A quick introduction to stack functions

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I aim to explain parts of my papers [4, 5] on 'stack functions', which are central to the sequels [6–9] and my work with Yinan Song [10–12], and are basically the same as the 'Hall algebras' used by Kontsevich and Soibelman [13].

1 Introduction to Artin $\mathbb{K}$-stacks

Fix an algebraically closed field $\mathbb{K}$ throughout. There are four main classes of 'spaces' over $\mathbb{K}$ used in algebraic geometry, in increasing order of generality:

$$\mathbb{K}\text{-varieties} \subset \mathbb{K}\text{-schemes} \subset \text{algebraic } \mathbb{K}\text{-spaces} \subset \text{algebraic } \mathbb{K}\text{-stacks}.$$

Algebraic stacks (also known as Artin stacks) were introduced by Artin, generalizing *Deligne–Mumford stacks*. For a good introduction to algebraic stacks see Gómez [3], and for a thorough treatment see Laumon and Moret-Bailly [14]. We make the convention that all algebraic $\mathbb{K}$-stacks in this paper are *locally of finite type*, and $\mathbb{K}$-substacks are *locally closed*.

Algebraic $\mathbb{K}$-stacks form a 2-*category*. That is, we have *objects* which are $\mathbb{K}$-stacks $\mathfrak{F}, \mathfrak{G}$, and also two kinds of morphisms, 1-*morphisms* $\phi, \psi : \mathfrak{F} \to \mathfrak{G}$ between $\mathbb{K}$-stacks, and 2-*morphisms* $A : \phi \to \psi$ between 1-morphisms. An analogy to keep in mind is a 2-category of categories, where objects are categories, 1-morphisms are functors between the categories, and 2-morphisms are isomorphisms (natural transformations) between functors.

We define the set of $\mathbb{K}$-*points* of a stack.

Definition 1.1. Let $\mathfrak{F}$ be a $\mathbb{K}$-stack. Write $\mathfrak{F}(\mathbb{K})$ for the set of 2-isomorphism classes $[x]$ of 1-morphisms $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$. Elements of $\mathfrak{F}(\mathbb{K})$ are called $\mathbb{K}$-*points*, or *geometric points*, of $\mathfrak{F}$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism then composition with ϕ induces a map of sets $\phi_* : \mathfrak{F}(\mathbb{K}) \to \mathfrak{G}(\mathbb{K})$.

For a 1-morphism $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$, the *stabilizer group* $\operatorname{Iso}_{\mathbb{K}}(x)$ is the group of 2-morphisms $x \to x$. When $\mathfrak{F}$ is an algebraic $\mathbb{K}$-stack, $\operatorname{Iso}_{\mathbb{K}}(x)$ is an *algebraic* $\mathbb{K}$-*group*. We say that $\mathfrak{F}$ *has affine geometric stabilizers* if $\operatorname{Iso}_{\mathbb{K}}(x)$ is an affine algebraic $\mathbb{K}$-group for all 1-morphisms $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$.

As an algebraic $\mathbb{K}$-group up to isomorphism, $\operatorname{Iso}_{\mathbb{K}}(x)$ depends only on the isomorphism class $[x] \in \mathfrak{F}(\mathbb{K})$ of x in $\operatorname{Hom}(\operatorname{Spec}\mathbb{K}, \mathfrak{F})$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism, composition induces a morphism of algebraic $\mathbb{K}$-groups $\phi_* : \operatorname{Iso}_{\mathbb{K}}([x]) \to \operatorname{Iso}_{\mathbb{K}}\big(\phi_*([x])\big)$, for $[x] \in \mathfrak{F}(\mathbb{K})$.

One important difference in working with 2-categories rather than ordinary categories is that in diagram-chasing one only requires 1-morphisms to be 2-*isomorphic* rather than *equal*. The simplest kind of *commutative diagram* is:

$$\begin{array}{ccccc} & & \mathfrak{G} & & \\ & \overset{\phi}{\nearrow} & \Downarrow F & \overset{\psi}{\searrow} & \\ \mathfrak{F} & & \xrightarrow[\chi]{\hspace{3em}} & & \mathfrak{H}, \end{array}$$

by which we mean that $\mathfrak{F},\mathfrak{G},\mathfrak{H}$ are $\mathbb{K}$-stacks, ϕ,ψ,χ are 1-morphisms, and $F : \psi\circ\phi \rightarrow \chi$ is a 2-isomorphism. Usually we omit F, and mean that $\psi\circ\phi \cong \chi$.

Definition 1.2. Let $\phi : \mathfrak{F} \rightarrow \mathfrak{H}$, $\psi : \mathfrak{G} \rightarrow \mathfrak{H}$ be 1-morphisms of $\mathbb{K}$-stacks. Then one can define the *fibre product stack* $\mathfrak{F} \times_{\phi,\mathfrak{H},\psi} \mathfrak{G}$, or $\mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G}$ for short, with 1-morphisms $\pi_{\mathfrak{F}}, \pi_{\mathfrak{G}}$ fitting into a commutative diagram:

$$\begin{array}{ccccc} & \overset{\pi_{\mathfrak{F}}}{\nearrow} & \mathfrak{F} & \overset{\phi}{\searrow} & \\ \mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G} & & \Downarrow & & \mathfrak{H}. \\ & \underset{\pi_{\mathfrak{G}}}{\searrow} & \mathfrak{G} & \underset{\psi}{\nearrow} & \end{array} \tag{1}$$

A commutative diagram

$$\begin{array}{ccccc} & \overset{\theta}{\nearrow} & \mathfrak{F} & \overset{\phi}{\searrow} & \\ \mathfrak{E} & & \Downarrow & & \mathfrak{H} \\ & \underset{\eta}{\searrow} & \mathfrak{G} & \underset{\psi}{\nearrow} & \end{array}$$

is a *Cartesian square* if it is isomorphic to (1), so there is a 1-isomorphism $\mathfrak{E} \cong \mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G}$. Cartesian squares may also be characterized by a universal property.

2 Constructible functions on stacks

Next we discuss *constructible functions* on $\mathbb{K}$-stacks, following [4]. For this section we need $\mathbb{K}$ to have *characteristic zero*.

Definition 2.1. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack. We call $C \subseteq \mathfrak{F}(\mathbb{K})$ *constructible* if $C = \bigcup_{i\in I} \mathfrak{F}_i(\mathbb{K})$, where $\{\mathfrak{F}_i : i \in I\}$ is a finite collection of finite type algebraic $\mathbb{K}$-substacks $\mathfrak{F}_i$ of $\mathfrak{F}$. We call $S \subseteq \mathfrak{F}(\mathbb{K})$ *locally constructible* if $S \cap C$ is constructible for all constructible $C \subseteq \mathfrak{F}(\mathbb{K})$.

A function $f : \mathfrak{F}(\mathbb{K}) \rightarrow \mathbb{Q}$ is called *constructible* if $f(\mathfrak{F}(\mathbb{K}))$ is finite and $f^{-1}(c)$ is a constructible set in $\mathfrak{F}(\mathbb{K})$ for each $c \in f(\mathfrak{F}(\mathbb{K})) \setminus \{0\}$. A function $f : \mathfrak{F}(\mathbb{K}) \rightarrow \mathbb{Q}$ is called *locally constructible* if $f \cdot \delta_C$ is constructible for all constructible $C \subseteq \mathfrak{F}(\mathbb{K})$, where δ_C is the characteristic function of C. Write $\mathrm{CF}(\mathfrak{F})$ and $\mathrm{LCF}(\mathfrak{F})$ for the $\mathbb{Q}$-vector spaces of $\mathbb{Q}$-valued constructible and locally constructible functions on $\mathfrak{F}$.

Following [4, Def.s 4.8, 5.1 & 5.5] we define *pushforwards* and *pullbacks* of constructible functions along 1-morphisms.

Definition 2.2. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $C \subseteq \mathfrak{F}(\mathbb{K})$ be constructible. Then [4, Def. 4.8] defines the *naïve Euler* *characteristic* $\chi^{\rm na}(C)$ of C. It is called *naïve* as it takes no account of stabilizer groups. For $f\in{\rm CF}(\mathfrak F)$, define $\chi^{\rm na}(\mathfrak F,f)$ in $\mathbb Q$ by

$$\chi^{\rm na}(\mathfrak F,f)=\textstyle\sum_{c\in f(\mathfrak F(\mathbb K))\setminus\{0\}}c\,\chi^{\rm na}\bigl(f^{-1}(c)\bigr).$$

Let $\mathfrak F,\mathfrak G$ be algebraic $\mathbb K$-stacks with affine geometric stabilizers, and $\phi:\mathfrak F\rightarrow\mathfrak G$ a representable 1-morphism. Then for any $x\in\mathfrak F(\mathbb K)$ we have an injective morphism $\phi_*:{\rm Iso}_{\mathbb K}(x)\rightarrow{\rm Iso}_{\mathbb K}\bigl(\phi_*(x)\bigr)$ of affine algebraic $\mathbb K$-groups. The image $\phi_*\bigl({\rm Iso}_{\mathbb K}(x)\bigr)$ is an affine algebraic $\mathbb K$-group closed in ${\rm Iso}_{\mathbb K}\bigl(\phi_*(x)\bigr)$, so the quotient ${\rm Iso}_{\mathbb K}\bigl(\phi_*(x)\bigr)/\phi_*\bigl({\rm Iso}_{\mathbb K}(x)\bigr)$ exists as a quasiprojective $\mathbb K$-variety. Define a function $m_\phi:\mathfrak F(\mathbb K)\rightarrow\mathbb Z$ by $m_\phi(x)=\chi\bigl({\rm Iso}_{\mathbb K}(\phi_*(x))/\phi_*({\rm Iso}_{\mathbb K}(x))\bigr)$ for $x\in\mathfrak F(\mathbb K)$.

For $f\in{\rm CF}(\mathfrak F)$, define ${\rm CF}^{\rm stk}(\phi)f:\mathfrak G(\mathbb K)\rightarrow\mathbb Q$ by

$${\rm CF}^{\rm stk}(\phi)f(y)=\chi^{\rm na}\bigl(\mathfrak F,m_\phi\cdot f\cdot\delta_{\phi_*^{-1}(y)}\bigr)\quad\text{for $y\in\mathfrak G(\mathbb K)$,}$$

where $\delta_{\phi_*^{-1}(y)}$ is the characteristic function of $\phi_*^{-1}(\{y\})\subseteq\mathfrak G(\mathbb K)$ on $\mathfrak G(\mathbb K)$. Then ${\rm CF}^{\rm stk}(\phi):{\rm CF}(\mathfrak F)\rightarrow{\rm CF}(\mathfrak G)$ is a $\mathbb Q$-linear map called the *stack pushforward*.

Let $\theta:\mathfrak F\rightarrow\mathfrak G$ be a finite type 1-morphism. If $C\subseteq\mathfrak G(\mathbb K)$ is constructible then so is $\theta_*^{-1}(C)\subseteq\mathfrak F(\mathbb K)$. It follows that if $f\in{\rm CF}(\mathfrak G)$ then $f\circ\theta_*$ lies in ${\rm CF}(\mathfrak F)$. Define the *pullback* $\theta^*:{\rm CF}(\mathfrak G)\rightarrow{\rm CF}(\mathfrak F)$ by $\theta^*(f)=f\circ\theta_*$. It is a linear map.

Here [4, Th.s 5.4, 5.6 & Def. 5.5] are some properties of these.

Theorem 2.3. *Let $\mathfrak E,\mathfrak F,\mathfrak G,\mathfrak H$ be algebraic $\mathbb K$-stacks with affine geometric stabilizers, and $\beta:\mathfrak F\rightarrow\mathfrak G$, $\gamma:\mathfrak G\rightarrow\mathfrak H$ be* 1*-morphisms. Then*

$$\begin{aligned}{\rm CF}^{\rm stk}(\gamma\circ\beta)&={\rm CF}^{\rm stk}(\gamma)\circ{\rm CF}^{\rm stk}(\beta):{\rm CF}(\mathfrak F)\rightarrow{\rm CF}(\mathfrak H),&&(2)\\ (\gamma\circ\beta)^*&=\beta^*\circ\gamma^*:{\rm CF}(\mathfrak H)\rightarrow{\rm CF}(\mathfrak F),&&(3)\end{aligned}$$

supposing β,γ representable in (2)*, and of finite type in* (3)*. If*

$$\begin{CD}\mathfrak E @>\eta>> \mathfrak G\\ @V\theta VV @VV\psi V\\ \mathfrak F @>\phi>> \mathfrak H\end{CD}\qquad\text{\it is a Cartesian square with η,ϕ representable and θ,ψ of finite type, then the following commutes:}\qquad\begin{CD}{\rm CF}(\mathfrak E) @>{\rm CF}^{\rm stk}(\eta)>> {\rm CF}(\mathfrak G)\\ @A\theta^*AA @AA\psi^*A\\ {\rm CF}(\mathfrak F) @>{\rm CF}^{\rm stk}(\phi)>> {\rm CF}(\mathfrak H).\end{CD}\qquad(4)$$

As discussed in [4, §3.3] for the $\mathbb K$-scheme case, equation (2) is *false* for algebraically closed fields $\mathbb K$ of characteristic $p>0$. This is my reason for restricting to $\mathbb K$ of characteristic zero in those parts of my papers dealing with constructible functions. In [4, §5.3] we extend Definition 2.2 and Theorem 2.3 to *locally constructible functions.*

3 Stack functions

Stack functions are a universal generalization of constructible functions introduced in [5, §3]. Here [5, Def. 3.1] is the basic definition. Throughout $\mathbb K$ is algebraically closed of arbitrary characteristic, except when we specify ${\rm char}\,\mathbb K=0$.

Definition 3.1. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers. Consider pairs $(\mathfrak{R},\rho)$, where $\mathfrak{R}$ is a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers and $\rho:\mathfrak{R}\rightarrow\mathfrak{F}$ is a 1-morphism. We call two pairs $(\mathfrak{R},\rho)$, $(\mathfrak{R}',\rho')$ *equivalent* if there exists a 1-isomorphism $\iota:\mathfrak{R}\rightarrow\mathfrak{R}'$ such that $\rho'\circ\iota$ and ρ are 2-isomorphic 1-morphisms $\mathfrak{R}\rightarrow\mathfrak{F}$. Write $[(\mathfrak{R},\rho)]$ for the equivalence class of $(\mathfrak{R},\rho)$. If $(\mathfrak{R},\rho)$ is such a pair and $\mathfrak{S}$ is a closed $\mathbb{K}$-substack of $\mathfrak{R}$ then $(\mathfrak{S},\rho|_{\mathfrak{S}})$, $(\mathfrak{R}\setminus\mathfrak{S},\rho|_{\mathfrak{R}\setminus\mathfrak{S}})$ are pairs of the same kind.

Define $\underline{\rm SF}(\mathfrak{F})$ to be the $\mathbb{Q}$-vector space generated by equivalence classes $[(\mathfrak{R},\rho)]$ as above, with for each closed $\mathbb{K}$-substack $\mathfrak{S}$ of $\mathfrak{R}$ a relation

$$[(\mathfrak{R},\rho)]=[(\mathfrak{S},\rho|_{\mathfrak{S}})]+[(\mathfrak{R}\setminus\mathfrak{S},\rho|_{\mathfrak{R}\setminus\mathfrak{S}})]. \tag{5}$$

Define $\mathrm{SF}(\mathfrak{F})$ to be the $\mathbb{Q}$-vector space generated by $[(\mathfrak{R},\rho)]$ with ρ *representable*, with the same relations (5). Then $\mathrm{SF}(\mathfrak{F})\subseteq\underline{\rm SF}(\mathfrak{F})$.

Elements of $\underline{\rm SF}(\mathfrak{F})$ will be called *stack functions*. In [5, Def. 3.2] we relate $\mathrm{CF}(\mathfrak{F})$ and $\mathrm{SF}(\mathfrak{F})$.

Definition 3.2. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers, and $C\subseteq\mathfrak{F}(\mathbb{K})$ be constructible. Then $C=\coprod_{i=1}^n\mathfrak{R}_i(\mathbb{K})$, for $\mathfrak{R}_1,\ldots,\mathfrak{R}_n$ finite type $\mathbb{K}$-substacks of $\mathfrak{F}$. Let $\rho_i:\mathfrak{R}_i\rightarrow\mathfrak{F}$ be the inclusion 1-morphism. Then $[(\mathfrak{R}_i,\rho_i)]\in\mathrm{SF}(\mathfrak{F})$. Define $\bar\delta_C=\sum_{i=1}^n[(\mathfrak{R}_i,\rho_i)]\in\mathrm{SF}(\mathfrak{F})$. We think of this stack function as the analogue of the characteristic function $\delta_C\in\mathrm{CF}(\mathfrak{F})$ of C. Define a $\mathbb{Q}$-linear map $\iota_{\mathfrak{F}}:\mathrm{CF}(\mathfrak{F})\rightarrow\mathrm{SF}(\mathfrak{F})$ by $\iota_{\mathfrak{F}}(f)=\sum_{0\ne c\in f(\mathfrak{F}(\mathbb{K}))}c\cdot\bar\delta_{f^{-1}(c)}$. For $\mathbb{K}$ of characteristic zero, define a $\mathbb{Q}$-linear map $\pi_{\mathfrak{F}}^{\rm stk}:\mathrm{SF}(\mathfrak{F})\rightarrow\mathrm{CF}(\mathfrak{F})$ by

$$\pi_{\mathfrak{F}}^{\rm stk}\bigl(\textstyle\sum_{i=1}^n c_i[(\mathfrak{R}_i,\rho_i)]\bigr)=\textstyle\sum_{i=1}^n c_i\,\mathrm{CF}^{\rm stk}(\rho_i)1_{\mathfrak{R}_i},$$

where $1_{\mathfrak{R}_i}$ is the function 1 in $\mathrm{CF}(\mathfrak{R}_i)$. Then [5, Prop. 3.3] shows $\pi_{\mathfrak{F}}^{\rm stk}\circ\iota_{\mathfrak{F}}$ is the identity on $\mathrm{CF}(\mathfrak{F})$. Thus, $\iota_{\mathfrak{F}}$ is injective and $\pi_{\mathfrak{F}}^{\rm stk}$ is surjective. In general $\iota_{\mathfrak{F}}$ is far from surjective, and $\underline{\rm SF},\mathrm{SF}(\mathfrak{F})$ are much larger than $\mathrm{CF}(\mathfrak{F})$.

All the operations of constructible functions in §2 extend to stack functions.

Definition 3.3. Define *multiplication* '$\cdot$' on $\underline{\rm SF}(\mathfrak{F})$ by

$$[(\mathfrak{R},\rho)]\cdot[(\mathfrak{S},\sigma)]=[(\mathfrak{R}\times_{\rho,\mathfrak{F},\sigma}\mathfrak{S},\rho\circ\pi_{\mathfrak{R}})]. \tag{6}$$

This extends to a $\mathbb{Q}$-bilinear product $\underline{\rm SF}(\mathfrak{F})\times\underline{\rm SF}(\mathfrak{F})\rightarrow\underline{\rm SF}(\mathfrak{F})$ which is commutative and associative, and $\mathrm{SF}(\mathfrak{F})$ is closed under '$\cdot$'. Let $\phi:\mathfrak{F}\rightarrow\mathfrak{G}$ be a 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. Define the *pushforward* $\phi_*:\underline{\rm SF}(\mathfrak{F})\rightarrow\underline{\rm SF}(\mathfrak{G})$ by

$$\phi_*:\textstyle\sum_{i=1}^m c_i[(\mathfrak{R}_i,\rho_i)]\longmapsto\textstyle\sum_{i=1}^m c_i[(\mathfrak{R}_i,\phi\circ\rho_i)]. \tag{7}$$

If ϕ is representable then ϕ_* maps $\mathrm{SF}(\mathfrak{F})\rightarrow\mathrm{SF}(\mathfrak{G})$. For ϕ of finite type, define *pullbacks* $\phi^*:\underline{\rm SF}(\mathfrak{G})\rightarrow\underline{\rm SF}(\mathfrak{F})$, $\phi^*:\mathrm{SF}(\mathfrak{G})\rightarrow\mathrm{SF}(\mathfrak{F})$ by

$$\phi^*:\textstyle\sum_{i=1}^m c_i[(\mathfrak{R}_i,\rho_i)]\longmapsto\textstyle\sum_{i=1}^m c_i[(\mathfrak{R}_i\times_{\rho_i,\mathfrak{G},\phi}\mathfrak{F},\pi_{\mathfrak{F}})]. \tag{8}$$

The *tensor product* $\otimes:\underline{\mathrm{SF}}(\mathfrak{F})\times\underline{\mathrm{SF}}(\mathfrak{G})\rightarrow\underline{\mathrm{SF}}(\mathfrak{F}\times\mathfrak{G})$ or $\mathrm{SF}(\mathfrak{F})\times\mathrm{SF}(\mathfrak{G})\rightarrow\mathrm{SF}(\mathfrak{F}\times\mathfrak{G})$ is

$$\bigl(\textstyle\sum_{i=1}^m c_i[(\mathfrak{R}_i,\rho_i)]\bigr)\otimes\bigl(\textstyle\sum_{j=1}^n d_j[(\mathfrak{S}_j,\sigma_j)]\bigr)=\textstyle\sum_{i,j} c_i d_j[(\mathfrak{R}_i\times\mathfrak{S}_j,\rho_i\times\sigma_j)]. \tag{9}$$

Here [5, Th. 3.5] is the analogue of Theorem 2.3.

Theorem 3.4. *Let $\mathfrak{E},\mathfrak{F},\mathfrak{G},\mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\beta:\mathfrak{F}\rightarrow\mathfrak{G}$, $\gamma:\mathfrak{G}\rightarrow\mathfrak{H}$ be 1-morphisms. Then*

$$\begin{aligned} &(\gamma\circ\beta)_*=\gamma_*\circ\beta_*:\underline{\mathrm{SF}}(\mathfrak{F})\rightarrow\underline{\mathrm{SF}}(\mathfrak{H}), && (\gamma\circ\beta)_*=\gamma_*\circ\beta_*:\mathrm{SF}(\mathfrak{F})\rightarrow\mathrm{SF}(\mathfrak{H}),\\ &(\gamma\circ\beta)^*=\beta^*\circ\gamma^*:\underline{\mathrm{SF}}(\mathfrak{H})\rightarrow\underline{\mathrm{SF}}(\mathfrak{F}), && (\gamma\circ\beta)^*=\beta^*\circ\gamma^*:\mathrm{SF}(\mathfrak{H})\rightarrow\mathrm{SF}(\mathfrak{F}),\end{aligned}$$

for β,γ representable in the second equation, and of finite type in the third and fourth. If $f,g\in\underline{\mathrm{SF}}(\mathfrak{G})$ and β is finite type then $\beta^*(f\cdot g)=\beta^*(f)\cdot\beta^*(g)$. If

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\eta} & \mathfrak{G}\\ \downarrow\theta & & \psi\downarrow\\ \mathfrak{F} & \xrightarrow{\phi} & \mathfrak{H}\end{array}$$

is a Cartesian square with θ,ψ of finite type, then the following commutes:

$$\begin{array}{ccc} \underline{\mathrm{SF}}(\mathfrak{E}) & \xrightarrow{\eta_*} & \underline{\mathrm{SF}}(\mathfrak{G})\\ \uparrow\theta^* & & \psi^*\uparrow\\ \underline{\mathrm{SF}}(\mathfrak{F}) & \xrightarrow{\phi_*} & \underline{\mathrm{SF}}(\mathfrak{H}).\end{array}$$

The same applies for $\mathrm{SF}(\mathfrak{E}),\ldots,\mathrm{SF}(\mathfrak{H})$ if η,ϕ are representable.

In [5, Prop. 3.7 & Th. 3.8] we relate pushforwards and pullbacks of stack and constructible functions using $\iota_{\mathfrak{F}},\pi^{\mathrm{stk}}_{\mathfrak{F}}$.

Theorem 3.5. *Let $\mathbb{K}$ have characteristic zero, $\mathfrak{F},\mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\phi:\mathfrak{F}\rightarrow\mathfrak{G}$ be a 1-morphism. Then*

(a) $\phi^*\circ\iota_{\mathfrak{G}}=\iota_{\mathfrak{F}}\circ\phi^*:\mathrm{CF}(\mathfrak{G})\rightarrow\mathrm{SF}(\mathfrak{F})$ *if ϕ is of finite type;*

(b) $\pi^{\mathrm{stk}}_{\mathfrak{G}}\circ\phi_*=\mathrm{CF}^{\mathrm{stk}}(\phi)\circ\pi^{\mathrm{stk}}_{\mathfrak{F}}:\mathrm{SF}(\mathfrak{F})\rightarrow\mathrm{CF}(\mathfrak{G})$ *if ϕ is representable; and*

(c) $\pi^{\mathrm{stk}}_{\mathfrak{F}}\circ\phi^*=\phi^*\circ\pi^{\mathrm{stk}}_{\mathfrak{G}}:\mathrm{SF}(\mathfrak{G})\rightarrow\mathrm{CF}(\mathfrak{F})$ *if ϕ is of finite type.*

In [5, §3] we extend all the material on $\underline{\mathrm{SF}},\mathrm{SF}(\mathfrak{F})$ to *local stack functions* $\underline{\mathrm{LSF}},\mathrm{LSF}(\mathfrak{F})$, the analogues of locally constructible functions. The main differences are in which 1-morphisms must be of finite type.

4 Motivic invariants of Artin stacks

In [5, §4] we extend *motivic* invariants of quasiprojective $\mathbb{K}$-varieties to Artin stacks. We need the following data, [5, Assumptions 4.1 & 6.1].

Assumption 4.1. Suppose Λ is a commutative $\mathbb{Q}$-algebra with identity 1, and

$$\Upsilon:\{\text{isomorphism classes } [X] \text{ of quasiprojective } \mathbb{K}\text{-varieties } X\}\longrightarrow\Lambda$$

a map for $\mathbb{K}$ an algebraically closed field, satisfying the following conditions:

(i) If $Y \subseteq X$ is a closed subvariety then $\Upsilon([X]) = \Upsilon([X \setminus Y]) + \Upsilon([Y])$;

(ii) If X, Y are quasiprojective $\mathbb{K}$-varieties then $\Upsilon([X \times Y]) = \Upsilon([X])\Upsilon([Y])$;

(iii) Write $\ell = \Upsilon([\mathbb{K}])$ in Λ, regarding $\mathbb{K}$ as a $\mathbb{K}$-variety, the affine line (not the point $\operatorname{Spec}\mathbb{K}$). Then ℓ and $\ell^k - 1$ for $k = 1, 2, \ldots$ are invertible in Λ.

Suppose Λ° is a $\mathbb{Q}$-subalgebra of Λ containing the image of Υ and the elements ℓ^{-1} and $(\ell^k + \ell^{k-1} + \cdots + 1)^{-1}$ for $k = 1, 2, \ldots$, but *not* containing $(\ell - 1)^{-1}$. Let Ω be a commutative $\mathbb{Q}$-algebra, and $\pi : \Lambda^\circ \to \Omega$ a surjective $\mathbb{Q}$-algebra morphism, such that $\pi(\ell) = 1$. Define

$$\Theta : \{\text{isomorphism classes } [X] \text{ of quasiprojective } \mathbb{K}\text{-varieties } X\} \longrightarrow \Omega$$

by $\Theta = \pi \circ \Upsilon$. Then $\Theta([\mathbb{K}]) = 1$.

We chose the notation 'ℓ' as in motivic integration $[\mathbb{K}]$ is called the *Tate motive* and written $\mathbb{L}$. We have $\Upsilon\bigl([\mathrm{GL}(m, \mathbb{K})]\bigr) = \ell^{m(m-1)/2} \prod_{k=1}^{m} (\ell^k - 1)$, so (iii) ensures $\Upsilon([\mathrm{GL}(m, \mathbb{K})])$ is invertible in Λ for all $m \geqslant 1$. Here [5, Ex.s 4.3 & 6.3] is an example of suitable $\Lambda, \Upsilon, \ldots$; more are given in [5, §4.1 & §6.1].

Example 4.2. Let $\mathbb{K}$ be an algebraically closed field. Define $\Lambda = \mathbb{Q}(z)$, the algebra of rational functions in z with coefficients in $\mathbb{Q}$. For any quasiprojective $\mathbb{K}$-variety X, let $\Upsilon([X]) = P(X; z)$ be the *virtual Poincaré polynomial* of X. This has a complicated definition in [5, Ex. 4.3] which we do not repeat, involving Deligne's weight filtration when $\operatorname{char}\mathbb{K} = 0$ and the action of the Frobenius on l-adic cohomology when $\operatorname{char}\mathbb{K} > 0$. If X is smooth and projective then $P(X; z)$ is the ordinary Poincaré polynomial $\sum_{k=0}^{2\dim X} b^k(X) z^k$, where $b^k(X)$ is the k^{th} Betti number in l-adic cohomology, for l coprime to $\operatorname{char}\mathbb{K}$. Also $\ell = P(\mathbb{K}; z) = z^2$.

Let Λ° be the subalgebra of $P(z)/Q(z)$ in Λ for which $z \pm 1$ do not divide $Q(z)$. Here are two possibilities for Ω, π. Assumption 4.1 holds in each case.

(a) Set $\Omega = \mathbb{Q}$ and $\pi : f(z) \mapsto f(-1)$. Then $\Theta([X]) = \pi \circ \Upsilon([X])$ is the *Euler characteristic* of X.

(b) Set $\Omega = \mathbb{Q}$ and $\pi : f(z) \mapsto f(1)$. Then $\Theta([X]) = \pi \circ \Upsilon([X])$ is the *sum of the virtual Betti numbers* of X.

We need a few facts about *algebraic* $\mathbb{K}$*-groups*. A good reference is Borel [1]. Following Borel, we define a $\mathbb{K}$-*variety* to be a $\mathbb{K}$-scheme which is reduced, separated, and of finite type, but *not* necessarily irreducible. An algebraic $\mathbb{K}$-group is then a $\mathbb{K}$-variety G with identity $1 \in G$, multiplication $\mu : G \times G \to G$ and inverse $i : G \to G$ (as morphisms of $\mathbb{K}$-varieties) satisfying the usual group axioms. We call G *affine* if it is an affine $\mathbb{K}$-variety. *Special* $\mathbb{K}$-groups are studied by Serre and Grothendieck in [2, §§1, 5].

Definition 4.3. An algebraic $\mathbb{K}$-group G is called *special* if every principal G-bundle is Zariski locally trivial. Properties of special $\mathbb{K}$-groups can be found in [2, §§1.4, 1.5 & 5.5] and [5, §2.1]. In [5, Lem. 4.6] we show that if Assumption 4.1 holds and G is special then $\Upsilon([G])$ is invertible in Λ.

In [5, Th. 4.9] we extend Υ to Artin stacks, using Definition 4.3.

Theorem 4.4. *Let Assumption 4.1 hold. Then there exists a unique morphism of $\mathbb{Q}$-algebras $\Upsilon' : \underline{\mathrm{SF}}(\operatorname{Spec}\mathbb{K}) \to \Lambda$ such that if G is a special algebraic $\mathbb{K}$-group acting on a quasiprojective $\mathbb{K}$-variety X then $\Upsilon'\bigl([[X/G]]\bigr) = \Upsilon([X])/\Upsilon([G])$.*

Thus, if $\mathfrak{R}$ is a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers the theorem defines $\Upsilon'([\mathfrak{R}]) \in \Lambda$. Taking Λ, Υ as in Example 4.2 yields the *virtual Poincaré function* $P(\mathfrak{R}; z) = \Upsilon'([\mathfrak{R}])$ of $\mathfrak{R}$, a natural extension of virtual Poincaré polynomials to stacks. Clearly, Theorem 4.4 only makes sense if $\Upsilon([G])^{-1}$ exists for all special $\mathbb{K}$-groups G. This excludes the Euler characteristic $\Upsilon = \chi$, for instance, since $\chi([\mathbb{K}^\times]) = 0$ is not invertible. We overcome this in [5, §6] by defining a finer extension of Υ to stacks that keeps track of maximal tori of stabilizer groups, and allows $\Upsilon = \chi$. This can then be used with Θ, Ω in Assumption 4.1.

5 Stack functions over motivic invariants

In [5, §4–§6] we integrate the stack functions of §3 with the motivic invariant ideas of §4 to define more stack function spaces.

Definition 5.1. Let Assumption 4.1 hold, and $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers. Consider pairs $(\mathfrak{R}, \rho)$, with equivalence, as in Definition 3.1. Define $\underline{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda)$ to be the Λ-module generated by equivalence classes $[(\mathfrak{R}, \rho)]$, with the following relations:

- (i) Given $[(\mathfrak{R}, \rho)]$ as above and $\mathfrak{S}$ a closed $\mathbb{K}$-substack of $\mathfrak{R}$ we have $[(\mathfrak{R}, \rho)] = [(\mathfrak{S}, \rho|_{\mathfrak{S}})] + [(\mathfrak{R} \setminus \mathfrak{S}, \rho|_{\mathfrak{R}\setminus\mathfrak{S}})]$, as in (5).
- (ii) Let $\mathfrak{R}$ be a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers, U a quasiprojective $\mathbb{K}$-variety, $\pi_{\mathfrak{R}} : \mathfrak{R} \times U \to \mathfrak{R}$ the natural projection, and $\rho : \mathfrak{R} \to \mathfrak{F}$ a 1-morphism. Then $[(\mathfrak{R} \times U, \rho \circ \pi_{\mathfrak{R}})] = \Upsilon([U])[(\mathfrak{R}, \rho)]$.
- (iii) Given $[(\mathfrak{R}, \rho)]$ as above and a 1-isomorphism $\mathfrak{R} \cong [X/G]$ for X a quasiprojective $\mathbb{K}$-variety and G a special algebraic $\mathbb{K}$-group acting on X, we have $[(\mathfrak{R}, \rho)] = \Upsilon([G])^{-1}[(X, \rho \circ \pi)]$, where $\pi : X \to \mathfrak{R} \cong [X/G]$ is the natural projection 1-morphism.

Define a $\mathbb{Q}$-linear projection $\Pi^{\Upsilon,\Lambda}_{\mathfrak{F}} : \underline{\mathrm{SF}}(\mathfrak{F}) \to \underline{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda)$ by

$$\Pi^{\Upsilon,\Lambda}_{\mathfrak{F}} : \textstyle\sum_{i\in I} c_i[(\mathfrak{R}_i, \rho_i)] \longmapsto \sum_{i\in I} c_i[(\mathfrak{R}_i, \rho_i)],$$

using the embedding $\mathbb{Q} \subseteq \Lambda$ to regard $c_i \in \mathbb{Q}$ as an element of Λ.

We also define variants of these: spaces $\underline{\bar{\mathrm{SF}}}, \bar{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda)$, $\underline{\bar{\mathrm{SF}}}, \bar{\mathrm{SF}}(\mathfrak{F}, \Upsilon, \Lambda^\circ)$ and $\underline{\bar{\mathrm{SF}}}, \bar{\mathrm{SF}}(\mathfrak{F}, \Theta, \Omega)$, which are the Λ, Λ°- and Ω-modules respectively generated by $[(\mathfrak{R}, \rho)]$ as above, with ρ representable for $\bar{\mathrm{SF}}(\mathfrak{F}, *, *)$, and with relations (i),(ii) above but (iii) replaced by a finer, more complicated relation [5, Def. 5.17(iii)]. There are natural projections $\Pi^{\Upsilon,\Lambda}_{\mathfrak{F}}, \bar\Pi^{\Upsilon,\Lambda}_{\mathfrak{F}}, \bar\Pi^{\Upsilon,\Lambda^\circ}_{\mathfrak{F}}, \bar\Pi^{\Theta,\Omega}_{\mathfrak{F}}$ between various of the spaces. We can also define *local stack function* spaces $\underline{\mathrm{LSF}}, \underline{\mathrm{L\bar{S}F}}, \mathrm{L\bar{S}F}(\mathfrak{F}, *, *)$.

In [5] we give analogues of Definitions 3.2 and 3.3 and Theorems 3.4 and 3.5 for these spaces. For the analogue of $\pi_{\mathfrak{F}}^{\rm stk}$, suppose $\mathrm{X}:\Lambda^\circ\rightarrow\mathbb{Q}$ or $\mathrm{X}:\Omega\rightarrow\mathbb{Q}$ is an algebra morphism with $\mathrm{X}\circ\Upsilon([U])=\chi([U])$ or $\mathrm{X}\circ\Theta([U])=\chi([U])$ for varieties U, where χ is the Euler characteristic. Define $\bar\pi_{\mathfrak{F}}^{\rm stk}:\bar{\rm SF}(\mathfrak{F},\Upsilon,\Lambda^\circ)\rightarrow{\rm CF}(\mathfrak{F})$ or $\bar\pi_{\mathfrak{F}}^{\rm stk}:\bar{\rm SF}(\mathfrak{F},\Theta,\Omega)\rightarrow{\rm CF}(\mathfrak{F})$ by

$$\bar\pi_{\mathfrak{F}}^{\rm stk}\bigl(\textstyle\sum_{i=1}^n c_i[(\mathfrak{R}_i,\rho_i)]\bigr)=\textstyle\sum_{i=1}^n \mathrm{X}(c_i)\,{\rm CF}^{\rm stk}(\rho_i)1_{\mathfrak{R}_i}.$$

The operations '$\cdot$', $\phi_*,\phi^*,\otimes$ on $\underline{\rm SF}(*,\Upsilon,\Lambda),\ldots,\bar{\rm SF}(*,\Theta,\Omega)$ are given by the same formulae. The important point is that (6)–(9) are compatible with the relations defining $\underline{\rm SF}(*,\Upsilon,\Lambda),\ldots,\bar{\rm SF}(*,\Theta,\Omega)$, or they would not be well-defined.

In [5, Prop. 4.14] we identify $\underline{\rm SF}(\operatorname{Spec}\mathbb{K},\Upsilon,\Lambda)$. The proof involves showing that Υ' in Theorem 4.4 is compatible with Definition 5.1(i)–(iii) and so descends to $\Upsilon':\underline{\rm SF}(\operatorname{Spec}\mathbb{K},\Upsilon,\Lambda)\rightarrow\Lambda$, which is the inverse of i_Λ.

Proposition 5.2. *The map* $i_\Lambda:\Lambda\rightarrow\underline{\rm SF}(\operatorname{Spec}\mathbb{K},\Upsilon,\Lambda)$ *taking* $i_\Lambda:c\mapsto c[\operatorname{Spec}\mathbb{K}]$ *is an isomorphism of algebras.*

Here [5, Prop.s 5.21 & 5.22] is a useful way of representing these spaces.

Proposition 5.3. $\underline{\bar{\rm SF}},\bar{\rm SF}(\mathfrak{F},\Upsilon,\Lambda)$*,* $\underline{\bar{\rm SF}},\bar{\rm SF}(\mathfrak{F},\Upsilon,\Lambda^\circ)$ *and* $\underline{\bar{\rm SF}},\bar{\rm SF}(\mathfrak{F},\Theta,\Omega)$ *are generated over* Λ,Λ° *and* Ω *respectively by elements* $[(U\times[\operatorname{Spec}\mathbb{K}/T],\rho)]$*, for* U *a quasiprojective* $\mathbb{K}$*-variety and* T *an algebraic* $\mathbb{K}$*-group isomorphic to* $(\mathbb{K}^\times)^k\times K$ *for* $k\geqslant 0$ *and* K *finite abelian.*

Suppose $\sum_{i\in I}c_i[(U_i\times[\operatorname{Spec}\mathbb{K}/T_i],\rho_i)]=0$ *in one of these spaces, where* I *is finite set,* $c_i\in\Lambda,\Lambda^\circ$ *or* Ω*,* U_i *is a quasiprojective* $\mathbb{K}$*-variety and* T_i *an algebraic* $\mathbb{K}$*-group isomorphic to* $(\mathbb{K}^\times)^{k_i}\times K_i$ *for* $k_i\geqslant 0$ *and* K_i *finite abelian, with* $T_i\not\cong T_j$ *for* $i\neq j$*. Then* $c_j[(U_j\times[\operatorname{Spec}\mathbb{K}/T_j],\rho_j)]=0$ *for all* $j\in I$*.*

In [5, §5.2] we define operators $\Pi^\mu,\Pi^{\rm vi}_n,\hat\Pi^\nu_{\mathfrak{F}}$ on $\underline{\rm SF}(\mathfrak{F}),\underline{\rm SF}(\mathfrak{F},*,*)$ (but not on $\underline{\rm SF}(\mathfrak{F},\Upsilon,\Lambda)$). Very roughly speaking, $\Pi^{\rm vi}_n$ projects $[(\mathfrak{R},\rho)]\in\underline{\rm SF}(\mathfrak{F})$ to $[(\mathfrak{R}_n,\rho)]$, where $\mathfrak{R}_n$ is the $\mathbb{K}$-substack of points $r\in\mathfrak{R}(\mathbb{K})$ whose stabilizer groups $\operatorname{Iso}_{\mathbb{K}}(r)$ have *rank* n, that is, maximal torus $(\mathbb{K}^\times)^n$.

Unfortunately, it is more complicated than this. The right notion is not the actual rank of stabilizer groups, but the *virtual rank*. This is a difficult idea which treats $r\in\mathfrak{R}(\mathbb{K})$ with nonabelian stabilizer group $G=\operatorname{Iso}_{\mathbb{K}}(r)$ as a linear combination of points with 'virtual ranks' in the range $\operatorname{rk}C(G)\leqslant n\leqslant\operatorname{rk}G$. Effectively this *abelianizes stabilizer groups*, that is, using virtual rank we can treat $\mathfrak{R}$ as though its stabilizer groups were all abelian, essentially tori $(\mathbb{K}^\times)^n$.

Here is a way to interpret the spaces of Definition 5.1, explained in [5]. In §2, pushforwards ${\rm CF}^{\rm stk}(\phi):{\rm CF}(\mathfrak{F})\rightarrow{\rm CF}(\mathfrak{G})$ are defined by 'integration' over the fibres of ϕ, using the Euler characteristic χ as measure. In the same way, given Λ,Υ as in Assumption 4.1 we could consider Λ-valued constructible functions ${\rm CF}(\mathfrak{F})_\Lambda$, and define a pushforward $\phi_*:{\rm CF}(\mathfrak{F})_\Lambda\rightarrow{\rm CF}(\mathfrak{G})_\Lambda$ by 'integration' using Υ as measure, instead of χ. But then $(\psi\circ\phi)_*=\psi_*\circ\phi_*$ may no longer hold, as this depends on properties of χ on non-Zariski-locally-trivial fibrations which are false for other Υ such as virtual Poincaré polynomials.

The space $\underline{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda)$ is very like $\mathrm{CF}(\mathfrak{F})_\Lambda$ with pushforwards ϕ_* defined using Υ, but satisfies $(\psi\circ\phi)_*=\psi_*\circ\phi_*$ and other useful functoriality properties. So we can use it as a substitute for $\mathrm{CF}(\mathfrak{F})$. The spaces $\bar{\underline{\mathrm{SF}}},\bar{\mathrm{SF}}(\mathfrak{F},*,*)$ are similar but also keep track of information on the maximal tori of stabilizer groups.

6 'Virtual rank' and projections Π^{vi}_n on $\underline{\mathrm{SF}}(\mathfrak{F})$

The most difficult part of [5] is [5, §5–§6], which discusses 'virtual rank' of stack functions, and defines projections Π^{vi}_n on $\underline{\mathrm{SF}}(\mathfrak{F})$ to stack functions of 'virtual rank n'. These are important in [4,6,6,7] to define a *Lie subalgebra* $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$ of the Ringel–Hall algebra $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$, of stack functions with 'virtual rank 1', that is, stack functions 'supported on virtual indecomposables'.

The reason the Π^{vi}_n are important in Ringel–Hall algebra questions is that they have a deep, nontrivial compatibility with multiplication $*$ in $\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$. This is difficult to state, but is (partially) explained in [7, §5.2–§5.3]. The simplest instance of it is that $\mathrm{SF}^{\mathrm{ind}}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})$, which is just $\Pi^{\mathrm{vi}}_1\big(\mathrm{SF}_{\mathrm{al}}(\mathfrak{Obj}_{\mathcal{A}})\big)$, is closed under the Lie bracket $[f,g]=f*g-g*f$.

The action of Π^{vi}_n on a stack function $[(\mathfrak{R},\rho)]$ depends on the *stabilizer groups* of $\mathfrak{R}$. Thus, they are truly a phenomenon to do with Artin stacks, and have no analogue in the world of schemes.

As motivation we first introduce 'real rank' projections Π^{re}_n, which project $[(\mathfrak{R},\rho)]$ to $[(\mathfrak{R}_n,\rho)]$, where $\mathfrak{R}_n$ is the substack of $\mathfrak{R}$ whose stabilizer groups have rank n (that is, the maximal torus of the stabilizer groups has dimension n). If all stabilizer groups of $\mathfrak{R}$ are abelian, then Π^{vi}_n coincides with Π^{re}_n on $[(\mathfrak{R},\rho)]$. But if $\mathfrak{R}$ has nonabelian stabilizer groups, then the Π^{vi}_n treat points of $\mathfrak{R}$ as if they were $\mathbb{Q}$-linear combinations of points with ranks.

6.1 Real rank and projections Π^{re}_n

We define a family of commuting projections $\Pi^{\mathrm{re}}_n:\underline{\mathrm{SF}}(\mathfrak{F})\rightarrow\underline{\mathrm{SF}}(\mathfrak{F})$ for $n=0,1,\ldots$ which project to the part of $\underline{\mathrm{SF}}(\mathfrak{F})$ spanned by $[(\mathfrak{R},\rho)]$ such that the stabilizer group $\mathrm{Aut}_{\mathbb{K}}(r)$ has rank n for all $r\in\mathfrak{R}(\mathbb{K})$. The superscript 're' is short for 'real', meaning that the Π^{re}_n decompose $\underline{\mathrm{SF}}(\mathfrak{F})$ by the real (actual) rank of stabilizer groups.

Definition 6.1. If $\mathfrak{R}$ is an algebraic $\mathbb{K}$-stack and $r\in\mathfrak{R}(\mathbb{K})$ then $\mathrm{Aut}_{\mathbb{K}}(r)$ is an algebraic $\mathbb{K}$-group, so the rank $\mathrm{rk}(\mathrm{Aut}_{\mathbb{K}}(r))$ is well-defined. There is a natural topology on $\mathfrak{R}(\mathbb{K})$, in which the open sets are $\mathfrak{U}(\mathbb{K})$ for open $\mathbb{K}$-substacks $\mathfrak{U}\subseteq\mathfrak{R}$. In this topology the function $r\mapsto\mathrm{rk}(\mathrm{Aut}_{\mathbb{K}}(r))$ is *upper semicontinuous.* Thus, there exist locally closed $\mathbb{K}$-substacks $\mathfrak{R}_n$ in $\mathfrak{R}$ for $n=0,1,\ldots$, such that $\mathfrak{R}(\mathbb{K})=\coprod_{n\geqslant 0}\mathfrak{R}_n(\mathbb{K})$, and $r\in\mathfrak{R}(\mathbb{K})$ has $\mathrm{rk}(\mathrm{Aut}_{\mathbb{K}}(r))=n$ if and only if $r\in\mathfrak{R}_n(\mathbb{K})$. If $\mathfrak{R}$ is of finite type then $\mathfrak{R}_n=\emptyset$ for $n\gg 0$.

Now let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers, and $\underline{\mathrm{SF}}(\mathfrak{F})$ be as in §3. Define $\mathbb{Q}$-linear maps $\Pi^{\mathrm{re}}_n:\underline{\mathrm{SF}}(\mathfrak{F})\rightarrow\underline{\mathrm{SF}}(\mathfrak{F})$ for $n=0,1,\ldots$ on the generators $[(\mathfrak{R},\rho)]$ of $\underline{\mathrm{SF}}(\mathfrak{F})$ by $\Pi^{\mathrm{re}}_n:[(\mathfrak{R},\rho)]\mapsto[(\mathfrak{R}_n,\rho|_{\mathfrak{R}_n})]$, for $\mathfrak{R}_n$ defined as above. If $\mathfrak{S}$ is a closed substack of $\mathfrak{R}$ it is easy to see that $\mathfrak{S}_n$ is a closed substack of $\mathfrak{R}_n$ and $(\mathfrak{R} \setminus \mathfrak{S})_n = \mathfrak{R}_n \setminus \mathfrak{S}_n$. Thus, Π_n^{re} is compatible with the relations (5) in $\underline{\mathrm{SF}}(\mathfrak{F})$, and is well-defined. If $\rho : \mathfrak{R} \to \mathfrak{F}$ is representable then so is $\rho|_{\mathfrak{R}_n}$, so the restriction to $\mathrm{SF}(\mathfrak{F})$ maps $\Pi_n^{\mathrm{re}} : \mathrm{SF}(\mathfrak{F}) \to \mathrm{SF}(\mathfrak{F})$.

Here are some easy properties of the Π_n^{re}:

Proposition 6.2. *In the situation above, we have:*

(i) $(\Pi_n^{\mathrm{re}})^2 = \Pi_n^{\mathrm{re}}$*, so that* Π_n^{re} *is a projection, and* $\Pi_m^{\mathrm{re}} \circ \Pi_n^{\mathrm{re}} = 0$ *for* $m \neq n$.

(ii) *For all* $f \in \underline{\mathrm{SF}}(\mathfrak{F})$ *we have* $f = \sum_{n \geqslant 0} \Pi_n^{\mathrm{re}}(f)$*, where the sum makes sense as* $\Pi_n^{\mathrm{re}}(f) = 0$ *for* $n \gg 0$.

(iii) *If* $\phi : \mathfrak{F} \to \mathfrak{G}$ *is a* 1*-morphism of algebraic* $\mathbb{K}$*-stacks with affine geometric stabilizers then* $\Pi_n^{\mathrm{re}} \circ \phi_* = \phi_* \circ \Pi_n^{\mathrm{re}} : \underline{\mathrm{SF}}(\mathfrak{F}) \to \underline{\mathrm{SF}}(\mathfrak{G})$.

(iv) *If* $f \in \underline{\mathrm{SF}}(\mathfrak{F})$, $g \in \underline{\mathrm{SF}}(\mathfrak{G})$ *then* $\Pi_n^{\mathrm{re}}(f \otimes g) = \sum_{m=0}^{n} \Pi_m^{\mathrm{re}}(f) \otimes \Pi_{n-m}^{\mathrm{re}}(g)$.

6.2 Operators Π^μ and projections Π_n^{vi}

Now we define some linear maps $\Pi^\mu : \underline{\mathrm{SF}}(\mathfrak{F}) \to \underline{\mathrm{SF}}(\mathfrak{F})$.

Definition 6.3. A *weight function* μ is a map

$$\mu : \bigl\{\mathbb{K}\text{-groups } (\mathbb{K}^\times)^k \times K,\ k \geqslant 0,\ K \text{ finite abelian, up to isomorphism}\bigr\} \to \mathbb{Q}.$$

For any algebraic $\mathbb{K}$-stack $\mathfrak{F}$ with affine geometric stabilizers, we will define a linear map $\Pi^\mu : \underline{\mathrm{SF}}(\mathfrak{F}) \to \underline{\mathrm{SF}}(\mathfrak{F})$. Now $\underline{\mathrm{SF}}(\mathfrak{F})$ is generated by elements $[(\mathfrak{R}, \rho)]$ with $\mathfrak{R}$ 1-isomorphic to a global quotient $[X/G]$, for X a quasiprojective $\mathbb{K}$-variety and G a special algebraic $\mathbb{K}$-group, with maximal torus T^G. Define $C_G(T^G) = \bigl\{\gamma \in G : \gamma\delta = \delta\gamma \ \forall \delta \in T^G\bigr\}$ to be the *centralizer* of T^G in G, and $N_G(T^G) = \bigl\{\gamma \in G : \gamma T^G = T^G \gamma\bigr\}$ to be the *normalizer* of T^G in G, and $W(G, T^G) = N_G(T^G)/C_G(T^G)$ to be the *Weyl group* of G. Then $W(G, T^G)$ acts on T^G by conjugation: if $w = \gamma C_G(T^G)$ for $\gamma \in N_G(T^G)$ then we define $w \cdot \delta = \gamma\delta\gamma^{-1}$ in T^G for all $\delta \in T^G$.

Every algebraic $\mathbb{K}$-subgroup L of T^G is of the form $(\mathbb{K}^\times)^k \times K$ for $k \geqslant 0$, K a finite abelian group. Let $\mathcal{S}(T^G)$ be the set of subsets of T^G defined by Boolean operations upon subgroups L of T^G. Given a weight function μ as above, define a *measure* $\mathrm{d}\mu : \mathcal{S}(T^G) \to \mathbb{Q}$ to be additive upon disjoint unions of sets in $\mathcal{S}(T^G)$, and to satisfy $\mathrm{d}\mu(L) = \mu(L)$ for all algebraic $\mathbb{K}$-subgroups L of T^G. Now define

$$\Pi^\mu\bigl([(\mathfrak{R}, \rho)]\bigr) = \int_{t \in T^G} \frac{|\{w \in W(G, T^G) : w \cdot t = t\}|}{|W(G, T^G)|} \bigl[\bigl([X^{\{t\}}/C_G(\{t\})], \rho \circ \iota^{\{t\}}\bigr)\bigr] \mathrm{d}\mu. \tag{10}$$

Here $X^{\{t\}}$ is the subvariety of X fixed by t, and $\iota^{\{t\}} : [X^{\{t\}}/C_G(\{t\})] \to [X/G]$ is the obvious 1-morphism of Artin stacks.

The point of this is that the integrand in (10), regarded as a function of $t \in T^G$, is a constructible function taking only finitely many values, and the level sets of the function lie in $\mathcal{S}(T^G)$, so they are measurable with respect to $\mathrm{d}\mu$, and the integral is well-defined.

As the integrand in (10) is invariant under the action of $W(G,T^G)$, we can simplify (10) by pushing the integration down to $T^G/W(G,T^G)$:

$$\Pi^\mu\bigl([(\mathfrak{R},\rho)]\bigr) = \int_{tW(G,T^G)\in T^G/W(G,T^G)} \bigl[\bigl([X^{\{t\}}/C_G(\{t\})],\rho\circ\iota^{\{t\}}\bigr)\bigr]\mathrm{d}\mu. \qquad (11)$$

If $\mathfrak{R}$ has abelian stabilizer groups, then $\Pi^\mu\bigl([(\mathfrak{R},\rho)]\bigr)$ simply weights each point r of $\mathfrak{R}$ by $\mu(\mathrm{Stab}_{\mathfrak{R}}(r))$. However, if $\mathfrak{R}$ has nonabelian stabilizer groups, then $\Pi^\mu\bigl([(\mathfrak{R},\rho)]\bigr)$ replaces each point r with stabilizer group G by a $\mathbb{Q}$-linear combination of points with stabilizer groups $C_G(\{t\})$ for $t\in T^G$, where the $\mathbb{Q}$-coefficients depend on the values of μ on subgroups of T^G.

Then [5, Th.s 5.11 & 5.12] shows:

Theorem 6.4. *In the situation above, $\Pi^\mu\bigl([(\mathfrak{R},\rho)]\bigr)$ is independent of the choices of X,G,T^G and 1-isomorphism $\mathfrak{R}\cong[X/G]$, and Π^μ extends to unique linear maps $\Pi^\mu:\underline{\mathrm{SF}}(\mathfrak{F})\to\underline{\mathrm{SF}}(\mathfrak{F})$ and $\Pi^\mu:\mathrm{SF}(\mathfrak{F})\to\mathrm{SF}(\mathfrak{F})$.*

Theorem 6.5. (a) *Π^1 defined using $\mu\equiv 1$ is the identity on $\underline{\mathrm{SF}}(\mathfrak{F})$.*

(b) *If $\phi:\mathfrak{F}\to\mathfrak{G}$ is a 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers then $\Pi^\mu\circ\phi_*=\phi_*\circ\Pi^\mu:\underline{\mathrm{SF}}(\mathfrak{F})\to\underline{\mathrm{SF}}(\mathfrak{G})$.*

(c) *If μ_1,μ_2 are weight functions as in Definition 6.3 then $\mu_1\mu_2$ is also a weight function and $\Pi^{\mu_2}\circ\Pi^{\mu_1}=\Pi^{\mu_1}\circ\Pi^{\mu_2}=\Pi^{\mu_1\mu_2}$.*

Definition 6.6. For $n\geqslant 0$, define Π_n^{vi} to be the operator Π^{μ_n} defined with weight μ_n given by $\mu_n([H])=1$ if $\dim H=n$ and $\mu_n([H])=0$ otherwise, for all $\mathbb{K}$-groups $H\cong(\mathbb{K}^\times)^k\times K$ with K a finite abelian group.

The analogue of Proposition 6.2 holds for the Π_n^{vi}. Note that (i) follows from Theorem 6.5(c), (ii) from Theorem 6.5(a), and (iii) from Theorem 6.5(b).

Proposition 6.7. *In the situation above, we have:*

(i) *$(\Pi_n^{\mathrm{vi}})^2=\Pi_n^{\mathrm{vi}}$, so that Π_n^{vi} is a projection, and $\Pi_m^{\mathrm{vi}}\circ\Pi_n^{\mathrm{vi}}=0$ for $m\neq n$.*

(ii) *For all $f\in\underline{\mathrm{SF}}(\mathfrak{F})$ we have $f=\sum_{n\geqslant 0}\Pi_n^{\mathrm{vi}}(f)$, where the sum makes sense as $\Pi_n^{\mathrm{vi}}(f)=0$ for $n\gg 0$.*

(iii) *If $\phi:\mathfrak{F}\to\mathfrak{G}$ is a 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers then $\Pi_n^{\mathrm{vi}}\circ\phi_*=\phi_*\circ\Pi_n^{\mathrm{vi}}:\underline{\mathrm{SF}}(\mathfrak{F})\to\underline{\mathrm{SF}}(\mathfrak{G})$.*

(iv) *If $f\in\underline{\mathrm{SF}}(\mathfrak{F})$, $g\in\underline{\mathrm{SF}}(\mathfrak{G})$ then $\Pi_n^{\mathrm{vi}}(f\otimes g)=\sum_{m=0}^n\Pi_m^{\mathrm{vi}}(f)\otimes\Pi_{n-m}^{\mathrm{vi}}(g)$.*

Now the operators Π_n^{vi} do not make sense on the spaces $\underline{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda)$ of §5, because they are not compatible with the relation Definition 5.1(iii). So in [5, §5.3] we define stack function spaces $\bar{\underline{\mathrm{SF}}},\bar{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda)$ in which Definition 5.1(iii) is replaced by a finer, more complicated relation compatible with the Π_n^{vi}, so that Π_n^{vi} is well-defined on $\bar{\underline{\mathrm{SF}}}(\mathfrak{F},\Upsilon,\Lambda)$. These have a nice way of representing elements of $\bar{\underline{\mathrm{SF}}},\bar{\mathrm{SF}}(\mathfrak{F},\Upsilon,\Lambda)$:

Proposition 6.8. *$\underline{\bar{\rm SF}}(\mathfrak{F},\Upsilon,\Lambda)$ and $\bar{\rm SF}(\mathfrak{F},\Upsilon,\Lambda)$ are generated over Λ by elements $[(U\times[\operatorname{Spec}\mathbb{K}/T],\rho)]$, for U a quasiprojective $\mathbb{K}$-variety and T an algebraic $\mathbb{K}$-group isomorphic to $(\mathbb{K}^\times)^k\times K$ for $k\geqslant 0$ and K finite abelian.*

We can do Ringel–Hall algebras using the spaces $\bar{\rm SF}(\mathfrak{F},\Upsilon,\Lambda)$. They should be useful for studying algebra morphisms of Kontsevich–Soibelman type, and more generally, because Proposition 6.8 will allow us to assume that our stack functions all have stabilizer groups $(\mathbb{K}^\times)^k$. In effect, this is like being able to assume that all the sheaves on the Calabi–Yau 3-fold that you ever have to deal with are of the form $E_1\oplus\cdots\oplus E_k$ with E_i simple and $\operatorname{Hom}(E_i,E_j)=0$ for $i\neq j$, so that $\operatorname{Aut}(E_1\oplus\cdots\oplus E_k)=(\mathbb{K}^\times)^k$. So, we eliminate questions of how to deal with more complicated stabilizer groups of sheaves.

References

[1] A. Borel, *Linear Algebraic Groups*, second edition, Graduate Texts in Math. 126, Springer-Verlag, New York, 1991.

[2] *Anneaux de Chow et applications*, Séminaire C. Chevalley, 2e année, École Normale Supérieure, Secrétariat mathématique, Paris, 1958.

[3] T.L. Gómez, *Algebraic stacks*, Proc. Indian Acad. Sci. Math. Sci. 111 (2001), 1–31. math.AG/9911199.

[4] D.D. Joyce, *Constructible functions on Artin stacks*, J. London Math. Soc. 74 (2006), 583–606. math.AG/0403305.

[5] D.D. Joyce, *Motivic invariants of Artin stacks and 'stack functions'*, Quarterly Journal of Mathematics 58 (2007), 345-392. math.AG/0509722.

[6] D.D. Joyce, *Configurations in abelian categories. I. Basic properties and moduli stacks*, Advances in Mathematics 203 (2006), 194–255. math.AG/0312190.

[7] D.D. Joyce, *Configurations in abelian categories. II. Ringel–Hall algebras*, Advances in Math, 210 (2007), 635–706. math.AG/0503029.

[8] D.D. Joyce, *Configurations in abelian categories. III. Stability conditions and invariants*, math.AG/0410267, version 4, 2006.

[9] D.D. Joyce, *Configurations in abelian categories. IV. Invariants and changing stability conditions*, math.AG/0410268, version 4, 2006.

[10] D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants. I. An invariant counting stable pairs*, arXiv:0810.5645, 2008.

[11] D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants. II. Multiplicative identities for Behrend functions*, in preparation, 2008.

[12] D. Joyce and Y. Song, *A theory of generalized Donaldson–Thomas invariants. III. Invariants and transformation formulae*, in preparation, 2008.

[13] M. Kontsevich and Y. Soibelman, *Stability structures, motivic Donaldson–Thomas invariants and cluster transformations*, arXiv:0811.2435, 2008.

[14] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergeb. der Math. und ihrer Grenzgebiete 39, Springer-Verlag, Berlin, 2000.

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