perfect complexes on X, and $\boldsymbol{\mathcal{M}}_\alpha^{\rm st}(\tau)\subseteq\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)\subset\boldsymbol{\mathcal{M}}$ the open substacks of Gieseker (semi)stable coherent sheaves on X with Chern character $\alpha\in H^{\rm even}(X,\mathbb{Q})$. Pantev–Toën–Vaquié–Vezzosi 2013 show $\boldsymbol{\mathcal{M}}$ has a -2-shifted symplectic structure. If $\boldsymbol{\mathcal{M}}_\alpha^{\rm st}(\tau)=\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)$ can be lifted to a moduli scheme, and can be given an *orientation*, Borisov–Joyce 2017 and Oh–Thomas 2023 show $\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)$ has a *virtual class* $[\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)]_{\rm virt}$ in $H_*(\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau),\mathbb{Z})$, which is used to define Donaldson–Thomas type *DT4 invariants* of X. This talk is about whether orientations exist on $\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)$, and what data you need to define canonical orientations.

It makes sense to study orientations on the full moduli stack $\boldsymbol{\mathcal{M}}$, and then restrict them to the substacks $\boldsymbol{\mathcal{M}}_\alpha^{\rm ss}(\tau)\subset\boldsymbol{\mathcal{M}}$.

Definition (Borisov–Joyce 2017)

The -2-shifted symplectic structure on $\boldsymbol{\mathcal{M}}$ gives a quasi-isomorphism $\mathbb{L}_{\boldsymbol{\mathcal{M}}}\rightarrow\mathbb{T}_{\boldsymbol{\mathcal{M}}}[2]$, and thus an isomorphism of line bundles $\Phi:\det(\mathbb{L}_{\boldsymbol{\mathcal{M}}})\rightarrow\det(\mathbb{T}_{\boldsymbol{\mathcal{M}}})=\det(\mathbb{L}_{\boldsymbol{\mathcal{M}}})^*$. An *orientation* on $\boldsymbol{\mathcal{M}}$ is an isomorphism $\phi:\det(\mathbb{L}_{\boldsymbol{\mathcal{M}}})\rightarrow\mathcal{O}_{\boldsymbol{\mathcal{M}}}$ with $\Phi=\phi^*\circ\phi$.

The Cao–Gross–Joyce orientability theorem is wrong!

Theorem 1 (Cao–Gross–Joyce 2020)

Let X be a compact Calabi–Yau 4-fold. Then the moduli stack $\boldsymbol{\mathcal{M}}$ of perfect complexes on X is orientable.

Unfortunately, there is a mistake in the proof. The theorem itself may be false, though we don't have a counterexample. I apologize for this.

Outline of proof in Cao–Gross–Joyce:

Step 1: Let $P \to X$ be a principal $\mathrm{U}(m)$-bundle, $m \geqslant 4$. Define moduli spaces $\mathcal{B}_P$ of all connections on P. Define a principal $\mathbb{Z}_2$-bundle $O_P \to \mathcal{B}_P$ of orientations on $\mathcal{B}_P$, using gauge theory.
Prove O_P is trivializable, that is, $\mathcal{B}_P$ is orientable. (This proof wrong.)
If X is a $\mathrm{Spin}(7)$-manifold, orientations of $\mathcal{B}_P$ restrict to orientations of moduli spaces $\mathcal{M}_P$ of $\mathrm{Spin}(7)$-*instantons* on P.

Step 2: Define map of topological classifying spaces $\Psi : \boldsymbol{\mathcal{M}}^{\mathrm{cla}}_{\mathrm{ch}=\mathrm{ch}\,P} \to \mathcal{B}^{\mathrm{cla}}_P$. Show orientations of $\mathcal{B}_P$ pull back along Ψ to orientations of $\boldsymbol{\mathcal{M}}_{\mathrm{ch}=\mathrm{ch}\,P}$. Hence $\mathcal{B}_P$ orientable implies $\boldsymbol{\mathcal{M}}$ orientable. (This proof is correct, as far as we know.)

How to fix the mistake in Cao–Gross–Joyce

Markus Upmeier and myself have developed a new theory for studying orientability and canonical orientations for moduli spaces $\mathcal{B}_P$, where X is a compact spin n-manifold with $n \equiv 1, 7, 8 \mod 8$, and G is a Lie group, and $P \to X$ is a principal G-bundle, and $\mathcal{B}_P$ is the moduli space (topological stack) of all connections ∇ on P, and orientations on $\mathcal{B}_P$ mean orientations of the (positive) Dirac operator on X twisted by $(\mathrm{ad}(P), \nabla)$. If X is a $\mathrm{Spin}(7)$-manifold, orientations on $\mathcal{B}_P$ restrict to orientations on moduli spaces of $\mathrm{Spin}(7)$-instantons on X. If X is a Calabi–Yau 4-fold and $G = \mathrm{U}(m)$, orientations on $\mathcal{B}_P$ restrict to Borisov–Joyce orientations on moduli spaces of rank m algebraic vector bundles on X.

When $n = 8$ (also $n = 7$) we give sufficient conditions on X for orientability of $\mathcal{B}_P$ for many G, including $G = \mathrm{U}(m)$ (necessary and sufficient if $G = E_8$). If these sufficient conditions hold, the problem with Step 1 of Cao–Gross–Joyce is fixed, and we deduce the Cao–Gross–Joyce orientability theorem under this extra condition. We also specify data (a *flag structure*) which determines canonical orientations.

2. First look at the methods in the proof

A principal G-bundle $P \to X$ is topologically equivalent to a map $\phi_P : X \to BG$, where BG is the classifying space of X. Thus $[X, \phi_P]$ is an element of the *spin bordism group* $\Omega_n^{\mathrm{Spin}}(BG)$. Orientability of $\mathcal{B}_P$ depends on the monodromy of $O_P \to \mathcal{B}_P$ around a loop $\gamma : \mathcal{S}^1 \to \mathcal{B}_P$. Then γ is equivalent to a principal G-bundle $Q \to X \times \mathcal{S}^1$, giving a map $\phi_Q : X \times \mathcal{S}^1 \to BG$, and a spin bordism class $[X \times \mathcal{S}^1, \phi_Q]$ in $\Omega_{n+1}^{\mathrm{Spin}}(BG)$. Now ϕ_Q is equivalent to a map $\psi_Q : X \to \mathcal{L}BG$, where $\mathcal{L}BG$ is the loop space of BG, so Q determines a bordism class $[X, \psi_Q]$ in $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$, and $[X \times \mathcal{S}^1, \phi_Q]$ is the image of $[X, \psi_Q]$ under a natural map $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG) \to \Omega_{n+1}^{\mathrm{Spin}}(BG)$.

It turns out that orientation problems for $\mathcal{B}_P$ factor via $\Omega_n^{\mathrm{Spin}}(BG)$, $\Omega_{n+1}^{\mathrm{Spin}}(BG)$, $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$ in a certain sense. For given X, we can show that $\mathcal{B}_P$ is orientable for all principal G-bundles $P \to X$ if and only if certain 'bad' classes α in $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$ cannot be written $\alpha = [X, \psi]$. If there are no bad classes we get orientability for all X, P (this often happens for $n = 7$). We need to compute $\Omega_n^{\mathrm{Spin}}(BG)$, $\Omega_{n+1}^{\mathrm{Spin}}(BG)$, $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BG)$ using algebraic topology.

If $\iota : G \to H$ is a morphism of Lie groups of 'complex type', and $P \to X$ is a principal G-bundle, then $Q = (P \times H)/G$ is a principal H-bundle, and an orientation for $\mathcal{B}_Q$ induces one for $\mathcal{B}_P$. Using complex type morphisms $\mathrm{SU}(8) \hookrightarrow E_8$ and $\mathrm{SU}(m) \hookrightarrow \mathrm{SU}(m')$ for $m \leqslant m'$, we can show that if X is a spin 8-manifold then orientability of $\mathcal{B}_Q$ for all principal E_8-bundles $Q \to X$ implies orientability of $\mathcal{B}_P$ for all principal $\mathrm{U}(m)$-bundles $P \to X$. Thus, to solve the CY4 orientability problem, it is enough to understand orientability for E_8-bundles.

There is a 16-connected map $BE_8 \to K(\mathbb{Z}, 4)$, where $K(\mathbb{Z}, 4)$ is the Eilenberg–MacLane space classifying $H^4(-, \mathbb{Z})$, so $\Omega_n^{\mathrm{Spin}}(BE_8) \cong \Omega_n^{\mathrm{Spin}}(K(\mathbb{Z}, 4))$ for $n < 16$, and $\Omega_n^{\mathrm{Spin}}(\mathcal{L}BE_8) \cong \Omega_n^{\mathrm{Spin}}(\mathcal{L}K(\mathbb{Z}, 4))$ for $n < 15$. Using this, we can reduce orientability questions for E_8-bundles to conditions that can be computed using *cohomology* and *cohomology operations* on X, in particular Steenrod squares. The proofs involve lots of complicated calculations of bordism groups in Algebraic Topology, spectral sequences, etc.

3. Statement of main results: orientability

I'll explain only results in 8 dimensions relevant to DT4 invariants, and a bit extra on $\mathrm{Spin}(7)$ instantons. They are part of a bigger theory, which also includes results on orientability of moduli spaces of submanifolds, such as Cayley 4-folds in $\mathrm{Spin}(7)$-manifolds.

Let X be a compact oriented spin 8-manifold. Impose the condition:

(*) Let $\alpha \in H^3(X,\mathbb{Z})$, and write $\bar\alpha \in H^3(X,\mathbb{Z}_2)$ for its mod 2 reduction, and $\mathrm{Sq}^2(\bar\alpha) \in H^5(X,\mathbb{Z}_2)$ for its Steenrod square. Then $\int_X \bar\alpha \cup \mathrm{Sq}^2(\bar\alpha) = 0$ in $\mathbb{Z}_2$ for all $\alpha \in H^3(X,\mathbb{Z})$.

Theorem 2

Suppose X satisfies condition $(*)$, and let G be a compact Lie group on the list, for all $m \geqslant 1$

$$E_8,\ E_7,\ E_6,\ G_2,\ \mathrm{Spin}(3),\ \mathrm{SU}(m),\ \mathrm{U}(m),\ \mathrm{Spin}(2m). \qquad (1)$$

Then $\mathcal{B}_P$ is orientable for every principal G-bundle $P \to X$.
For $G = E_8$, this holds **if and only if** $(*)$ *holds.*

We do this by applying our general orientability theory for $G = E_8$ by studying $\Omega_n^{\mathrm{Spin}}(K(\mathbb{Z},4))$ and $\Omega_n^{\mathrm{Spin}}(\mathcal{L}K(\mathbb{Z},4))$. The other cases are deduced from $G = E_8$ using complex type morphisms.

The case $G = E_8$ and Step 2 of Cao–Gross–Joyce implies:

Corollary 3

Suppose a Calabi–Yau 4-fold X satisfies condition $(*)$. Then the moduli stack $\boldsymbol{\mathcal{M}}$ of perfect complexes on X is orientable in the sense of Borisov–Joyce 2017.

Example 4

Let $X \subset \mathbb{CP}^5$ be a smooth sextic. Then $H^3(X,\mathbb{Z}) = 0$ by the Lefschetz Hyperplane Theorem. So $(*)$ and Corollary 3 hold.

Corollary 5

Suppose a compact $\mathrm{Spin}(7)$-manifold (X,Ω) satisfies condition $(*)$, and G lies on the list (1), and $P \to X$ is a principal G-bundle. Then the moduli space $\mathcal{M}_P^{\mathrm{irr}}$ of irreducible $\mathrm{Spin}(7)$-instanton connections on P is orientable. (Here $\mathcal{M}_P^{\mathrm{irr}}$ is a smooth manifold if Ω is generic, and a derived manifold otherwise.)

4. Statement of main results: canonical orientations

Suppose now that $(*)$ holds, so we have orientability of moduli spaces $\mathcal{B}_P$ or $\boldsymbol{\mathcal{M}}$ on X. What extra choices do we need to make on X to define *canonical orientations* on $\mathcal{B}_P$ or $\boldsymbol{\mathcal{M}}$?

Definition

Let X be a spin 8-manifold, and $P \to X$ a principal G-bundle, and $O_P \to \mathcal{B}_P$ be the orientation bundle. Define the *normalized orientation bundle* $\check{O}_P \to \mathcal{B}_P$ by $\check{O}_P = O_P \otimes_{\mathbb{Z}_2} \mathrm{Or}(O_{X\times G}|_{[\nabla_0]})$, where $\mathrm{Or}(O_{X\times G}|_{[\nabla_0]})$ is the $\mathbb{Z}_2$-torsor of orientations of $\mathcal{B}_{X\times G}$ for the trivial G-bundle $X \times G \to X$ at the trivial connection ∇_0. A trivialization of $\mathrm{Or}(O_{X\times G}|_{[\nabla_0]})$ is an orientation for $\mathrm{ind}(\not{D}_X^+) \otimes \mathfrak{g}$, where $\not{D}_X^+$ is the positive Dirac operator of X, $\mathrm{ind}(\not{D}_X^+)$ its orientation torsor as a Fredholm operator, $\mathfrak{g}$ the Lie algebra of G.

We show normalized orientations on $\mathcal{B}_P$ are determined by a choice of *flag structure* (next slide). Orientations on $\mathcal{B}_P$ also need an orientation on $\mathrm{ind}(\not{D}_X^+)\otimes\mathfrak{g}$. If X is a Calabi–Yau 4-fold, there is a natural orientation for $\mathrm{ind}(\not{D}_X^+)$, so we don't need this second choice.

Flag structures – first idea

Joyce 2018 and Joyce–Upmeier 2023 introduced *flag structures* on 7-manifolds, and used them to define orientations on moduli spaces of associative 3-folds and G_2-instantons on compact G_2-manifolds. We define a related (but more complicated) notion of flag structure F for compact spin 8-manifolds X satisfying condition $(*)$, as a choice of natural trivialization of an orientation functor associated to X (more details later). We can write a flag structure F as $(F_\alpha : \alpha \in H^4(X,\mathbb{Z}))$, where each F_α lies in a $\mathbb{Z}_2$-torsor. Thus, the set of flag structures on X is a torsor for $\mathrm{Map}(H^4(X,\mathbb{Z}),\mathbb{Z}_2)$. By imposing extra conditions we can cut this down to a finite choice of flag structures.

If X is a Calabi–Yau 4-fold, the orientation on $\boldsymbol{\mathcal{M}}$ at a perfect complex $[\mathcal{E}^\bullet] \in \boldsymbol{\mathcal{M}}$ depends on F_α for $\alpha = c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2$. There is a canonical choice for F_0. Hence, if $c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2 = 0$, there is a canonical choice of orientation on the connected component of $\boldsymbol{\mathcal{M}}$ containing $\mathcal{E}^\bullet$. Thus we deduce:

Theorem 6

Suppose a Calabi–Yau 4-fold X satisfies condition $(*)$. Choose a flag structure F on X. Then we can construct a canonical orientation on the moduli stack $\boldsymbol{\mathcal{M}}$ of perfect complexes on X. On the open and closed substack $\boldsymbol{\mathcal{M}}_{c_2-c_1^2=0} \subset \boldsymbol{\mathcal{M}}$ of perfect complexes $\mathcal{E}^\bullet$ with $c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2 = 0$, we can define the canonical orientation without choosing a flag structure.

The second part resolves a paradox. There are several conjectures in the literature by Bojko, Cao, Kool, Maulik, Toda, ..., of the form

$$\text{Conventional invariants of } X \simeq \text{DT4 invariants of } X, \qquad (2)$$

where the left hand side, involving Gromov–Witten invariants etc., needs no choice of orientation, but the right hand side needs a Borisov–Joyce orientation to determine the sign. All these conjectures are really about sheaves on points and curves — Hilbert schemes of points, MNOP, DT-PT, etc. — and so involve only complexes $\mathcal{E}^\bullet$ with $c_2(\mathcal{E}^\bullet) - c_1(\mathcal{E}^\bullet)^2 = 0$ in $H^4(X, \mathbb{Z})$.

5. Our orientability theory. Bordism categories.

We'll now explain our orientability theory for gauge theory moduli spaces $\mathcal{B}_P$ for principal G-bundles $P \to X$. This works if X is a compact spin n-manifold with $n \equiv 1, 7, 8 \mod 8$, and any Lie group G. There is a parallel theory for orientations of moduli spaces of submanifolds $N \subset X$, such as associative 3-folds in G_2-manifolds or Cayley 4-folds in $\mathrm{Spin}(7)$-manifolds, in which BG is related by a Thom space. The basic ideas are:

- $\mathfrak{Bord}_X(BG)$ is a category with objects principal G-bundles $P \to X$.
- There is a functor $F_X : \mathfrak{Bord}_X(BG) \to \mathrm{s}\text{-}\mathbb{Z}_2\text{-}\mathrm{tor}$ mapping P to the $\mathbb{Z}_2$-torsor of orientations on the moduli space $\mathcal{A}_P$ of all connections on P, without quotienting by gauge transformations $\mathcal{G}_P$. Here $\mathrm{s}\text{-}\mathbb{Z}_2\text{-}\mathrm{tor}$ is the category of (super) $\mathbb{Z}_2$-torsors.
- An orientation on $\mathcal{B}_P = \mathcal{A}_P/\mathcal{G}_P$ for all G-bundles $P \to X$ is equivalent to a natural isomorphism $\omega : F_X \Rightarrow \mathbb{1}_X$, where $\mathbb{1}_X : \mathfrak{Bord}_X(BG) \to \mathbb{Z}_2\text{-}\mathrm{tor}$ is the constant functor with value $\mathbb{Z}_2$. So $\mathcal{B}_P$ is orientable if and only such a natural isomorphism ω exists.
- *Flag structures* are essentially equivalent to such ω when $G = E_8$.

- We define a *bordism category* $\mathfrak{Bord}_n^{\mathrm{Spin}}(BG)$. Objects of $\mathfrak{Bord}_n^{\mathrm{Spin}}(BG)$ are pairs (X,P) of a compact spin n-manifold X and a principal G-bundle $P\rightarrow X$. Morphisms $[Y,Q]:(X_0,P_0)\rightarrow(X_1,P_2)$ are equivalence classes of pairs (Y,Q), where Y is a spin bordism from X_0 to X_1 (that is, Y is a compact spin $(n+1)$-manifold with $\partial Y=-X_0\amalg X_1$) and $Q\rightarrow Y$ is a principal G-bundle extending $P_0\amalg P_1\rightarrow X_0\amalg X_1$.
- There is an obvious functor $\Pi_X:\mathfrak{Bord}_X(BG)\rightarrow\mathfrak{Bord}_n^{\mathrm{Spin}}(BG)$ mapping $P\mapsto(X,P)$.
- There is a symmetric monoidal structure $\otimes$ on $\mathfrak{Bord}_n^{\mathrm{Spin}}(BG)$ with $(X_0,P_0)\otimes(X_1,P_1)=(X_0\amalg X_1,P_0\amalg P_1)$. This makes $\mathfrak{Bord}_n^{\mathrm{Spin}}(BG)$ into a *Picard groupoid* (abelian 2-group), a categorified notion of an abelian group.
- A Picard groupoid $\mathcal{P}$ is classified by abelian groups $\pi_0(\mathcal{P})$ and $\pi_1(\mathcal{P})=\mathrm{Hom}_{\mathcal{P}}(0_{\mathcal{P}},0_{\mathcal{P}})$, and a linear quadratic form $q:\pi_0(\mathcal{P})\rightarrow\pi_1(\mathcal{P})$. We have $\pi_i(\mathfrak{Bord}_n^{\mathrm{Spin}}(BG))=\Omega_{n+i}^{\mathrm{Spin}}(BG)$ for $i=0,1$, and $q:[X,P]\mapsto[X\times\mathcal{S}^1,P\times\mathcal{S}^1]$. Thus, if we can compute $\Omega_m^{\mathrm{Spin}}(BG)$ for $m=n,n+1$, we understand $\mathfrak{Bord}_n^{\mathrm{Spin}}(BG)$.

- It turns out that the orientation functor F_X factors as

$$\begin{array}{ccc} \mathfrak{Bord}_X(BG) & \xrightarrow{\quad F_X \quad} & \\ \Pi_X \downarrow & \Downarrow & \searrow \\ \mathfrak{Bord}_n^{\mathrm{Spin}}(BG) & \xrightarrow{\quad \mathrm{O} \quad} & \text{s-}\mathbb{Z}_2\text{-tor}, \end{array}$$

where O is a morphism of Picard groupoids. This depends on a nontrivial analytic fact (Upmeier 2021), needed to define O, that orientation problems of this type have a bordism-invariance property.

- Morphisms of Picard groupoids $F : \mathcal{P} \to \mathcal{P}'$ are classified by group morphisms $\pi_i(F) : \pi_i(\mathcal{P}) \to \pi_i(\mathcal{P}')$ for $i = 0, 1$ satisfying $q' \circ \pi_0(F) = \pi_1(F) \circ q$. Thus, to understand the functor O, we have to compute the morphisms $\pi_i(\mathrm{O}) : \Omega_{n+i}^{\mathrm{Spin}}(BG) \to \mathbb{Z}_2$ for $i = 0, 1$.
- If $P \to X$ is a principal G-bundle, then $\mathcal{B}_P$ is orientable if and only if the following composition is trivial:

$$\mathrm{Aut}_{\mathfrak{Bord}_X(BG)}(P) \xrightarrow{\Pi_X} \begin{array}{l}\mathrm{Aut}_{\mathfrak{Bord}_n^{\mathrm{Spin}}(BG)}((X,P)) \\ = \Omega_{n+1}^{\mathrm{Spin}}(BG)\end{array} \xrightarrow{\pi_1(\mathrm{O})} \begin{array}{l}\mathrm{Aut}_{\text{s-}\mathbb{Z}_2\text{-tor}}(F_X(P)) \\ = \mathbb{Z}_2.\end{array}$$

• Thus, $\mathcal{B}_P$ is orientable if and only if $\Pi_X(\mathrm{Aut}_{\mathfrak{Bord}_X(BG)}(P))$ lies in $\mathrm{Ker}\big(\pi_1(O) : \Omega_{n+1}^{\mathrm{Spin}}(BG) \to \mathbb{Z}_2\big)$. We can hope to compute $\Omega_{n+1}^{\mathrm{Spin}}(BG)$ and $\pi_1(O)$ by algebraic-topological techniques.

• Elements of $\Pi_X(\mathrm{Aut}_{\mathfrak{Bord}_X(BG)}(P))$ are of the form $[X \times \mathcal{S}^1, Q]$, and so lie in the image of a natural morphism $\Xi : \Omega_n^{\mathrm{Spin}}(\mathcal{L}BG) \to \Omega_{n+1}^{\mathrm{Spin}}(BG)$. If $\mathrm{Im}\,\Xi \subseteq \mathrm{Ker}\,\pi_1(O)$ then $\mathcal{B}_P$ is orientable for *all* compact spin n-manifolds X and principal G-bundles $P \to X$, and vice versa.

This holds if $n = 7$ and G lies on the list (1), and using this we can prove strong orientability results for moduli spaces of G_2-instantons on compact G_2-manifolds.

• Unfortunately, $\mathrm{Im}\,\Xi \not\subseteq \mathrm{Ker}\,\pi_1(O)$ when $n = 8$ and $G = E_8$ or $G = \mathrm{U}(m)$ for $m \geqslant 4$. So for spin 8-manifolds, and compact Calabi–Yau 4-folds, to determine orientability we need to test whether $\Pi_X(\mathrm{Aut}_{\mathfrak{Bord}_X(BG)}(P))$ lies in $\mathrm{Ker}\,\pi_1(O)$ separately for each X. (The answer is independent of P, at least when $P = \mathrm{U}(m)$ for $m \geqslant 4$.)