Orientations on moduli spaces of coherent sheaves on Calabi-Yau 4-folds. II

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Geometry meets Physics: Calabi-Yau fourfolds and beyond
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Spin(7)-Instantons

Let X be an 8-dimensional $\mathrm{Spin}(7)$-manifold with admissible 4-form Φ (for example, a Calabi–Yau 4-fold).

- Φ induces an orientation and a Riemannian metric on X.
- There is a splitting of 2-forms

$$\Lambda^2 = \Lambda^2_7 \oplus \Lambda^2_{21}$$

into the eigenspaces of $\alpha \mapsto *(\alpha \wedge \Phi)$.

Definition

Let $P \to X$ be a principal G-bundle. The moduli space $\mathcal{M}_P^{\mathrm{Spin}(7)}$ of $\mathrm{Spin}(7)$-instantons is the quotient stack of

$$\mathcal{A}_P^{\mathrm{Spin}(7)} = \{A \in \mathcal{A}_P \mid *(F_A \wedge \Phi) = -F_A\}$$

by the action of the gauge group $\mathcal{G}_P = \mathrm{Aut}(P)$.

Orientation bundles

The tangent space $T_A\mathcal{M}_P^{\mathrm{Spin}(7)}$ is given by the deformation complex

$$C_A^\bullet : \quad \Omega^0(\mathfrak{g}_P) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_P) \xrightarrow{\pi_7^2 \circ d_A} \Omega_7^2(\mathfrak{g}_P).$$

Definition

Let $\mathcal{G} \curvearrowright \mathcal{A}$ and $\{C_A^\bullet\}_{A\in\mathcal{A}}$ be a $\mathcal{G}$-equivariant family of elliptic complexes of differential operators.

The orientation bundle is a principal $\mathbb{Z}_2$-bundle $O_\mathcal{A} \to \mathcal{A}$ with fibers

$$O_\mathcal{A}\big|_A = \left(\bigotimes\nolimits_{p\in\mathbb{Z}} \det H^p(C_A^\bullet)^{(-1)^p}\right) \setminus \{0\} \Big/ \mathbb{R}_{>0}.$$

Since $O_\mathcal{A}$ is $\mathcal{G}$-equivariant, it descends to the quotient

$$O_\mathcal{B} \longrightarrow \mathcal{B} := \mathcal{A}/\!\!/\mathcal{G}.$$

- $\mathcal{B}$ is orientable if $O_\mathcal{B}$ is trivial ($\Leftrightarrow$ $O_\mathcal{A}$ is $\mathcal{G}$-equivariantly trivial).
- An orientation is a choice of trivialization.

Simplification 1

The deformation complex is homologous to a twisted Dirac operator

$$D_A^+ : \Gamma(\Sigma_X^+ \otimes_{\mathbb{R}} \mathfrak{g}_P) \longrightarrow \Gamma(\Sigma_X^- \otimes_{\mathbb{R}} \mathfrak{g}_P).$$

The orientation bundle of $\mathcal{M}_P^{\mathrm{Spin}(7)}$ extends to $\mathcal{B}_P$ as the orientation bundle $O_{\mathcal{B}_P}$ of the $\mathcal{G}_P$-equivariant family $\{D_A^+\}_{A \in \mathcal{A}_P}$. In particular,

$$\text{orientation of } \mathcal{B}_P \implies \text{orientation of } \mathcal{M}_P^{\mathrm{Spin}(7)}.$$

Moreover, $\mathcal{B}_P$ and $O_{\mathcal{B}_P}$ make sense for *every* compact spin 8-manifold X.

Definition

The orientation $\mathbb{Z}_2$-torsor is the $\mathbb{Z}_2$-torsor of global sections $\Gamma(O_{\mathcal{A}_P})$, where we use that $\mathcal{A}_P$ is contractible.

- Orientations of $\mathcal{B}_P$ are just fixed points of the $\mathcal{G}_P$-action on $\Gamma(O_{\mathcal{A}_P})$.

Orientability and classical indices

A gauge transformation $\gamma : P \to P$ determines a principal G-bundle

$$Q = (P \times \mathbb{R})/_{\gamma}\,\mathbb{Z} \longrightarrow X \times S^1.$$

The action of γ on the orientation $\mathbb{Z}_2$-torsor $\Gamma(O_{\mathcal{A}_P})$ equals the skew-index of the 9-dimensional skew-adjoint Dirac operator twisted by Q.

The skew-index is bordism-invariant, so orientability of $\mathcal{B}_P$ depends on

$$\Omega_8^{\mathrm{Spin}}(\mathcal{L}BG) \xrightarrow{\ \xi_8\ } \Omega_9^{\mathrm{Spin}}(BG) \xrightarrow{\text{skew-ind}} \mathbb{Z}_2.$$

- For $G = E_8$ we show $\Omega_9^{\mathrm{Spin}}(BG) \cong \mathbb{Z}_2$ and that skew-ind is an isomorphism.
- The map ξ_8 maps a principal E_8-bundle $Q \to X \times S^1$ to

$$\int_X \bar\beta \cup \mathrm{Sq}^2(\bar\beta) \in \mathbb{Z}_2, \qquad \alpha = \beta \times [S^1] + \gamma \times 1 \in H^4(X \times S^1; \mathbb{Z}),$$

where α is the characteristic class of Q.

Simplification 2

Besides classical indices, orientations are also bordism invariant, done suitably.

Picard groupoids

Definition

A Picard groupoid is a monoidal category $(\mathcal{C}, \otimes, 1)$ with symmetry isomorphisms $\sigma_{x,y} : x \otimes y \to y \otimes x$ such that

- every morphism is invertible,
- for every object x there exists an object y such that $x \otimes y \cong 1$.

A morphism of Picard groupoids is a symmetric monoidal functor $F : \mathcal{C} \to \mathcal{C}'$.

Example: graded torsors

Let π_0, π_1 be abelian groups.

- A π_0-graded π_1-torsor is a pair (x, S), where $x \in \pi_0$ and S is a set with a free, transitive π_1-action; these form a monoidal groupoid $\pi_0 /\!\!/ \pi_1$.
- Given a skew-symmetric bilinear map $\sigma : \pi_0 \times \pi_0 \to \pi_1$, we can make $\pi_0 /\!\!/ \pi_1$ into a Picard groupoid with symmetry isomorphisms

$$s_0 \otimes_{\pi_1} s_1 \longmapsto \sigma(x_0, x_1) \cdot s_1 \otimes_{\pi_1} s_0.$$

In particular, $\mathbb{Z}$-graded $\mathbb{Z}_2$-torsors are a Picard groupoid with $\sigma \neq 0$.

Classification theorem for Picard groupoids

- Up to equivalence, a Picard groupoid $\mathcal{C}$ is classified by the abelian groups

$$\pi_0 = \mathrm{Ob}(\mathcal{C})/\cong, \qquad \pi_1 = \mathrm{Aut}(\mathbb{I}),$$

  and the linear quadratic form

$$q : \pi_0 \longrightarrow \pi_1, \quad [x] \longmapsto \sigma_{x,x} \in \mathrm{Aut}_{\mathcal{C}}(x) \cong \pi_1.$$

- Given Picard groupoids $\mathcal{C}$, $\mathcal{C}'$ and group morphisms

$$f_0 : \pi_0 \longrightarrow \pi_0', \quad f_1 : \pi_1 \longrightarrow \pi_1',$$

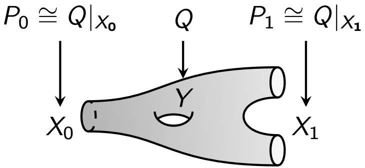
  there exists a morphism $F : \mathcal{C} \to \mathcal{C}'$ of Picard groupoids with $\pi_i(F) = f_i$ if and only if $q' \circ f_0 = f_1 \circ q$.
- A pair of morphisms F, G of Picard groupoids has a difference class $\omega(F, G) \in H^2_{\mathrm{sym}}(\pi_0, \pi_1')$ in group cohomology. Moreover,

$$F \cong G \iff \omega(F, G) = 0.$$

- Given morphisms of Picard groupoids F, G, the space of natural isomorphisms $F \cong G$ is a torsor over $H^1(\pi_0, \pi_1')$.

Bordism category $\mathrm{Bord}_n^{\mathrm{Spin}}(BG)$

- Objects are compact spin n-manifolds X with principal G-bundle $P \to X$.
- Morphisms $(X_0, P_0) \to (X_1, P_1)$ are bordisms



- $\otimes = \amalg$

Variant: If we allow only bordisms with $Y = X \times [0,1]$, the same construction yields the loop bordism category $\mathrm{Bord}_n^{\mathrm{Spin}}(\mathcal{L}BG)$. Fixing X throughout, we get a non-monoidal bordism category $\mathrm{Bord}_X(BG)$.

The bordism category $\mathrm{Bord}_n^{\mathrm{Spin}}(BG)$ is a Picard groupoid with

$$\pi_0 = \Omega_n^{\mathrm{Spin}}(BG), \quad \pi_1 = \Omega_{n+1}^{\mathrm{Spin}}(BG),$$

and $q = \alpha_1 : \Omega_n^{\mathrm{Spin}}(BG) \to \Omega_{n+1}^{\mathrm{Spin}}(BG)$. Similarly for $\mathcal{L}BG$.

Computing the homotopy type of bordism categories

I spent a long time computing the groups $\Omega_n^{\mathrm{Spin}}(BG)$ and $\Omega_{n-1}^{\mathrm{Spin}}(\mathcal{L}BG)$ for various Lie groups G and $n \leq 9$ using the Atiyah–Hirzebruch spectral sequence.

For example,

$$\Omega_8^{\mathrm{Spin}}(BE_8) = \mathbb{Z}\langle\zeta_2, \zeta_3\rangle,$$
$$\Omega_9^{\mathrm{Spin}}(BE_8) = \mathbb{Z}_2\langle\alpha_1\zeta_2\rangle,$$

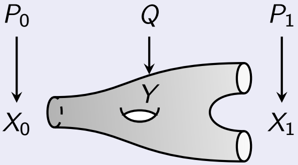
where

$$\zeta_2 = [X_2, u_2], \qquad X_2 = \mathbb{HP}^2, \qquad u_2 = \mathrm{Pd}([\overline{\mathbb{HP}}^1]),$$
$$\zeta_3 = [X_3, u_3], \qquad X_3 = \mathbb{CP}^3 \times S^1_{\mathrm{b}} \times S^1_{\mathrm{b}}, \qquad u_3 = \mathrm{Pd}([\mathbb{CP}^2] \times 1 \times 1)$$

(principal E_8-bundles amount to degree 4 cohomology classes).

Theorem (Upmeier 2023, special case)

A bordism



induces an isomorphism

$$(Y, Q)_* : \Gamma(O_{\mathcal{A}_{P_0}}) \to \Gamma(O_{\mathcal{A}_{P_1}})$$

of orientation $\mathbb{Z}_2$*-torsors. Hence there is a morphism of Picard groupoids*

$$O_8^{BG} : \mathrm{Bord}_8^{\mathrm{Spin}}(BG) \longrightarrow \mathbb{Z}\text{-graded } \mathbb{Z}_2\text{-torsors}, \quad (X, P) \longmapsto \big(\mathrm{ind}(D_A^+), \Gamma(O_{\mathcal{A}_P})\big),$$

called the orientation functor. It has invariants

$$\pi_0(O_8^{BG}) = \mathrm{ind} : \Omega_8^{\mathrm{Spin}}(BG) \to \mathbb{Z}, \qquad \pi_1(O_8^{BG}) = \text{skew-ind} : \Omega_9^{\mathrm{Spin}}(BG) \to \mathbb{Z}_2.$$

The action of $\gamma \in \mathcal{G}_P$ corresponds to the functoriality in the special case of cylinders $Y = X \times [0, 1]$.

Flag structures – fixing orientations

Theorem

The orientation functor maps $\pi_0(O_8^{BG}) : \zeta_2 \mapsto 1, \zeta_3 \mapsto 0$ *and* $\pi_1(O_8^{BG}) : \alpha_1\zeta_2 \mapsto 1$.

- Let $G = E_8$. Fix once and for all a morphism of Picard groupoids

$$\mathrm{Bord}_8^{\mathrm{Spin}}(BG) \xrightarrow{H_8^{BG}} \mathbb{Z}_2\text{-graded } \mathbb{Z}_2\text{-torsors}$$

  with $\pi_i(H_8^{BG}) = \pi_i(O_8^{BG})$, $i = 0, 1$, called the flag functor.
- As $H^2_{\mathrm{sym}} = 0$ here, there is an isomorphism $O_8^{BG} \cong H_8^{BG}$ (fix once and for all).
- Suppose that $\int_X \bar{\beta} \cup \mathrm{Sq}^2(\bar{\beta}) = 0$ for all $\beta \in H^3(X; \mathbb{Z})$. Then

$$\mathrm{Bord}_X(BG) \xrightarrow{I_X} \mathrm{Bord}_8^{\mathrm{Spin}}(BG) \xrightarrow{H_8^{BG}} \mathbb{Z}_2\text{-torsors}$$

  factors through $\mathrm{Bord}_8^{\mathrm{Spin}}(\mathcal{L}BG)$ and is therefore naturally isomorphic to the trivial functor $\mathbb{I}_{\mathbb{Z}_2}$ A flag structure is a choice of natural isomorphism

$$F_X : H_8^{BG} \circ I_X \cong \mathbb{I}_{\mathbb{Z}_2}.$$

  It determines an orientation of $O_{\mathcal{B}_P}$ for every principal E_8-bundle $P \to X$.