D-manifolds and derived differential geometry.

Dominic Joyce, Oxford

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Based on survey paper:

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and preliminary version of book

which may be downloaded from

`people.maths.ox.ac.uk/`

`~joyce/dmanifolds.html.`

These slides available at

`people.maths.ox.ac.uk/~joyce/talks.html.`

1. Introduction

I will describe a new class of geometric objects I call *d-manifolds* — 'derived' smooth manifolds. Some properties of d-manifolds:

• They form a *strict* 2-*category* $\mathbf{dMan}$. That is, we have objects $\boldsymbol{X}$, the d-manifolds, 1-morphisms $\boldsymbol{f},\boldsymbol{g}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$, the smooth maps, and also 2-morphisms $\eta:\boldsymbol{f}\Rightarrow\boldsymbol{g}$.

• Smooth manifolds embed into d-manifolds as a full (2)-subcategory.

• There are also 2-categories $\mathbf{dMan^b}$ of d-manifolds *with boundary* and $\mathbf{dMan^c}$ of *d-manifolds with corners*, and orbifold versions $\mathbf{dOrb},\mathbf{dOrb^b}$, $\mathbf{dOrb^c}$ of these, *d-orbifolds*.

• Many concepts of differential geometry extend nicely to d-manifolds: submersions, immersions, orientations, submanifolds, transverse fibre products, cotangent bundles, ...

• Almost any moduli space used in any enumerative invariant problem over $\mathbb{R}$ or $\mathbb{C}$ has a d-manifold or d-orbifold structure, natural up to equivalence. There are truncation functors to d-manifolds and d-orbifolds from structures currently used – $\mathbb{C}$-schemes with obstruction theories, Kuranishi spaces, polyfolds.

• Virtual classes/cycles/chains can be constructed for compact oriented d-manifolds and d-orbifolds.

So, d-manifolds and d-orbifolds provide a unified framework for studying enumerative invariants and moduli spaces. They also have other applications, and are interesting and beautiful in their own right.

D-manifolds and d-orbifolds are related to other classes of spaces already studied, in particular to the *Kuranishi spaces* of Fukaya–Oh–Ohta–Ono in symplectic geometry, and to David Spivak's *derived manifolds*, from Jacob Lurie's 'derived algebraic geometry' programme.

2. D-spaces and d-manifolds

Algebraic geometry (based on algebra and polynomials) has excellent tools for studying singular spaces – the theory of schemes.

In contrast, conventional differential geometry (based on smooth real functions and calculus) deals well with nonsingular spaces – manifolds – but poorly with singular spaces.

There is a little-known theory of schemes in differential geometry, C^∞*-schemes*, going back to Lawvere, Dubuc, Moerdijk and Reyes, . . . in synthetic differential geometry in the 1960s-1980s. This will be the foundation of our d-manifolds.

2.1. C^∞-rings

Let X be a manifold, and $C^\infty(X)$ the set of smooth functions $c : X \to \mathbb{R}$. Then $C^\infty(X)$ is an $\mathbb{R}$-*algebra*, by adding and multiplying smooth functions. But there are many more operations on $C^\infty(X)$, e.g. if $c : X \to \mathbb{R}$ is smooth then $\exp(c) : X \to \mathbb{R}$ is smooth, giving $\exp : C^\infty(X) \to C^\infty(X)$, algebraically independent of addition and multiplication.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth. Define $\Phi_f : C^\infty(X)^n \to C^\infty(X)$ by

$$\Phi_f(c_1, \dots, c_n)(x) = f\bigl(c_1(x), \dots, c_n(x)\bigr)$$

for all $x \in X$. Addition comes from $f : \mathbb{R}^2 \to \mathbb{R}$, $f : (c_1, c_2) \mapsto c_1 + c_2$, multiplication from $(c_1, c_2) \mapsto c_1 c_2$.

Definition. A C^∞-*ring* is a set $\mathfrak{C}$ together with n-fold operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ for all smooth maps $f : \mathbb{R}^n \to \mathbb{R}$, $n \geqslant 0$, satisfying the following conditions:
Let $m, n \geqslant 0$, and $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $g : \mathbb{R}^m \to \mathbb{R}$ be smooth functions. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(x_1, \ldots, x_n) = g(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1 \ldots, x_n)),$$

for $(x_1, \ldots, x_n) \in \mathbb{R}^n$. Then for all $c_1, \ldots, c_n$ in $\mathfrak{C}$ we have

$$\Phi_h(c_1, \ldots, c_n) = \\ \Phi_g(\Phi_{f_1}(c_1, \ldots, c_n), \ldots, \Phi_{f_m}(c_1, \ldots, c_n)).$$

Also defining $\pi_j : (x_1, \ldots, x_n) \mapsto x_j$ for $j = 1, \ldots, n$ we have $\Phi_{\pi_j} : (c_1, \ldots, c_n) \mapsto c_j$.
A *morphism* of C^∞-rings is $\phi : \mathfrak{C} \to \mathfrak{D}$ with $\Phi_f \circ \phi^n = \phi \circ \Phi_f : \mathfrak{C}^n \to \mathfrak{D}$ for all smooth $f : \mathbb{R}^n \to \mathbb{R}$. Write $\mathbf{C^\infty Rings}$ for the category of C^∞-rings.

Then $C^\infty(X)$ is a C^∞-ring for any manifold X, and from $C^\infty(X)$ we can recover X up to isomorphism. If $f : X \to Y$ is smooth then $f^* : C^\infty(Y) \to C^\infty(X)$ is a morphism of C^∞-rings. This gives a *full and faithful functor* $F : \mathbf{Man} \to \mathbf{C^\infty Rings}^{\mathbf{op}}$ by $F : X \mapsto C^\infty(X)$, $F : f \mapsto f^*$.

Thus, we think of manifolds as examples of C^∞-rings, and C^∞-rings as generalizations of manifolds. But there are many more C^∞-rings than manifolds, e.g. $C^0(X)$ is a C^∞-ring for any topological space X.

2.2. C^∞-schemes

We can now develop the whole machinery of scheme theory in algebraic geometry, replacing rings or algebras by C^∞-rings throughout — see my arXiv:1001.0023.

We obtain a category $\mathbf{C^\infty Sch}$ of C^∞-*schemes* $\underline{X} = (X, \mathcal{O}_X)$, which are topological spaces X equipped with a sheaf of C^∞-rings $\mathcal{O}_X$ locally modelled on the spectrum of a C^∞-ring.

If X is a manifold, define a C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ by $\mathcal{O}_X(U) = C^\infty(U)$ for all open $U \subseteq X$. This defines a full and faithful embedding $\mathbf{Man} \hookrightarrow \mathbf{C^\infty Sch}$.

We also define *vector bundles*, *coherent sheaves* coh($\underline{X}$) and *quasi-coherent sheaves* qcoh($\underline{X}$), and the *cotangent sheaf* $T^*\underline{X}$ on $\underline{X}$. Then qcoh($\underline{X}$) is an abelian category.

Some differences with conventional algebraic geometry:

• affine schemes are Hausdorff. No need to introduce étale topology.

• partitions of unity exist subordinate to any open cover of a (nice) C^∞-scheme $\underline{X}$.

• C^∞-rings such as $C^\infty(\mathbb{R}^n)$ are not noetherian as $\mathbb{R}$-algebras. Causes problems with coherent sheaves: coh($\underline{X}$) is not closed under kernels, so not an abelian category.

2.3. The 2-category of d-spaces

We define d-manifolds as a 2-subcategory of a larger 2-category of *d-spaces*. These are 'derived' versions of C^∞-schemes.

Definition. A *d-space* is a is a quintuple $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ where $\underline{X} = (X, \mathcal{O}_X)$ is a separated, second countable, locally fair C^∞-scheme, $\mathcal{O}'_X$ is a second sheaf of C^∞-rings on X, and $\mathcal{E}_X$ is a quasicoherent sheaf on $\underline{X}$, and $\imath_X : \mathcal{O}'_X \to \mathcal{O}_X$ is a surjective morphism of sheaves of C^∞-rings whose kernel $\mathcal{I}_X$ is a sheaf of *square zero ideals* in $\mathcal{O}'_X$, and $\jmath_X : \mathcal{E}_X \to \mathcal{I}_X$ is a surjective morphism in qcoh($\underline{X}$), so we have an exact sequence of sheaves on X:

$$\mathcal{E}_X \xrightarrow{\jmath_X} \mathcal{O}'_X \xrightarrow{\imath_X} \mathcal{O}_X \longrightarrow 0.$$

A 1-*morphism* $\boldsymbol{f} : \boldsymbol{X} \to \boldsymbol{Y}$ is a triple $\boldsymbol{f} = (\underline{f}, f', f'')$, where $\underline{f} = (f, f^\sharp) : \underline{X} \to \underline{Y}$ is a morphism of C^∞-schemes and $f' : f^{-1}(\mathcal{O}'_Y) \to \mathcal{O}'_X$, $f'' : \underline{f}^*(\mathcal{E}_Y) \to \mathcal{E}_X$ are sheaf morphisms such that the following commutes:

$$\begin{array}{ccccccc} f^{-1}(\mathcal{E}_Y) & \xrightarrow{f^{-1}(\jmath_Y)} & f^{-1}(\mathcal{O}'_Y) & \xrightarrow{f^{-1}(\imath_Y)} & f^{-1}(\mathcal{O}_Y) & \longrightarrow & 0 \\ \downarrow f'' & & \downarrow f' & & \downarrow f^\sharp & & \\ \mathcal{E}_X & \xrightarrow{\jmath_X} & \mathcal{O}'_X & \xrightarrow{\imath_X} & \mathcal{O}_X & \longrightarrow & 0. \end{array}$$

Let $\boldsymbol{f}, \boldsymbol{g} : \boldsymbol{X} \to \boldsymbol{Y}$ be 1-morphisms with $\boldsymbol{f} = (\underline{f}, f', f'')$, $\boldsymbol{f} = (\underline{g}, g', g'')$. Suppose $\underline{f} = \underline{g}$. A 2-*morphism* $\eta : \boldsymbol{f} \Rightarrow \boldsymbol{g}$ is a morphism

$$\eta : f^{-1}(\Omega_{\mathcal{O}'_Y}) \otimes_{f^{-1}(\mathcal{O}'_Y)} \mathcal{O}_X \longrightarrow \mathcal{E}_X$$

in qcoh($\underline{X}$), where $\Omega_{\mathcal{O}'_Y}$ is the sheaf of cotangent modules of $\mathcal{O}'_Y$, such that $g' = f' + \jmath_X \circ \eta \circ \Pi_{XY}$ and $g'' = f'' + \eta \circ \underline{f}^*(\phi_Y)$, for natural morphisms Π_{XY}, ϕ_Y.

Theorem 1. *This defines a strict* 2*-category* $\mathbf{dSpa}$*. All fibre products exist in* $\mathbf{dSpa}$.

We can map $\mathbf{C^\infty Sch}$ into $\mathbf{dSpa}$ by taking a C^∞-scheme $\underline{X} = (X, \mathcal{O}_X)$ to the d-space $\boldsymbol{X} = (\underline{X}, \mathcal{O}_X, 0, \mathrm{id}_{\mathcal{O}_X}, 0)$, with exact sequence

$$0 \xrightarrow{\;0\;} \mathcal{O}_X \xrightarrow{\mathrm{id}_{\mathcal{O}_X}} \mathcal{O}_X \longrightarrow 0.$$

This embeds $\mathbf{C^\infty Sch}$, and hence manifolds $\mathbf{Man}$, as discrete 2-subcategories of $\mathbf{dSpa}$. For *transverse* fibre products of manifolds, the fibre products in $\mathbf{Man}$ and $\mathbf{dSpa}$ agree.

2.4. The 2-subcategory of d-manifolds

Definition. A d-space $\boldsymbol{X}$ is a *d-manifold of dimension* $n \in \mathbb{Z}$ if $\boldsymbol{X}$ may be covered by open d-subspaces $\boldsymbol{Y}$ equivalent in $\mathbf{dSpa}$ to a fibre product $\boldsymbol{U} \times_{\boldsymbol{W}} \boldsymbol{V}$, where $\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{W}$ are manifolds without boundary and $\dim \boldsymbol{U} + \dim \boldsymbol{V} - \dim \boldsymbol{W} = n$. We allow $n < 0$.

Think of a d-manifold $\boldsymbol{X} = (\underline{X}, \mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X)$ as a 'classical' C^∞-scheme $\underline{X}$, with extra 'derived' data $\mathcal{O}'_X, \mathcal{E}_X, \imath_X, \jmath_X$.

Write $\mathbf{dMan}$ for the full 2-subcategory of d-manifolds in $\mathbf{dSpa}$. It is not closed under fibre products in $\mathbf{dSpa}$, but we can say:

Theorem 2. *All fibre products of the form* $\boldsymbol{X} \times_{\boldsymbol{Z}} \boldsymbol{Y}$ *with* $\boldsymbol{X}, \boldsymbol{Y}$ *d-manifolds and* $\boldsymbol{Z}$ *a manifold exist in the 2-category* $\mathbf{dMan}$.

2.5. Gluing by equivalences

Theorem 3. *Let* $\boldsymbol{X}, \boldsymbol{Y}$ *be d-manifolds,* $\emptyset \neq \boldsymbol{U} \subseteq \boldsymbol{X}$, $\emptyset \neq \boldsymbol{V} \subseteq \boldsymbol{Y}$ *be open, and* $\boldsymbol{f} : \boldsymbol{U} \to \boldsymbol{V}$ *an equivalence. Suppose* $Z = X \cup_{U=V} Y$ *is Hausdorff. Then there exists a d-manifold* $\boldsymbol{Z}$*, unique up to equivalence in* $\mathbf{dMan}$*, open* $\hat{\boldsymbol{X}}, \hat{\boldsymbol{Y}} \subseteq \boldsymbol{Z}$ *with* $\boldsymbol{Z} = \hat{\boldsymbol{X}} \cup \hat{\boldsymbol{Y}}$*, equivalences* $\boldsymbol{g} : \boldsymbol{X} \to \hat{\boldsymbol{X}}$ *and* $\boldsymbol{h} : \boldsymbol{Y} \to \hat{\boldsymbol{Y}}$*, and a* 2*-morphism* $\eta : \boldsymbol{g}|_{\boldsymbol{U}} \Rightarrow \boldsymbol{h} \circ \boldsymbol{f}$*.*

Theorem 3 extends to gluing families of d-manifolds $\boldsymbol{X}_i : i \in I$ by equivalences on overlaps $\boldsymbol{X}_i \cap \boldsymbol{X}_j$, with (weak) conditions on overlaps $\boldsymbol{X}_i \cap \boldsymbol{X}_j \cap \boldsymbol{X}_k$. This is very useful for proving existence of d-manifold structures on moduli spaces.

2.6. D-manifold bordism

Let Y be a manifold. Define the *bordism group* $B_k(Y)$ to have elements $\sim$-equivalence classes $[X,f]$ of pairs (X,f), where X is a compact oriented k-manifold and $f : X \to Y$ is smooth, and $(X,f) \sim (X',f')$ if there exists a compact oriented $(k+1)$-manifold with boundary W and a smooth map $e : W \to Y$ with $\partial W \cong X \amalg -X'$ and $e|_{\partial W} \cong f \amalg f'$. It is an abelian group, with $[X,f]+[X',f']=[X \amalg X', f \amalg f']$.

Similarly, define the *derived bordism group* $dB_k(Y)$ to have elements $\approx$-equivalence classes $[\boldsymbol{X},\boldsymbol{f}]$ of pairs (X,f), where $\boldsymbol{X}$ is a compact oriented d-manifold with $\operatorname{vdim}\boldsymbol{X}=k$ and $\boldsymbol{f}:\boldsymbol{X}\to\boldsymbol{Y}=F_{\mathbf{Man}}^{\mathbf{dMan}}(Y)$ is a 1-morphism in $\mathbf{dMan}$, and $(\boldsymbol{X},\boldsymbol{f})\approx(\boldsymbol{X}',\boldsymbol{f}')$ if there exists a compact oriented d-manifold with boundary $\boldsymbol{W}$ with $\operatorname{vdim}\boldsymbol{W}=k+1$ and a 1-morphism $\boldsymbol{e}:\boldsymbol{W}\to\boldsymbol{Y}$ in $\mathbf{dMan^b}$ with $\partial\boldsymbol{W}\simeq\boldsymbol{X}\amalg-\boldsymbol{X}'$ and $\boldsymbol{e}|_{\partial\boldsymbol{W}}\cong\boldsymbol{f}\amalg\boldsymbol{f}'$. It is an abelian group, with $[\boldsymbol{X},\boldsymbol{f}]+[\boldsymbol{X}',\boldsymbol{f}']=[\boldsymbol{X}\amalg\boldsymbol{X}',\boldsymbol{f}\amalg\boldsymbol{f}']$.

There is a natural morphism $\Pi_{\text{bo}}^{\text{dbo}}$: $B_k(Y) \to dB_k(Y)$ mapping $[X, f] \mapsto [F_{\mathbf{Man}}^{\mathbf{dMan}}(X), F_{\mathbf{Man}}^{\mathbf{dMan}}(f)]$.

Theorem 4. $\Pi_{\text{bo}}^{\text{dbo}} : B_k(Y) \to dB_k(Y)$ *is an isomorphism for all* k, *with* $dB_k(Y) = 0$ *for* $k < 0$.

This holds because every d-manifold can be perturbed to a manifold.

Composing $(\Pi_{\text{bo}}^{\text{dbo}})^{-1}$ with the projection $B_k(Y) \to H_k(Y, \mathbb{Z})$ gives a morphism $\Pi_{\text{dbo}}^{\text{hom}} : dB_k(Y) \to H_k(Y, \mathbb{Z})$. We can interpret this as a *virtual class map* for compact oriented d-manifolds. Virtual classes (in homology over $\mathbb{Q}$) also exist for compact oriented d-orbifolds.

2.7. Why is a 2-category enough?

Usually in derived algebraic geometry, one considers an ∞-category of objects (derived stacks, etc.). But we work in a 2-category, effectively a truncation of Spivak's ∞-category of derived manifolds.

Here are two reasons why this truncation does not lose important information. Firstly, d-manifolds correspond to *quasi-smooth* derived schemes $\boldsymbol{X}$, whose cotangent complex $\mathbb{L}_{\boldsymbol{X}}$ lies in degrees $[-1,0]$. So $\mathbb{L}_{\boldsymbol{X}}$ lies in a 2-category of complexes, not an ∞-category. Note that $\boldsymbol{f}:\boldsymbol{X}\rightarrow\boldsymbol{Y}$ is étale in $\mathbf{dMan}$ iff $\Omega_{\boldsymbol{f}}:\boldsymbol{f}^*(\mathbb{L}_{\boldsymbol{Y}})\rightarrow\mathbb{L}_{\boldsymbol{X}}$ is an equivalence.

Secondly, the existence of *partitions of unity* in differential geometry means that our structure sheaves $\mathcal{O}_{\boldsymbol{X}}$ are 'fine' or 'soft', which simplifies behaviour. Partitions of unity are also essential in gluing by equivalences in $\mathbf{dMan}$, as in Theorem 3. Our '2-category style derived geometry' probably would not work very well in a conventional algebro-geometric context, rather than a differential-geometric one.