Introduction to Riemannian holonomy groups and calibrated geometry

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1. Holonomy groups

Let M^n be a manifold of dimension n. Let $x \in M$. Then $T_x M$ is the *tangent space* to M at x.
Let g be a Riemannian metric on M.
Let ∇ be the *Levi-Civita connection* of g.
Let $R(g)$ be the *Riemann curvature* of g.

Fix $x \in M$. The *holonomy group* $\mathrm{Hol}(g)$ of g is the set of isometries of T_xM given by *parallel transport* using ∇ about closed loops γ in M based at x. It is a subgroup of $O(n)$. Up to conjugation, it is independent of the base-point x.

Berger's classification

Let M be simply-connected and g be irreducible and nonsymmetric. Then Hol(g) is one of $SO(m)$, $U(m)$, $SU(m)$, $Sp(m)$, $Sp(m)Sp(1)$ for $m \geqslant 2$, or G_2 or $Spin(7)$. We call G_2 and $Spin(7)$ the *exceptional holonomy groups*. Dim(M) is 7 when Hol(g) is G_2 and 8 when Hol(g) is $Spin(7)$.

Understanding Berger's list

The four *inner product algebras* are

$\mathbb{R}$ — *real numbers*.

$\mathbb{C}$ — *complex numbers*.

$\mathbb{H}$ — *quaternions*.

$\mathbb{O}$ — *octonions*,

or *Cayley numbers*.

Here $\mathbb{C}$ is not ordered,

$\mathbb{H}$ is not commutative,

and $\mathbb{O}$ is not associative.

Also we have $\mathbb{C} \cong \mathbb{R}^2$, $\mathbb{H} \cong \mathbb{R}^4$ and $\mathbb{O} \cong \mathbb{R}^8$, with $\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$.

Group	**Acts on**
$SO(m)$	$\mathbb{R}^m$
$O(m)$	$\mathbb{R}^m$
$SU(m)$	$\mathbb{C}^m$
$U(m)$	$\mathbb{C}^m$
$Sp(m)$	$\mathbb{H}^m$
$Sp(m)Sp(1)$	$\mathbb{H}^m$
G_2	$\mathrm{Im}\,\mathbb{O} \cong \mathbb{R}^7$
$Spin(7)$	$\mathbb{O} \cong \mathbb{R}^8$

Thus there are two holonomy groups for each of $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

2. Calibrations

Let (M, g) be a Riemannian manifold. An *oriented tangent k-plane* V on M is an oriented vector subspace V of some tangent space T_xM to M with $\dim V = k$. Each has a *volume form* vol_V defined using g.

A *calibration* on M is a closed k-form φ with $\varphi|_V \leqslant \mathrm{vol}_V$ for every oriented tangent k-plane V on M.

Let N be an oriented k-fold in M with $\dim N = k$. We call N *calibrated* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

If N is compact then $\mathrm{vol}(N) \geqslant [\varphi] \cdot [N]$, and if N is compact and calibrated then $\mathrm{vol}(N) = [\varphi] \cdot [N]$, where $[\varphi] \in H^k(M, \mathbb{R})$ and $[N] \in H_k(M, \mathbb{Z})$.

Thus calibrated submanifolds are volume-minimizing in their homology class, and are *minimal submanifolds*.

Calibrations on $\mathbb{R}^n$

Let $(\mathbb{R}^n, g)$ be Euclidean, and φ be a constant k-form on $\mathbb{R}^n$ with $\varphi|_V \leqslant \mathrm{vol}_V$ for all oriented k-planes V in $\mathbb{R}^n$.

Let $\mathcal{F}_\varphi$ be the set of oriented k-planes V in $\mathbb{R}^n$ with $\varphi|_V = \mathrm{vol}_V$. Then an oriented k-fold N in $\mathbb{R}^n$ is a φ-submanifold iff $T_xN \in \mathcal{F}_\varphi$ for all $x \in N$.

For φ to be interesting, $\mathcal{F}_\varphi$ must be fairly large, or there will be few φ-submanifolds.

Calibrations and special holonomy metrics

Let $G \subset \mathrm{O}(n)$ be the holonomy group of a Riemannian metric. Then G acts on $\Lambda^k(\mathbb{R}^n)^*$. Suppose $\varphi_0 \in \Lambda^k(\mathbb{R}^n)^*$ is nonzero and G-invariant. Rescale φ_0 so that $\varphi_0|_V \leqslant \mathrm{vol}_V$ for all oriented k-planes $V \subset \mathbb{R}^n$, and $\varphi_0|_U = \mathrm{vol}_U$ for some U. Then $U \in \mathcal{F}_{\varphi_0}$, so by G-invariance $\mathcal{F}_{\varphi_0}$ contains the G-orbit of U. Usually $\mathcal{F}_{\varphi_0}$ is 'fairly big'.

Let (M, g) be have holonomy G. Then there is constant k-form φ on M corresponding to the G-invariant k-form φ_0. It is a *calibration* on M.

At each $x \in M$ the family of oriented tangent k-planes V with $\varphi|_V = \mathrm{vol}_V$ is $\mathcal{F}_{\varphi_0}$, which is 'fairly big'. So we expect many φ-submanifolds N in M.

Thus manifolds with special holonomy often have interesting calibrations.

Here are some examples:

- *complex submanifolds* of *Kähler manifolds* (with holonomy $U(m)$).
- *Special Lagrangian m-folds* in *Calabi–Yau m-folds* (with holonomy SU(m), and real dimension $2m$).
- *associative* 3*-folds* and *coassociative* 4*-folds* in 7-manifolds with holonomy G_2.
- *Cayley* 4*-folds* in 8-manifolds with holonomy Spin(7).

3. Compact calibrated submanifolds

Let (M, J, g) be a Calabi–Yau m-fold with complex volume form Ω. Then $\mathrm{Re}\,\Omega$ is a *calibration* on M. Its calibrated submanifolds are called *special Lagrangian m-folds*, or *SL m-folds* for short.

What can we say about *compact* SL m-folds in M?

Let (M, J, g, Ω) be a Calabi–Yau m-fold and N a compact SL m-fold in M. Let $\mathcal{M}_N$ be the moduli space of *SL deformations* of N. We ask:

1. Is $\mathcal{M}_N$ a manifold, and of what dimension?

2. Does N persist under deformations of (J, g, Ω)?

3. Can we compactify $\mathcal{M}_N$ by adding a 'boundary' of *singular* SL m-folds? If so, what are the singularities like?

These questions concern the *deformations* of SL m-folds, *obstructions* to their existence, and their *singularities*.

Questions 1 and 2 are fairly well understood, and we shall discuss them in this lecture. Question 3 will be discussed tomorrow.

3.1 Deformations of compact SL m-folds

Robert McLean proved the following result.

Theorem. *Let* (M, J, g, Ω) *be a Calabi–Yau* m*-fold, and* N *a compact SL* m*-fold in* M*. Then the moduli space* $\mathcal{M}_N$ *of SL deformations of* N *is a smooth manifold of dimension* $b^1(N)$*, the first Betti number of* N*.*

Here is a sketch of the proof. Let $\nu \longrightarrow N$ be the *normal bundle* of N in M. Then J identifies $\nu \cong TN$ and g identifies $TN \cong T^*N$. So $\nu \cong T^*N$. We can identify a small *tubular neighbourhood* T of N in M with a neighbourhood of the zero section in ν, identifying ω on M with the symplectic structure on T^*N.

Let $\pi : T \longrightarrow N$ be the obvious projection.

Then graphs of small 1-forms α on N are identified with submanifolds N' in $T \subset M$ close to N. Which α correspond to *SL m-folds* N'?

Well, N' is special Lagrangian iff $\omega|_{N'} \equiv \operatorname{Im}\Omega|_{N'} \equiv 0$.

Now $\pi|_{N'} : N' \to N$ is a diffeomorphism, so this holds iff $\pi_*\bigl(\omega|_{N'}\bigr) = \pi_*\bigl(\operatorname{Im}\Omega|_{N'}\bigr) = 0$.

We regard $\pi_*\bigl(\omega|_{N'}\bigr)$ and $\pi_*\bigl(\operatorname{Im}\Omega|_{N'}\bigr)$ as functions of α.

Calculation shows that $\pi_*\bigl(\omega|_{N'}\bigr) = \mathrm{d}\alpha$ and $\pi_*\bigl(\mathrm{Im}\,\Omega|_{N'}\bigr) = F(\alpha, \nabla\alpha)$, where F is nonlinear. Thus, $\mathcal{M}_N$ is locally the set of small 1-forms α on N with $\mathrm{d}\alpha \equiv 0$ and $F(\alpha, \nabla\alpha) \equiv 0$. Now $F(\alpha, \nabla\alpha) \approx \mathrm{d}(*\alpha)$ for small α. So $\mathcal{M}_N$ is locally approximately the set of 1-forms α with $\mathrm{d}\alpha = \mathrm{d}(*\alpha) = 0$. But by Hodge theory this is the de Rham group $H^1(N, \mathbb{R})$, of dimension $b^1(N)$.

3.2 Obstructions to existence of SL m-folds

Let M be a C-Y m-fold. Then an m-fold N in M is SL iff $\omega|_N \equiv \mathrm{Im}\,\Omega|_N = 0$. This holds only if $[\omega|_N] = [\mathrm{Im}\,\Omega|_N] = 0$ in $H^*(N, \mathbb{R})$. So we have:

Lemma. *Let M be a Calabi–Yau m-fold, and N a compact m-fold in M. Then N is isotopic to an SL m-fold N' in M only if $[\omega|_N] = 0$ and $[\mathrm{Im}\,\Omega|_N] = 0$ in $H^*(N, \mathbb{R})$.*

The Lemma is a *necessary* condition for a C-Y m-fold to have an SL m-fold in a given deformation class. Locally, it is also *sufficient*.

Theorem. *Let* $M_t : t \in (-\epsilon, \epsilon)$ *be a family of Calabi–Yau* m-*folds, and* N_0 *a compact SL* m-*fold of* M_0. *If* $[\omega_t|_{N_0}] = [\mathrm{Im}\,\Omega_t|_{N_0}] = 0$ *in* $H^*(N_0, \mathbb{R})$ *for all* t, *then* N_0 *extends to a family* $N_t : t \in (-\delta, \delta)$ *of SL* m-*folds in* M_t, *for* $0 < \delta \leqslant \epsilon$.

3.3 Coassociative 4-folds

Let (M, g) have holonomy G_2. Then M has a constant 3-form φ and 4-form $*\varphi$. They are calibrations, whose calibrated submanifolds are called *associative 3-folds* and *coassociative 4-folds*. A 4-fold N in M is coassociative iff $\varphi|_N \equiv 0$. Also, if N is coassociative then the normal bundle ν is isomorphic to $\Lambda^2_+ T^*N$, the self-dual 2-forms.

Using this, McLean proved:

Theorem. *Let* (M, g) *be a 7-manifold with holonomy* G_2*, and* N *a compact coassociative 4-fold in* M*. Then the moduli space* $\mathcal{M}_N$ *of coassociative deformations of* N *is a smooth manifold of dimension* $b^2_+(N)$*.*

Roughly, nearby coassociative 4-folds correspond to small closed forms in $\Lambda^2_+ T^*N$, which are $H^2_+(N, \mathbb{R})$ by Hodge theory.

3.4 Associative 3-folds and Cayley 4-folds

Associative 3-folds in 7-manifolds with holonomy G_2, and *Cayley 4-folds* in 8-manifolds with holonomy Spin(7), cannot be defined by the vanishing of closed forms. This gives their deformation theory a different character. Here is how the theories work.

Let N be a compact associative 3-fold or Cayley 4-fold in M. Then there are vector bundles $E, F \to N$ and a first order elliptic operator
$D_N : C^\infty(E) \to C^\infty(F)$.
The *kernel* $\operatorname{Ker} D_N$ is the set of *infinitesimal deformations* of N. The *cokernel* $\operatorname{Coker} D_N$ is the *obstruction space*. The *index* of D_N is $\operatorname{ind}(D_N) = \dim \operatorname{Ker} D_N - \dim \operatorname{Coker} D_N$.

In the associative case $\mathrm{ind}(D_N) = 0$, and in the Cayley case $\mathrm{ind}(D_N) = \tau(N) - \frac{1}{2}\chi(N) - \frac{1}{2}[N] \cdot [N]$, where τ is the signature and χ the Euler characteristic. Generically $\mathrm{Coker}\, D_N = 0$, and then $\mathcal{M}_N$ is locally a manifold with dimension $\mathrm{ind}(D_N)$. If $\mathrm{Coker}\, D_N \neq 0$, then $\mathcal{M}_N$ may be singular, or have a different dimension.

Note that the special Lagrangian and coassociative cases are unusual: there are *no* obstructions, and the moduli space is *always* a manifold of given dimension, without genericity assumptions.
This is a minor mathematical miracle.