Difficult problems in special Lagrangian geometry

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Almost Calabi-Yau m-folds

An *almost Calabi-Yau* m*-fold* (M, J, g, Ω) is a compact complex m-fold (M, J) with a Kähler metric g with Kähler form ω, and a nonvanishing holomorphic $(m, 0)$-form Ω, the *holomorphic volume form*. It is a *Calabi-Yau* m*-fold* if $|\Omega|^2 \equiv 2^m$. Then $\nabla\Omega = 0$, the holonomy group $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$, and g is Ricci-flat.

Special Lagrangian m-folds

Let (M, J, g, Ω) be an almost Calabi-Yau m-fold. Let N be a real m-submanifold of M. We call N *special Lagrangian* (*SL*) if $\omega|_N \equiv \mathrm{Im}\,\Omega|_N \equiv 0$, and *SL with phase* $\mathrm{e}^{i\theta}$ if $\omega|_N \equiv (\cos\theta\,\mathrm{Im}\,\Omega - \sin\theta\,\mathrm{Re}\,\Omega)|_N \equiv 0$. If (M, J, g, Ω) is a Calabi-Yau m-fold then $\mathrm{Re}\,\Omega$ is a *calibration* on (M, g), and N is an SL m-fold iff it is calibrated with respect to $\mathrm{Re}\,\Omega$.

Deformations of compact SL m-folds

Robert McLean proved the following result.

Theorem. *Let* (M, J, g, Ω) *be an almost Calabi–Yau* m*-fold, and* N *a compact SL* m*-fold in* M*. Then the moduli space* $\mathcal{M}_N$ *of SL deformations of* N *is a smooth manifold of dimension* $b^1(N)$*, the first Betti number of* N*.*

Here is a sketch of the proof. Let $\nu \to N$ be the *normal bundle* of N in M. Then J identifies $\nu \cong TN$ and g identifies $TN \cong T^*N$. So $\nu \cong T^*N$. We can identify a small *tubular neighbourhood* T of N in M with a neighbourhood of the zero section in ν, identifying ω on M with the symplectic structure on T^*N.
Let $\pi : T \to N$ be the obvious projection.

Then graphs of small 1-forms α on N are identified with submanifolds N' in $T \subset M$ close to N. Which α correspond to *SL m-folds* N'?

Well, N' is special Lagrangian iff $\omega|_{N'} \equiv \mathrm{Im}\,\Omega|_{N'} \equiv 0$.

Now $\pi|_{N'} : N' \to N$ is a diffeomorphism, so this holds iff $\pi_*(\omega|_{N'}) = \pi_*(\mathrm{Im}\,\Omega|_{N'}) = 0$.

We regard $\pi_*(\omega|_{N'})$ and $\pi_*(\mathrm{Im}\,\Omega|_{N'})$ as functions of α.

Calculation shows that $\pi_*(\omega|_{N'}) = \mathrm{d}\alpha$ and $\pi_*(\mathrm{Im}\,\Omega|_{N'}) = F(\alpha, \nabla\alpha)$, where F is nonlinear. Thus, $\mathcal{M}_N$ is locally the set of small 1-forms α on N with $\mathrm{d}\alpha \equiv 0$ and $F(\alpha, \nabla\alpha) \equiv 0$. Now $F(\alpha, \nabla\alpha) \approx \mathrm{d}(*\alpha)$ for small α. So $\mathcal{M}_N$ is locally approximately the set of 1-forms α with $\mathrm{d}\alpha = \mathrm{d}(*\alpha) = 0$. But by Hodge theory this is the de Rham group $H^1(N, \mathbb{R})$, of dimension $b^1(N)$.

Obstructions to existence of SL m-folds

Let M be an almost C-Y m-fold. An m-fold N in M is SL iff $\omega|_N \equiv \operatorname{Im}\Omega|_N \equiv 0$, so only if $[\omega|_N] = [\operatorname{Im}\Omega|_N] = 0$ in $H^*(N,\mathbb{R})$. Thus we have:

Lemma. *Let M be an almost Calabi–Yau m-fold, and N a compact m-fold in M. Then N is isotopic to an SL m-fold N' in M only if $[\omega|_N]=0$ and $[\operatorname{Im}\Omega|_N]=0$ in $H^*(N,\mathbb{R})$.*

The Lemma is a *necessary* condition for an almost C-Y m-fold to have an SL m-fold in a given deformation class. Locally, it is also *sufficient*.

Theorem. *Let* $M_t : t \in (-\epsilon, \epsilon)$ *be a family of almost C-Y* m-*folds, and* N_0 *a compact SL* m-*fold of* M_0. *If* $[\omega_t|_{N_0}] = [\mathrm{Im}\,\Omega_t|_{N_0}] = 0$ *in* $H^*(N_0, \mathbb{R})$ *for all* t, *then* N_0 *extends to a family* $N_t : t \in (-\delta, \delta)$ *of SL* m-*folds in* M_t, *for* $0 < \delta \leqslant \epsilon$.

Singular SL m-folds

Two main approaches so far:

• *Geometric Measure Theory* (Harvey, Lawson, Schoen, Wolfson). Study SL *integral currents* N, measure-theoretic generalizations of submanifolds with good *compactness properties*: in compact M, set of N with $\mathrm{vol}(N) \leqslant C$ is compact. Singularities may be very bad, not well understood. Deformation theory very bad.

• *SL m-folds with isolated conical singularities (ICS)* (Joyce). Study SL m-folds N in M with only singularities $x_1, \ldots, x_n$, N modelled on SL cone C_i in $T_{x_i}M$ near x_i, for $C_i \setminus \{0\}$ nonsingular. Good deformation-obstruction theory. Can *desingularize* them by gluing in Asymptotically Conical SL m-folds in $\mathbb{C}^m$ at $x_1, \ldots, x_n$.
Problem: generalize to other classes of SL singularities, e.g. nonisolated conical, $m \geqslant 4$.

Generic codimension of singularities

Given an SL m-fold N with ICS in M, we have moduli spaces $\mathcal{M}_N$ of deformations of N, and $\mathcal{M}_{\tilde{N}}$ of desingularizations $\tilde{N}$ of N made by gluing in Asymptotically Conical $L_1, \ldots, L_n$. Here $\mathcal{M}_N$ is part of the *boundary* of $\mathcal{M}_{\tilde{N}}$. When M is a *generic* almost C-Y m-fold $\mathcal{M}_N$, $\mathcal{M}_{\tilde{N}}$ are smooth of known dimension.

Call $\dim\mathcal{M}_{\tilde N} - \dim\mathcal{M}_N$ the *index* of the singularities of N. It is the sum over i of s-ind(C_i) and topological terms from L_i. In a dim k family $\mathcal{B}$ of SL m-folds in a generic almost C-Y m-fold M, only singularities with index $\leqslant k$ occur. For SYZ in generic M we need to know about singularities with index 1,2,3 (and 4).

Problem: classify singularities with small index.

Mirror Symmetry

String theorists believe that each Calabi–Yau 3-fold M has a quantization, a *SCFT*.

Calabi–Yau 3-folds $M, \hat{M}$ are a *mirror pair* if their SCFT's are related by a certain involution of SCFT structure. Then invariants of $M, \hat{M}$ are related in surprising ways. For instance,

$H^{1,1}(M) \cong H^{2,1}(\hat{M})$ and

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Using physics, Strominger, Yau and Zaslow proposed:

The SYZ Conjecture. *Let $M, \hat{M}$ be mirror Calabi–Yau 3-folds. There is a compact 3-manifold B and continuous, surjective fibrations $f : M \to B$ and $\hat{f} : \hat{M} \to B$, such that*

(i) *For b in a dense $B_0 \subset B$, the fibres $f^{-1}(b), \hat{f}^{-1}(b)$ are 'dual' SL 3-tori T^3 in $M, \hat{M}$.*

(ii) *For $b \notin B_0$, $f^{-1}(b), \hat{f}^{-1}(b)$ are singular SL 3-folds in $M, \hat{M}$.*

Hard problem: construct SL fibration $f : M \to B$, with singular fibres, of a compact, holonomy SU(3) Calabi–Yau 3-fold M.

Lagrangian fibrations are fairly well understood globally (Gross, Ruan). U(1)-invariant local models in $\mathbb{C}^3$ known for singularities of f (Joyce), expected to be generic. N.B. f not smooth, only continuous.

Let N be a Lagrangian in a Calabi–Yau m-fold M. Then the *Mean Curvature Flow (MCF)* applied to N decreases vol(N), and stays within Hamiltonian equivalent Lagrangians N_t. Smooth N fixed by MCF are Lagrangian and minimal (among all submanifolds), so SL m-folds.

Hard problem: study blow up of Lagrangian MCF in C-Y 3-folds. Does generic N flow to union of SL 3-folds?

If N is a smooth Lagrangian in a C-Y m-fold M, then N is minimal among Lagrangians iff minimal among all submanifolds iff SL m-fold. Suggests *Schoen–Wolfson programme*: take a class of Lagrangians $\mathcal{L}$ in M, e.g. those in a homology class α in $H_m(M,\mathbb{Z})$. Minimize volume in $\mathcal{L}$ to get limit Lagrangian integral current N. Prove N is SL current, or sum of SL currents with different phases $\mathrm{e}^{i\theta}$.

S-W programme suggests SL m-folds are very abundant!

Problems with S-W:

• Must choose $\mathcal{L}$ large enough so good limit N exists.

• If N singular, minimal among Lagrangians does not imply minimal, only Hamiltonian stationary. So, need to understand Hamiltonian stationary, non SL singularities. Progress only when $m = 2$ so far.

The Fukaya category.
Homological Mirror Symmetry (Kontsevich) says $M, \hat{M}$ mirror means $D^b(F(M))$ equivalent to $D^b(\text{coh}(\hat{M}))$ as triangulated categories. Here $D^b(F(M))$ is the (derived) Fukaya category. Objects are (complexes of) graded Lagrangians N in M with *unobstructed Floer homology*. Morphisms $\text{Hom}(N_1, N_2)$ are *Floer homology* $HF^0(N_1, N_2)$.

Conjecture: complex structure on M induces a *stability condition* Z on $D^b(F(M))$ (Bridgeland). Lagrangian N is Z-stable iff N Hamiltonian equivalent to SL 3-fold N'.
Compare: holomorphic vector bundles on Kähler manifold polystable (algebraic condition) iff have a Hermitian–Einstein connection (existence of solution of nonlinear p.d.e.).

Theorem (Thomas). *A Hamiltonian equivalence class of Lagrangians N in M with unobstructed HF^* contains at most one SL m-fold.*

Every object in $D^b(F(M))$ decomposes uniquely into Z-(semi)stable objects. So, conjecture implies there are enough SL m-folds to generate $D^b(F(M))$; again, SL m-folds are very abundant.

Principle: for many problems (SYZ, S-W, . . .), should restrict to SL m-folds N with *unobstructed* HF^*.

Question: does this simplify the singular behaviour of N, or limits of such N?

Problem: Fix the definition of $D^b(F(M))$, to include immersed and some kinds of singular Lagrangians. Otherwise conjecture cannot be true.

Conjecture (Joyce). There should exist interesting invariants $I^\alpha(M)$ of almost Calabi–Yau 3-folds M 'counting' SL homology 3-spheres N in M with class $\alpha \in H_3(M, \mathbb{Z})$ with flat U(1)-connections. Should be independent of Kähler class of M, and transform by known law under deformation of complex structure of M. Expected to be mirror to extension of Donaldson–Thomas invariants.

Conclusions. All these conjectures assert some deep existence, uniqueness and stability properties of SL m-folds. SL m-folds (with unobstructed HF^*) cannot pop in and out of existence in a chaotic way; rather, they do so by very ordered, algebraic criteria.

It may be possible to classify the most common singularities of SL 3-folds in generic almost C-Y 3-folds, and so understand these properties.