Singularities of special Lagrangian submanifolds and SYZ

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Almost Calabi-Yau m-folds

An *almost Calabi-Yau m-fold* (M, J, g, Ω) is a compact complex m-fold (M, J) with a Kähler metric g with Kähler form ω, and a nonvanishing holomorphic $(m, 0)$-form Ω, the *holomorphic volume form*.
It is a *Calabi-Yau m-fold* if $|\Omega|^2 \equiv 2^m$. Then $\nabla\Omega = 0$ and g is Ricci-flat.

Special Lagrangian m-folds

Let (M, J, g, Ω) be an almost Calabi-Yau m-fold. Let N be a real m-submanifold of M. We call N *special Lagrangian* (*SL*) if $\omega|_N \equiv \mathrm{Im}\,\Omega|_N \equiv 0$.

If (M, J, g, Ω) is a Calabi-Yau m-fold then $\mathrm{Re}\,\Omega$ is a *calibration* on (M, g), and N is an SL m-fold iff it is calibrated with respect to $\mathrm{Re}\,\Omega$.

Singular SL m-folds

General singularities of SL m-folds may be very bad, and difficult to study. Would like a class of singular SL m-folds with nice, well-behaved singularities to study in depth. Would like these to occur often in real life, i.e. of finite codimension in the space of all SL m-folds. SL m-folds with *isolated conical singularities* (*ICS*) are such a class.

Let N be an SL m-fold in M whose only singular points are $x_1, \ldots, x_n$. Near x_i we can identify M with $\mathbb{C}^m \cong T_{x_i}M$, and N near x_i approximates an SL m-fold in $\mathbb{C}^m$ with singularity at 0. We say N has *isolated conical singularities* if near x_i it converges with order $O(r^{\mu_i})$ for $\mu_i > 1$ to an SL cone C_i in $\mathbb{C}^m$ nonsingular except at 0.

SL m-folds with ICS have a rich theory.

• **Examples.** Many examples of SL cones C_i in $\mathbb{C}^m$ have been constructed. Rudiments of classification for $m = 3$.

• **Regularity near** $x_1, \ldots, x_n$**.** Let $\iota : N \to M$ be the inclusion. If $\nabla^k \iota$ converges to C_i near x_i with order $O(r^{\mu_i - k})$ for $k = 0, 1$ then it does so for all $k \geqslant 0$.

• **Deformation theory.** The moduli space $\mathcal{M}_N$ of deformations of N is locally homeomorphic to $\Phi^{-1}(0)$, for smooth $\Phi : \mathcal{I} \to \mathcal{O}$ and fin. dim. vector spaces $\mathcal{I}, \mathcal{O}$ with $\mathcal{I}$ the image of $H^1_{\mathrm{cs}}(N', \mathbb{R})$ in $H^1(N', \mathbb{R})$, $N' = N \setminus \{x_1, \ldots, x_n\}$, and $\dim \mathcal{O} = \sum_{i=1}^n \text{s-ind}(C_i)$. Here $\text{s-ind}(C_i) \in \mathbb{N}$ is the *stability index*, the obstructions from C_i. If $\text{s-ind}(C_i) = 0$ for all i then $\mathcal{M}_N$ is smooth.

• **Desingularization.** Let C be an SL cone in $\mathbb{C}^m$, nonsingular except at 0. A nonsingular SL m-fold L in $\mathbb{C}^m$ is *Asymptotically Conical* (*AC*) C if L converges to C at infinity with order $O(r^\lambda)$ for $\lambda < 1$. Then tL converges to C as $t \to 0_+$. Thus, AC SL m-folds model how families of nonsingular SL m-folds develop singularities modelled on C.

If N is an SL m-fold with ICS at $x_1, \ldots, x_n$ and cones C_i, and $L_1, \ldots, L_n$ are AC SL m-folds in $\mathbb{C}^m$ with cones C_i, then under cohomological conditions we can construct a family of compact nonsingular SL m-folds $\tilde{N}_t$ for small $t > 0$ converging to N as $t \to 0$, by gluing tL_i into N at x_i, all i.

• **Generic codimension of singularities.** Given an SL m-fold N with ICS in M, we have moduli spaces $\mathcal{M}_N$ of deformations of N, and $\mathcal{M}_{\tilde{N}}$ of desingularizations $\tilde{N}$ of N made by gluing in $L_1, \ldots, L_n$. Here $\mathcal{M}_N$ is part of the *boundary* of $\mathcal{M}_{\tilde{N}}$. If M is a *generic* almost C-Y m-fold then $\mathcal{M}_N$, $\mathcal{M}_{\tilde{N}}$ are smooth with known dimension.

Call dim $\mathcal{M}_{\tilde{N}}$ − dim $\mathcal{M}_N$ the *index* of the singularities of N. It is the sum over i of s-ind(C_i) and topological terms from L_i. In a dim k family $\mathcal{B}$ of SL m-folds in a generic almost C-Y m-fold M, only singularities with index $\leqslant k$ occur. For SYZ in generic M we need to know about singularities with index 1,2,3 (and 4).

Problem: classify singularities with small index.

Mirror Symmetry

String theorists believe that each Calabi–Yau 3-fold X has a quantization, a *SCFT*.

Calabi–Yau 3-folds $X, \hat{X}$ are a *mirror pair* if their SCFT's are related by a certain involution of SCFT structure.

Then invariants of $X, \hat{X}$ are related in surprising ways. For instance,

$H^{1,1}(X) \cong H^{2,1}(\hat{X})$ and

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Using physics, Strominger, Yau and Zaslow proposed:

The SYZ Conjecture. *Let $X, \hat{X}$ be mirror Calabi–Yau 3-folds. There is a compact 3-manifold B and continuous, surjective $f : X \to B$ and $\hat{f} : \hat{X} \to B$, such that*

(i) *For b in a dense $B_0 \subset B$, the fibres $f^{-1}(b), \hat{f}^{-1}(b)$ are dual SL 3-tori T^3 in $X, \hat{X}$.*

(ii) *For $b \notin B_0$, $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are singular SL 3-folds in $X, \hat{X}$.*

We call $f, \hat{f}$ *special Lagrangian fibrations*, and $\Delta = B \backslash B_0$ the *discriminant*.

In (i), the nonsingular fibres $T, \hat{T}$ of $f, \hat{f}$ are supposed to be *dual tori*. Topologically, this means an isomorphism $H^1(T, \mathbb{Z}) \cong H_1(\hat{T}, \mathbb{Z})$. But the metrics on $T, \hat{T}$ should really be dual as well. This only makes sense in the 'large complex structure limit', when the fibres are small and nearly flat.

U(1)-**invariant SL 3-folds**

Let U(1) act on $\mathbb{C}^3$ by
$(z_1, z_2, z_3) \mapsto (\mathrm{e}^{i\theta} z_1, \mathrm{e}^{-i\theta} z_2, z_3)$.
Let N be a U(1)-invariant SL 3-fold. Then locally we can write N in the form

$$\Bigl\{(z_1, z_2, z_3) : |z_1|^2 - |z_2|^2 = 2a, \quad z_1 z_2 = v(x,y) + iy, \quad z_3 = x + iu(x,y),\ x, y \in \mathbb{R}\Bigr\},$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$ satisfy

$$u_x = v_y \quad \text{and} \quad v_x = -2(v^2 + y^2 + a^2)^{1/2} u_y. \qquad (*)$$

Since $u_x = v_y$, there exists a potential function f with $u = f_y$ and $v = f_x$. The 2nd equation of $(*)$ becomes

$$f_{xx}+2(f_x^2+y^2+a^2)^{1/2}f_{yy} = 0. \tag{+}$$

This is a second-order quasi-linear equation. When $a \neq 0$ it is locally uniformly elliptic. When $a=0$ it is non-uniformly elliptic, except at *singular points* $f_x = y = 0$.

Theorem A. Let S be a compact domain in $\mathbb{R}^2$ satisfying some convexity conditions.
Let $\phi \in C^{3,\alpha}(\partial S)$.
If $a \neq 0$ there exists a unique $f \in C^{3,\alpha}(S)$ satisfying $(+)$ with $f|_{\partial S} = \phi$. If $a = 0$ there exists a unique $f \in C^1(S)$ satisfying $(+)$ with weak second derivatives, with $f|_{\partial S} = \phi$.
Also f depends continuously in $C^1(S)$ on a, ϕ.

Theorem A shows that the Dirichlet problem for (+) is uniquely solvable in certain convex domains. The induced solutions $u, v \in C^0(S)$ of $(*)$ yield U(1)-invariant SL 3-folds in $\mathbb{C}^3$ satisfying certain boundary conditions over ∂S. When $a \neq 0$ these SL 3-folds are nonsingular, when $a = 0$ they are singular when $v = y = 0$.

Theorem B.
Let $\phi, \phi' \in C^{3,\alpha}(\partial S)$, let $a \in \mathbb{R}$ and let $f, f' \in C^{3,\alpha}(S)$ or $C^1(S)$ be the solutions of $(+)$ from Theorem A with $f|_{\partial S} = \phi$, $f'|_{\partial S} = \phi'$. Let $u = f_y$, $v = f_x$, $u' = f'_y$, $v' = f'_x$. Suppose $\phi - \phi'$ has $k+1$ local maxima and $k+1$ local minima on ∂S. Then $(u, v) - (u', v')$ has no more than k zeroes in S°, counted with multiplicity.

Theorem C.
Let $u, v \in C^0(S)$ be a singular solution of $(*)$ with $a = 0$, e.g. from Theorem A. Then **either** $u(x,y) \equiv u(x,-y)$ and $v(x,y) \equiv -v(x,-y)$, so that u, v is singular on the x-axis, **or** the singularities $(x,0)$ of u, v in S° are *isolated*, with a *multiplicity* $n > 0$. Multiplicity n singularities occur in codimension n of boundary data. All multiplicities occur.

Theorem D.
Let $U \subset \mathbb{R}^3$ be open, S as above, and $\Phi : U \to C^{3,\alpha}(\partial S)$ continuous such that if $(a,b,c) \neq (a,b',c') \in U$ then $\Phi(a,b,c) - \Phi(a,b',c')$ has 1 local maximum and 1 local minimum.
For $\alpha = (a,b,c) \in U$, let $f_\alpha \in C^1(S)$ be the solution of (+) from Theorem A with $f_\alpha|_{\partial S} = \Phi(\alpha)$.

Set $u_\alpha = (f_\alpha)_y$ and $v_\alpha = (f_\alpha)_x$. Let N_α be the SL 3-fold

$$\bigl\{(z_1,z_2,z_3) : |z_1|^2 - |z_2|^2 = 2a,$$
$$z_1 z_2 = v_\alpha(x,y) + iy,$$
$$z_3 = x + iu_\alpha(x,y),\ (x,y) \in S^\circ\bigr\}.$$

Then there exists an open $V \subset \mathbb{C}^3$ and a continuous map $F : V \to U$ with $F^{-1}(\alpha) = N_\alpha$. This is a U(1)-*invariant special Lagrangian fibration*. It can include *singular fibres*, of every multiplicity $n > 0$.

Example. Define $f : \mathbb{C}^3 \to \mathbb{R} \times \mathbb{C}$ by $f(z_1, z_2, z_3) = (a, b)$, where $2a = |z_1|^2 - |z_2|^2$ and

$$b = \begin{cases} z_3, & z_1 = z_2 = 0, \\ z_3 + \bar{z}_1 \bar{z}_2 / |z_1|, & a \geqslant 0,\ z_1 \neq 0, \\ z_3 + \bar{z}_1 \bar{z}_2 / |z_2|, & a < 0. \end{cases}$$

Then f is a piecewise-smooth SL fibration of $\mathbb{C}^3$. It is not smooth on $|z_1| = |z_2|$.
The fibres $f^{-1}(a, b)$ are T^2-cones when $a = 0$, and non-singular $\mathcal{S}^1 \times \mathbb{R}^2$ when $a \neq 0$.

Conclusions

Using these SL fibrations as local models, if X is a *generic* ACY 3-fold and $f : X \to B$ an SL fibration, I predict:

• f is only piecewise smooth.

• All fibres have finitely many singular points.

• Δ is codim 1 in B. Generic singularities are modelled on the example above.

• Some codim 2 singularities are also locally U(1)-invariant.

• Codim 3 singularities are not locally U(1)-invariant.

• If $f : X \to B$, $\hat{f} : \hat{X} \to B$ are dual SL fibrations of mirror C-Y 3-folds, the discriminants $\Delta, \hat{\Delta}$ have different topology near codim 3 singular fibres, so $\Delta \neq \hat{\Delta}$.

This contradicts some statements of the SYZ Conjecture.

I regard SYZ as primarily a limiting statement about the 'large complex structure limit'.