Configurations in abelian categories: introduction, and Ringel–Hall algebras

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1. The basic idea

Let $\mathcal{A}$ be an abelian category. We will define *configurations* (σ,ι,π) in $\mathcal{A}$, collections of objects and morphisms in $\mathcal{A}$ attached to a *finite poset* $(I,\preceq)$, satisfying axioms. They are a new tool for describing *how an object in* $\mathcal{A}$ *breaks up into subobjects*. They are useful for studying *stability conditions* on $\mathcal{A}$.

We shall define *moduli stacks* $\mathfrak{Obj}_{\mathcal{A}}$ and $\mathfrak{M}(I,\preceq)_{\mathcal{A}}$ of objects and $(I,\preceq)$-configurations in $\mathcal{A}$, and many *1-morphisms* between them.

Pushforwards and pullbacks along 1-morphisms give linear maps on *constructible* and *stack functions* $\mathrm{CF},\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathrm{CF},\mathrm{SF}(\mathfrak{M}(I,\preceq)_{\mathcal{A}})$.

Combining these gives algebraic operations on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ and $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, in particular an associative multiplication $*$ making $\mathrm{CF},\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ into *infinite-dimensional algebras*.

2. Configurations

Let $\mathcal{A}$ be an abelian category and $X \in \mathcal{A}$. A *subobject* $S \subset X$ is an equivalence class of injective $i : S \to X$. Call $0 \neq X \in \mathcal{A}$ *simple* if the only subobjects $S \subset X$ are $0, X$.

Jordan-Hölder Theorem. *For $\mathcal{A}$ of finite length and X in $\mathcal{A}$, there exist subobjects $0 = A_0 \subset A_1 \subset \cdots \subset A_n = X$ with $S_k = A_k / A_{k-1}$ simple, and n, S_k unique up to order, iso.*

Let $S_1, \ldots, S_n$ be *pairwise non-isomorphic*. Write $\{S_1, \ldots, S_n\} = \{S^i : i \in I\}$, for I a finite *indexing set*, $|I| = n$. Then for each composition series $0 = B_0 \subset B_1 \subset \cdots \subset B_n = X$ with $T_k = B_k/B_{k-1}$, there is a unique *bijection* $\phi : I \to \{1, \ldots, n\}$ with $S^i \cong T_{\phi(i)}$, all $i \in I$.
Define a *partial order* $\preceq$ on I by $i \preceq j$ if $\phi(i) \leqslant \phi(j)$ for all ϕ from composition series as above.

Call $J \subseteq I$ an *s-set* if $i \in I$, $j \in J$ and $i \preceq j \Rightarrow i \in J$.
Call $J \subseteq I$ an *f-set* if $i \in I$, $h, j \in J$ and $h \preceq i \preceq j \Rightarrow i \in J$.
There are *1-1 correspondences*:

- *subobjects* $S \subset X \leftrightarrow$ *s-sets* $J \subseteq I$, where S has simple factors S^j, $j \in J$. If $S, T \leftrightarrow J, K$ then $S \subset T \Leftrightarrow J \subseteq K$.
- *factors* $F = T/S$ for $S \subset T \subset X$ $\leftrightarrow$ *f-sets* $J \subseteq I$, where F has simple factors S^j, $j \in J$.
- *composition series* $0 = B_0 \subset B_1 \subset \cdots \subset B_n = X$ $\leftrightarrow$ *bijections* $\phi : I \to \{1, \ldots, n\}$ with $i \preceq j \Rightarrow \phi(i) \leqslant \phi(j)$.

Definition. Let $(I,\preceq)$ be a finite poset. Write $\mathcal{F}_{(I,\preceq)}$ for the set of f-sets of I. Define $\mathcal{G}_{(I,\preceq)}$ to be the subset of $(J,K)\in\mathcal{F}_{(I,\preceq)}\times\mathcal{F}_{(I,\preceq)}$ such that $J\subseteq K$, and if $j\in J$ and $k\in K$ with $k\preceq j$, then $k\in J$. Define $\mathcal{H}_{(I,\preceq)}=\{(K,K\setminus J):(J,K)\in\mathcal{G}_{(I,\preceq)}\}$.

Define an $(I,\preceq)$-*configuration* (σ,ι,π) in an abelian category $\mathcal{A}$ to be maps $\sigma:\mathcal{F}_{(I,\preceq)}\rightarrow\mathrm{Obj}(\mathcal{A})$, $\iota:\mathcal{G}_{(I,\preceq)}\rightarrow\mathrm{Mor}(\mathcal{A})$, and $\pi:\mathcal{H}_{(I,\preceq)}\rightarrow\mathrm{Mor}(\mathcal{A})$, where $\iota(J,K),\pi(J,K)$ are morphisms $\sigma(J)\rightarrow\sigma(K)$.

These should satisfy the conditions:

(A) Let $(J,K)\in\mathcal{G}_{(I,\preceq)}$ and set $L=K\setminus J$.
Then the following is exact in $\mathcal{A}$:

$$0\longrightarrow\sigma(J)\xrightarrow{\iota(J,K)}\sigma(K)\xrightarrow{\pi(K,L)}\sigma(L)\longrightarrow 0.$$

(B) If $(J,K)\in\mathcal{G}_{(I,\preceq)}$ and $(K,L)\in\mathcal{G}_{(I,\preceq)}$
then $\iota(J,L)=\iota(K,L)\circ\iota(J,K)$.

(C) If $(J,K)\in\mathcal{H}_{(I,\preceq)}$ and $(K,L)\in\mathcal{H}_{(I,\preceq)}$
then $\pi(J,L)=\pi(K,L)\circ\pi(J,K)$.

(D) If $(J,K)\in\mathcal{G}_{(I,\preceq)}$ and $(K,L)\in\mathcal{H}_{(I,\preceq)}$ then

$$\pi(K,L)\circ\iota(J,K)=\iota(J\cap L,L)\circ\pi(J,J\cap L).$$

This encodes the properties of the set of subobjects $S \subset X$ when X has nonisomorphic simple factors.

Theorem 1. *Let $\mathcal{A}$ have finite length, $X \in \mathcal{A}$ have nonisomorphic simple factors $\{S^i : i \in I\}$, and $\preceq$ be as before. Then there exists an $(I, \preceq)$-configuration (σ, ι, π) with $\sigma(I) = X$, unique up to isomorphism, such that if a subobject $S \subset X$ has simple factors $\{S^j : j \in J\}$, then S is represented by $\iota(J, I) : \sigma(J) \to X$.*

Quotient configurations

Let $(I,\preceq)$, $(K,\trianglelefteq)$ be finite posets, and $\phi : I \to K$ surjective with $i \preceq j$ implies $\phi(i) \trianglelefteq \phi(j)$. Let (σ,ι,π) be an $(I,\preceq)$-configuration. Define the *quotient* $(K,\trianglelefteq)$*-configuration* $(\tilde\sigma,\tilde\iota,\tilde\pi)$ to be $(\sigma\circ\phi^*,\iota\circ\phi^*,\pi\circ\phi^*)$, where $\phi^* : \mathcal{F}_{(K,\trianglelefteq)},\mathcal{G}_{(K,\trianglelefteq)},\mathcal{H}_{(K,\trianglelefteq)} \to \mathcal{F}_{(I,\preceq)},\mathcal{G}_{(I,\preceq)},\mathcal{H}_{(I,\preceq)}$ pulls back subsets of K to subsets of I.

Subconfigurations

Let J be an f-set in $(I,\preceq)$. Then $(J,\preceq)$ is a poset with $\mathcal{F}_{(J,\preceq)} \subseteq \mathcal{F}_{(I,\preceq)}$, etc. Define the $(J,\preceq)$-*subconfiguration* (σ',ι',π') of (σ,ι,π) to be $(\sigma|_{\mathcal{F}_{(J,\preceq)}},\iota|_{\mathcal{G}_{(J,\preceq)}},\pi|_{\mathcal{H}_{(J,\preceq)}})$.

We can also combine configurations by *substituting* one in another.

Examples. A $(\{1\},\leqslant)$-configuration is an *object* $\sigma(\{1\})$ in $\mathcal{A}$.
A $(\{1,2\},\leqslant)$-configuration (σ,ι,π) is a *short exact sequence*
$0\rightarrow\sigma(\{1\})\xrightarrow{\iota}\sigma(\{1,2\})\xrightarrow{\pi}\sigma(\{2\})\rightarrow 0.$
Essentially this says $\sigma(\{1,2\})$ has a *subobject* $\sigma(\{1\})\subset\sigma(\{1,2\})$.
A $(\{1,2,3\},\leqslant)$-configuration (σ,ι,π) is equivalent to a *pair of subobjects* $\sigma(\{1\})\subset\sigma(\{1,2\})\subset\sigma(\{1,2,3\})$.
The $(\{1,2\},\leqslant)$-*subconfiguration* is the subobject $\sigma(\{1\})\subset\sigma(\{1,2\})$.
Define $\phi:\{1,2,3\}\rightarrow\{1,2\}$ by $1\mapsto 1$, $2,3\mapsto 2$. Then the *quotient* $(\{1,2\},\leqslant)$-*configuration* is the subobject $\sigma(\{1\})\subset\sigma(\{1,2,3\})$.

3. Moduli stacks

Let $\mathbb{K}$ be an algebraically closed field. *Artin* $\mathbb{K}$*-stacks* $\mathfrak{F}$ are a very general kind of space in algebraic geometry, useful for moduli problems. They include $\mathbb{K}$*-schemes.*

Write $\mathrm{Sch}_{\mathbb{K}}$ for the 2-*category of* $\mathbb{K}$*-schemes*, with the *étale topology*. Then a $\mathbb{K}$-stack is a *sheaf of groupoids* on $\mathrm{Sch}_{\mathbb{K}}$.

For a $\mathbb{K}$-stack $\mathfrak{F}$, write $\mathfrak{F}(\mathbb{K})$ for the set of *geometric points* of $\mathfrak{F}$. Then each $x \in \mathfrak{F}(\mathbb{K})$ has a *stabilizer group* $\mathrm{Iso}_{\mathbb{K}}(x)$. If $\mathfrak{F}$ is a $\mathbb{K}$-*scheme* then $\mathrm{Iso}_{\mathbb{K}}(x) = \{1\}$ for all x.

Let $\mathcal{A}$ be a $\mathbb{K}$-linear abelian category. To form moduli stacks in $\mathcal{A}$ we need some *extra data*. Let $\mathfrak{F}_{\mathcal{A}}$ be a *sheaf of exact categories* on $\mathrm{Sch}_{\mathbb{K}}$ with $\mathfrak{F}_{\mathcal{A}}(\mathrm{Spec}\,\mathbb{K}) = \mathcal{A}$. If $U \in \mathrm{Sch}_{\mathbb{K}}$, we interpret $\mathfrak{F}_{\mathcal{A}}(U)$ as the exact category of *families of objects and morphisms* in $\mathcal{A}$ parametrized by the *base* $\mathbb{K}$*-scheme* U.

If $\mathcal{A}, \mathfrak{F}_{\mathcal{A}}$ satisfy some conditions then for finite posets $(I, \preceq)$ we define the *moduli* $\mathbb{K}$*-stack of* $(I, \preceq)$*-configurations* $\mathfrak{M}(I, \preceq)$. Here $\mathfrak{M}(I, \preceq)(U)$ is the groupoid of $(I, \preceq)$-configs in the exact category $\mathfrak{F}_{\mathcal{A}}(U)$.

Then $\mathfrak{M}(I, \preceq)(\mathbb{K})$ is the set of iso. classes $[(\sigma, \iota, \pi)]$ of $(I, \preceq)$-configurations (σ, ι, π) in $\mathcal{A}$, and $\mathrm{Iso}_{\mathbb{K}}\big([(\sigma, \iota, \pi)]\big) = \mathrm{Aut}\big((\sigma, \iota, \pi)\big)$.

We also define many 1-*morphisms* between the $\mathfrak{M}(I,\preceq)$ and $\mathfrak{Obj}_{\mathcal{A}}$. E.g., if $J \subseteq I$ is an f-set, $S(I,\preceq,J) : \mathfrak{M}(I,\preceq) \to \mathfrak{M}(J,\preceq)$ takes $(I,\preceq)$-configs to $(J,\preceq)$-subconfigs, and $\boldsymbol{\sigma}(J) : \mathfrak{M}(I,\preceq) \to \mathfrak{Obj}_{\mathcal{A}}$ takes (σ,ι,π) to $\sigma(J)$. These 1-morphisms often form *Cartesian squares*.

Examples. We can define $\mathcal{A}, \mathfrak{F}_{\mathcal{A}}$ satisfying the conditions, and get well-defined moduli stacks $\mathfrak{M}(I,\preceq)$, when

- $\mathcal{A} =$ mod-$\mathbb{K}Q$, the abelian category of $\mathbb{K}$-representations of a (finite) *quiver* Q.
- $\mathcal{A} =$ mod-$\mathbb{K}Q/I$, representations of a *quiver with relations* (Q,I).
- $\mathcal{A} = \text{coh}(P)$, *coherent sheaves* on a *projective* $\mathbb{K}$*-scheme* P.

4. Recap of last seminar

Constructible functions on stacks satisfy:

• To each $\mathbb{K}$-stack $\mathfrak{F}$ with affine stabilizers, associate a $\mathbb{Q}$-algebra $\mathrm{CF}(\mathfrak{F})$.

• Constructible $S \subseteq \mathfrak{F}(\mathbb{K})$ have *characteristic functions* $\delta_S \in \mathrm{CF}(\mathfrak{F})$.

• To each finite type 1-morphism $\phi : \mathfrak{F} \to \mathfrak{G}$ associate a *pullback* algebra morphism $\phi^* : \mathrm{CF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{F})$, with $(\psi\circ\phi)^* = \phi^*\circ\psi^*$.

• When $\mathrm{char}\,\mathbb{K} = 0$, to each representable 1-morphism $\phi : \mathfrak{F} \to \mathfrak{G}$ associate a linear *pushforward* $\mathrm{CF}^{\mathrm{stk}}(\phi) : \mathrm{CF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{F})$, with $\mathrm{CF}^{\mathrm{stk}}(\psi \circ \phi) = \mathrm{CF}^{\mathrm{stk}}(\psi) \circ \mathrm{CF}^{\mathrm{stk}}(\phi)$.

• In a *Cartesian square* of Artin $\mathbb{K}$-stacks

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\eta} & \mathfrak{G} \\ \downarrow \theta & & \downarrow \psi \\ \mathfrak{F} & \xrightarrow{\phi} & \mathfrak{H}, \end{array}$$

the following commutes:

$$\begin{array}{ccc} \mathrm{CF}(\mathfrak{E}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\eta)} & \mathrm{CF}(\mathfrak{G}) \\ \uparrow \theta^* & & \uparrow \psi^* \\ \mathrm{CF}(\mathfrak{F}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\phi)} & \mathrm{CF}(\mathfrak{H}). \end{array}$$

Stack functions also have these properties.

5. Ringel–Hall algebras

Let $\mathcal{A}, \mathfrak{F}_{\mathcal{A}}$ be as usual. Write $\mathfrak{Obj}_{\mathcal{A}}$ for the *moduli stack of objects* in $\mathcal{A}$. Then there are 1-*morphisms* $\boldsymbol{\sigma}(\{1\})$, $\boldsymbol{\sigma}(\{2\})$, $\boldsymbol{\sigma}(\{1,2\}) : \mathfrak{M}(\{1,2\},\leqslant) \rightarrow \mathfrak{Obj}_{\mathcal{A}}$ taking a $(\{1,2\},\leqslant)$-config (σ,ι,π) to $\sigma(\{1\}),\sigma(\{2\}),\sigma(\{1,2\})$. Define a *multiplication* $*$ on $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ by $f * g = \mathrm{CF}^{\mathrm{stk}}\Big(\boldsymbol{\sigma}(\{1,2\})\Big)\Big[\boldsymbol{\sigma}(\{1\})^*(f) \cdot \boldsymbol{\sigma}(\{2\})^*(g)\Big]$. Similarly, on $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ define

$$f * g = \boldsymbol{\sigma}(\{1,2\})_*\Big[\Big(\boldsymbol{\sigma}(\{1\}) \times \boldsymbol{\sigma}(\{2\})\Big)^*(f \otimes g)\Big].$$

This is essentially the *Ringel–Hall algebra* idea. In physics terms, think of them as *algebras of BPS states*.

To prove $*$ is *associative*, consider the commutative diagram of 1-morphisms:

$$
\begin{array}{ccccc}
\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}} & \xleftarrow{\mathrm{id}\times\boldsymbol{\sigma}(2)\times\boldsymbol{\sigma}(3)} & \mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{M}(\{2,3\},\leqslant)_{\mathcal{A}} & \xrightarrow{\mathrm{id}\times\boldsymbol{\sigma}(2,3)} & \mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}} \\
\Big\uparrow{\scriptstyle \boldsymbol{\sigma}(1)\times\boldsymbol{\sigma}(2)\times\mathrm{id}} & & \Big\uparrow{\scriptstyle \alpha} & & \Big\uparrow{\scriptstyle \boldsymbol{\sigma}(1)\times\boldsymbol{\sigma}(2)} \\
\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}} & \xleftarrow{\beta} & \mathfrak{M}(\{1,2,3\},\leqslant)_{\mathcal{A}} & \xrightarrow{\gamma} & \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}} \\
\Big\downarrow{\scriptstyle \boldsymbol{\sigma}(1,2)\times\mathrm{id}} & & \Big\downarrow{\scriptstyle \delta} & & \Big\downarrow{\scriptstyle \boldsymbol{\sigma}(1,2)} \\
\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}} & \xleftarrow{\boldsymbol{\sigma}(1)\times\boldsymbol{\sigma}(2)} & \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}} & \xrightarrow{\boldsymbol{\sigma}(1,2)} & \mathfrak{Obj}_{\mathcal{A}}.
\end{array}
$$

The top right and bottom left squares are *Cartesian*, so the following commutes:

$$
\begin{array}{ccccc}
\mathrm{CF}\big(\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}\big) & \xrightarrow{(\mathrm{id}\times\boldsymbol{\sigma}(2)\times\boldsymbol{\sigma}(3))^*} & \mathrm{CF}\big(\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{M}(\{2,3\},\leqslant)_{\mathcal{A}}\big) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\mathrm{id}\times\boldsymbol{\sigma}(2,3))} & \mathrm{CF}\big(\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}\big) \\
\Big\downarrow{\scriptstyle (\boldsymbol{\sigma}(1)\times\boldsymbol{\sigma}(2)\times\mathrm{id})^*} & & \Big\downarrow{\scriptstyle \alpha^*} & & \Big\downarrow{\scriptstyle (\boldsymbol{\sigma}(1)\times\boldsymbol{\sigma}(2))^*} \\
\mathrm{CF}\big(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}\big) & \xrightarrow{\beta^*} & \mathrm{CF}\big(\mathfrak{M}(\{1,2,3\},\leqslant)_{\mathcal{A}}\big) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\gamma)} & \mathrm{CF}\big(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}\big) \\
\Big\downarrow{\scriptstyle \mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(1,2)\times\mathrm{id})} & & \Big\downarrow{\scriptstyle \mathrm{CF}^{\mathrm{stk}}(\delta)} & & \Big\downarrow{\scriptstyle \mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(1,2))} \\
\mathrm{CF}\big(\mathfrak{Obj}_{\mathcal{A}}\times\mathfrak{Obj}_{\mathcal{A}}\big) & \xrightarrow{(\boldsymbol{\sigma}(1)\times\boldsymbol{\sigma}(2))^*} & \mathrm{CF}\big(\mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}\big) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\boldsymbol{\sigma}(1,2))} & \mathrm{CF}\big(\mathfrak{Obj}_{\mathcal{A}}\big).
\end{array}
$$

Applying the two routes round the outside to $f\otimes g\otimes h$ proves $(f*g)*h=f*(g*h)$.

We can translate work by many authors into the configurations framework to give *geometric realizations* of interesting algebras such as *universal enveloping algebras* of *Kac–Moody algebras* $U(\mathfrak{g})$ as algebras of constructible functions on $\mathfrak{Obj}_{\mathcal{A}}$, where $\mathcal{A}$ is mod-$\mathbb{K}Q$ or mod-$\mathbb{K}Q/I$ for a *quiver* Q. We can also do a lot more. There are other ways to use configuration 1-morphisms to define associative multiplications on $\mathrm{CF}(\mathfrak{M}(I,\preceq))$, and *comultiplications* to make *Hopf algebras*. The *Drinfeld double* construction has a configuration explanation, I believe. And so on.

6. Indecomposables and Lie algebras

Call $X \in \mathcal{A}$ *indecomposable* if $X \not\cong 0$ and $X \not\cong Y \oplus Z$ for any $Y, Z \not\cong 0$. Any $X \in \mathcal{A}$ has $X \cong X_1 \oplus \cdots \oplus X_n$ for X_a indecomposable and unique up to order, isomorphism.

Write $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ for the subspace of $f \in \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ supported on points $[X]$ for X indecomposable. If $f, g \in \mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ then $f * g$ is supported on $[X]$ with 1 or 2 indecomposable factors, and $(f*g)([X \oplus Y]) = f(X)g(Y) + f(Y)g(X)$ for indecomposable $X \not\cong Y$. So $[f, g] = f * g - g * f$ lies in $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$, which is a *Lie algebra*.

Stack functions supported on indecomposables are *not* closed under $[\,,\,]$, but there is a Lie subalgebra $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ of stack functions supported on 'virtual indecomposables' (rather complicated!).

7. Algebra morphisms from $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$

Recall from last seminar: let Υ be a *motivic invariant* of $\mathbb{K}$-varieties with values in a $\mathbb{Q}$-algebra Λ, $\ell = \Upsilon(\mathbb{K})$, ℓ and $\ell^k - 1$, $k \geqslant 1$ invertible in Λ. We extend Υ uniquely to $\Upsilon'(\mathfrak{F})$ for finite type $\mathbb{K}$-stacks $\mathfrak{F}$, such that $\Upsilon'([X/G]) = \Upsilon(X)\Upsilon(G)^{-1}$ for X a variety and G a special $\mathbb{K}$-group. Example: $\Upsilon(X)$ can be the *virtual Poincaré polynomial* $P_X(z)$, Λ the $\mathbb{Q}$-algebra of rational functions in z.

For such Υ, Λ, define a $\mathbb{Q}$-linear map
$\Pi_\Lambda : \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}) \rightarrow \Lambda$ by
$\Pi_\Lambda : [(\mathfrak{R}, \rho)] \mapsto \Upsilon'(\mathfrak{R})$.

Write $K(\mathcal{A})$ for K-theory of $\mathcal{A}$. Suppose $\chi : K(\mathcal{A}) \times K(\mathcal{A}) \to \mathbb{Z}$ is biadditive with

$$\dim \mathrm{Hom}(X,Y) - \dim \mathrm{Ext}^1(X,Y) = \chi([X],[Y])$$

for all $X, Y \in \mathcal{A}$. This holds for $\mathcal{A} =$ mod-$\mathbb{K}Q$ and $\mathcal{A} = \mathrm{coh}(P)$, P smooth curve. Write $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}$ for the substack of $[X] \in \mathfrak{Obj}_{\mathcal{A}}$ in class $\alpha \in K(\mathcal{A})$. Then we prove:

Theorem. *Let $f, g \in \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$ be supported on $\mathfrak{Obj}_{\mathcal{A}}^{\alpha}, \mathfrak{Obj}_{\mathcal{A}}^{\beta}$ for $\alpha, \beta \in K(\mathcal{A})$. Then $\Pi_\Lambda(f * g) = \ell^{-\chi(\beta,\alpha)} \Pi_\Lambda(f) \Pi_\Lambda(g)$ in Λ.*

Can use this identity to define an *algebra morphism* $\Phi^\Lambda : \mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}}) \to A(\mathcal{A}, \Lambda, \chi)$, where $A(\mathcal{A}, \Lambda, \chi)$ is the Λ-algebra with Λ-basis a^α, $\alpha \in K(\mathcal{A})$, and multiplication $a^\alpha \star a^\beta = \ell^{-\chi(\beta,\alpha)} a^{\alpha+\beta}$, by
$\Phi^\Lambda(f) = \sum_{\alpha \in K(\mathcal{A})} \Pi_\Lambda(f|_{\mathfrak{Obj}_{\mathcal{A}}^{\alpha}}) a^\alpha$.

Sketch proof: Can write the support of $f \otimes g$ as a finite disjoint union of substacks $\mathfrak{F}_i \subset \mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}$, with vector spaces H_i, E_i such that for all $([X],[Y]) \in \mathfrak{F}_i(\mathbb{K})$ we have $\mathrm{Hom}(Y,X) \cong H_i$ and $\mathrm{Ext}^1(Y,X) = E_i$. So $\dim H_i - \dim E_i = \chi(\beta,\alpha)$.

Can also arrange $\mathfrak{F}_i \cong [X_i/G_i]$ for G_i special. Then the fibre product

$\mathfrak{G}_i = \mathfrak{F}_i \times_{\mathfrak{Obj}_{\mathcal{A}} \times \mathfrak{Obj}_{\mathcal{A}}} \mathfrak{M}(\{1,2\},\leqslant)_{\mathcal{A}}$ is 1-isomorphic to $[X_i \times E_i/G_i \ltimes H_i]$, since $(\{1,2\},\leqslant)$-configurations over $([X],[Y])$ are parametrized by $\mathrm{Ext}^1(Y,X)$ and have $\mathrm{Hom}(Y,X)$ in their stabilizer group.

Thus $\Upsilon'(\mathfrak{F}_i) = \Upsilon(X_i)\Upsilon(G_i)^{-1}$ and $\Upsilon'(\mathfrak{G}_i) = \Upsilon(X_i)\Upsilon(E_i)\Upsilon(G_i)^{-1}\Upsilon(H_i)^{-1}$, and $\Upsilon(E_i) = \ell^{\dim E_i}$, $\Upsilon(H_i) = \ell^{\dim H_i}$, so $\Upsilon'(\mathfrak{G}_i) = \ell^{-\chi(\beta,\alpha)}\Upsilon'(\mathfrak{F}_i)$.

If P is a *Calabi–Yau 3-fold* then for biadditive $\bar\chi : K(\mathrm{coh}(P))\times K(\mathrm{coh}(P))\rightarrow\mathbb{Z}$ and all $X,Y\in\mathrm{coh}(P)$ we have

$$\dim\mathrm{Hom}(X,Y)-\dim\mathrm{Ext}^1(X,Y)$$
$$-\dim\mathrm{Hom}(Y,X)+\dim\mathrm{Ext}^1(Y,X)=\bar\chi([X],[Y]).$$

We can construct a *Lie algebra morphism* $\Psi^\Omega:\mathrm{SF}_{\mathrm{al}}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathrm{coh}(P)})\rightarrow C(\mathrm{coh}(P),\Omega,\frac{1}{2}\bar\chi)$ to an explicit algebra, in a similar way.

These Φ^Λ,Ψ^Ω will be used next seminar to define interesting invariants 'counting' τ-semistable objects in $\mathcal{A}$. Writing $\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau)$ for the moduli space of τ-semistable objects in class $\alpha\in K(\mathcal{A})$, stack functions like $\bar\delta_{\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau)}$ satisfy identities in the algebra $\mathrm{SF}(\mathfrak{Obj}_{\mathcal{A}})$, so Φ^Λ being a morphism implies *multiplicative identities* on the invariants $I^\alpha(\tau)=\Pi_\Lambda(\bar\delta_{\mathrm{Obj}_{\mathrm{ss}}^\alpha(\tau)})$ in Λ.