Kuranishi (co)homology: a new tool in symplectic geometry. III. Effective Kuranishi (co)homology, integrality, and Kuranishi (co)bordism

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III.1. Introduction

In Lecture II, I explained how to define Kuranishi (co)homology KH_*, $KH^*(Y;R)$ for R a $\mathbb{Q}$-*algebra*. We have $KH_*(Y;R) \cong H_*^{\rm si}(Y;R)$, singular homology, and $KH^*(Y;R) \cong H^*_{\rm cs}(Y;R)$, compactly-supported cohomology.

We now discuss how to define Kuranishi (co)homology theories which work for R *any commutative ring*, such as $R=\mathbb{Z}$, not just $\mathbb{Q}$-algebras, and are isomorphic to $H_*^{\rm si}(Y;R)$, $H^*_{\rm cs}(Y;R)$. We call these *effective Kuranishi (co)homology* $KH_*^{\rm ef}(Y;R)$, $KH^*_{\rm ec}(Y;R)$.

These (co)homology theories will be useful for studying *integrality questions*, for instance, under what circumstances Gromov–Witten invariants can be defined in $H^*(M;\mathbb{Z})$ rather than $H^*(M;\mathbb{Q})$, and the Integrality Conjecture for Gopakumar–Vafa invariants.

However, there are disadvantages to working over $\mathbb{Z}$. Some good properties of Kuranishi (co)homology *can only work over a* $\mathbb{Q}$*-algebra*, so any theory which works over $\mathbb{Z}$ will not have them.

Features of Kuranishi (co)homology which cannot work over $R = \mathbb{Z}$

• Let X be a compact oriented Kuranishi space without boundary, and $\boldsymbol{f} : X \to Y$ strongly smooth. Then there exists gauge-fixing data $\boldsymbol{G}$ for $(X, \boldsymbol{f})$, and $[X, \boldsymbol{f}, \boldsymbol{G}]$ is a cycle in Kuranishi homology. The homology class $\bigl[[X, \boldsymbol{f}, \boldsymbol{G}]\bigr] \in KH_*(Y; \mathbb{Q})$ is identified with the *virtual class* of $(X, \boldsymbol{f})$.

If X has *nontrivial orbifold groups*, virtual classes are generally defined only over $\mathbb{Q}$, not $\mathbb{Z}$. So we cannot form $[[X, \boldsymbol{f}, \boldsymbol{G}]] \in KH_*^{\mathrm{ef}}(Y;\mathbb{Z})$.

Conclusion: in effective Kuranishi (co)homology, not all Kuranishi spaces are allowed as (co)chains; there must be restrictions on the orbifold groups and orbifold strata. So, can't use every curve moduli space as a (co)chain, there will be restrictions.

• Kuranishi cochains $KC^*(Y;R)$ have a cup product $\cup$ which is associative and supercommutative.

Now *Steenrod squares* are invariants in algebraic topology defined using the failure of the cup product for $H^*(Y;Z)$ to be supercommutative at the cochain level. They imply that it is not possible to define a cohomology theory computing $H^*(Y;\mathbb{Z})$ with a supercommutative cup product on cochains.

Conclusion: on effective Kuranishi cochains, the cup product cannot be supercommutative.

Parts of the proof of $KH_*(Y;R) \cong H_*^{\mathrm{si}}(Y;R)$ which require R a $\mathbb{Q}$-algebra

In proving $KH_*(Y;R) \cong H_*^{\mathrm{si}}(Y;R)$ we used $\mathbb{Q} \subseteq R$ in two different ways:

(a) Relation (iv) in $KC_*(Y;R)$ says that if Γ is a finite subgroup of $\mathrm{Aut}(X, \boldsymbol{f}, \boldsymbol{G})$ then

$$[X/\Gamma, \boldsymbol{\pi}_*(\boldsymbol{f}), \boldsymbol{\pi}_*(\boldsymbol{G})] = \frac{1}{|\Gamma|}[X, \boldsymbol{f}, \boldsymbol{G}].$$

This makes sense only if $1/|\Gamma| \in R$, so $\mathbb{Q} \subseteq R$.

We use (iv) like this: given an arbitrary chain $[X, \boldsymbol{f}, \boldsymbol{G}]$ we 'cut' X into small pieces X_c, $c \in C$ with $X_c = \acute{X}_c/\Gamma_c$, for $\acute{X}_c$ a Kuranishi space with *trivial stabilizers* (i.e. all orbifold groups are $\{1\}$). Then we replace $\sum_{c\in C}[X_c, \boldsymbol{f}_c, \boldsymbol{G}_c]$ by $\sum_{c\in C} 1/|\Gamma_c|[\acute{X}_c, \acute{\boldsymbol{f}}_c, \acute{\boldsymbol{G}}_c]$.

Conclusion: we can't use relation (iv) in effective Kuranishi (co)homology.

(b) At various points in the proof we have to modify chains $[X, \boldsymbol{f}, \boldsymbol{G}]$ to perturb them into manifolds, triangulate by simplices, etc. These modifications must be preserved by the symmetries $\mathrm{Aut}(X, \boldsymbol{f}, \boldsymbol{G})$. But this is not always possible with just one modification. So we choose an arbitrary modification, and then average over its images under $\mathrm{Aut}(X, \boldsymbol{f}, \boldsymbol{G})$. To average we divide by $|\mathrm{Aut}(X, \boldsymbol{f}, \boldsymbol{G})|$. Thus need $1/|\mathrm{Aut}(X, \boldsymbol{f}, \boldsymbol{G})| \in R$, so $\mathbb{Q} \subseteq R$.

Note: This was why we needed Aut(X, f, G) finite, so why we introduced gauge-fixing data. Also, this is the same reason Fukaya–Ono use *multisections*, not single-valued sections.

Conclusion. We need to ensure Aut$(X, f, G) = \{1\}$ for effective Kuranishi (co)chains, not just Aut(X, f, G) finite. (It is enough for this to hold for (X, f, G) 'connected'.)

III.2. Stabilizer groups and effective orbifolds

Let V be an orbifold. Then each $v \in V$ has *stabilizer group* or *orbifold group* $\mathrm{Stab}_V(v)$, a finite group, and V near v is locally modelled on $\mathbb{R}^n / \mathrm{Stab}_V(v)$ near 0, where $\mathrm{Stab}_V(v)$ acts linearly on $\mathbb{R}^n$, $n = \dim V$.

Note: we do not require $\mathrm{Stab}_V(v)$ to act *effectively* on $\mathbb{R}^n$. For instance, $\mathrm{Stab}_V(v)$ could act trivially on $\mathbb{R}^n$. So we cannot regard $\mathrm{Stab}_V(v)$ as a subgroup of $\mathrm{GL}(n, \mathbb{R})$.

We call an orbifold *effective* if $\mathrm{Stab}_V(v)$ acts effectively on $\mathbb{R}^n$ for all $v \in V$. Equivalently, an orbifold V is effective if generic points $v \in V$ have $\mathrm{Stab}_V(v) = \{1\}$.
For example, if Γ is a finite group then $\{0\}/\Gamma$ is a 0-dimensional orbifold, a single point with stabilizer group Γ, which is effective if and only if $\Gamma = \{1\}$.

Suppose V is a compact, oriented n-orbifold without boundary. Then we can form the *fundamental class* $[V]$ in singular homology. We have $[V] \in H_n(V;\mathbb{Z})$ if V is effective, but $[V] \in H_n(V;\mathbb{Q})$ if V is not effective. This is because when we triangulate V by simplices $\sigma : \Delta_n \to V$, if generic points in V have stabilizer Γ, then the simplices must be weighted by $\pm 1/|\Gamma|$. These weights lie in $\mathbb{Z}$ if V is effective, so $\Gamma = \{1\}$, and in $\mathbb{Q}$ otherwise.

Conclusion. To do homology over $\mathbb{Z}$, we need *effective* orbifolds.

III.3. Orbifold strata

If V is an orbifold, we may write $V = \coprod_{\Gamma} V^{\Gamma}$, where the disjoint union is over all isomorphism classes of finite groups Γ, and $V^{\Gamma} = \{v \in V : \mathrm{Stab}_V(v) \cong \Gamma\}$. This is called the *orbifold stratification* of V.

This definition of V^{Γ} is not very useful, for two reasons. Firstly, V^{Γ} is not closed in V. Secondly, V^{Γ} can be a union of manifolds of different dimensions. If $v \in V^{\Gamma}$ then V is modelled on $\mathbb{R}^n/\Gamma$ near V, and V^{Γ} is modelled on the fixed points $\mathrm{Fix}(\Gamma)$ of Γ in $\mathbb{R}^n$, which depends on the *representation* of Γ on $\mathbb{R}^n$.

So, make new definition of orbifold strata $V^{\Gamma,\rho}$, including a representation ρ of Γ.

Let Γ be a finite group, and W be a finite-dimensional representation of Γ (real, for now). Call W *trivial* if Γ acts trivially, and *nontrivial* if $\mathrm{Fix}(\Gamma) = \{0\}$. Then every representation W can be written uniquely as $W = W^{\mathrm{tr}} \oplus W^{\mathrm{nt}}$, a direct sum of a trivial and nontrivial representation.

Let ρ be an *isomorphism class of nontrivial representations* of Γ, and V an orbifold.
As a set, define the *orbifold stratum* $V^{\Gamma,\rho}$ to be

$$V^{\Gamma,\rho} = \{\mathsf{Stab}_V(v) \cdot (v, \lambda) : v \in V, \lambda : \Gamma \to \mathsf{Stab}_V(v) \text{ is an injective group morphism}, [(T_v V)^{\mathsf{nt}}] = \rho\},$$

where λ makes $T_v V$ into a Γ-representation, $T_v V = (T_v V)^{\mathsf{tr}} \oplus (T_v V)^{\mathsf{nt}}$ is its splitting into trivial and nontrivial representations, and $[(T_v V)^{\mathsf{nt}}]$ is its isomorphism class.

Define $\iota^{\Gamma,\rho} : V^{\Gamma,\rho} \to V$ by $\iota^{\Gamma,\rho} : \mathrm{Stab}_V(v) \cdot (v, \lambda) \mapsto v$.

Then we have:

Proposition. *$V^{\Gamma,\rho}$ has the structure of an orbifold, with $\dim V^{\Gamma,\rho} = \dim V - \dim \rho$, and $\iota^{\Gamma,\rho} : V^{\Gamma,\rho} \to V$ is a proper, finite immersion.*

Here $\iota^{\Gamma,\rho}$ *proper* implies $\iota^{\Gamma,\rho}(V^{\Gamma,\rho})$ is closed in V.

An orbifold V is *effective* iff $V^{\Gamma,\rho} = \emptyset$ unless ρ is an effective representation of Γ, for all Γ, ρ. So, can characterize effective orbifolds by their orbifold strata.

III.4. Orbifold strata of Kuranishi spaces

Let X be a Kuranishi space. We will define the *orbifold strata* $X^{\Gamma,\rho}$ of X. Let $p \in X$ and (V_p, E_p, s_p, ψ_p) be a Kuranishi neighbourhood of $p \in X$. Set $v = \psi_p^{-1}(p)$ in V_p. Then $\mathrm{Stab}_{V_p}(v)$ is a finite group with representations on the vector spaces $T_v V_p$ and $E_p|_v$.

We need to think of $T_v V_p \ominus E_p|_v$ as a formal difference of representations of $\mathrm{Stab}_{V_p}(v)$, that is, a *virtual representation*.

A *virtual vector space* $W_1 \ominus W_2$ is a formal difference of finite-dimensional vector spaces W_1, W_2. We call $W_1 \ominus W_2$ and $W_1' \ominus W_2'$ *equivalent* if $W_1 \oplus A \cong W_1' \oplus B$ and $W_2 \oplus A \cong W_2' \oplus B$ for some finite-dimensional vector spaces A, B.

Write $\mathrm{vdim}(W_1 \ominus W_2) = \dim W_1 - \dim W_2$.

If Γ is a finite group, *virtual* Γ-*representations* and *equivalence* are the same with Γ-representations, not vector spaces.

Equivalence classes of virtual Γ-representations lie in a lattice $\mathbb{Z}^l$, the Grothendieck group $K_0(\text{mod-}\Gamma)$.

Let ρ be an *equivalence class of virtual nontrivial* Γ*-representations*. Let X be a Kuranishi space. As a set, define

$$
\begin{aligned}
X^{\Gamma,\rho} = \{ & \mathrm{Stab}_X(p)\cdot(p,\lambda) : p \in X, \\
& \lambda:\Gamma\to\mathrm{Stab}_X(p) \text{ is an injective} \\
& \qquad\qquad \text{group morphism}, \\
& \qquad [(T_vV_p)^{\mathrm{nt}} \ominus (E_p|_v)^{\mathrm{nt}}] = \rho\},
\end{aligned}
$$

where $(V_p,\ldots,\psi_p)$ is a Kuranishi neighbourhood for p, and $v=\psi_p^{-1}(p)$ in V_p, and $\lambda:\Gamma\to\mathrm{Stab}_X(p)=\mathrm{Stab}_{V_p}(v)$ makes $T_vV_p, E_p|_v$ into Γ-representations, and $(T_vV_p)^{\mathrm{nt}}, (E_p|_v)^{\mathrm{nt}}$ are their nontrivial parts.

Define $\iota^{\Gamma,\rho} : X^{\Gamma,\rho} \to X$ by $\iota^{\Gamma,\rho}$: $\mathrm{Stab}_X(p)\cdot(p,\lambda) \mapsto p$.

Then we have:

Proposition. $X^{\Gamma,\rho}$ *has the structure of a Kuranishi space, with* $\mathrm{vdim}\, X^{\Gamma,\rho} = \mathrm{vdim}\, X - \mathrm{vdim}\, \rho$, *and* $\iota^{\Gamma,\rho}$ *lifts to a proper, finite, strongly smooth map* $\boldsymbol{\iota}^{\Gamma,\rho} : \boldsymbol{X}^{\Gamma,\rho} \to \boldsymbol{X}$.

To prove this, note that the condition $\left[(T_vV_p)^{\mathrm{nt}} \ominus (E_p|_v)^{\mathrm{nt}}\right] = \rho$ is preserved by coordinate changes $(\phi_{pq}, \hat\phi_{pq})$, as going from $(V_q,\ldots,\psi_q)$ to $(V_p,\ldots,\psi_p)$ adds the same Γ-representation to T_vV_q and $E_q|_v$.

Also note that as $\operatorname{vdim}\rho$ can be positive, negative or zero, can have $\operatorname{vdim} X^{\Gamma,\rho} < \operatorname{vdim} X$ or $\operatorname{vdim} X^{\Gamma,\rho} > \operatorname{vdim} X$ or $\operatorname{vdim} X^{\Gamma,\rho} = \operatorname{vdim} X$.

If X is a compact oriented Kuranishi space without boundary, $\boldsymbol{f} : X \rightarrow Y$ is strongly smooth, and $VC(X, \boldsymbol{f})$ is a virtual class for X in the homology of Y, can show that $VC(X, \boldsymbol{f}) \in H_*^{\mathrm{si}}(Y; \mathbb{Z})$ if $\operatorname{vdim} X^{\Gamma,\rho} \leqslant \operatorname{vdim} X - 2$ for all $\Gamma \neq \{1\}$ and ρ with $X^{\Gamma,\rho} \neq \emptyset$. So, integrality of virtual classes fails due to orbifold strata $X^{\Gamma,\rho}$ with $\operatorname{vdim} X^{\Gamma,\rho} > \operatorname{vdim} X - 2$.

III.5. Effective Kuranishi spaces

Let X be a Kuranishi space. We call X *effective* if for all $p \in X$, if (V_p, E_p, s_p, ψ_p) is a Kuranishi neighbourhood in the germ at p in X, and $v = \psi_p^{-1}(p)$ in V_p, then $\mathrm{Stab}_{V_p}(v)$ acts *effectively* on $T_v V_p$ and trivially on $E_p|_v$.

If $\lambda : \Gamma \to \mathrm{Stab}_X(p)$ is an injective group morphism, this implies that Γ acts effectively on $T_v V_p$ and trivially on $E_p|_v$. Hence $(T_v V_p)^{\mathrm{nt}}$ is an effective Γ-representation and $(E_p|_v)^{\mathrm{nt}} = 0$.

Thus $\rho = \big[(T_vV_p)^{\mathrm{nt}} \ominus (E_p|_v)^{\mathrm{nt}}\big]$ is the equivalence class of an effective Γ-representation, not a virtual representation. Hence, if X is an effective Kuranishi space then $X^{\Gamma,\rho} = \emptyset$ unless ρ is the equivalence class of an effective Γ-representation. If $\Gamma \neq \{1\}$ this implies $\dim \rho > 0$. If X is orientable we also exclude the case $\dim \rho = 1$. Therefore if $X^{\Gamma,\rho} \neq \emptyset$ and $\Gamma \neq \{1\}$ then $\mathrm{vdim}\, X^{\Gamma,\rho} \leqslant \mathrm{vdim}\, X - 2$. This was the condition to define virtual class for X over $\mathbb{Z}$. So, for effective Kuranishi spaces, can define virtual classes over $\mathbb{Z}$.

Another feature of effective Kuranishi spaces: if (V_p, E_p, s_p, ψ_p) is a Kuranishi neighbourhood near p on an effective Kuranishi space X, then the stabilizers of V_p act trivially on the fibres of E_p (at least near $v = \psi_p^{-1}(p)$). Hence, E_p is a vector bundle, not just an orbibundle. Let $\tilde{s}_p$ be a generic small perturbation of s_p. Then $\tilde{s}_p$ is *transverse*, and $(\tilde{s}_p)^{-1}(0)$ is an effective suborbifold of V_p. Therefore, an effective Kuranishi space X *can be perturbed to an effective orbifold* $\tilde{X}$*, by a single-valued perturbation*. Also effective orbifolds have virtual chains over $\mathbb{Z}$.

III.6. Effective Kuranishi homology

We now have the ingredients for $KH_*^{\mathrm{ef}}(Y;R)$. Let Y be an orbifold and R a commutative ring. We define *effective Kuranishi chains* $KC_k(Y;R)$ to be spanned over R by isomorphism classes $[X,\boldsymbol{f},\underline{\boldsymbol{G}}]$, where X is a compact, oriented, *effective* Kuranishi space, and $\boldsymbol{f}: X \to Y$ is strongly smooth, and $\underline{\boldsymbol{G}}$ is *effective gauge-fixing data* for $(X,\boldsymbol{f})$. This is like gauge-fixing data, but with stronger conditions that imply $\mathrm{Aut}(X,\boldsymbol{f},\underline{\boldsymbol{G}}) = \{1\}$ for $(X,\boldsymbol{f},\boldsymbol{G})$ *connected*. We impose relations (i)–(iii) of §II.4, but not (iv).

This gives a homology theory isomorphic to $H_*^{\mathrm{si}}(Y;R)$. Can also define *effective Kuranishi cohomology* $KH^*_{\mathrm{ec}}(Y;R)$, where the cochains $KC^*_{\mathrm{ec}}(Y;R)$ are spanned by $[X,\boldsymbol{f},\underline{C}]$ with $\boldsymbol{f}:X\rightarrow Y$ a cooriented, *coeffective* strong submersion. (*Coeffective* is a relative version of effective). Can prove Poincaré duality and $KH^*_{\mathrm{ec}}(Y;R)\cong H^*_{\mathrm{cs}}(Y;R)$ only for Y a *manifold*, as Poincaré duality over $\mathbb{Z}$ fails for orbifolds.

The cup product $\cup$ on $KC^*_{\mathrm{ec}}(Y;R)$ is associative, but not supercommutative, because products of effective co-gauge-fixing data are not commutative.

III.7. Classical bordism

Let Y be a manifold or orbifold, and R a commutative ring. Define the *classical bordism groups* $B_k(Y;R)$ for $k \in \mathbb{Z}$ to be the R-modules of finite R-linear combinations of isomorphism classes $[X,f]$ for X a compact, oriented k-manifold without boundary and $f : X \to Y$ a smooth map, with relations:

(i) $[X,f]+[X',f'] = [X \amalg X', f \amalg f']$ for all classes $[X,f],[X',f']$; and

(ii) let Z be a compact, oriented $(k+1)$-manifold with boundary but without corners, and $g : Z \to Y$ be smooth. Then $[\partial Z, g|_{\partial Z}] = 0$.

Then classical bordism groups are a *generalized homology theory*. Usually $B_k(Y;\mathbb{Z})$ is written $MSO_k(Y)$. There is a corresponding generalized cohomology theory called *cobordism*, written $MSO^k(Y)$. It has an algebraic topology definition in terms of limits of homotopy groups, but no good definition using differential geometry, as far as I know.

III.7. Kuranishi bordism

Let Y be an orbifold. Motivated by classical bordism, consider pairs $(X, \boldsymbol{f})$, where X is a compact oriented Kuranishi space without boundary or corners, and $\boldsymbol{f} : X \rightarrow Y$ is strongly smooth. An *isomorphism* between pairs $(X, \boldsymbol{f}), (\tilde{X}, \tilde{\boldsymbol{f}})$ is an orientation-preserving strong diffeomorphism $\boldsymbol{i} : X \rightarrow \tilde{X}$ with $\boldsymbol{f} = \tilde{\boldsymbol{f}} \circ \boldsymbol{i}$. Write $[X, \boldsymbol{f}]$ for the isomorphism class of $(X, \boldsymbol{f})$. Let R be a commutative ring and $k \in \mathbb{Z}$.

Define the *Kuranishi bordism group* $KB_k(Y;R)$ of Y to be the R-module of finite R-linear combinations of isomorphism classes $[X,\boldsymbol{f}]$ for which $\operatorname{vdim} X=k$, with the relations:

(i) $[X,\boldsymbol{f}]+[X',\boldsymbol{f}']=[X\amalg X',\boldsymbol{f}\amalg\boldsymbol{f}']$ for all classes $[X,\boldsymbol{f}],[X',\boldsymbol{f}']$; and

(ii) let W be a compact oriented Kuranishi space with boundary but without corners, with $\operatorname{vdim} W=k+1$, and $\boldsymbol{e}:W\rightarrow Y$ be strongly smooth. Then $[\partial W,\boldsymbol{e}|_{\partial Z}]=0$.

Define $\Pi_{\mathrm{bo}}^{\mathrm{Kb}}:B_k(Y;R)\rightarrow KB_k(Y;R)$ by $\Pi_{\mathrm{bo}}^{\mathrm{Kb}}:[X,f]\mapsto[X,f]$. Define $\Pi_{\mathrm{Kb}}^{\mathrm{Kh}}:KB_k(Y;R)\rightarrow KH_k(Y;R\otimes_{\mathbb{Z}}\mathbb{Q})$ by $\Pi_{\mathrm{Kb}}^{\mathrm{Kh}}:[X,\boldsymbol{f}]\mapsto\big[[X,\boldsymbol{f},\boldsymbol{G}]\big]$, where $\boldsymbol{G}$ is any gauge-fixing data for $(X,\boldsymbol{f})$.

III.8. Kuranishi cobordism

Similarly, following the definition of Kuranishi cohomology, consider pairs $(X, \boldsymbol{f})$, where X is a compact Kuranishi space without boundary or corners, and $\boldsymbol{f} : X \to Y$ is a cooriented strong submersion. Define the *Kuranishi cobordism group* $KB^k(Y;R)$ of Y to be the R-module generated by isomorphism classes $[X, \boldsymbol{f}]$ for which $\mathrm{vdim}\, X = \dim Y - k$, with relations (i),(ii) as above.

Define $\Pi_{\mathrm{Kcb}}^{\mathrm{Kch}} : KB^k(Y;R) \to KH^k(Y;R \otimes_{\mathbb{Z}} \mathbb{Q})$ by $\Pi_{\mathrm{Kcb}}^{\mathrm{Kch}} : [X, \boldsymbol{f}] \mapsto \big[[X, \boldsymbol{f}, \boldsymbol{C}]\big]$, where $\boldsymbol{C}$ is any co-gauge-fixing data for $(X, \boldsymbol{f})$.

As for Kuranishi (co)homology, can define cup and cap products on Kuranishi (co)bordism, pushforwards on bordism, pullbacks on cobordism – the whole homology/cohomology package.

Can also define other kinds of Kuranishi bordism. In particular, define *effective Kuranishi bordism* $KB_k^{\text{eb}}(Y;R)$ as for $KB_k(Y;R)$ but with X,W effective, and *effective Kuranishi cobordism* $KB^k_{\text{ecb}}(Y;R)$ as for $KB^k(Y;R)$ but with $\boldsymbol{f}: X \to Y$, $\boldsymbol{e}: W \to Y$ coeffective.

Then we have projections $\Pi^{\mathsf{ef}}_{\mathsf{eb}} : KB^{\mathsf{eb}}_k(Y;R) \to KH^{\mathsf{ef}}_k(Y;R)$ and $\Pi^{\mathsf{ec}}_{\mathsf{ecb}} : KB^k_{\mathsf{ecb}}(Y;R) \to KH^k_{\mathsf{ec}}(Y;R)$. Using the isomorphisms between (effective) Kuranishi homology and singular homology, we see that we have projections $KB_*(Y;\mathbb{Z}) \to H^{\mathsf{si}}_*(Y;\mathbb{Q})$ and $KB^{\mathsf{eb}}_*(Y;\mathbb{Z}) \to H^{\mathsf{si}}_*(Y;\mathbb{Z})$.

III.9. Projections $\Pi^{\Gamma,\rho}$

Let Γ be a finite group and ρ an isomorphism class of nontrivial virtual representations of Γ. Then for each Kuranishi space X we have an *orbifold stratum* $X^{\Gamma,\rho}$, with strongly smooth map $\boldsymbol{\iota}^{\Gamma,\rho} : X^{\Gamma,\rho} \to X$.

We would like to define a projection $\Pi^{\Gamma,\rho} : KB_k(Y;R) \to KB_{k-\dim\rho}(Y;R)$ by $\Pi^{\Gamma,\rho} : [X,\boldsymbol{f}] \mapsto [X^{\Gamma,\rho}, \boldsymbol{f}\circ\boldsymbol{\iota}^{\Gamma,\rho}]$.

There is one problem: we need to make an *orientation* on $X^{\Gamma,\rho}$ from the orientation on X. This is possible if $|\Gamma|$ is odd, and $\Pi^{\Gamma,\rho}$ is well-defined.

We can show that the projections $\Pi^{\mathsf{Kh}}_{\mathsf{Kb}} \circ \Pi^{\Gamma,\rho} : KB_k(Y;\mathbb{Z}) \to KH_{k-\dim\rho}(Y;\mathbb{Q}) \cong H^{\mathsf{si}}_{k-\dim\rho}(Y;\mathbb{Q})$ are linearly independent. Therefore $KB_k(Y;\mathbb{Z})$ is *huge*. Even for Y a single point, $KB_k(Y;\mathbb{Z})$ has at least one generator over $\mathbb{Z}$ for each isomorphism class of finite groups Γ with $|\Gamma|$ odd, and ρ with $\dim\rho = k$.

III.10. Gromov–Witten type invariants

Let (M,ω) be a compact symplectic manifold, J an almost complex structure compatible with ω, $\beta \in H_2(M;\mathbb{Z})$, and $g,m \geqslant 0$. Then the moduli space $\bar{\mathcal{M}}_{g,m}(M,J,\beta)$ of genus g stable J-holomorphic curves in class β in M with m marked points is a compact oriented Kuranishi space with strong submersions $\mathbf{ev}_i : \bar{\mathcal{M}}_{g,m}(M,J,\beta) \rightarrow M$. The G–W type invariant $GW_{g,m}^{\mathrm{Kb}}(\beta) = [\bar{\mathcal{M}}_{g,m}(M,J,\beta), \mathbf{ev}_1 \times \cdots \times \mathbf{ev}_m]$ in $KB_*(M^m;\mathbb{Z})$ is well-defined and independent of J and other choices.

These G–W invariants in $KB_*(M^m;\mathbb{Z})$ project to $KH_*(M^m;\mathbb{Q}) \cong H_*^{\mathrm{si}}(M^m;\mathbb{Q})$, and their images are the symplectic G–W invariants of Fukaya–Ono. So they are refinements of conventional G–W invariants.

Two important points:

(a) since the groups $KB_*(M^m;\mathbb{Z})$ the invariants lie in are huge, these invariants *contain more information* than conventional G–W invariants, including information 'counting' J-hol curves with symmetry group Γ.

(b) as they lie in groups defined over $\mathbb{Z}$, not $\mathbb{Q}$, they are a tool for studying *integrality properties* of G–W invariants.

I have a sketch proof (with many holes) of the Integrality Conjecture for Gopakumar–Vafa invariants using similar ideas. Actually I need to use *almost complex Kuranishi bordism*, involving an extra 'almost complex structure' on the Kuranishi spaces to do this.

The main idea is to 'blow up' a Kuranishi space at its orbifold strata to get an *effective* Kuranishi space. This gives a functor $B : KB_k(Y;\mathbb{Z}) \to KB_k^{\mathrm{eb}}(Y;\mathbb{Z})$. Then we map $KB_k^{\mathrm{eb}}(Y;\mathbb{Z}) \to KH_k^{\mathrm{ef}}(Y;\mathbb{Z}) \cong H_k^{\mathrm{si}}(Y;\mathbb{Z})$.