Holomorphic generating functions for invariants counting sheaves on Calabi–Yau 3-folds

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1. Introduction

If (M, ω) is a compact symplectic manifold, one can define *Gromov–Witten invariants* $\Phi_A(\alpha, \beta, \gamma)$ of M. It is natural to encode these in a *holomorphic generating function* $\mathcal{S} : H^{\text{ev}}(M, \mathbb{C}) \to \mathbb{C}$ called the *Gromov–Witten potential*, given by a power series with coefficients the $\Phi_A(\alpha, \beta, \gamma)$. Identities on the $\Phi_A(\alpha, \beta, \gamma)$ imply that $\mathcal{S}$ satisfies a p.d.e., the *WDVV equation*. This p.d.e. can be interpreted as the flatness of a 1-parameter family of connections defined using $\mathcal{S}$, which make $H^{\text{ev}}(M, \mathbb{C})$ into a *Frobenius manifold*.

This talk will tell a similar story. Given an abelian category $\mathcal{A}$ satisfying some conditions, we form a complex manifold Stab($\mathcal{A}$) of slope stability conditions Z on $\mathcal{A}$. In previous work I defined systems of invariants $\epsilon^\alpha(Z)$ 'counting' Z-semistable objects in $\mathcal{A}$ in class α in $K(\mathcal{A})$. The $\epsilon^\alpha(Z)$ live in an *infinite-dimensional Lie algebra* $\mathcal{L}$.
We combine the $\epsilon^\alpha(Z)$ into *holomorphic generating functions* f^α : Stab($\mathcal{A}$) $\rightarrow \mathcal{L}$. The $\epsilon^\alpha(Z)$ are locally constant in Z except that they change *discontinuously* on real hypersurfaces in Stab($\mathcal{A}$). Requiring f^α to be continuous and holomorphic determines the form of f^α essentially uniquely. Remarkably, the f^α turn out to satisfy a p.d.e., which implies the flatness of an $\mathcal{L}$-valued connection on Stab($\mathcal{A}$).

This story should extend to *triangulated categories* $\mathcal{T}$ such as *derived categories* $D^b(\mathcal{A})$, with Bridgeland stability conditions. My motivation for this is as follows. Let P be a Calabi–Yau 3-fold, $\mathcal{T} = D^b(\text{coh}(P))$ its derived category, and $\text{Stab}(\mathcal{T})$ the complex manifold of Bridgeland stability conditions. Then one should define invariants $J^\alpha(Z) \in \mathbb{Q}$ for $Z \in \text{Stab}(\mathcal{T})$ 'counting' Z-semistable complexes in class α in $K(\mathcal{T})$, generalizing Donaldson–Thomas invariants. This work shows how to combine the $J^\alpha(Z)$ into holomorphic generating functions $f^\alpha : \text{Stab}(\mathcal{T}) \to \mathbb{C}$ satisfying a p.d.e., which define an interesting geometric structure on $\text{Stab}(\mathcal{T})$. I think this is some new thing in Homological Mirror Symmetry, and I hope String Theorists will be able to explain it.

2. The general set up

Let $\mathcal{A}$ be an abelian category, and $K(\mathcal{A})$ the quotient of the Grothendieck group $K_0(\mathcal{A})$ by some fixed subgroup, such that if $X \in \mathcal{A}$ and $[X] = 0$ in $K(\mathcal{A})$ then $X \cong 0$. Define the *positive cone* in $K(\mathcal{A})$:

$$C(\mathcal{A}) = \bigl\{[X] \in K(\mathcal{A}) : X \in \mathcal{A}, \;\; X \not\cong 0\bigr\}.$$

Let $c, r : K(\mathcal{A}) \to \mathbb{R}$ be group homomorphisms with $r(\alpha) > 0$ for all $\alpha \in C(\mathcal{A})$. Define the *slope* $\mu : C(\mathcal{A}) \to \mathbb{R}$ by $\mu(\alpha) = c(\alpha)/r(\alpha)$. Define the *central charge* $Z \in \mathrm{Hom}(K(\mathcal{A}), \mathbb{C})$ by $Z(\alpha) = -c(\alpha) + ir(\alpha)$. It maps $C(\mathcal{A})$ to the upper half plane $H = \{x + iy : x \in \mathbb{R}, \, y > 0\}$ in $\mathbb{C}$. Write $\mathrm{Stab}(\mathcal{A})$ for the complex manifold of such Z.

An object $X \not\cong 0$ in $\mathcal{A}$ is called *Z-semistable* if for all subobjects $0 \neq S \subset X$ we have $\mu([S]) \leqslant \mu([X])$.

Then my papers I–IV provide lots of ways of defining the following general structure:

• an *associative algebra* $\mathcal{H}$ with (generally noncommutative) multiplication $*$.

• a splitting $\mathcal{H} = \bigoplus_{\alpha \in C(\mathcal{A}) \cup \{0\}} \mathcal{H}^\alpha$ with $\mathcal{H}^\alpha * \mathcal{H}^\beta \subseteq \mathcal{H}^{\alpha+\beta}$ and $1 \in \mathcal{H}^0$.

• a *Lie subalgebra* $\mathcal{L} \subset \mathcal{H}$ with Lie bracket $[f,g] = f * g - g * f$.

• a splitting $\mathcal{L} = \bigoplus_{\alpha \in C(\mathcal{A})} \mathcal{L}^\alpha$ with $\mathcal{L}^\alpha \subseteq \mathcal{H}^\alpha$ and $[\mathcal{L}^\alpha, \mathcal{L}^\beta] \subseteq \mathcal{L}^{\alpha+\beta}$.

• elements $\epsilon^\alpha(Z) \in \mathcal{L}^\alpha$ for $\alpha \in C(\mathcal{A})$ and $Z \in \mathrm{Stab}(\mathcal{A})$, such that for all $\alpha, Z, \tilde{Z}$

$$\epsilon^\alpha(\tilde{Z}) = \sum_{\substack{\alpha_1,\ldots,\alpha_n \in C(\mathcal{A}): \\ \alpha_1+\cdots+\alpha_n=\alpha}} \begin{array}{l} U(\alpha_1,\ldots,\alpha_n; Z, \tilde{Z}) \\ \epsilon^{\alpha_1}(Z) * \cdots * \epsilon^{\alpha_n}(Z), \end{array} \qquad (1)$$

for combinatorial coefficients $U(\cdots) \in \mathbb{Q}$. Here (1) is a *Lie algebra identity* in $\mathcal{L}$.

What this means: we form an *Artin stack* $\mathfrak{Obj}_{\mathcal{A}}$ of objects in $\mathcal{A}$. Then for $\alpha \in C(\mathcal{A})$, the moduli space $\mathrm{Obj}^{\alpha}_{\mathrm{ss}}(Z)$ of Z-semistable objects in class α is a *constructible set* in $\mathfrak{Obj}_{\mathcal{A}}$, so its characteristic function $\delta^{\alpha}_{\mathrm{ss}}(Z)$ is a *constructible function* in $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$.

We can make $\mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}})$ into an *algebra*. The subspace $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$ supported on indecomposables is a *Lie subalgebra*.

Roughly, we form an algebra $\mathcal{H}$ with Lie subalgebra $\mathcal{L}$, and an algebra morphism $\Phi : \mathrm{CF}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathcal{H}$ taking $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}}) \to \mathcal{L}$. Then the $\epsilon^{\alpha}(Z)$ are got by applying Φ to modified versions of $\delta^{\alpha}_{\mathrm{ss}}(Z)$ lying in $\mathrm{CF}^{\mathrm{ind}}(\mathfrak{Obj}_{\mathcal{A}})$. Equation (1) comes from an expression for $\delta^{\alpha}_{\mathrm{ss}}(\tilde{Z})$ in terms of the $\delta^{\alpha_i}_{\mathrm{ss}}(Z)$. This is all rather oversimplified.

A motivating example.

Let P be a Calabi–Yau 3-fold, and $\mathcal{A} = \mathrm{coh}(P)$ the coherent sheaves on P. We can take $K(\mathcal{A}) \subset H^{\mathrm{ev}}(P, \mathbb{Z})$ using the Chern character. There is a natural biadditive, antisymmetric form $\chi : K(\mathcal{A}) \times K(A) \to \mathbb{Z}$. Define $\mathcal{L}$ to be the $\mathbb{C}$-Lie algebra with basis e^α for $\alpha \in C(\mathcal{A})$ and $[e^\alpha, e^\beta] = \chi(\alpha, \beta) e^{\alpha+\beta}$, and $\mathcal{L}^\alpha = \mathbb{C} \cdot e^\alpha$. Define $\mathcal{H} = U(\mathcal{L})$, the universal enveloping algebra of $\mathcal{L}$.

For Gieseker rather than slope stability conditions Z, I defined invariants $J^\alpha(Z)$ in $\mathbb{Q}$ 'counting' Z-semistable sheaves in class α, similar to Donaldson–Thomas invariants. Setting $\epsilon^\alpha(Z) = J^\alpha(Z) e^\alpha$, these transform according to (1). I expect this to extend to Bridgeland stability on $D^b(\mathrm{coh}(P))$.

3. Setting up the problem

Let $\mathcal{A}, K(\mathcal{A}), C(\mathcal{A}), \mathcal{H}, \mathcal{L}, \text{Stab}(\mathcal{A}), \epsilon^\alpha(Z)$ be as above. For $\alpha \in C(\mathcal{A})$, consider the function $f^\alpha : \text{Stab}(\mathcal{A}) \to \mathcal{H}^\alpha$ given by

$$f^\alpha(Z) = \sum_{\substack{\alpha_1,\ldots,\alpha_n \in C(\mathcal{A}):\\ \alpha_1+\cdots+\alpha_n=\alpha}} \begin{array}{l} F_n(Z(\alpha_1),\ldots,Z(\alpha_n)) \\ \epsilon^{\alpha_1}(Z) * \cdots * \epsilon^{\alpha_n}(Z), \end{array} \qquad (2)$$

for $F_n : (\mathbb{C}^\times)^n \to \mathbb{C}$, where $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. We shall find functions F_n so that f^α is *continuous* and *holomorphic* in Z, despite the fact that the $\epsilon^{\alpha_i}(Z)$ change *discontinuously* across real hypersurfaces in Stab($\mathcal{A}$), according to (1). We also require:

(a) $F_1 \equiv (2\pi i)^{-1}$;
(b) $F_n(z_1,\ldots,z_n) \equiv F_n(\lambda z_1,\ldots,\lambda z_n)$;
(c) $|F_n(z_1,\ldots,z_n)| = o(|z_k|^{-1})$ as $z_k \to 0$;
(d) $\sum_{\sigma\in S_n} F_n(z_{\sigma(1)},\ldots,z_{\sigma(n)}) \equiv 0$.

There are *unique* F_n satisfying all this.

Here (d) follows from a condition on F_n making (2) into a *Lie algebra equation*, so that f^α actually maps Stab($\mathcal{A}$) $\rightarrow \mathcal{L}^\alpha$.

The simplest wall-crossing behaviour of the $\epsilon^\alpha(Z)$, encoded in (1), is that $\epsilon^{\alpha+\beta}(Z)$ jumps by $\epsilon^\alpha(Z)*\epsilon^\beta(Z)-\epsilon^\beta(Z)*\epsilon^\alpha(Z)$ across the hypersurface $Z(\beta)/Z(\alpha) \in (0,\infty)$.

So, given $\alpha_1,\ldots,\alpha_n \in C(\mathcal{A})$, across the hypersurface $Z(\alpha_{l+1})/Z(\alpha_l) \in (0,\infty)$, the term $\epsilon^{\alpha_1}(Z) * \cdots * \epsilon^{\alpha_{l-1}}(Z) * \epsilon^{\alpha_l+\alpha_{l+1}}(Z)$ $\epsilon^{\alpha_{l+2}}(Z) * \cdots * \epsilon^{\alpha_n}(Z)$ jumps by

$\epsilon^{\alpha_1}(Z) * \cdots * \epsilon^{\alpha_n}(Z) - \epsilon^{\alpha_1}(Z) * \cdots * \epsilon^{\alpha_{l-1}}(Z) *$ $\epsilon^{\alpha_{l+1}}(Z) * \epsilon^{\alpha_l} * \epsilon^{\alpha_{l+2}}(Z) * \cdots * \epsilon^{\alpha_n}(Z)$.

The function f^α in (2) is continuous under this transition if $F_n(z_1,\ldots,z_n)$ jumps by $F_{n-1}(z_1,\ldots,z_{l-1},z_l+z_{l+1},z_{l+2},\ldots,z_n)$ across the hypersurface $z_{l+1}/z_l \in (0,\infty)$.

We summarize the conditions on F_n:

Proposition 1. *For f^α to be continuous and holomorphic, we need F_n to be continuous and holomorphic on the set*

$$\Big\{(z_1, \ldots, z_n) \in (\mathbb{C}^\times)^n : z_{k+1}/z_k \notin (0, \infty) \\ \text{for all } 1 \leqslant k < n\Big\}. \tag{3}$$

Near a point on only one hypersurface $z_{l+1}/z_l \in (0, \infty)$ in $(\mathbb{C}^\times)^n$, the function $F_n(z_1, \ldots, z_n) - \eta(z_{l+1}/z_l) F_{n-1}(z_1, \ldots, z_{l-1}, z_l + z_{l+1}, z_{l+2}, \ldots, z_n)$ must be continuous and holomorphic, where $\eta(z) = \frac{1}{2}$ if $\mathrm{Im}(z) < 0$, $\eta(z) = 0$ if $\mathrm{Im}(z) = 0$, and $\eta(z) = -\frac{1}{2}$ if $\mathrm{Im}(z) > 0$.

Along intersections of two or more hypersurfaces $z_{l+1}/z_l \in (0, \infty)$ in $(\mathbb{C}^\times)^n$, the F_n satisfy more complicated conditions.

Here is a uniqueness result:

Theorem 2. *There is at most one family of functions F_n satisfying Proposition 1 and conditions* (a)–(d) *above.*

The proof is by induction on n. Let F_n, F_n' for $n \geqslant 1$ be two families satisfying the conditions, and suppose by induction that $F_k \equiv F_k'$ for $k < n$. This holds for $n = 2$ by (a). Then we find that $f = F_n - F_n' : (\mathbb{C}^\times)^n \to \mathbb{C}$ is holomorphic. By (b) it pulls back to the complement of some hyperplanes in $\mathbb{CP}^{n-1}$, and by (c) it extends over these hyperplanes. Thus f comes from a holomorphic function on $\mathbb{CP}^{n-1}$, so $f \equiv c$. Finally (d) shows $c = 0$, so $F_n \equiv F_n'$.

4. A p.d.e. on the f^α

We will now *guess* a p.d.e. which the f^α will satisfy. From Proposition 1 and (a)–(d) we find that $F_1(z_1) = (2\pi i)^{-1}$ and $F_2(z_1, z_2) = (2\pi i)^{-2}\big(\log(z_2/z_1) - \pi i\big)$ on $z_2/z_1 \notin (0,\infty)$, with $\operatorname{Im}\log(z_2/z_1) \in (0, 2\pi)$. Consider a situation with classes $\beta, \gamma, \beta+\gamma$ in $C(\mathcal{A})$ which do not otherwise split as sums of elements in $C(\mathcal{A})$. Then (2) gives $f^\beta(Z) = (2\pi i)^{-1}\epsilon^\beta(Z)$, $f^\gamma = (2\pi i)^{-1}$ $\epsilon^\gamma(Z)$ and $f^{\beta+\gamma}(Z) = (2\pi i)^{-1}\epsilon^{\beta+\gamma}(Z) +$ $(2\pi i)^{-2}(\log(Z(\gamma)/Z(\beta)) - \pi i)[\epsilon^\beta(Z), \epsilon^\gamma(Z)]$ away from $Z(\gamma)/Z(\beta) \in (0, \infty)$.

These satisfy the p.d.e.

$$\begin{aligned} \mathrm{d}f^{\beta+\gamma}(Z) = {} & \big[f^\beta(Z), f^\gamma(Z)\big] \otimes \\ & \big(\mathrm{d}Z(\gamma)/Z(\gamma) - \mathrm{d}Z(\beta)/Z(\beta)\big). \end{aligned} \tag{4}$$

This motivates the following:

We now guess the f^α should satisfy the p.d.e. in $\mathcal{L}$-valued 1-forms on Stab($\mathcal{A}$):

$$\mathrm{d}f^\alpha(Z) = -\sum_{\beta,\gamma\in C(\mathcal{A}):\alpha=\beta+\gamma} [f^\beta(Z), f^\gamma(Z)] \otimes \frac{\mathrm{d}Z(\beta)}{Z(\beta)}. \quad (5)$$

This has some very nice properties:

• For (5) to hold, the r.h.s. must be closed. But taking d of the r.h.s. and using (5) to substitute for $\mathrm{d}f^\beta$ and $\mathrm{d}f^\gamma$, everything cancels to give zero. Thus, (5) is its own consistency condition!

• Set $\Gamma(Z) = \sum_{\alpha\in C(\mathcal{A})} f^\alpha(Z) \otimes \mathrm{d}Z(\alpha)/Z(\alpha)$. This is an $\mathcal{L}$-valued connection matrix, with curvature $R_\Gamma = \mathrm{d}\Gamma + \frac{1}{2}\Gamma\wedge\Gamma = 0$ by (5), so $\mathrm{d}+\Gamma$ is a *flat connection*.

• Define a section s : Stab($\mathcal{A}$) $\to \mathcal{L}$ by $s(Z) = \sum_{\alpha\in C(\mathcal{A})} f^\alpha(Z)$. Then (5) implies s is *constant* under the flat connection $\mathrm{d}+\Gamma$.

Substituting (2) into (5) and rearranging, we can show the f^α satisfy (5) provided the F_n satisfy the p.d.e.

$$\mathrm{d}F_n(z_1,\ldots,z_n) = \sum_{k=1}^{n-1} F_k(z_1,\ldots,z_k)F_{n-k}(z_{k+1},\ldots,z_n)\cdot \left[\frac{\mathrm{d}z_{k+1}+\cdots+\mathrm{d}z_n}{z_{k+1}+\cdots+z_n} - \frac{\mathrm{d}z_1+\cdots+\mathrm{d}z_k}{z_1+\cdots+z_k}\right] \quad (6)$$

in the domain (3), oversimplifying a bit.

• The 1-form $[\cdots]$ in (6) restricts to zero on $z_1+\cdots+z_n = 0$, so $\mathrm{d}F_n|_{z_1+\cdots+z_n=0} \equiv 0$, and F_n is constant on $z_1+\cdots+z_n = 0$. Then (d) shows this constant is zero.

• As for (5), taking d of the r.h.s. of (6) and substituting in (6) for $\mathrm{d}F_k$ and $\mathrm{d}F_{n-k}$ gives 0. Therefore, if (6) holds for $n < m$, then the r.h.s. of (6) is closed for $n = m$. This is the basis of an inductive construction for F_n satisfying (6).

Proposition 3. *There exists a unique family of functions F_n for $n \geqslant 1$, defined on the domain* (3)*, satisfying $F_1 \equiv (2\pi i)^{-1}$ on* (3)*, equation* (6) *and $F_n|_{z_1+\cdots+z_n=0} \equiv 0$.*

The proof is by induction on n. Having constructed $F_1, \ldots, F_{m-1}$, the r.h.s. of (6) is closed on (3) for $n = m$, by (6) for $n < m$. It is the pull-back of a closed 1-form on a connected, simply-connected region of $\mathbb{CP}^{m-1}$, so it's exact, and is $\mathrm{d}F_m$ for F_m unique up to addition of a constant. Requiring $F_m|_{z_1+\cdots+z_m=0} \equiv 0$ fixes the constant.

Note that $(z_1 + \cdots + z_k)^{-1}$ in (6) causes no singularities in F_n, since $F_k|_{z_1+\cdots+z_k=0} \equiv 0$ implies $(z_1 + \cdots + z_k)^{-1} F_k$ extends holomorphically over $z_1 + \cdots + z_k = 0$. The same holds for $(z_{k+1} + \cdots + z_n)^{-1}$.

Surprisingly, we can now prove:

Theorem 4. *The functions* F_n *on* (3) *in Proposition 3 extend to functions* F_n *on* $(\mathbb{C}^\times)^n$ *satisfying Proposition 1 and* (a)–(d) *above. By Theorem 2, they are the unique functions which do this.*

The main point of the proof is to show that $F_n(z_1,\ldots,z_n)$ jumps by $F_{n-1}(z_1,\ldots, z_{l-1}, z_l + z_{l+1}, z_{l+2},\ldots,z_n)$ across the hypersurface $z_{l+1}/z_l \in (0,\infty)$ in $(\mathbb{C}^\times)^n$, as in Proposition 1. Write $D_{l,n}(z_1,\ldots,z_n)$ for the difference of the limiting values of $F_n(z_1,\ldots,z_n)$ from the two sides of $z_{l+1}/z_l \in (0,\infty)$, so that $D_{l,n}$ is a function on the hypersurface $z_{l+1}/z_l \in (0,\infty)$ in $(\mathbb{C}^\times)^n$.

Then taking the difference of (6) on both sides of $z_{l+1}/z_l \in (0, \infty)$ gives

$$\mathrm{d}D_{l,n}(z_1, \ldots, z_n) = \sum_{k=1}^{l-1} F_k(z_1, \ldots, z_k) D_{l-k,n-k}(z_{k+1}, \ldots, z_n) \cdot \left[\frac{\mathrm{d}z_{k+1} + \cdots + \mathrm{d}z_n}{z_{k+1} + \cdots + z_n} - \frac{\mathrm{d}z_1 + \cdots + \mathrm{d}z_k}{z_1 + \cdots + z_k}\right] + \sum_{k=l+1}^{n-1} D_{l,k}(z_1, \ldots, z_k) F_{n-k}(z_{k+1}, \ldots, z_n) \cdot \left[\frac{\mathrm{d}z_{k+1} + \cdots + \mathrm{d}z_n}{z_{k+1} + \cdots + z_n} - \frac{\mathrm{d}z_1 + \cdots + \mathrm{d}z_k}{z_1 + \cdots + z_k}\right], \tag{7}$$

and also $D_{l,n}(z_1, \ldots, z_n)|_{z_1 + \cdots + z_n = 0} \equiv 0$. Comparing this with (6), we prove by induction that $D_{l,n}(z_1, \ldots, z_n) = F_{n-1}(z_1, \ldots, z_{l-1}, z_l + z_{l+1}, z_{l+2}, \ldots, z_n)$, where for the first case $D_{1,2}$ we use the explicit formulae for F_1, F_2 above.

5. Discussion

We have now shown that two *completely different* conditions on the f^α lead to the same unique family of functions F_n. That is, requiring the f^α to be holomorphic and continuous, plus some minor conditions, is equivalent to requiring the f^α to satisfy the p.d.e. (5). So (5) emerges from nowhere, as a consequence of the f^α being holomorphic and continuous.

Actually, we have used the triangulated category case to prove this. For the abelian case we only need F_n to be defined on H^n, not $(\mathbb{C}^\times)^n$, where $H = \{x + iy : x \in \mathbb{R}, y > 0\}$, and then the conditions are not strong enough to define the F_n uniquely.

6. The Calabi–Yau 3-fold case

In the Calabi–Yau 3-fold example above, as f^α : Stab($\mathcal{A}$) $\rightarrow \mathcal{L}^\alpha$ and $\mathcal{L}^\alpha = \mathbb{C} \cdot e^\alpha$, we can write $f^\alpha = F^\alpha e^\alpha$, for holomorphic functions F^α : Stab($\mathcal{A}$) $\rightarrow \mathbb{C}$ for $\alpha \in C(\mathcal{A})$. Equation (5) reduces to

$$\mathrm{d}F^\alpha(Z) = -\sum_{\beta,\gamma\in C(\mathcal{A}):\alpha=\beta+\gamma} \chi(\beta,\gamma) F^\beta(Z) F^\gamma(Z) \frac{\mathrm{d}Z(\beta)}{Z(\beta)}. \quad (8)$$

For the triangulated category $\mathcal{T}$ case we extend this from $\alpha, \beta, \gamma \in C(\mathcal{A})$ to $\alpha, \beta, \gamma \in K(\mathcal{T}) \setminus \{0\}$, with $F^{-\alpha} \equiv F^\alpha$.

Thus, we conjecture there should be holomorphic functions F^α : Stab($\mathcal{T}$) $\rightarrow \mathbb{C}$ that encode generalizations of Donaldson–Thomas invariants, satisfy (8) with $K(\mathcal{T}) \setminus \{0\}$ in place of $C(\mathcal{A})$, and give a flat $\mathcal{L}$-connection on Stab($\mathcal{T}$). What is the meaning of this in String Theory?