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Deformations of special Lagrangian submanifolds

Stephen P. Marshall
Wadham College

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Abstract

In this thesis we study the deformations of special Lagrangian submanifolds $X \subseteq M$ sitting inside a Calabi-Yau manifold (M, g, J, Ω). Let N be the normal bundle of X, and identify $N \cong T^*X$ via the complex structure J and induced metric on X. Then using the exponential map one can identify small 1-forms ξ on X with submanifolds $X_\xi \subseteq M$ close to X.

In the case that X is compact, McLean [50, Theorem 3-6], showed that the small 1-forms ξ parameterising special Lagrangian submanifolds $X_\xi \subseteq M$ form a smooth manifold $\mathcal{M} \subseteq C^\infty(T^*X)$ of dimension $b^1(X)$, the first Betti number of X. We give a full proof of this result, including the necessary details which were absent from [50]. In fact our result Theorem 3.21 is an extension of the original McLean theorem, in that we show that the special Lagrangian deformations $\mathcal{M}$ persist under (certain types of) perturbations of the ambient Calabi-Yau structure.

We then go on to consider the situation when $X \subseteq \mathbb{C}^n$ is non-compact, but asymptotic to a cone $C \subseteq \mathbb{C}^n$ at a specified rate $\tilde{\alpha} < 1$ of decay. Provided that $\tilde{\alpha}$ is not too negative, it turns out that for almost all $\tilde{\alpha}$ there is again a smooth manifold $\mathcal{M}_{\tilde{\alpha}} \subseteq C^\infty(T^*X)$ parameterising the special Lagrangian submanifolds $X_\xi \subseteq \mathbb{C}^n$ which are near to X and decay towards C at rate $\tilde{\alpha}$. The main result here is Theorem 6.45, which also gives the dimensions of the smooth manifold $\mathcal{M}_{\tilde{\alpha}}$. It turns out that for small rates of decay, $\dim \mathcal{M}_{\tilde{\alpha}}$ depends only on the topology of X, whereas for higher rates $\dim \mathcal{M}_{\tilde{\alpha}}$ will also depend on analytic data got from the link $\Sigma := S^{2n-1} \cap C$ of the cone C. Along the way to proving Theorem 6.45 we develop a theory of analysis for asymptotically conical Riemannian manifolds, expanding on the existing theory of Lockhart and McOwen [46] and Lockhart [45] for damped Sobolev spaces. In particular, in Section 6.1.1 we give the relevant details for damped Hölder spaces. We finish in Section 6.3 by applying our theory to some specific examples, and prove the existence of special Lagrangian submanifolds in $X_\xi \subseteq \mathbb{C}^n$ which were previously unknown.

Contents

Chapter 1

Introduction

1.1 Motivation and previous work

The notion of calibrated geometry was first introduced by Harvey and Lawson in the foundational paper [21]. They define a *calibration* to be a closed r-form ϕ on a Riemannian manifold (M, g) such that the length of $\phi|_V$ is less than or equal to 1 for all r-planes V in TM. Then an r-dimensional submanifold $X \subseteq M$ is *calibrated* when $\left|\phi|_{T_x X}\right|_g = 1$ for all $x \in X$. It follows quickly from the definitions that calibrated submanifolds $X \subseteq M$ are *minimal*, and this fact makes the subject of calibrated geometry worthy of serious study.

The basic example of a calibration is the $2l$-form $\frac{1}{l!}\omega^l$ on a Kähler manifold (M, J, g) with Kähler form ω. It turns out that the calibrated submanifolds in this case are the complex submanifolds $X \subseteq M$ of complex dimension l. One may then ask if there are any further interesting examples of calibrations, and indeed Harvey and Lawson [21] provide us with some. The *special Lagrangian* calibration on Calabi-Yau manifolds is the first they consider: it turns out that if (M, J, g, Ω) is a Calabi-Yau manifold of real dimension $2n$, then the n-form $\operatorname{Re}\Omega$ is a calibration on M, and the calibrated submanifolds $X \subseteq M$ are called *special Lagrangian*.

To any Riemannian manifold (M, g) of dimension m we can associate the *holonomy group* $\operatorname{Hol}(g) \leqslant \mathrm{O}(m)$. Supposing that there exists a non-zero $\phi_0 \in \Lambda^r(\mathbb{R}^m)^*$ which is fixed by the action of $\operatorname{Hol}(g)$, we can, by rescaling ϕ_0 if necessary, easily construct using parallel translation an r-form ϕ on M which is covariant constant and has length less than or equal to 1 on each r-plane V in TM. It follows that ϕ will be a calibration on (M, g). In practice, all of the calibrations that one meets come from some reduced holonomy group in this way. Applying this general principle to the above examples, note that: a Kähler manifold is precisely a Riemannian manifold with holonomy group contained in $\mathrm{U}(n)$, where $n = 2m$, and a Calabi-Yau manifold is precisely a Riemannian manifold with holonomy group contained in $\mathrm{SU}(n)$. The other calibrations that Harvey and Lawson consider also live on Riemannian manifolds with reduced holonomy: namely the *associative* and *coassociative* calibrations on a 7-manifold of holonomy G_2, which have respectively $r = 3, 4$, and the *Cayley* calibration on an 8-manifold with holonomy $\mathrm{Spin}(7)$, which has $r = 4$.

Although the full theory of calibrated geometry is very important, the special Lagrangian calibration itself has received a great deal of attention over the past few years. One of the main reasons for this has been the attempt by Strominger, Yau and Zaslow [58] to explain the idea of *mirror symmetry* in terms of special Lagrangian submanifolds. We shall briefly explain some of the ideas involved.

String theory is a branch of theoretical physics in which particles are modelled not as points but as 1-dimensional loops, or "strings" propagating in some ambient space N. In the most popular version of string theory, the space N has 10 dimensions, and locally looks like $\mathbb{R}^4 \times M$ where $\mathbb{R}^4$ is Minkowski 4-space and M is a compact, 6-dimensional Calabi-Yau manifold. The physics of the space N is encapsulated in a complicated mathematical object one can associate to M called a *superconformal field theory*, or SCFT for short, and then the properties of the Calabi-Yau manifold M get translated over to properties of the SCFT.

It turns out that there is a simple automorphism which one can apply to a given SCFT, and we say that Calabi-Yau manifolds M, $\hat{M}$ are a *mirror pair* if their SCFTs are associated by this automorphism. Using physical arguments, one can now deduce all sorts of miraculous relations between the Calabi-Yau manifolds M and $\hat{M}$, because their SCFTs can be identified in the way described above. This is the phenomenon of mirror symmetry. We now give an example of the sort of thing physicists believe to be true. Denote the Hodge numbers of a compact complex manifold by $h^{p,q}$. Then the Hodge diamond of a compact, 6-dimensional Calabi-Yau manifold M has the form

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{2,1} & & h^{2,1} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

so that the only independent Hodge numbers of M are $h^{1,1}$ and $h^{2,1}$. Mirror symmetry predicts that if M and $\hat{M}$ are a mirror pair of Calabi-Yau manifolds then $h^{1,1}(M) = h^{2,1}(\hat{M})$ and $h^{1,1}(\hat{M}) = h^{2,1}(M)$, so that the mirror transform $M \mapsto \hat{M}$ interchanges odd and even cohomology. That is an intriguing result.

Unfortunately, the transition between a Calabi-Yau manifold M and its SCFT is not yet a process which is mathematically understood, and so the whole theory of mirror symmetry as explained in the previous paragraph is lacking any rigorous proof. As mathematicians, we would like to fill this gap, and in particular, given a compact, 6-dimensional Calabi-Yau manifold M, we would like to know how to build a second compact, 6-dimensional Calabi-Yau manifold $\hat{M}$, with M and $\hat{M}$ related in the ways the physicists predict. One (among others: see for example [39]) conjectural recipe is that given in [58].

Strominger, Yau and Zaslow argue using physics that certain compact 6-dimensional Calabi-Yau manifolds M (those "near the large complex structure limit") should admit a fibration $M \to B$, with generic fibre a special Lagrangian 3-torus $T^3 \to M$, and then the mirror Calabi-Yau manifold $\hat{M}$ is got by "dualising" these fibres $T^3 \to M$ in some way. The proposal of Strominger, Yau and Zaslow has come to be known as the *SYZ Conjecture*, and even its precise formulation has not yet been worked out. In any case, to make some progress towards understanding mirror symmetry via the SYZ Conjecture, we need to understand fibrations of Calabi-Yau manifolds by special Lagrangian submanifolds, and in particular, tori. The cases we are most interested in have $\mathrm{Hol}(M, g) = \mathrm{SU}(3)$. It turns out that in this situation, if $\pi : M \to B$ is a fibration as above, then there must exist fibres $\pi^{-1}(b) \subseteq M$ which are singular. If we let $B_0 \subseteq B$ denote the subset of $b \in B$ such that $\pi^{-1}(b)$ is non-singular, then one part of understanding the global properties of the fibration $\pi : M \to B$ is to work out what happens to the fibres $\pi^{-1}(b)$ as $b \in B_0$ approaches the singular locus $B \setminus B_0 \subseteq B$.

Joyce, in a recent series of articles [28], [31], [33], [37] has begun a programme to help understand the issues raised in the previous paragraph, and we now outline some of that programme. For a more detailed description of the following discussion, see in particular [31]. Suppose that M is any Calabi-Yau manifold, and that $X \subseteq M$ is a compact special Lagrangian submanifold. Then by the McLean Theorem ([50, Theorem 3-6], [5, Theorem 2.2.27] and Section 3.2 below) the submanifold $X \subseteq M$ is contained in a smooth, connected, moduli space $\mathcal{M}$ of compact special Lagrangian submanifolds $X_\xi \subseteq M$, and the dimension of $\mathcal{M}$ is $b^1(X)$, the first Betti number of X. We now think about *compactifying* the manifold $\mathcal{M}$, by adding a boundary $\partial\mathcal{M}$ consisting of singular special Lagrangian submanifolds $X_{\mathrm{sing}} \subseteq M$, and ask what happens to the non-singular elements of $\mathcal{M}$ as they approach the singular elements of the boundary $\partial\mathcal{M}$.

If $\dim M = 2n$ then the manifold $\mathbb{C}^n$ is itself (non-compact) Calabi-Yau, and is an approximate local model for the Calabi-Yau manifold M. Let $X_{\mathrm{sing}} \subseteq M$ be an element of $\partial\mathcal{M}$, and pick a singular point $p \in X_{\mathrm{sing}}$. Using Geometric Measure Theory, as in the book [17] of Federer, we can define the *tangent cone* $C \subseteq \mathbb{C}^n$ to X_{sing} at p, and moreover, C will be a special Lagrangian cone in $\mathbb{C}^n$. Provided the singularity at p is not too badly behaved, there will be an open neighbourhood U of $p \in M$ such that $U \cap X_{\mathrm{sing}}$ looks approximately like the cone $C \subseteq \mathbb{C}^n$. We further assume, for simplicity, that p

is the only singular point of X_{sing}, so that in particular, p is isolated, and therefore $0 \in C$ is also an isolated singular point.

In vague terms, we expect to be able to *desingularise* $X_{\text{sing}} \subseteq M$ by cutting out the open subset $U \cap X_{\text{sing}} \subseteq X_{\text{sing}}$, and gluing back in its place a submanifold $X_{\text{ac}} \subseteq \mathbb{C}^n$ which is asymptotic to the cone C in some way. The result after this gluing process will be a *non-singular* special Lagrangian submanifold $X_\xi \subseteq M$, an element of the moduli space $\mathcal{M}$ which is close to X_{sing} in $\overline{\mathcal{M}} = \mathcal{M} \cup \partial\mathcal{M}$.

Suppose that there is a smooth portion W of the boundary $\partial\mathcal{M}$ whose elements have 1 isolated singularity, modelled on a cone C as above. Then we expect that, near W, the smooth manifold $\mathcal{M}$ looks like $W \times \frac{\mathcal{M}_{\text{ac}}}{\mathbb{C}^n}$, where $\mathcal{M}_{\text{ac}}$ is some moduli space of special Lagrangian submanifolds in $\mathbb{C}^n$, which are asymptotic to the cone C, at some prescribed decay rate, and $\mathbb{C}^n$ acts on this space via translations. In particular, we would have

$$b^1(X_\xi) = \dim \mathcal{M} = \dim \partial\mathcal{M} + \dim \mathcal{M}_{\text{ac}} - 2n$$

for special Lagrangian submanifolds $X_\xi \in \mathcal{M}$ near the open subset W of the boundary $\partial\mathcal{M}$.

It follows from the above discussion that, for a given special Lagrangian cone $C \subseteq \mathbb{C}^n$, possibly singular at 0, but elsewhere smooth, we are interested in the set $\mathcal{M}_{\text{ac}}$ of special Lagrangian submanifolds in $\mathbb{C}^n$ which are asymptotic to the cone C, with some prescribed rate of decay. In particular, we hope to prove that this set $\mathcal{M}_{\text{ac}}$ is a smooth manifold, and then to calculate the dimension $\dim \mathcal{M}_{\text{ac}}$. These issues provide the primary motivation for this thesis.

1.2 The main results

Our first result is a rigorous proof of McLean's Theorem, which tells us that compact special Lagrangian submanifolds live in smooth moduli spaces. No doubt McLean was aware of the details of his proof, but these are lacking in the published version [50, Theorem 3-6]. The purpose of providing an explicit proof is therefore to fill an existing gap in the literature, but also the version we shall give is an extension of the original theorem, in that we take into account perturbations of the ambient Calabi-Yau structure. Our result is given as Theorem 3.21, and we include here a brief version.

Theorem 1.1 (Extended McLean Theorem) *Let M be a manifold and $\big(J(p), g(p), \Omega(p)\big)$ a smooth family of Calabi-Yau structures on M, parameterised by $p \in \mathbb{R}^m$, and suppose that $X \subseteq M$ is a compact submanifold which is special Lagrangian with respect to $\big(J(0), g(0), \Omega(0)\big)$. If $[\omega(p)|_X] = [\operatorname{Im}\Omega(p)|_X] = 0$ in $H^*(X)$ for all $p \in \mathbb{R}^m$ then there exist an open subset $W \subseteq \mathbb{R}^m$ containing 0, and a family $(\mathcal{M}_p)_{p\in W}$ of smooth manifolds, each with dimension $\dim \mathcal{M}_p = b^1(X)$, such that for each $p \in W$, $\mathcal{M}_p$ parameterised the smooth submanifolds $X_\xi \subseteq M$ near to X which are special Lagrangian with respect to $\big(J(p), g(p), \Omega(p)\big)$. Moreover the total space $\mathcal{M} := \bigcup_{p\in W} \mathcal{M}_p$ is smooth.*

Another reason for giving a detailed proof of Theorem 1.1 is that, at least conceptually, it can be carried straight over to the asymptotically conical case that we shall consider later in Chapter 6.

We note here that Baier [5] has also provided a proof of the original theorem of McLean, but he uses a different method to ours in Section 3.2 below. Also, there have been versions of McLean's Theorem for compact special Lagrangian submanifolds X, with boundary $\partial X \neq \emptyset$ satisfying certain conditions: see the article [9, Main Theorem] of Butscher.

Before stating our main result, we shall give some definitions. Let Σ be a compact manifold, with connected components

$$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_L.$$

By a *manifold with ends* we mean a connected manifold X which off some compact subset $X_0 \subseteq X$ is diffeomorphic to the product $(T, \infty) \times \Sigma$, where $T \in \mathbb{R}$. We shall always consider $X \setminus X_0$ and $(T, \infty) \times \Sigma$ as being identified by some fixed diffeomorphism. For the purposes of this thesis, we shall always insist that a manifold X with ends has dimension $\dim X =: n \geqslant 3$. In the theory we give, this condition is important, and we explain why in the description of Chapter 6 below.

Suppose further that $C \subseteq \mathbb{C}^n$ is a cone, smooth away from 0, and that $\Sigma = S^{2n-1} \cap C$. Then there is an embedding

$$\begin{aligned} i_C : \mathbb{R} \times \Sigma &\rightarrow \mathbb{C}^n \\ (t, \sigma) &\mapsto e^t \sigma \end{aligned}$$

with image $C \setminus \{0\}$, a smooth submanifold of $\mathbb{C}^n$. If $\tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_L) \in \mathbb{R}^L$, with $\tilde{\alpha}_j < 1$ for each $1 \leqslant j \leqslant L$, and $X \subseteq \mathbb{C}^n$ is a submanifold, we shall say that X is *strongly asymptotically conical with cone C and rate $\tilde{\alpha}$* if for each $1 \leqslant j \leqslant L$ we have

$$|i_X - i_C| = O(e^{\tilde{\alpha}_j t}) \tag{1.1}$$

on $(T, \infty) \times \Sigma_j$, together with appropriate decay in the derivatives of $i_X - i_C$. In equation (1.1) we use i_X to denote the inclusion $X \setminus X_0 \subseteq \mathbb{C}^n$, considered as a map $(T, \infty) \times \Sigma \rightarrow \mathbb{C}^n$.

In the above situation, the link Σ of C inherits a Riemannian metric g_Σ from the Euclidean metric on $\mathbb{C}^n$. For each $1 \leqslant j \leqslant L$, let μ_j denote the first positive element of the spectrum of the Laplacian $\Delta : C^\infty(\Sigma_j) \rightarrow C^\infty(\Sigma_j)$, and then define λ_j to be the unique positive real number such that $\lambda_j(\lambda_j + 2 - n) = \mu_j$, together with $\lambda = (\lambda_1, \ldots, \lambda_L)$. Our main result, Theorem 6.45, can now be summarised as follows:

Theorem 1.2 (Deformation Theorem for AC special Lagrangians in $\mathbb{C}^n$) *Suppose that $X \subseteq \mathbb{C}^n$ is special Lagrangian and strongly asymptotically conical with cone $C \subseteq \mathbb{C}^n$ and rate $\tilde{\alpha} \in \mathbb{R}^L$, where $\tilde{\alpha} < 1$. Then, provided that $\tilde{\alpha} > 1 - n - \lambda$, and $\tilde{\alpha}$ is generic, there is a smooth moduli space $\mathcal{M}_{\tilde{\alpha}}$ of special Lagrangian submanifolds $X_\xi \subseteq \mathbb{C}^n$ which are strongly asymptotically conical with cone C and rate $\tilde{\alpha}$. Moreover,*

$$\dim \mathcal{M}_{\tilde{\alpha}} = \begin{cases} b^1_c(X) - L + 1 & \text{if } 1 - n - \lambda < \tilde{\alpha} < 1 - n \\ b^1_c(X) & \text{if } 1 - n < \tilde{\alpha} < -1 \\ b^1(X) + L - 1 & \text{if } -1 < \tilde{\alpha} < \lambda \\ b^1(X) + L - 1 + \chi(\tilde{\alpha} + 1) & \text{if } \lambda < \tilde{\alpha} < 1 \end{cases}$$

where $\chi(\tilde{\alpha} + 1) > 0$ is an analytic term got from the spectrum of the Laplacian of the link Σ, acting on functions, and we write $\beta < \delta$ whenever $\beta_j < \delta_j$ for each $1 \leqslant j \leqslant L$.

Related results have been proved recently by Pacini [55], but this work was carried out independently, and nearing completion at the time [55] was published. However, our methods differ somewhat in that we use different analytic machinery to reach our respective goals. The author's route is via a Hölder space version of the Lockhart-McOwen Theory of [45] and [46], and essentially deals with non-compact manifolds, without boundary. Pacini, on the other hand, uses the pseudo-differential operator theory of Melrose, as described in [51], where the emphasis is on Sobolev spaces over compact manifolds with boundary.

In Pacini's work, the ambient space M containing the submanifold X is taken to be asymptotically conical, rather than just $\mathbb{C}^n$, so in that respect, is more general than the work of the author. However, the theory presented here has advantages over that of [55] in that more general growth rates are considered (he proves the case which, in our notation, is $\tilde{\alpha} = -1 + \varepsilon$ for small $\varepsilon > 0$, and states the corresponding result for the case $\tilde{\alpha} = -\frac{n}{2}$ when $M = \mathbb{C}^n$). Also, Pacini seems to encounter some kind of obstruction at the boundary ∂X of his compact manifold X, so that he cannot infer his deformed submanifolds X_ξ have the smoothness one would hope for on their boundary ∂X_ξ.

Theorem 1.2 provides us with new examples of special Lagrangian submanifolds of $\mathbb{C}^n$. For example, if $a_1, \ldots, a_8, b \in \mathbb{R}$ with $b \neq 0$ then the $\mathrm{U}(1)^7$-invariant special Lagrangian submanifold

$$X := \left\{ (z_1, \ldots, z_8) \in \mathbb{C}^8 : \begin{array}{l} |z_1|^2 - a_1 = \cdots = |z_8|^2 - a_8 \\ \mathrm{Im}(iz_1 \ldots z_8) = b \end{array} \right\}$$

of Harvey and Lawson [21] moves in a smooth moduli space $\mathcal{M}_{\tilde{\alpha}}$ of special Lagrangian submanifolds, where

$$\dim \mathcal{M}_{\tilde{\alpha}} = 376.$$

This can be quickly deduced from the relevant example of Section 6.3.4. Here $\mathcal{M}_{\tilde{\alpha}}$ consists of the special Lagrangian submanifolds $X_\xi \subseteq M$ which are near X and strongly asymptotically conical with cone

$$C := \left\{ (z_1, \dots, z_8) \in \mathbb{C}^8 : \begin{array}{c} |z_1| = \dots = |z_8| \\ \mathrm{Im}(iz_1 \dots z_8) = 0 \end{array} \right\}$$

and rate $\sqrt{24} - 4 < \tilde{\alpha} < 1$.

1.3 An overview of the chapters

Chapter 2

In Section 2.1 we describe some standard material from functional analysis and differential geometry. The most important results here are the Implicit Function Theorems 2.10 and 2.11, which we use to show the moduli spaces $\mathcal{M}$ and $\mathcal{M}_{\tilde{\alpha}}$ of our main theorems are in fact smooth.

Section 2.2 is a description of some of the relevant aspects of submanifold theory. We start by stating the Tubular Neighbourhood Theorem 2.15, which gives us our basic method for deforming submanifolds. The latter half of Section 2.2 is aimed at demonstrating why special Lagrangian submanifolds must necessarily be smooth. The route is via calibrated geometry, minimal submanifolds and then the usual elliptic regularity results from PDE theory. We need this material to show that the *points* of our moduli spaces $\mathcal{M}$ and $\mathcal{M}_{\tilde{\alpha}}$ are smooth as maps between manifolds.

In Section 2.3 we outline some of the relevant theory from Calabi-Yau and special Lagrangian geometry. After some basic definitions and examples, we go on to demonstrate why the infinitesimal deformations of a special Lagrangian submanifold $X \subseteq M$ may be thought of as the closed and coclosed 1-forms on X. This gives us a good idea of what the tangent spaces, and hence dimension, of our moduli spaces $\mathcal{M}$ and $\mathcal{M}_{\tilde{\alpha}}$ should be: using the Implicit Function Theorems 2.10 and 2.11 we aim to write $\mathcal{M}$ and $\mathcal{M}_{\tilde{\alpha}}$ as the graph of some smooth map on this tangent space, at least locally.

Chapter 3

In Chapter 3 we give a proof of Theorem 1.1.

The first half, Section 3.1, is again standard. We give it for completeness, and as a model for the non-compact setting we shall consider. The main results here are the embedding and compactness results Theorem 3.2 and Theorem 3.3, and then the elliptic estimates of Theorem 3.4. The essential arguments in each of these theorems come from PDE theory, and yet with them one can prove powerful results such as the "Fredholm Alternative", Theorem 3.8, which has many applications in geometry. We give a proof of Theorem 3.8, our motivation being that the methods used carry over to the non-compact case we shall consider in Section 4.3.3.

In Section 3.1.3 we give an application of our general theory, and describe Hodge Theory for compact, Riemannian manifolds (X, g). Then one can deduce, for example, that

$$\dim\big\{\xi \in C^\infty(T^*X) : \mathrm{d}_g^*\xi = \mathrm{d}\xi = 0\big\} = \dim H^1(X) =: b^1(X) \tag{1.2}$$

the 1st Betti number of X. We included proofs in Section 3.1.3 so that the reader can see the techniques required to obtain results such as (1.2). These techniques are simply not available in the non-compact case, and the corresponding theorems will not hold. We come back to this point in Section 6.1.3, where we try to establish just what we *can* say.

Armed with the preliminary material of Chapter 2 and Section 3.1, we go on in Section 3.2 to prove our McLean-type result. The main result here is Theorem 3.21, which is an expanded, more precise, version of Theorem 1.1 given above. The general idea is that, if $X \subseteq M$ is a submanifold which is special Lagrangian with respect to $\big(J(0), g(0), \Omega(0)\big)$, then one can identify small 1-forms ξ on X with submanifolds X_ξ which are close to X. One then defines a "deformation" map F whose value on a pair (p, ξ) measures how far away the submanifold X_ξ is from being special Lagrangian with respect to $\big(J(p), g(p), \Omega(p)\big)$, so that the fibre $F^{-1}(0)$ is our total space of special Lagrangian deformations. We set things up in such a way so that F is a smooth map between open subsets of Banach spaces, with derivative $F'(0)$ that is surjective, and with kernel

$$\operatorname{Ker} F'(0) = \left\{\xi \in C^\infty(T^*X) : \mathrm{d}_g^*\xi = \mathrm{d}\xi = 0\right\} \oplus \mathbb{R}^m.$$

Now applying the Inverse Function Theorem shows that $F^{-1}(0)$ is a smooth manifold in a neighbourhood of $(0,0)$, with dimension $b^1(X) + m$.

Chapter 4

For the rest of the thesis, we leave the compact case behind, and concentrate on the situation in which X is a manifold with ends, in the sense described above. The general idea of Chapter 4 is to provide an exposition of some material from the literature, and then to adapt this material to a situation which shall useful for us later. Section 4.2 only is meant to contain the Fredholm material that we shall quote, the source being the Lockhart-McOwen Theory of [45] and [46], but also in that section we include additional useful results, not to be found in these papers.

We begin in Section 4.1 by establishing our basic objects of study, together with notation we shall use time and again in the sequel. Supposing that X is a manifold with ends, we define the notion of an *admissible vector bundle* E on X: these are the basic vector bundles we shall work with. It turns out that bundles of tensors $(\otimes^s TX) \otimes (\otimes^r T^*X)$, and forms $\Lambda^r T^*X$, are admissible, and this certainly covers the applications we have in mind. We then go on in Section 4.2.1 to define Banach spaces $W^p_{k,\beta}(E)$ and $B^{k,a}_\beta(E)$, consisting of sections of an admissible bundle E. We call these spaces *damped Sobolev spaces* and *damped Hölder spaces*, respectively. The indices p, k, a mean pretty much what they do in the compact case, and one can think of the index β as imposing a constraint to decay at rate $O(e^{\beta t})$ on the subset $X \setminus X_0$. Actually, β is an L-tuple $\beta = (\beta_1, \ldots, \beta_L)$, and each β_j reflects the order of decay on the component $(T, \infty) \times \Sigma_j$. We also quote in Section 4.2.1 some embedding and compactness results from the literature for the spaces $W^p_{k,\beta}(E)$, $B^{k,a}_\beta(E)$. These results are the first part of our "tool-kit" for analysis on manifolds with ends.

It turns out that one *must* work with the damped spaces of the previous paragraph if one wishes to have a good Fredholm theory for differential operators on X: to quote an example of Lockhart [45, Equation (0.2)], if h is a metric on X, tending towards a cylindrical metric $\tilde{h}$ on the ends X_∞ of X, then

$$\Delta^r_h : L^2_2(\Lambda^r T^*X) \to L^2(\Lambda^r T^*X) \tag{1.3}$$

is Fredholm precisely when $H^r(\Sigma) = H^{r-1}(\Sigma) = 0$, so that the usual L^2-Sobolev spaces on X are too restrictive for Fredholm theory. We will consider similar sorts of issues, for asymptotically conical metrics, in Section 6.1.2.

In Section 4.2.2 and Section 4.2.3 we describe the core of the Lockhart-McOwen Theory. Suppose that P is an elliptic differential operator, of order $l \geqslant 1$, acting between admissible bundles E and F. In the paper [46], Lockhart and McOwen prove that the operator P has a *good* Fredholm theory, provided that P tends towards some elliptic, translation invariant operator P_∞, on the ends of X. For such operators, they give elliptic estimates for the maps

$$P : W^p_{k+l,\beta}(E) \to W^p_{k,\beta}(F), \tag{1.4}$$

together with an explicit characterisation of when the map (1.4) is Fredholm. It turns out that (1.4) is Fredholm for β in an open, dense, subset $\mathbb{R}^L \setminus \mathcal{D}(P)$ of $\mathbb{R}^L$ which is independent of p and k. Lockhart and McOwen further give a "jumping" formula telling us how the index of (1.4) changes as β moves from one connected component of $\mathbb{R}^L \setminus \mathcal{D}(P)$ to another. It turns out that the subset $\mathcal{D}(P)$ and the size of these "jumps" can be got in a very explicit way from the limit operator P_∞. This theory will be invaluable for us in Chapter 5. Note that Lockhart and McOwen do not say anything about operators P as above acting between the damped Hölder spaces. Indeed, in the literature generally there tends to be a bias towards the use of Sobolev spaces of some type, rather than Hölder spaces. In Section 6.1.1 we attempt to redress the balance, at least for Fredholm theory on manifolds with ends.

In Section 4.3 we convert the Lockhart-McOwen Theory of Section 4.2 over to the case of *asymptotically conical* manifolds, and specialise to the case where the admissible bundles E, F on X are made up of tensors or forms. Most of this conversion is nothing more that a change of language, but we give it anyway because the applications we have in mind are best phrased in this new terminology. In Section 4.3.1 we define new classes $L^p_{k,\beta}(E)$, $C^{k,a}_\beta(E)$ of Banach spaces of sections. These spaces can be given a coordinate-free description in terms of an asymptotically conical metric on X. We therefore refer to the $L^p_{k,\beta}(E)$, $C^{k,a}_\beta(E)$ as *conical damped Sobolev spaces* and *conical damped Hölder spaces*, respectively. As before, the indices p, k, a have their usual significance, and β refers to the order of growth of our tensors or forms on X, as measured using an asymptotically conical metric g. It follows that the growth rate $\beta = -\frac{n}{2}$ corresponds to the usual L^2-Sobolev spaces, built using g.

Our definition of asymptotically conical is quite weak, requiring (for metrics, at least) only rate $o(1)$ decay: this is all that is needed to apply the material of 4.2. Later, in Section 5.1.4, we shall consider stronger rates of decay for our asymptotically conical metrics, namely $O(e^{\alpha t})$, where $\alpha < 0$. We shall need these stronger decay rates to deduce certain things which do not seem to be obtainable with mere $o(1)$ decay. In Section 4.3.2 we define the notion of an *asymptotically conical operator* Q. Essentially, this is just an operator corresponding to the asymptotically translation invariant operators of Section 4.2.3, together with an additional damping factor $O(e^{-\gamma t})$. We call γ the *rate* of the operator Q.

We end Chapter 4 by giving a useful characterisation of the image of an asymptotically conical operator

$$Q : L^p_{k+l,\beta+\gamma}(E) \to L^p_{k,\beta}(F). \tag{1.5}$$

This is in the literature, at least implicitly: see the paper [10] of Cantor. The method of proof uses the Lockhart-McOwen material in an essential way. It also turns out that a key part of the proof is the fact that the Sobolev spaces $L^p_{k,\beta}(F)$ are *reflexive*. Specifically, we can write $L^p_{0,\beta}(F)^* \cong L^{p'}_{0,-\beta-n}(F)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then we can proceed pretty much as for the compact case. Note however that, in considering the operator

$$Q : C^{k+l,a}_{\beta+\gamma}(E) \to C^{k,a}_\beta(F) \tag{1.6}$$

we have neither the Lockhart-McOwen Theory, nor the reflexivity we require.

Chapter 5

Here we give some applications of the machinery from the previous chapter. We suppose that g is an asymptotically conical metric on our manifold with ends X, and start by looking at the Laplacian Δ^0_g of g, acting on functions. It turns out that this operator is asymptotically conical, with rate 2, so that we have a map

$$\Delta^0_g : L^p_{k+2,\beta+2}(X) \to L^p_{k,\beta}(X). \tag{1.7}$$

A key issue shall be when the map (1.7) is Fredholm, and this boils down to working out the exceptional set $\mathcal{D}(\Delta^0_g) \subseteq \mathbb{R}^L$ that is mentioned above. It turns out that

$$\mathcal{D}(\Delta^0_g) = \big(\mathcal{D}(P_\infty, 1) \times \mathbb{R}^{L-1}\big) \cup \big(\mathbb{R} \times \mathcal{D}(P_\infty, 2) \times \mathbb{R}^{L-2}\big) \cup \cdots \cup \big(\mathbb{R}^{L-1} \times \mathcal{D}(P_\infty, L)\big)$$

where each $\mathcal{D}(P_\infty, j)$ consists of points $\beta_j + 2 \in \mathbb{R}$ such that $(\beta_j + 2)(\beta_j + n)$ lies in the spectrum of the Laplacian on component Σ_j of Σ. We then compute the relevant index "jumps", and these turn out to be equal to the dimensions of the corresponding eigenspaces. Next, using the Maximum Principle, together with elliptic regularity and the embedding theorems, we show that (1.7) is injective for all $\beta+2 < 0$, and then the material of Section 4.3.3 shows that (1.7) is surjective for all $\beta+2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$ with $\beta + 2 > 2 - n$. It follows that proving the existence of g-harmonic functions on X, with growth rates $\beta + 2 \geqslant 0$, comes down to finding eigenfunctions for the Laplacian on the compact Riemannian manifold (Σ, g_Σ).

The main reason for the interest in the Laplacian Δ^0_g is that, if h is a harmonic function, then $\mathrm{d}h$ is a closed and coclosed 1-form on X: we shall be interested in these objects in the sequel, where $X \subseteq \mathbb{C}^n$ is a special Lagrangian submanifold. Actually, we must take growth rates $\beta + 2$ into account, and so consider

$$K_{\beta+1} := \big\{\xi \in C^\infty_{\beta+1}(T^*X) : \mathrm{d}^*_g\xi = 0 \text{ and } \mathrm{d}\xi = 0\big\},$$

a finite-dimensional vector space, containing $\mathrm{d}h$ for all h in the kernel of the map (1.7).

Much of Chapter 5 is aimed at acquiring a good understanding of the space $K_{\beta+1}$, for as large a range of $\beta + 1$ as is possible. The exact elements of $K_{\beta+1}$ will be the of the form $\mathrm{d}h$ for some harmonic function $h : X \to \mathbb{R}$. It turns out that most of these harmonic functions *will* be in the kernel of the map (1.7), but there will be others if $2 - n < \beta + 2 < 0$: there exist non-constant harmonic functions $h_1, \ldots, h_L : X \to \mathbb{R}$ such that each h_j tends to δ_{jk} on the kth end of X, at rate $O(e^{(\beta+2)t})$, for any $2 - n < \beta + 2 < 0$. Therefore the exterior derivatives $\mathrm{d}h_j$ have decay $\beta + 1$ for each $2 - n < \beta + 2 < 0$. Essentially, the vector space $\mathrm{Span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_L\}$ is the kernel of the natural map $\phi_1 : H^1_c(X) \to H^1(X)$.

The number of non-exact elements of $K_{\beta+1}$ is measured by how much of the cohomology $H^1(X)$ the space $K_{\beta+1}$ represents. We prove that this is the whole group if $\beta + 2 > 0$, and the image of the natural map $H^1_c(X) \to H^1(X)$ if $2 - n < \beta + 2 < 0$.

Using the above facts, together with some additional considerations, we build up a detailed picture of $K_{\beta+1}$ for a large range of $\beta + 2$. It turns out that for $\beta + 2 < 2 - n$, or for $\beta + 2$ in $\mathcal{D}(\Delta^0_g)$, we need to assume stronger decay assumptions on our metric g, and we introduce the notion of *strongly* asymptotically conical metrics in Section 5.1.4, together with some useful consequences. The final results for the chapter are then given in Section 5.2.3.

Chapter 6

In Chapter 6 we bring together all of the above results, to deduce the deformation theorem for strongly asymptotically conical special Lagrangian submanifolds $X \subseteq \mathbb{C}^n$. The corresponding result in Chapter 6 is Theorem 6.45, which is a more accurate and complete version of the Theorem 1.2 given above. Besides the deformation theorem, we also give some Hodge-theoretic results which should have useful applications in the analysis of Laplacians, and related operators, on asymptotically conical Riemannian manifolds.

We begin in Section 6.1.1 by deriving a Fredholm theory for asymptotically conical operators

$$Q : C^{k+l,a}_{\beta+\gamma}(E) \to C^{k,a}_{\gamma}(F).$$

The results we obtain are very general, and so should have many applications. It is nice to know that the conical damped Hölder spaces admit a good Fredholm theory, as in the case of the $L^p_{k,\beta}(E)$ spaces. Note that we cannot find this material in the literature. The main issue is the existence of a Green's function for the corresponding limit operator Q_∞ on the full cone $\mathbb{R} \times \Sigma$, and this is given in [49]. The Fredholm theory for (conical) damped Hölder spaces can then be deduced using the techniques that are normally only applied in the situation of (conical) damped Sobolev spaces, as in [6]. Because of the differences between the two types of spaces, we need to adapt some of the techniques for our own ends.

We also see in Section 6.1.1 why we have insisted on X having dimension at least 3. This is essentially because the Green's function for the Laplacian on $\mathbb{R}^n$ has different behaviour in the cases $n = 2$ and $n \geqslant 3$. If we allow $\dim X = 2$, then many of the results of Chapter 5 will fail to hold.

Note that the material of Section 6.1.1 could also have been placed at the end of Chapter 4, but we prefer its present location: our particular methods for Chapter 5 do not rely on the material of Section 6.1.1, which we view as a second, separate, application of the material of Chapter 4. The main place we need the Fredholm results for conical damped Hölder spaces is Section 6.2, and so we keep the relevant theory nearby.

In Section 6.1.2 we give the results of some explicit details that have been worked out privately by the author, and we hope they will have useful applications in the future. They are the calculations of the exceptional sets $\mathcal{D}(Q) \subseteq \mathbb{R}^L$ for certain asymptotically conical operators Q. We give these exceptional sets in the cases where Q is taken to be the Laplacian Δ^r_g of an asymptotically conical metric g, on the bundle of r-forms $\Lambda^r T^* X$, and also where Q is the operator $\mathrm{d}^*_g + \mathrm{d}$ on the odd, even and total exterior bundles over X. The results we obtain are analogous to the example of Lockhart given in equation (1.3) above.

Other applications of the material on exceptional sets are given in Section 6.1.3. Here we attempt to develop Hodge theoretic results for our non-compact manifold X, under assumptions which are as general as possible. One can think of the results of Section 3.1.3, for compact manifolds, as being a collection of lemmas which piece together to give grand results such as the Hodge Decomposition, as in equation (3.31) for example. In Section 6.1.3 we take each of these lemmas, and see how far they will generalise on an asymptotically conical manifold X. For example, as described in Lemma 6.24, we deduce that any form $C^0_{\beta+2}(\Lambda^r T^*X)$ which is harmonic must necessarily be closed and coclosed if $\beta+2<-r$ or $\beta+2<r-n$. This is a big improvement on the usual integration by parts argument got from L^2-decay, corresponding to growth rate $\beta+2=1-\frac{n}{2}$.

We now come to the proof of our main result, and this is given in Section 6.2. We begin in Section 6.2.1 by describing our basic objects of study, namely submanifolds $X\subseteq\mathbb{R}^m$, which have some prescribed rate of convergence towards a cone $C\subseteq\mathbb{R}^m$. We call these types of submanifold X either *asymptotically conical with cone* C, or, for $\tilde{\alpha}\in\mathbb{R}^L$ with $\tilde{\alpha}<1$, *strongly asymptotically conical with cone* C *and rate* $\tilde{\alpha}$: the latter rate of convergence is stronger than the former. It turns out that the metric g induced on the submanifold X then decays towards the metric $\tilde{g}$ induced on the cone C. Submanifolds which are asymptotically conical with cone C have metrics g which are asymptotically conical, so that we are in the regime of Chapters 4 and 5. Submanifolds which are strongly asymptotically conical, with cone C and rate $\tilde{\alpha}$, have metrics g which are strongly asymptotically conical, with rate $\alpha:=\tilde{\alpha}-1<0$, so that we may apply the additional theory of Section 5.1.4.

In Section 6.2.2 we consider asymptotically conical submanifolds $X\subseteq\mathbb{C}^n$ which are Lagrangian. Then, as in the compact case, we can identify the normal bundle N of X with the cotangent bundle T^*X, and take a tubular neighbourhood $\tilde{U}\subseteq N$ of X, corresponding to some $U\subseteq T^*X$. The tubular neighbourhood $\tilde{U}\cong U$ allows us to identify submanifolds $X_\xi\subseteq\mathbb{C}^n$ which are “near” to X, with “small” 1-forms on our submanifold X. Moreover, we can relate the decay rate of the form ξ in the conical damped Hölder spaces $C^{k,a}_\beta(T^*X)$ with the decay of the submanifold X_ξ towards the cone C.

Finally, in Section 6.2.3, we consider strongly asymptotically conical special Lagrangian submanifolds. We bring together the results of the previous chapters to prove our main result, Theorem 6.45, which is a more complete and accurate version of Theorem 1.2 given above. The ideas here are exactly as for the compact case, the difference being that the details are harder. Using the Implicit Function Theorem 2.11, we write our moduli space $\mathcal{M}_{\tilde{\alpha}}$, in a neighbourhood of X, as the graph of a smooth function defined on an open subset of $K_{\tilde{\alpha}}=K_{\alpha+1}$. This further provides the dimension $\dim K_{\alpha+1}$ of $\mathcal{M}_{\tilde{\alpha}}$, which we have computed in Chapter 5.

We round off in Section 6.3 by applying our main theorem to examples of strongly asymptotically conical submanifolds $X\subseteq\mathbb{C}^n$. Due to the work of Joyce [26], [27], [28], [29], [30], [32], [34], [35], [36], [37], and others, there is a plethora of examples to apply our theory to. As we have indicated above, we show the existence of new families of asymptotically conical special Lagrangian submanifolds in $\mathbb{C}^n$, as well as proving results which show that other examples are, in certain circumstances, *isolated*, say modulo translations if we are considering growth rates $\alpha+1>0$.

1.4 Further work

We briefly outline some directions in which our deformation theory could be extended or improved.

1. As mentioned above, we could consider the situation in which the ambient manifold is not $\mathbb{C}^n$, but a general asymptotically conical Calabi-Yau manifold. Although Pacini [55] has proved a theorem along these lines, he only considers the growth rate $\alpha+1=-1+\varepsilon$. The proof of a result with a more general ambient space should not be too arduous, as the material of Chapter 4 and Chapter 5 will carry through unchanged. Of course, Section 6.2 will need modifying.

2. We could consider the deformation theory of *singular* special Lagrangian submanifolds in $\mathbb{C}^n$, with singularities modelled on special Lagrangian cones $C\subseteq\mathbb{C}^n$ with an isolated singularity at 0. This theory would share certain features with the material we present here, but there would be some additional issues to deal with.

3. We could investigate the deformations of asymptotically conical submanifolds got from the other Harvey and Lawson calibrations, mentioned in Section 1.1. Note that our class of asymptotically

conical operators is large enough for the theory of Chapter 4 and Chapter 5 to be applied in these situations also. A good place to start would be the coassociative case, whose deformation theory would have certain similarities with the special Lagrangian case we consider in this thesis.

4. Last (and probably least!) we could consider extending our deformation theory to more negative growth rates, and give dimension formulae as in Table 5.1 for growth rates $\beta+1$ less than $1-n-\lambda$, to use the notation of that chapter. In the same vein, but with entirely different methods of attack, we could attempt to remove the hypothesis $\alpha+2 \notin \mathcal{D}(\Delta^0_g)$ in the statement of Theorem 6.45.

1.5 Notation and conventions

Here are some conventions we adopt. We also give, in Table 1.1, a selection of the notation we shall use for the rest of this thesis. We urge the reader to consult Table 1.1 when confronted with a piece of notation they have not met, especially since some of these notations are not explicitly defined in the text.

1. All manifolds are smooth, connected and have empty boundary, unless explicitly stated otherwise.

2. If W is a vector space then we embed $\Lambda^r W$ inside $\otimes^r W$ via the map

$$w_1 \wedge \ldots \wedge w_r = \frac{1}{r!} \cdot \sum_{\sigma \in S_r} \mathrm{Sign}(\sigma) \cdot w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(r)} =: \mathrm{Alt}(\omega_1 \otimes \cdots \otimes \omega_r)$$

for all $w_1, \ldots, w_r \in W$. We denote the rth symmetric power of W by $\mathrm{Sym}^r W$, and embed $\mathrm{Sym}^r W$ inside $\otimes^r W$ via the map

$$w_1 \odot \cdots \odot w_r = \frac{1}{r!} \cdot \sum_{\sigma \in S_r} w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(r)} =: \mathrm{Sym}(\omega_1 \otimes \cdots \otimes \omega_r)$$

for all $w_1, \ldots, w_r \in W$. These conventions extends to vector bundles over manifolds, especially when we view r-forms as multilinear mappings on tangent spaces.

3. Suppose that $\beta_j, T \in \mathbb{R}$ and $f : [T, \infty) \to \mathbb{R}$ is some function. We write $f(t) = O(e^{\beta_j t})$ to mean that $|e^{-\beta_j t} f(t)|$ is bounded on $[T, \infty)$. A stronger requirement is $f(t) = o(e^{\beta_j t})$ which means that $e^{-\beta_j t} f(t) \to 0$ as $t \to \infty$.

4. When $\beta = (\beta_1, \ldots, \beta_L)$, $\delta = (\delta_1, \ldots, \delta_L)$ are elements of $\mathbb{R}^L$ we write $\beta \leqslant \delta$ if $\beta_j \leqslant \delta_j$ for each $1 \leqslant j \leqslant L$, and similarly for the strict inequality. Also, if $u \in \mathbb{R}$ we sometimes abuse notation and write $\beta + u$ to mean the L-tuple with jth entry $\beta_j + u$.

5. If $1 < p < \infty$ we define the *dual exponent* $1 < p' < \infty$ by $\frac{1}{p} + \frac{1}{p'} = 1$.

6. If (X, g) is a Riemannian manifold we denote the Laplace operator, acting on r-forms, by Δ^r_g. If (Σ, g_Σ) is a compact, Riemannian manifold, we denote the spectrum of the operator $\Delta^r_{g_\Sigma}$ by $\mathrm{Spec}(\Sigma, g_\Sigma, r)$, which is a discrete, countable subset of $[0, \infty)$.

7. Suppose an open subset $G \subseteq \mathbb{R}^n$ has coordinates $(x_1, \ldots, x_n)$. Given $1 \leqslant j \leqslant n$ we define $\partial_j := \frac{\partial}{\partial x_j}$ a differential operator on functions $G \to \mathbb{R}$. A *multi-index* is some n-tuple of non-negative integers $\lambda = (\lambda_1, \ldots, \lambda_n)$. Given such a λ we define $|\lambda| = \sum_{j=1}^n \lambda_j$ and $\partial^\lambda := \partial_1^{\lambda_1} \ldots \partial_n^{\lambda_n}$ a differential operator of order $|\lambda|$. If $w \in \mathbb{R}^n$ we denote by $|w|$ the usual Euclidean norm of w, and further define the product $w^\lambda := w_1^{\lambda_1} \ldots w_n^{\lambda_n}$, with the convention that $0^0 = 1$.

8. If X is a manifold with dimension $\dim X = n$ we use the notation $\Lambda^* T^* X := \oplus_{r=0}^n \Lambda^r T^* X$ to denote the full exterior bundle over X. Also, we write $\Lambda^{odd} T^* X := \oplus_{k \geqslant 0} \Lambda^{2k+1} T^* X$ and $\Lambda^{even} T^* X := \oplus_{k \geqslant 0} \Lambda^{2k} T^* X$ to denote odd and even exterior bundles over X.

9. If X is a manifold we denote the rth de Rham cohomology group of X by

$$H^r(X) := \frac{\mathrm{Ker}\big(\mathrm{d} : C^\infty(\Lambda^r T^*X) \to C^\infty(\Lambda^{r+1}T^*X)\big)}{\mathrm{Im}\big(\mathrm{d} : C^\infty(\Lambda^{r-1}T^*X) \to C^\infty(\Lambda^r T^*X)\big)}$$

and the rth compactly supported de Rham cohomology group of X by X by

$$H^r_c(X) := \frac{\mathrm{Ker}\big(\mathrm{d} : C^\infty_c(\Lambda^r T^*X) \to C^\infty_c(\Lambda^{r+1}T^*X)\big)}{\mathrm{Im}\big(\mathrm{d} : C^\infty_c(\Lambda^{r-1}T^*X) \to C^\infty_c(\Lambda^r T^*X)\big)}.$$

10. If X is a topological space we denote the rth real singular homology group of X by $H_r(X)$ and if $A \subseteq X$ is a subset the rth real singular relative homology group of the pair (X, A) by $H_r(X, A)$.

Notation	Meaning	Definition
	Global notation	
λ	multi-index	1.5
$\|\lambda\|$	size of a multi-index λ	1.5
∂^λ	differential operator, of order $\|\lambda\|$	1.5
p, q	elements of $(1, \infty)$	—
p', q'	dual exponents	1.5
a, b	elements of $(0, 1)$	—
i, j, k, m	non-negative integers	—
l	order of a differential operator	—
Y, Z	arbitrary manifolds	—
M	Calabi-Yau manifold	—
$2n$	$\dim M$	—
(J, g, Ω)	Calabi-Yau structure on M	—
ω	Kähler form on M	—
X	special Lagrangian submanifold of M	—
$H^r(X)$	de Rham cohomology of X	1.5
$H^r_c(X)$	compactly supported de Rham cohomology of X	1.5
$H_r(X)$	real singular homology of X	—
$H_r(X, A)$	real singular relative homology of (X, A)	—
r	covariant degree of a tensor on X	—
s	contravariant degree of a tensor on X	—
$C^k(E)$	class C^k sections of a vector bundle E	—
$C^k_c(E)$	compactly supported elements of $C^k(E)$	—
$\iota(\cdot)$	interior product on a differential form	—
$\sigma_P(\cdot)$	symbol of a differential operator P	2.1.2
∇_g	Levi-Civita connection of a metric g	—
Δ^r_g	Laplacian of a metric g, acting on r-forms	2.1.2
$\|\cdot\|_g$	fibre metric on a bundle got from g	2.1.2
$\flat_g$	isomorphism $T^*X \cong TX$ got via a metric g	—
$*_g$	Hodge star of a metric g	—
d	exterior derivative	—
d^*_g	formal adjoint of the exterior derivative got via a metric g	2.1.2
$L^1(E)$	integrable sections of a vector bundle E	2.1.2
$L^1_{loc}(E)$	locally integrable sections of a vector bundle E	2.1.2
$L^2(E)$	L^2-integrable sections of a vector bundle E	2.1.2
N	normal bundle of a submanifold	2.2.1
f_ξ	submanifold got from a normal vector field ξ	2.2.1
exp	exponential map	2.2.1
ξ^t	infinitesimal variation	2.2.2

Table 1.1: List of selected notation

Notation	Meaning	Definition
	Chapter 4 onwards	
$\alpha, \beta, \gamma, \delta$	elements of $\mathbb{R}^L$	—
T	an element of $\mathbb{R}$	—
Σ	compact manifold, $\dim \Sigma = n-1$	—
L	number of connected components Σ_j of Σ	—
g_Σ	a metric on Σ	—
X	manifold with ends	4.1
βt	special function on X	4.1
X_S	compact core of X at distance $S \geqslant 0$ from X_0	4.1
X_∞	the L ends $X \setminus X_0 \cong (T, \infty) \times \Sigma$ of X	4.1
n	$\dim X \geqslant 3$	—
E, F	admissible vector bundles on X	4.1
$e^{(r-s)t}$	special operator on tensors	4.1
$\tilde{h}$	cylindrical metric on X	4.2.1
h	asymptotically cylindrical metric on X	4.2.1
$W^p_{k,\beta}(E)$	damped Sobolev space for admissible bundles $E \to X$	4.2.1
$B^{k,a}_\beta(E)$	damped Hölder space for admissible bundles $E \to X$	4.2.1
P_∞	translation invariant operator on X	4.2.2
$P_\infty(w)$	operator pencil got from P_∞	4.2.2
$\mathcal{C}(P_\infty, j) \subseteq \mathbb{C}$	eigenvalues of $P_\infty(w)$ on jth end	4.2.2
$\mathcal{D}(P_\infty, j) \subseteq \mathbb{R}$	real parts of $\mathcal{C}(P_\infty, j)$	4.2.2
$d(j, w)$	multiplicity for a point $w \in \mathcal{C}(P_\infty, j)$	4.2.2
$N(\beta, \delta)$	index jump between δ and β	4.2.2
P	asymptotically translation invariant operator	4.2.3
$\tilde{g}$	conical metric on X	4.3.1
g	asymptotically conical metric on X	4.3.1
$L^p_{k,\beta}(E)$	conical damped Sobolev space for tensor or exterior bundles $E \to X$	4.3.1
$C^{k,a}_\beta(E)$	conical damped Hölder space for tensor or exterior bundles $E \to X$	4.3.1
Q	asymptotically conical operator on X	4.3.2
γ	rate of an asymptotically conical operator	4.3.2
$\mathrm{Ker}(\dagger)^{\mathrm{sup}}_{\mathrm{sub}}$	kernel of (†), where sup,sub are indices for the domain	4.2.2, 4.2.3, 4.3.2
$\mathrm{Im}(\dagger)^{\mathrm{sup}}_{\mathrm{sub}}$	image of (†), where sup,sub are indices for the domain	4.2.2, 4.2.3, 4.3.2
$\mathrm{Coker}(\dagger)^{\mathrm{sup}}_{\mathrm{sub}}$	cokernel of (†), where sup,sub are indices for the domain	4.2.2, 4.2.3, 4.3.2
$\mathrm{Ind}(\dagger)^{\mathrm{sup}}_{\mathrm{sub}}$	index of (†), where sup,sub are indices for the domain	4.2.2, 4.2.3, 4.3.2
$\mathcal{D}(\dagger) \subseteq \mathbb{R}^L$	exceptional set of the operator (†)	4.2.2, 4.2.3, 4.3.2
$\left(\mathbb{R}^L \setminus \mathcal{D}(\dagger)\right)_*$	connected component of $\mathbb{R}^L \setminus \mathcal{D}(\dagger)$ containing $*$	4.2.2, 4.2.3, 4.3.2

Notation	Meaning	Definition
	Chapter 3 only	
X	compact manifold	—
$W^p_k(E)$	Sobolev space for vector bundles $E \to X$	3.1.1
$C^{k,a}(E)$	Hölder space for vector bundles $E \to X$	3.1.1
$\mathcal{H}^r$	harmonic r-forms on a compact Riemannian manifold	3.1.3
$(\hat{J}, \hat{g}, \hat{\Omega})$	variation of Calabi-Yau structures on M	3.2.1
$\big(J(p), g(p), \Omega(p)\big)$	a point of $(\hat{J}, \hat{g}, \hat{\Omega})$	3.2.1
	Chapter 5 onwards	
$\mathrm{Spec}(\Sigma, g_\Sigma, r)$	spectrum of the operator $\Delta^r_{g_\Sigma}$ on a compact Riemannian manifold (Σ, g_Σ)	—
μ	typical eigenvalue of $\Delta^0_{g_\Sigma}$	—
$\mu_{j,i}$	elements of $\mathrm{Spec}(\Sigma_j, g_\Sigma, 0)$	5.1.1
μ_j	the first positive eigenvalue $\mu_{j,1}$ of $\Delta^0_{g_\Sigma}$ on Σ_j	5.1.1
$\lambda > 0$	L-tuple of first exceptional growth rates	5.1.1
$\chi(\beta+2)$	eigenvalue counter for rates $\leqslant O(e^{(\beta+2)t})$	5.1.1
$\hat{\chi}(\beta+2)$	eigenvalue counter for rates $\leqslant o(e^{(\beta+2)t})$	5.1.1
$\beta+2$	growth rate of a typical function	—
$\beta+1$	growth rate of a typical 1-form	—
ϕ_r, p_r, ∂_r	linear maps in a certain long exact sequence	5.1.2
f_c	for $c \in \mathbb{R}^L$, a function constant on the ends of X	5.1.2
h_j	g-harmonic function on X, tends to δ_{jk} on kth end of X	5.1.3
h^1_j	$\tilde{g}$-harmonic function on X, equal to δ_{jk} on kth end of X	5.1.3
h^2_j	$\tilde{g}$-harmonic function on X, equal to $e^{(2-n)t}\delta_{jk}$ on kth end of X	5.1.3
(a_{jk})	special $L \times L$ matrix	5.1.3
(b_{jk})	special $L \times L$ matrix	5.1.4
$\alpha < 0$	decay rate of a strongly asymptotically conical metric	5.1.4
f_j	special function on X	5.1.4
$K_{\beta+1}$	smooth closed and coclosed 1-forms with growth rate $O(e^{(\beta+1)t})$	5.2
$\psi_{\beta+1}$	representation map on $K_{\beta+1}$	5.2
$\hat{K}_{\beta+1}$	smooth closed and coclosed 1-forms with growth rate $o(e^{(\beta+1)t})$	5.2.3
$\hat{\psi}_{\beta+1}$	representation map on $\hat{K}_{\beta+1}$	5.2.3
	Chapter 6	
$\tilde{X} = \mathbb{R} \times \Sigma$	full cylinder on Σ	6.1.1
$\tilde{E}, \tilde{F}$	admissible bundles on $\tilde{X}$	6.1.1
$W^p_{k,\beta}(\tilde{E}), B^{k,a}_\beta(\tilde{E})$	Banach spaces of sections of $\tilde{E}$	6.1.1
$\hat{B}^{k,a}_\beta(\tilde{E})$	closure of $C^\infty_c(\tilde{E})$ in $B^{k,a}_\beta(\tilde{E})$	6.1.1
$I_r \subseteq (2-n, 0)$	good growth rate interval for the operator Δ^r_g	6.1.2
C	cone in $\mathbb{R}^m$ or $\mathbb{C}^n$	6.2.1
$\tilde{\alpha} = \alpha + 1$	decay rate of a strongly asymptotically conical submanifold	6.2.1
$d_{\mathrm{tr}}(\Sigma)$	dimension of deformations of X got from the $\mathbb{C}^n$ moment map	6.3
$d_{\mathrm{rot}}(\Sigma)$	dimension of deformations of X got from the $\mathrm{SU}(n)$ moment map	6.3

Chapter 2

Background material

2.1 Analytic result for general manifolds

2.1.1 Banach space theory

We state here the definitions and results we shall need from the theory of Banach spaces. All the results are proved in the books [43] of Lang, [54] of Murphy, or [56] of Rudin.

General theory

If $\mathcal{X}, \mathcal{Y}$ are Banach spaces we denote the Banach space of continuous linear maps $\mathcal{X} \to \mathcal{Y}$ by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. We denote the Banach space of continuous multilinear maps

$$\underbrace{\mathcal{X} \times \cdots \times \mathcal{X}}_{k \text{ factors}} \to \mathcal{Y}$$

by $\mathcal{B}^k(\mathcal{X}, \mathcal{Y})$. Note that a multilinear map $T : \mathcal{X} \times \cdots \times \mathcal{X} \to \mathcal{Y}$ is continuous precisely when there exists a $C > 0$ such that

$$\|T(x_1, \ldots, x_k)\| \leqslant C\|x_1\|_{\mathcal{X}} \ldots \|x_k\|_{\mathcal{X}}$$

for all $(x_1, \ldots, x_k) \in \mathcal{X} \times \cdots \times \mathcal{X}$. Furthermore, there is a canonical isomorphism

$$\mathcal{B}^k(\mathcal{X}, \mathcal{Y}) \cong \mathcal{B}\big(\mathcal{X}, \mathcal{B}(\mathcal{X}, \ldots \mathcal{B}(\mathcal{X}, \mathcal{B}(\mathcal{X}, \mathcal{Y}))\ldots)\big). \tag{2.1}$$

If $\mathcal{X}$ is a Banach space we define $\mathcal{X}^* := \mathcal{B}(\mathcal{X}, \mathbb{R})$. Besides the usual norm topology $\mathcal{T}^*$, there is a topology $\mathcal{T}^{w-*}$ on $\mathcal{X}^*$ called the *weak-* topology* which is uniquely characterised by the fact that $\phi_j \to \phi$ in $\mathcal{T}^{w-*}$ precisely when $\phi_j(x) \to \phi(x)$ for all $x \in \mathcal{X}$. Note that the weak-* topology is weaker than the usual norm topology on $\mathcal{X}^*$ in that $\mathcal{T}^{w-*} \subseteq \mathcal{T}^*$. Given $x \in \mathcal{X}$ we define an element $\tau(x) \in (\mathcal{X}^*)^*$ by $\tau(x)\phi = \phi(x)$ for all $\phi \in \mathcal{X}^*$. Then $\tau : \mathcal{X} \to (\mathcal{X}^*)^*$ is an isometric isomorphism onto a closed subspace of $(\mathcal{X}^*)^*$. In fact the image of τ consists precisely of those linear functionals $T : \mathcal{X}^* \to \mathbb{R}$ which are continuous relative to the weak-* topology on $\mathcal{X}^*$. We say that $\mathcal{X}$ is *reflexive* if τ has image $(\mathcal{X}^*)^*$, and in this case we normally identify $\mathcal{X}$ with $(\mathcal{X}^*)^*$ via the map τ. Note that a closed subspace of a reflexive space is always reflexive. If $\mathcal{A}_1 \subseteq \mathcal{X}$ we define

$$\mathcal{A}_1^\circ := \{\phi \in \mathcal{X}^* : \phi(x) = 0 \text{ for all } x \in \mathcal{A}_1\}$$

which is a weak-* closed linear subspace of $\mathcal{X}^*$, and if $\mathcal{A}_2 \subseteq \mathcal{X}^*$ we define

$$\mathcal{A}_2^\circ := \{x \in \mathcal{X} : \phi(x) = 0 \text{ for all } \phi \in \mathcal{A}_2\}$$

which is a closed linear subspace of $\mathcal{X}$. We then have the following result.

Proposition 2.1 *Let $\mathcal{A}_1 \leqslant \mathcal{X}$ be a linear subspace of a Banach space $\mathcal{X}$. Then $(\mathcal{A}_1^\circ)^\circ = \overline{\mathcal{A}_1}$ the norm closure of $\mathcal{A}_1$ in $\mathcal{X}$. Also if $\mathcal{A}_2 \leqslant \mathcal{X}^*$ is a linear subspace of $\mathcal{X}^*$ then $(\mathcal{A}_2^\circ)^\circ = \overline{\mathcal{A}_2}^{w-*}$ the weak-* closure of $\mathcal{A}_2$ in $\mathcal{X}^*$.*

We also have the following useful result.

Proposition 2.2 *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$. Then*

1. $\operatorname{Ker} T^* = (\operatorname{Im} T)^\circ$ *in* $\mathcal{Y}^*$*, and hence* $(\operatorname{Ker} T^*)^\circ = \overline{\operatorname{Im} T}$ *in* $\mathcal{Y}$.
2. $\operatorname{Ker} T = (\operatorname{Im} T^*)^\circ$ *in* $\mathcal{X}$*, and hence* $(\operatorname{Ker} T)^\circ = \overline{\operatorname{Im} T^*}^{w-*}$ *in* $\mathcal{X}^*$.

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces with $\mathcal{B} := \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \leqslant 1\}$ the closed unit ball in $\mathcal{X}$. We say that a linear map $T : \mathcal{X} \to \mathcal{Y}$ is *compact* if $\overline{T(\mathcal{B})}$ is a compact subset of $\mathcal{Y}$. The following lemma is then entirely straightforward.

Lemma 2.3 *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. Let $T : \mathcal{X} \to \mathcal{Y}$ be linear. Then T is compact if and only if for every bounded sequence $(x_j) \subseteq \mathcal{X}$ the sequence $(Tx_j) \subseteq \mathcal{Y}$ has a convergent subsequence.*

If $\mathcal{X}, \mathcal{Y}$ are Banach spaces then a *continuous embedding* from $\mathcal{X}$ into $\mathcal{Y}$ is a continuous, injective map $T : \mathcal{X} \to \mathcal{Y}$. A *compact embedding* from $\mathcal{X}$ into $\mathcal{Y}$ is a compact, injective map $T : \mathcal{X} \to \mathcal{Y}$. Note that any compact embedding is a continuous embedding, so that one notion is stronger than the other. In the sequel, quite often $\mathcal{Y}$ will be a vector subspace of $\mathcal{X}$ and the map T will simply be the inclusion. In this situation we shall write $\mathcal{Y} \leqslant \mathcal{X}$.

Finally for this section, we have the following result:

Proposition 2.4 *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces. If $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ and $\operatorname{Im} T \leqslant \mathcal{Y}$ has finite codimension then $\operatorname{Im} T \leqslant \mathcal{Y}$ is closed.*

Differential calculus

The theory we give here is an extension of the usual calculus techniques in Euclidean space to the possibly infinite-dimensional case of Banach spaces.

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces, $\mathcal{U} \subseteq \mathcal{X}$ an open subset and $F : \mathcal{U} \to \mathcal{Y}$ a C^0 (ie. continuous) map. Given $x \in \mathcal{U}$ we say that F is *differentiable* at x if there exists $T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$ such that

$$\big\|F(x+h) - F(x) - Th\big\|_{\mathcal{Y}} = o\big(\|h\|_{\mathcal{X}}\big)$$

as $\|h\|_{\mathcal{X}} \to 0$. Now such a T must be unique if it exists and we usually write $F'(x) := T$ the *derivative* of F at x. If $F'(x)$ exists for each $x \in \mathcal{U}$ then we have a map

$$\begin{array}{rcl} \mathcal{U} & \to & \mathcal{B}(\mathcal{X}, \mathcal{Y}) \\ x & \mapsto & F'(x) \end{array}$$

and we say that F is of class C^1 if this map is continuous. Clearly if $S : \mathcal{X} \to \mathcal{Y}$ is a bounded linear map then S is of class C^1 and $S'(x) = S$ for all $x \in \mathcal{X}$.

Continuing inductively, for $k \geqslant 1$ we say that F is of class C^{k+1} if F is of class C^k and the continuous map

$$\mathcal{U} \to \mathcal{B}\big(\mathcal{X}, \mathcal{B}(\mathcal{X}, \ldots \mathcal{B}(\mathcal{X}, \mathcal{B}(\mathcal{X}, \mathcal{Y}))\ldots)\big) \cong \mathcal{B}^k(\mathcal{X}, \mathcal{Y})$$

is of class C^1. Here we are using the identification (2.1). We say F is *smooth* or C^∞ if F is of class C^k for each $k \geqslant 0$.

Let $\mathcal{X}_1, \mathcal{X}_2$ be Banach spaces, with open subsets $\mathcal{U}_1 \subseteq \mathcal{X}_1, \mathcal{U}_2 \subseteq \mathcal{X}_2$ and $F : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{Y}$ a map. Let $(x_1, x_2) \in \mathcal{U}_1 \times \mathcal{U}_2$. If the map

$$\begin{array}{rcl} \mathcal{U}_1 & \to & \mathcal{Y} \\ u & \mapsto & F(u, x_2) \end{array}$$

is differentiable at x_1 we shall write its derivative as $F_1'(x_1, x_2) \in \mathcal{B}(\mathcal{X}_1, \mathcal{Y})$ the *partial derivative* of F at (x_1, x_2) in the $\mathcal{X}_1$ direction. If $F_1'(x_1, x_2)$ exists for each $(x_1, x_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ we have a map

$$\begin{array}{rcl} F_1' : \mathcal{U}_1 \times \mathcal{U}_2 & \to & \mathcal{B}(\mathcal{X}_1, \mathcal{Y}) \\ (x_1, x_2) & \mapsto & F_1'(x_1, x_2) \end{array}$$

the *first partial derivative* of F. Similarly we have the notion of $F_2' : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{B}(\mathcal{X}_2, \mathcal{Y})$, the *second partial derivative* of F.

For the rest of Section 2.1.1 we fix a $1 \leqslant k \leqslant \infty$. We now have the usual differential calculus theorems, extended to the situation of Banach spaces. All results are proved in Lang [43].

Theorem 2.5 (Chain Rule) *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Banach spaces and $\mathcal{U} \subseteq \mathcal{X}, \mathcal{V} \subseteq \mathcal{Y}$ open subsets. Let $F : \mathcal{U} \to \mathcal{V}$ and $G : \mathcal{V} \to \mathcal{Z}$ be maps. If F and G are of class C^k then $G \circ F : \mathcal{U} \to \mathcal{Z}$ is of class C^k and $(G \circ F)'(x) = G'(F(x)) \circ F'(x)$ for all $x \in \mathcal{U}$.*

Theorem 2.6 (Product Rule) *Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}$ be Banach spaces with open subsets $\mathcal{U}_1 \subseteq \mathcal{X}_1, \mathcal{U}_2 \subseteq \mathcal{X}_2$. If $F : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{Y}$ is a map then F is of class C^k precisely when both partial derivatives*

$$\begin{aligned} F_1' : \mathcal{U}_1 \times \mathcal{U}_2 &\to \mathcal{B}(\mathcal{X}_1, \mathcal{Y}) \\ F_2' : \mathcal{U}_1 \times \mathcal{U}_2 &\to \mathcal{B}(\mathcal{X}_2, \mathcal{Y}) \end{aligned}$$

exist and are of class C^{k-1}. In this case the bounded linear map $F'(x_1, x_2) : \mathcal{X}_1 \oplus \mathcal{X}_2 \to \mathcal{Y}$ acts as

$$F'(x_1, x_2)(v_1, v_2) = F_1'(x_1, x_2)v_1 + F_2'(x_1, x_2)v_2$$

for each $(x_1, x_2) \in \mathcal{U}_1 \times \mathcal{U}_2$ and $(v_1, v_2) \in \mathcal{X}_1 \oplus \mathcal{X}_2$.

The last part of the Product Rule 2.6 is saying

$$\begin{aligned} F_1'(x_1, x_2) &= F'(x_1, x_2)\big|_{\mathcal{X}_1} \\ F_2'(x_1, x_2) &= F'(x_1, x_2)\big|_{\mathcal{X}_2} \end{aligned}$$

for each $(x_1, x_2) \in \mathcal{U}_1 \times \mathcal{U}_2$.

Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and $\mathcal{U} \subseteq \mathcal{X}, \mathcal{V} \subseteq \mathcal{Y}$ be open subsets. We shall say that a map $F : \mathcal{U} \to \mathcal{V}$ is a *C^k-diffeomorphism* if F is bijective and the mappings $F : \mathcal{U} \to \mathcal{Y}$, $F^{-1} : \mathcal{V} \to \mathcal{X}$ are of class C^k. The following result is very useful.

Theorem 2.7 (Inverse Function Theorem) *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces with $\mathcal{U} \subseteq \mathcal{X}$ an open neighbourhood of 0. If $F : \mathcal{U} \to \mathcal{Y}$ is a map of class C^k, such that the bounded linear map*

$$F'(0) : \mathcal{X} \to \mathcal{Y}$$

is a topological linear isomorphism of Banach spaces, then there exists an open subset $0 \in \mathcal{V} \subseteq \mathcal{U}$ with $F(\mathcal{V}) \subseteq \mathcal{Y}$ open, such that $F|_{\mathcal{V}} : \mathcal{V} \to F(\mathcal{V})$ is a C^k-diffeomorphism.

The Inverse Function Theorem 2.7 has the following immediate corollary:

Corollary 2.8 *Let $\mathcal{X}, \mathcal{Y}$ be Banach spaces and $\mathcal{U} \subseteq \mathcal{X}, \mathcal{V} \subseteq \mathcal{Y}$ be open subsets. If $F : \mathcal{U} \to \mathcal{V}$ is a C^1-diffeomorphism and is of class C^k then F is a C^k-diffeomorphism.*

Also, the following theorem can be useful to help invoke the Inverse Function Theorem 2.7.

Theorem 2.9 (Open Mapping Theorem) *Let $T : \mathcal{X} \to \mathcal{Y}$ be a bounded linear map between Banach spaces.*

1. *If T is surjective then T is an open mapping.*

2. *If T is bijective then $T : \mathcal{X} \to \mathcal{Y}$ is a topological linear isomorphism.*

Closely related to the Inverse Function Theorem 2.7 is the following result.

Theorem 2.10 (Implicit Function Theorem: version 1) *Let $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}$ be Banach spaces and $\mathcal{U}_1 \subseteq \mathcal{X}_1, \mathcal{U}_2 \subseteq \mathcal{X}_2$ open subsets both containing* 0. *Let*

$$\begin{array}{rcl} F : \mathcal{U}_1 \times \mathcal{U}_2 & \to & \mathcal{Y} \\ (0,0) & \mapsto & 0 \end{array}$$

be a map of class C^k such that the bounded linear map $F_2'(0,0) : \mathcal{X}_2 \to \mathcal{Y}$ is a topological, linear isomorphism. Then there exist open subsets $\mathcal{W}_1 \subseteq \mathcal{U}_1, \mathcal{W}_2 \subseteq \mathcal{U}_2$ both containing 0 *and a unique map $\chi : \mathcal{W}_1 \to \mathcal{W}_2$ such that*

$$F^{-1}(0) \cap (\mathcal{W}_1 \times \mathcal{W}_2) = \Big\{ \big(x_1, \chi(x_1)\big) : x_1 \in \mathcal{W}_1 \Big\}. \tag{2.2}$$

Moreover, the map χ is of class C^k.

Proof: Consider the C^k map $G : \mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{X}_1 \oplus \mathcal{Y}$ defined $G(x_1, x_2) := \big(x_1, F(x_1, x_2)\big)$. Then the derivative $G'(0,0) : \mathcal{X}_1 \oplus \mathcal{X}_2 \to \mathcal{X}_1 \oplus \mathcal{Y}$ acts as the matrix

$$G'(0,0) = \begin{pmatrix} \mathrm{id} & 0 \\ F_1'(0,0) & F_2'(0,0) \end{pmatrix} \tag{2.3}$$

on vectors $(v_1, v_2)^t \in \mathcal{X}_1 \oplus \mathcal{X}_2$. Now, the bounded linear map (2.3) is easily seen to be invertible, and therefore must be a homeomorphism by the Open Mapping Theorem 2.9. So by the Inverse Function Theorem 2.7 there exist open subsets $\mathcal{V}_1 \subseteq \mathcal{U}_1, \mathcal{V}_2 \subseteq \mathcal{U}_2$ both containing 0 such that $G|_{\mathcal{V}_1 \times \mathcal{V}_2} : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathcal{X}_1 \oplus \mathcal{Y}$ is a C^k-diffeomorphism onto an open subset $\Omega \subseteq \mathcal{X}_1 \oplus \mathcal{Y}$ containing 0. It follows that $G^{-1} : \Omega \to \mathcal{V}_1 \times \mathcal{V}_2$ is of class C^k and there exists a unique map $h : \Omega \to \mathcal{V}_2$ such that $G^{-1}(x_1, y) = \big(x_1, h(x_1, y)\big)$ for all $(x_1, y) \in \Omega$, and h is class C^k. Now put

$$\begin{array}{rcl} \mathcal{W}_1 & = & \big\{x_1 \in \mathcal{V}_1 : (x_1, 0) \in \Omega\big\} \\ \mathcal{W}_2 & = & \mathcal{V}_2 \end{array}$$

and define the class C^k map $\chi : \mathcal{W}_1 \to \mathcal{W}_2$ by $\chi(x_1) = h(x_1, 0)$ for all $x_1 \in \mathcal{W}_1$, so that equation (2.2) holds as required. Note that equation (2.2) determines χ uniquely on $\mathcal{W}_1$.

∎

For the purposes of the next result, if $\mathcal{X}$ is a Banach space we say that a closed subspace $\mathcal{K} \leqslant \mathcal{X}$ *splits* $\mathcal{X}$ if there exists a closed subspace $\mathcal{A} \leqslant \mathcal{X}$ such that $\mathcal{X} = \mathcal{K} \oplus \mathcal{A}$ as vector spaces (and hence as topological vector spaces, by the Open Mapping Theorem 2.9). We call $\mathcal{A}$ a *complementary subspace* for $\mathcal{K}$. Note that any finite-dimensional subspace will always split $\mathcal{X}$. This can be proved using the Hahn-Banach Theorem.

Theorem 2.11 (Implicit Function Theorem: version 2) *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces with $\mathcal{U} \subseteq \mathcal{X}$ an open neighbourhood of* 0. *Let $F : \mathcal{U} \to \mathcal{Y}$ be a map of class C^k, with $F(0) = 0$. Suppose the bounded linear map $F'(0) : \mathcal{X} \to \mathcal{Y}$ is surjective and has a kernel $\mathcal{K} \leqslant \mathcal{X}$ which splits $\mathcal{X}$ with a complementary subspace $\mathcal{A}$. Then there exist open subsets $\mathcal{W}_1 \subseteq \mathcal{K}$, $\mathcal{W}_2 \subseteq \mathcal{A}$ both containing* 0 *with $\mathcal{W}_1 \times \mathcal{W}_2 \subseteq \mathcal{U}$ and a unique map $\chi : \mathcal{W}_1 \to \mathcal{W}_2$ such that*

$$F^{-1}(0) \cap (\mathcal{W}_1 \times \mathcal{W}_2) = \Big\{ \big(x, \chi(x)\big) : x \in \mathcal{W}_1 \Big\}$$

in $\mathcal{X} = \mathcal{K} \oplus \mathcal{A}$. Moreover, the map χ is of class C^k.

Note that Theorem 2.11 follows quickly from Theorem 2.10.

2.1.2 Differential operators

Let $E, F \to X$ be a vector bundles over a manifold X and $\mathcal{U} = \{U_\nu : \nu \in \Lambda\}$ an open covering of X such that X, E, F are trivial over each $U \in \mathcal{U}$. A *smooth, linear, differential operator* of *order* $l \geqslant 1$ from E to F is a linear map

$$P : C_c^\infty(E) \to C_c^\infty(F)$$

such that for each $\nu \in \Lambda$, $1 \leqslant j \leqslant \operatorname{rank} E$, $1 \leqslant i \leqslant \operatorname{rank} F$ and multi-index λ with $0 \leqslant |\lambda| \leqslant l$ there exists a $P_{ij}^{\nu\lambda} \in C^\infty(U_\nu)$ such that

$$(P\xi)_i^\nu = \sum_{j=1}^{\operatorname{rank} E} \sum_{0 \leqslant |\lambda| \leqslant l} P_{ij}^{\nu\lambda} \partial^\lambda \xi_j^\nu \tag{2.4}$$

for each $\xi \in C_c^\infty(E)$, $\nu \in \Lambda$ and $1 \leqslant i \leqslant \operatorname{rank} F$. We require that not all the $P_{ij}^{\nu\lambda}$ with $|\lambda| = l$ be zero. In (2.4) the (ξ_j^ν) are the components of ξ in the given trivialisations of E, X over U_ν. By the Chain Rule 2.5 the above definition does not depend on our choice of open cover $\mathcal{U}$ of X.

A Riemannian metric g equips X with a Lebesgue measure $\mathrm{d}V_g$ and we have the corresponding space $L^1(X)$ of integrable functions $X \to \mathbb{R}$. Suppose we choose some fibre metric $(\ ,\ )_E$ on the bundle E, which induces a pointwise norm $|\ |_E$ on the fibres of E. We define $L^1(E)$ to be the vector space of sections ξ of E such that $|\xi|_E \in L^1(X)$, and further define $L^1_{loc}(E)$ to be the vector space of sections ξ of E such that $\phi\xi \in L^1(E)$ for all $\phi \in C_c^\infty(X)$. Then $L^1(E)$ is equipped with the norm

$$\|\xi\|_{L^1(E)} := \int_X |\xi|_E \mathrm{d}V_g. \tag{2.5}$$

We follow the usual convention of identifying sections of E that are equal almost everywhere.

Given $\xi_1, \xi_2 \in C_c^\infty(E)$ we may form the continuous, compactly supported function $(\xi_1, \xi_2)_E$ on X, which lies in $L^1(X)$. We define the L^2-*inner product* of ξ_1 and ξ_2 to be

$$\langle \xi_1 | \xi_2 \rangle_{L^2(E)} = \int_X (\xi_1, \xi_2)_E \mathrm{d}V_g. \tag{2.6}$$

The induced norm on $C_c^\infty(E)$ is denoted $\|\cdot\|_{L^2(E)}$ and the completion of $C_c^\infty(E)$ with respect to this norm is the Hilbert space $L^2(E)$, which will depend upon our choices of metric g and fibre metric $(\ ,\ )_E$.

Suppose now the bundle F is also equipped with a fibre metric $(\ ,\ )_F$. If $P : C_c^\infty(E) \to C_c^\infty(F)$ is a smooth, linear differential operator of order $l \geqslant 1$, there exists a unique map

$$P^* : C_c^\infty(F) \to C_c^\infty(E)$$

with the property

$$\langle \xi | P^* \eta \rangle_{L^2(E)} = \langle P\xi | \eta \rangle_{L^2(F)} \tag{2.7}$$

for all $\xi \in C_c^\infty(E)$ and $\eta \in C_c^\infty(F)$. The map P^* is also a smooth, linear differential operator of order l and is called the *formal adjoint* of the operator P. Note that the map P^* depends upon the choice of Riemannian metric on X and the fibre metrics on E, F. The process of taking formal adjoints satisfies the usual properties

$$\begin{aligned} (\mu_1 P_1 + \mu_2 Q_2)^* &= \mu_1 P_1^* + \mu_2 P_2^* \\ (P_1 P_2)^* &= P_2^* P_1^* \\ (P^*)^* &= P \end{aligned}$$

for all real μ_1, μ_2 and all suitable smooth, linear differential operators P_1, P_2, P. We say that an operator P is *self-adjoint* if $E = F$ and $P = P^*$ as linear maps $C_c^\infty(E) \to C_c^\infty(E)$.

Suppose that $\xi \in L^1_{loc}(E)$ and $\eta \in L^1_{loc}(F)$. We say that ξ is a *weak solution* of the equation $P\xi = \eta$ when

$$\langle \eta | \psi \rangle_{L^2(F)} = \langle \xi | P^* \psi \rangle_{L^2(E)} \tag{2.8}$$

for all $\psi \in C_c^\infty(F)$. This definition is motivated by the fact that equation (2.8) obviously holds for all $\psi \in C_c^\infty(F)$ when $\xi \in C_c^\infty(E)$, $\eta \in C_c^\infty(F)$ and $P\xi = \eta$ in the usual way.

An example of a smooth, linear differential operator is the usual exterior derivative

$$\mathrm{d} : C_c^\infty(\Lambda^r T^*X) \to C_c^\infty(\Lambda^{r+1} T^*X) \tag{2.9}$$

on the manifold X. The exterior derivative has order $l = 1$. A Riemannian metric g on X endows the bundles $\Lambda^r T^*X$ with a fibre metric and we denote the formal adjoint of d in this situation by d_g^*. One can easily check that

$$\mathrm{d}_g^*\xi = (-1)^{nr+n+1} *_g \mathrm{d} *_g \xi \tag{2.10}$$

for all $\xi \in C_c^\infty(\Lambda^r T^*X)$. Here $*_g$ is the Hodge star operator on the Riemannian manifold (X, g). Note that we do not require X to be oriented for equation (2.10) to make sense, as any sign ambiguity in the Hodge star $*_g : \Lambda^r T^*X \to \Lambda^{n-r} T^*X$ will be counted twice and therefore cancel. We now define the *Laplacian* on the Riemannian manifold (X, g) by

$$\Delta := \mathrm{d}\mathrm{d}_g^* + \mathrm{d}_g^*\mathrm{d} : C_c^\infty(\Lambda^r T^*X) \to C_c^\infty(\Lambda^r T^*X) \tag{2.11}$$

which is a smooth, linear, differential operator of order 2 that is self-adjoint. When we wish to indicate the metric g and the degree r of the forms on which the Laplacian acts we denote the operator (2.11) by Δ_g^r.

Suppose that P is a smooth, linear differential operator of order $l \geqslant 1$ which is given by equation (2.4) in the cover $\mathcal{U}$ of X. Define

$$L_{ij}^\nu(x, w) := \sum_{|\lambda|=l} P_{ij}^{\nu\lambda}(x) \cdot w^\lambda$$

for each $\nu \in \Lambda$, $1 \leqslant i \leqslant \operatorname{rank} F$, $1 \leqslant j \leqslant \operatorname{rank} E$, $x \in U_\nu$ and $w \in \mathbb{R}^n$. When $\operatorname{rank} E = \operatorname{rank} F$ and

$$\det\bigl(L_{ij}^\nu(x, w)\bigr) \neq 0$$

for each $\nu \in \Lambda$, $x \in U_\nu$ and $w \in \mathbb{R}^n \setminus \{0\}$ we say that the differential operator P is *elliptic*. This definition does not depend on our choice trivialisation $\mathcal{U}$, as can be seen by applying the Chain Rule 2.5.

The notion of ellipticity may be formulated in a coordinate free manner as we shall now describe. Given a smooth, linear, differential operator $P : C_c^\infty(E) \to C_c^\infty(F)$ of order $l \geqslant 1$ we can construct an object called the *symbol* σ_P of P. Now σ_P is a smooth section of the vector bundle $\mathrm{Sym}^l TX \otimes \mathrm{Hom}(E, F)$ and given $\eta \in C^\infty(T^*X)$ we can form $\sigma_P(\eta) \in C^\infty\bigl(\mathrm{Hom}(E, F)\bigr)$ by substitution into the first factor. In fact if P is given as in equation (2.4) in the local trivialisation U_ν then $\sigma_P(\eta) \in C^\infty\bigl(\mathrm{Hom}(E, F)\bigr)$ acts as the $\operatorname{rank} F \times \operatorname{rank} E$ matrix with (i, j) entry

$$\sum_{|\lambda|=l} P_{ij}^{\nu\lambda} \eta^\lambda \tag{2.12}$$

where we consider η as the n-tuple $(\eta_1, \ldots, \eta_n)$ with respect to the basis $\{\mathrm{d}x_1, \ldots, \mathrm{d}x_n\}$ of covectors on U_ν.

The symbol operation σ satisfies various desirable properties: for any $\eta \in C^\infty(T^*X)$ we have

$$\sigma_{\mu_1 P_1 + \mu_2 P_2}(\eta) = \mu_1 \sigma_{P_1}(\eta) + \mu_2 \sigma_{P_2}(\eta) \tag{2.13}$$

$$\sigma_{P_1}(\eta)\sigma_{P_2}(\eta) = \sigma_{P_1 P_2}(\eta) \tag{2.14}$$

for any real μ_1, μ_2 and suitable smooth, linear differential operators P_1, P_2. Note that for equation (2.13) to be valid P_1 and P_2 must have the same order. Also, if we equip X with a Riemannian metric and vector bundles E, F with fibre metrics then

$$\sigma_P(\eta)^* = \sigma_{P^*}(\eta) \tag{2.15}$$

in $C^\infty\big(\mathrm{Hom}(F,E)\big)$ for all $\eta \in C^\infty(T^*X)$ and differential operators $P : C_c^\infty(E) \to C_c^\infty(F)$.

The symbol σ_P encodes the highest order data in P and in fact P is elliptic if and only if for each $x \in X$ and non-zero $\eta_x \in T_x^*X$ the linear map $\sigma_P(\eta_x) : E_x \to F_x$ is an isomorphism. We can now deduce from simple linear algebra various lemmas about ellipticity, for example: P is elliptic if and only if P^* is elliptic. Also, it is easy to check that the Laplacian defined in equation (2.11) is elliptic. This follows from the following lemma of linear algebra.

Lemma 2.12 *If $U \xrightarrow{S} V \xrightarrow{T} W$ is an exact sequence of finite-dimensional inner product spaces and linear maps then $SS^* + T^*T : V \to V$ is a linear isomorphism.*

Proof: If $v \in V$ with $SS^*v + T^*Tv = 0$ then

$$\|S^*v\|^2 + \|Tv\|^2 = \langle SS^*v + T^*Tv | v\rangle = 0$$

so that $S^*v = 0$ in U and $Tv = 0$ in W. Then by exactness there exists $u \in U$ with $Su = v$ so that $S^*Su = 0$. This implies that $v = Su = 0$.

∎

Corollary 2.13 *The Laplacian $\Delta : C_c^\infty(\Lambda^r T^*X) \to C_c^\infty(\Lambda^r T^*X)$ is elliptic.*

Proof: Let $x \in X$ and $\eta \in C^\infty(T^*X)$. Then from the properties (2.13), (2.14), (2.15) of the symbol given above the linear map $\sigma_\Delta(\eta)_x : \Lambda^r T_x^*X \to \Lambda^r T_x^*X$ acts as

$$\sigma_\Delta(\eta)_x = \sigma_{\mathrm{d}\mathrm{d}_g^* + \mathrm{d}_g^*\mathrm{d}}(\eta)_x = \sigma_{\mathrm{d}}(\eta)_x \sigma_{\mathrm{d}}(\eta)_x^* + \sigma_{\mathrm{d}}(\eta)_x^* \sigma_{\mathrm{d}}(\eta)_x. \tag{2.16}$$

Now, it is easy to show that the symbol of the exterior derivative (2.9) acts as

$$\begin{array}{rcl} \sigma_{\mathrm{d}}(\eta)_x : \Lambda^r T_x^*X & \to & \Lambda^{r+1} T_x^*X \\ \xi_x & \mapsto & \eta_x \wedge \xi_x \end{array}$$

for all $\eta \in C^\infty(T^*X)$, $x \in X$ and $\xi_x \in \Lambda^r T_x^*X$. But the sequence of linear maps

$$\Lambda^{r-1} T_x^*X \xrightarrow{\eta_x\wedge} \Lambda^r T_x^*X \xrightarrow{\eta_x\wedge} \Lambda^{r+1} T_x^*X$$

is exact whenever $\eta_x \neq 0$, and so from Lemma 2.12 we deduce that (2.16) is a linear isomorphism whenever $\eta_x \neq 0$. Consequently Δ is elliptic, as required.

∎

Corollary 2.14 *The operator $\mathrm{d}_g^* + \mathrm{d} : C_c^\infty(\Lambda^* T^*X) \to C_c^\infty(\Lambda^* T^*X)$ is elliptic.*

Proof: If we consider the Laplacian Δ acting on the whole exterior bundle Λ^*T^*X then we have $(\mathrm{d}_g^* + \mathrm{d})^2 = \Delta$, an elliptic operator. The result now follows from property (2.14) of the symbol σ.

∎

2.2 Submanifolds

From now on, we adopt the convention that a *submanifold* of a manifold Z is a manifold Y together with an injective immersion $k : Y \to Z$ that is a homeomorphism onto its image. Then we can cover $k(Y) \subseteq Z$ with charts for Z that restrict to $\dim Y$ slices on $k(Y)$. This gives $k(Y)$ the structure of a manifold such that $k : Y \to k(Y)$ is a diffeomorphism.

We shall identify the submanifolds $k_1 : Y_1 \to Z$ and $k_2 : Y_2 \to Z$ of Z if there exists a diffeomorphism $\phi : Y_1 \to Y_2$ such that $k_1 = k_2 \circ \phi$. In each equivalence class of submanifolds there is a unique representative $i : A \to Z$ where i is the inclusion of some subset $A \subseteq Z$, and the differentiable structure on A comes from slices of charts for Z.

2.2.1 Tubular neighbourhoods

Let $k : Y \to Z$ be a submanifold of a manifold Z. Then we have the pulled-back tangent bundle $k^*TZ \to Y$ and the quotient

$$N := \frac{k^*TZ}{TY}$$

is called the *normal bundle* of Y in Z. Suppose that Z is equipped with a Riemannian metric g. Then we can identify canonically the fibre N_y of N over $y \in Y$ with the $g_{k(y)}$-orthogonal complement $(T_yY)^\perp$ of $T_yY \leqslant T_{k(y)}Z$, and hence the bundle N with $(TY)^\perp$ the subbundle g-orthogonal to $TY \leqslant k^*TZ$.

Now for each $y \in Y$ we have – using our metric g – the exponential map $\exp_{k(y)}$ which maps a sufficiently small open neighbourhood of $0 \in T_{k(y)}Z$ diffeomorphically onto an open neighbourhood of $k(y) \in Z$, and this certainly restricts to a diffeomorphism defined on the tangent vectors $g_{k(y)}$-normal to T_yY. Moreover, this normal geodesic flow can be pieced together to form a global diffeomorphism from an open subset of the normal bundle N onto an open subset of Z. This fact is the content of the following theorem, which is proved in the book [42, Chapter IV, Theorem 9] of Lang.

Theorem 2.15 (Tubular Neighbourhood Theorem) *Let $k : Y \to Z$ be a submanifold of a Riemannian manifold (Z, g) with $k(Y) \subseteq Z$ a closed subspace. Let N be the normal bundle of Y in Z. Then there exists an open subset $\tilde{U} \subseteq N$ containing the image of the zero section, such that the restriction*

$$\exp|_{\tilde{U}} : \tilde{U} \to Z$$

is a diffeomorphism onto an open subset of Z.

It follows from Theorem 2.15 that if $k(Y)$ is closed in Z then any normal vector field $\xi \in C^\infty(N)$ with $\xi_y \in \tilde{U}$ for all $y \in Y$ defines a submanifold $k_\xi : Y \to Z$ of Z where

$$k_\xi(y) := \exp_{k(y)}(\xi_y)$$

for each $y \in Y$. In this way we view "small" normal vector fields ξ as giving rise to submanifolds $k_\xi : Y \to Z$ that are "near" to $k : Y \to Z$. Note that $k_0 = k$.

2.2.2 Variations

Let $I \subseteq \mathbb{R}$ be an open interval and $V : I \times Y \to Z$ a map of manifolds. Suppose the maps $v_t : Y \to Z$ are defined by

$$v_t(y) := V(t, y)$$

for all $t \in I$ and $y \in Y$. Then we call V a *variation* of each v_t. In this situation it follows by definition that for all $s \in I$ and all forms θ on Z we have

$$\mathcal{L}_{\frac{\partial}{\partial t}}\big(V^*\theta\big)\Big|_{Y_s} = \frac{\partial}{\partial t}\Big(v_t^*\theta\Big)\Big|_{t=s}$$

where on the left hand side $\mathcal{L}$ denotes Lie derivative, t is the canonical coordinate on I and $Y_s := \{s\} \times Y \cong Y$. On the right hand side the derivatives are calculated pointwise on Y. Given $s \in I$ we define $\xi^s \in C^\infty(v_s^*TZ)$ by

$$\xi^s_y := \mathrm{d}V_{(s,y)}\left(\frac{\partial}{\partial t}\right)$$

for all $y \in Y$. We call ξ^s the *infinitesimal variation* of V at $s \in I$. The following lemma will be useful later, and is proved in the book [19, Proposition (I.b.5)] of Griffiths.

Lemma 2.16 *Refer to the above notation. Suppose $s \in I$ is such that the map $v_s : Y \to Z$ is a submanifold. Then*

$$\mathcal{L}_{\frac{\partial}{\partial t}}\big(V^*\theta\big)\Big|_{Y_s} = v_s^*\big(\iota(\xi^s)\mathrm{d}\theta + \mathrm{d}(\iota(\xi^s)\theta)\big)$$

for all $\theta \in C^\infty(\Lambda^rT^*Z)$. Here ξ^s is extended to any vector field on a small neighbourhood of $v_s(Y)$ in Z. The resulting right hand side is independent of our choice of extension.

2.2.3 Minimal submanifolds

In this section we describe the theory of minimal submanifolds that we shall need. The book [24] of Jost contains a more detailed treatment of this theory.

Let Y, Z be manifolds and $I \subseteq \mathbb{R}$ an open interval containing 0. Let

$$K : I \times Y \to Z$$

be a map with $K(t,y) =: k_t(y)$ for all $t \in I$ and $y \in Y$. Then K is a variation of each map $k_t : Y \to Z$, in the sense of Section 2.2.2. We say that K is a *local variation* of k_0 if

$$\mathrm{supp}_0 K := \overline{\{y \in Y : K(t,y) \neq K(0,y) \text{ for some } t \in I\}}$$

is compact.

Suppose now that Z has a Riemannian metric g and the mapping $k_0 : Y \to Z$ is an immersion over $\mathrm{supp}_0 K$. Then there exists an $\varepsilon > 0$ such that $k_t : Y \to Z$ is an immersion over $\mathrm{supp}_0 K$ for each $|t| < \varepsilon$, and consequently we have a metric $k_t^* g$ on $\mathrm{supp}_0 K \subseteq Y$ for each $|t| < \varepsilon$. Define the *variation of volume* for K at 0 to be

$$\mathrm{Var}_0(K) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathrm{supp}_0 K} \mathrm{d}V_t\right)\Bigg|_{t=0}$$

where $\mathrm{d}V_t$ is the Lebesgue measure on $\mathrm{supp}_0 K$ coming from $k_t^* g$. Now if $k : Y \to Z$ is a submanifold we shall say that k is *minimal* if

$$\mathrm{Var}_0(K) = 0$$

for all local variations K of $k_0 := k$. Note that composing k with a diffeomorphism $Y \to Y$ does not change whether or not a submanifold $k : Y \to Z$ is minimal. The following result gives a useful characterisation of minimal submanifolds, and is proved in [24, Section 3.6].

Proposition 2.17 *Let $k : Y \to Z$ be a submanifold of a Riemannian manifold (Z,g). Then k is minimal if and only if for all local coordinates $(y_1, \ldots, y_n)$ on Y and $(z_1, \ldots, z_m)$ on Z we have*

$$\Delta k_j - \sum_{\alpha,\beta=1}^{n} \sum_{i,l=1}^{m} (k^* g)^{\alpha\beta} (\Gamma^j_{il} \circ k) \frac{\partial k_i}{\partial y_\alpha} \frac{\partial k_l}{\partial y_\beta} = 0 \tag{2.17}$$

for $j = 1, \ldots, m$. Here Δ is the Laplacian of the metric $k^* g$ on Y and the Γ^j_{il} are the Christoffel symbols of the metric g on Z.

In Proposition 2.17 the two conditions are equivalent to the *mean curvature* of the submanifold $k : Y \to Z$ vanishing in all normal directions, but we shall not need this fact here. Also, maps $k : Y \to Z$ satisfying equations (2.17) are called *harmonic*. The m equations (2.17) above form a system of non-linear partial differential equations in the unknowns $(k_1, \ldots, k_m)$ which are functions of the independent variables $(x_1, \ldots, x_n)$. This system has special properties, as we shall see below. A computation of the Laplacian Δk_j in local coordinates gives

$$\begin{aligned}
\Delta k_j &= \frac{-1}{\sqrt{\det(k^* g)}} \sum_{\alpha,\beta=1}^{n} \frac{\partial}{\partial y_\beta}\left(\sqrt{\det(k^* g)}(k^* g)^{\alpha\beta} \frac{\partial k_j}{\partial y_\alpha}\right) \qquad (2.18)\\
&= -\sum_{\alpha,\beta=1}^{n}\left((k^* g)^{\alpha\beta} \frac{\partial^2 k_j}{\partial y_\alpha \partial y_\beta} + \frac{1}{\sqrt{\det(k^* g)}} \frac{\partial}{\partial y_\beta}\left(\sqrt{\det(k^* g)}(k^* g)^{\alpha\beta}\right) \frac{\partial k_j}{\partial y_\alpha}\right)
\end{aligned}$$

which contains terms

$$\frac{\partial}{\partial y_\beta}\left(\sqrt{\det(k^* g)}(k^* g)^{\alpha\beta}\right). \tag{2.19}$$

In actual fact, the terms in (2.19) are linear in the second order derivatives of $k_1, \ldots, k_m$, so that (2.17) is a quasi-linear set of equations. However, these equations cannot be quasi-linear elliptic because then regularity theory as in Morrey [52, Theorem 9.1] would imply that all solutions k would be smooth. This clearly cannot be the case as composing k with a C^2, but not smooth, diffeomorphism $Y \to Y$ would show.

2.2.4 Harmonic coordinates and regularity results

Although all the manifolds and mappings we have considered so far in Section 2.2 were smooth, the definitions given in Section 2.2.3 above will still make sense when the objects concerned have lower degrees of differentiability, and the corresponding propositions will remain true in these lower regularity cases. Given this, we now go on to describe why any minimal submanifold can in fact be given a smooth parameterisation. The material of this section is similar to that of DeTurck and Kazdan [15].

We say that a non-empty open subset $U \subseteq \mathbb{R}^n$ is a *domain* if U is bounded and convex. Let us fix a domain $G \subseteq \mathbb{R}^n$. If $D \subseteq \mathbb{R}^n$ is a domain we shall write

$$D \subset\subset G$$

to mean $\overline{D} \subseteq G$.

We define $C^k(G)$ to be the vector space of functions $G \to \mathbb{R}$ which are k times continuously differentiable. We then put

$$C^k(\overline{G}) := \big\{ u \in C^k(G) : u = v|_G \text{ for some } v \in C^k(W) \text{ where } G \subset\subset W \big\}$$

so that elements of $C^k(G)$ could tend to infinity at the boundary of G whereas elements of $C^k(\overline{G})$ do not. Note that if $u : G \to \mathbb{R}$ then $u \in C^k(G)$ precisely when $u \in C^k(\overline{D})$ for all $D \subset\subset G$. We put

$$\begin{aligned} C^\infty(G) &:= \bigcap_{k \geqslant 0} C^k(G) \\ C^\infty(\overline{G}) &:= \big\{ u \in C^\infty(G) : u = v|_G \text{ for some } v \in C^\infty(W) \text{ where } G \subset\subset W \big\} \end{aligned}$$

the smooth versions of the spaces defined above.

Given $u \in C^k(\overline{G})$ we define

$$\|u\|_{C^k(\overline{G})} := \sum_{0 \leqslant |\lambda| \leqslant k} \sup_G |\partial^\lambda u|$$

where the sum is taken over all multi-indices λ as given. With this norm $C^k(\overline{G})$ becomes a Banach space. For obvious reasons we call this Banach space a *C^k-space*.

Fix some small $\varepsilon > 0$. Then given a subset $A \subseteq \mathbb{R}^n$ and a function $u : A \to \mathbb{R}$ we define

$$[u]_{a;A} := \sup \left\{ \frac{|u(x) - u(y)|}{|x - y|^a} : x, y \in A \text{ with } 0 < |x - y| < \varepsilon \right\}$$

which may, or may not, be finite. We also define the *Hölder spaces*

$$\begin{aligned} C^{k,a}(\overline{G}) &:= \big\{ u \in C^k(\overline{G}) : [\partial^\lambda u]_{a;G} < \infty \text{ for all } |\lambda| = k \big\} \\ C^{k,a}(G) &:= \big\{ u \in C^k(G) : u \in C^{k,a}(\overline{D}) \text{ for all } D \subset\subset G \big\}. \end{aligned}$$

Note that $C^{k,a}(\overline{G})$ is equipped with a norm

$$\|u\|_{C^{k,a}(\overline{G})} := \|u\|_{C^k(\overline{G})} + \sum_{|\lambda| = k} [\partial^\lambda u]_{a;G}$$

which, it turns out, makes $C^{k,a}(\overline{G})$ into a Banach space. This Banach space is independent as a topological vector space of our choice of $\varepsilon > 0$. Also, functions $G \to \mathbb{R}$ which lie in $C^{k,a}(G)$ are said to be *of class $C^{k,a}$* or *having regularity $C^{k,a}$*.

Let (Z, g) be a Riemannian manifold of class C^2, with Laplace operator Δ acting on functions. We say that local coordinates $(x_1, \dots, x_n)$ on Z are *harmonic* if

$$\Delta x_j = 0 \tag{2.20}$$

for each $j = 1, \ldots, n$. In the local coordinates $(x_1, \ldots, x_n)$ these equations become

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\det(g)} \cdot g^{ji} \right) = 0 \tag{2.21}$$

for $j = 1, \ldots, n$. The following lemma, proved in [15, Lemma 1.2], tells us that harmonic coordinates always exist, and gives useful information about their regularity.

Lemma 2.18 *Let (Z, g) be a Riemannian manifold of class C^2. Let $k \geqslant 2$ and $(y_1, \ldots, y_n)$ be coordinates about a point $p \in Z$ such that the metric g has coefficients of class $C^{k,a}$ with respect to the (y_j). Then in a neighbourhood of p there exist harmonic coordinates (x_j) which are of class $C^{k+1,a}$ with respect to the coordinates (y_j). Moreover any harmonic coordinates about p have this regularity with respect to the coordinate system (y_j).*

Suppose now Y is a second C^2 manifold and the map $k : Y \to Z$ is a submanifold of class C^2. If $k : Y \to Z$ is minimal then we have equations (2.17) holding, which become

$$\sum_{\alpha,\beta=1}^{n} \left((k^*g)^{\alpha\beta} \frac{\partial^2 k_j}{\partial x_\alpha x_\beta} + \sum_{i,l=1}^{m} (k^*g)^{\alpha\beta} (\Gamma^j_{il} \circ k) \frac{\partial k_i}{\partial x_\alpha} \frac{\partial k_l}{\partial x_\beta} \right) = 0 \tag{2.22}$$

for $j = 1, \ldots, m$ in (k^*g)-harmonic coordinates (x_α) on Y. This is by equations (2.21) and the usual expression (2.18) for the Laplacian on functions in local coordinates.

The whole point of introducing harmonic coordinates on Y was to reduce the minimality equations (2.17) to the simpler form (2.22): we now observe that the equations (2.22) form a second order quasi-linear elliptic system, as in the article [52] of Morrey. We can now give the following regularity result, whose proof we give to indicate the ideas involved.

Proposition 2.19 *Let Y_1, Z be smooth manifolds and let g be a smooth metric on Z. Let $k_1 : Y_1 \to Z$ be a minimal submanifold with k_1 of class $C^{l,a}$ for some $l \geqslant 3$. Then there exists a smooth manifold Y_2 and a diffeomorphism $\phi : Y_1 \to Y_2$ of class C^l such that the mapping $k_2 : Y_2 \to Z$ defined by*

$$\begin{array}{ccc} Z & \xrightarrow{\text{id}} & Z \\ {\scriptstyle k_1}\uparrow & & \uparrow{\scriptstyle k_2} \\ Y_1 & \xrightarrow[\phi]{} & Y_2 \end{array}$$

is smooth.

It follows that $k_1(Y_1) \subseteq Z$ has the structure of a smooth manifold got from taking slices of charts for Z and the inclusion $i : k_1(Y_1) \to Z$ is a C^∞ submanifold.

Proof: Let Y_1 have the metric k_1^*g, which will have regularity $C^{l-1,a}$ in arbitrary coordinates $(y_1, \ldots, y_n)$ for Y_1. By Lemma 2.18 there exist harmonic coordinates $(x_1, \ldots, x_n)$ about each point $p \in Y$ which have regularity of class $C^{l,a}$ when expressed in terms of the (y_α). So the (x_α) don't necessarily lie in the same C^∞-structure as the (y_α), but they do lie in the same C^l-structure. We now attempt to build a new C^∞-structure on Y containing the harmonic coordinates (x_α). The components $(k_{1,j})$ of k_1 with respect to the coordinates (x_α) on Y and arbitrary coordinates on Z satisfy equations (2.22). These equations form a second order quasi-linear elliptic system which has smooth data coming from g on Z. Therefore by Morrey's regularity results [52, Theorem 9.1] we can conclude that the $(k_{1,j})$ must be smooth. So with respect to any harmonic coordinates on Y, the metric k_1^*g on Y is smooth. Now if $(\tilde{x}_1, \ldots, \tilde{x}_n)$ is a second set of harmonic coordinates on Y then the transition functions $x_\alpha = x_\alpha(\tilde{x}_1, \ldots, \tilde{x}_n)$ will satisfy the linear elliptic equations (2.20) which we now know have *smooth* data coming from the metric k_1^*g with respect to harmonic coordinates. Therefore by Morrey's regularity results for linear elliptic equations [53, Theorem 6.4.8] we can conclude transition functions between harmonic coordinates on Y_1 are smooth. We could also have seen this by applying the last part of Lemma 2.18.

Now we are done: let Y_2 be the set Y_1 with smooth structure the maximal smooth atlas containing all $(k_1^* g)$-harmonic coordinate systems, and define $\phi = \mathrm{id}$, $k_2 = k_1$ as functions between sets.

∎

Of course more general versions of Proposition 2.19 can be proved, but we shall not need them here.

2.2.5 Calibrations and calibrated submanifolds

The notion of *calibrated geometry* was first introduced by Harvey and Lawson in their foundational paper [21]. We include in this section the theory from calibrated geometry that we shall be needing later.

If W is a real vector space then an *r-plane* in W is a vector subspace $V \leqslant W$ such that $\dim V = r$. Suppose that (Z, g) is a Riemannian manifold. Given $z \in Z$ and an oriented r-plane $V \leqslant T_z Z$ the metric and orientation on V induces a linear isomorphism

$$\begin{array}{rcl} \Lambda^r V^* & \cong & \mathbb{R} \\ \mathrm{d}V_g & \leftrightarrow & 1 \end{array}$$

where $\mathrm{d}V_g$ is the volume form on V. In particular, we now have an order relation on $\Lambda^r V^*$. We say $\phi \in C^\infty(\Lambda^r T^* Z)$ is a *calibration* on (Z, g) if $\mathrm{d}\phi = 0$ and

$$\phi_z|_V \leqslant \mathrm{d}V_g \tag{2.23}$$

for all $z \in Z$ and oriented r-planes $V \leqslant T_z Z$. We then refer to the triple (Z, g, ϕ) as a *Riemannian manifold with calibration*. A closed r-form ϕ on Z is a calibration if and only if

$$\left|\phi_z|_V\right|_g \leqslant 1$$

for all $z \in Z$ and r-planes $V \leqslant T_z Z$, where $|\cdot|_g$ is the norm induced on $\Lambda^r V^*$ by g.

Let $\phi \in C^\infty(\Lambda^r T^* Z)$ be a calibration on (Z, g). Let $k : Y \to Z$ be an oriented submanifold with $\dim Y = r$. We say that $k : Y \to Z$ is a *calibrated submanifold* of (Z, g, ϕ) if

$$k^* \phi = \mathrm{d}V_g$$

in $C^\infty(\Lambda^r T^* Y)$. Here $\mathrm{d}V_g$ is the volume form on Y got from the orientation on Y and the restriction of the metric g to Y. This condition means that we have equality in the inequality (2.23) for each of the oriented r-planes $V = T_y Y \leqslant T_{k(y)} Z$.

Suppose that the r-form ϕ is a calibration on (Z, g), and that $k : Y \to Z$ is a compact oriented submanifold with $\dim Y = r$. Then ϕ defines a class $[\phi]$ in the rth de Rham cohomology group $H^r(Z)$ of Z and $k : Y \to Z$ defines a class $[k(Y)]$ in the rth real singular homology group $H_r(Z)$ of Z. If we denote the usual pairing of $[\phi]$ and $[k(Y)]$ by

$$[\phi] \cdot [k(Y)] := \int_Y k^* \phi$$

then we have

$$\mathrm{Vol}(Y) \;=\; \textstyle\int_Y \mathrm{d}V_g \;\geqslant\; \int_Y k^* \phi \;=\; [\phi] \cdot [k(Y)]$$

with equality if and only if $k : Y \to Z$ is a calibrated submanifold of (Z, g, ϕ). It follows that any compact calibrated submanifold $k : Y \to Z$ of (Z, g, ϕ) has minimal volume amongst the compact oriented submanifolds representing the homology class $[k(Y)] \in H_r(Z)$, and therefore $k : Y \to Z$ will be a minimal submanifold of the Riemannian manifold (Z, g). In fact, *any* calibrated submanifold of a Riemannian manifold with calibration will automatically be minimal, and this fact is the content of the following proposition.

Proposition 2.20 *Let (Z, g, ϕ) be a Riemannian manifold with calibration. Let $k : Y \to Z$ be a calibrated submanifold. Then k is a minimal submanifold of (Z, g).*

Proof: As $k : Y \to Z$ is calibrated, Y must be oriented and we can integrate top degree compactly supported forms over Y. If $K : I \times Y \to Z$ is any local variation of $k_0 := k$ then we have

$$\int_{\mathrm{supp}_0 K} \mathrm{d}V_0 = \int_{\mathrm{supp}_0 K} k_0^* \phi \tag{2.24}$$

since the submanifold $k_0 : Y \to Z$ is calibrated with respect to ϕ. Here $\mathrm{d}V_0$ is Lebesgue measure on $\mathrm{supp}_0 K$ coming from the metric $k_0^* g$. Also, given $t \in I$ the map K defines a homotopy between k_t and k_0, in the sense of Bott and Tu [8]. Since ϕ is closed it follows that there exist forms θ_t on Y such that

$$k_t^* \phi = k_0^* \phi + \mathrm{d}\theta_t \tag{2.25}$$

for each $t \in I$. This is by the homotopy invariance of cohomology. In fact, we may take

$$\theta_t = \int_0^t k_s^* \big(\iota(\xi^s)\phi\big) \mathrm{d}s \tag{2.26}$$

where the integrations are carried out pointwise on Y. To see this, consider

$$\begin{aligned} k_t^* \phi - k_0^* \phi &= \int_0^t \frac{\partial}{\partial t} \Big(k_t^* \phi \Big) \Big|_{t=s} \mathrm{d}s \\ &= \int_0^t \mathrm{d}\big(k_s^* (\iota(\xi^s)\phi) \big) \mathrm{d}s \\ &= \mathrm{d} \left(\int_0^t k_s^* \big(\iota(\xi^s)\phi\big) \mathrm{d}s \right) \end{aligned}$$

using the material of Section 2.2.2. Here ξ^s is the infinitesimal variation associated to K at $s \in I$, which vanishes on the boundary of $\mathrm{supp}_0 K$, since $K(t, x) = K(t, 0)$ for all $t \in I$ and $x \in \partial(\mathrm{supp}_0 K)$.

Now let us fix any small $|t|$ so that we have a metric $k_t^* g$ on $\mathrm{supp}_0 K$. Let $\mathrm{d}V_t$ be the associated Lebesgue measure on $\mathrm{supp}_0 K$. Then by equations (2.24) and (2.25) we have

$$\int_{\mathrm{supp}_0 K} \mathrm{d}V_0 = \int_{\mathrm{supp}_0 K} k_t^* \phi - \int_{\mathrm{supp}_0 K} \mathrm{d}\theta_t \leqslant \int_{\mathrm{supp}_0 K} \mathrm{d}V_t - \int_{\mathrm{supp}_0 K} \mathrm{d}\theta_t$$

since ϕ is a calibration on (Z, g). But now

$$\int_{\mathrm{supp}_0 K} \mathrm{d}\theta_t = \int_{\partial(\mathrm{supp}_0 K)} \theta_t = 0$$

by Stokes' Theorem and the fact θ_t vanishes on the boundary of $\mathrm{supp}_0 K$. It follows that $\mathrm{Var}_0(K) = 0$ and hence we are done.

∎

2.3 Calabi-Yau and special Lagrangian geometry

2.3.1 Basic definitions and examples

We approach the subject from the point of view of Riemannian geometry, and therefore define a *Calabi-Yau manifold* to be a Riemannian manifold (M, g) with holonomy group

$$\mathrm{Hol}(g) \leqslant \mathrm{SU}(n)$$

where $\dim M = 2n$ the real dimension of M. This means that firstly M admits a complex structure J with respect to which g is a Kähler metric. In particular if the non-degenerate 2-form ω on M is defined by

$$\omega(\xi_1, \xi_2) := \frac{1}{2} g(J\xi_1, \xi_2) \tag{2.27}$$

for all $\xi_1, \xi_2 \in C^\infty(TM)$, then ω is closed. We call ω the *Kähler form*. Secondly, the complex manifold (M, J) admits a nowhere vanishing type $(n, 0)$-form Ω that is covariant constant. Clearly any such Ω is unique up to scaling by non-zero complex numbers. Given this freedom we may in fact choose Ω so that

$$\frac{\omega^n}{n!} = (-1)^{\frac{n(n-1)}{2}} \frac{i^n}{2^n} \cdot \Omega \wedge \overline{\Omega} \tag{2.28}$$

and in this case, it turns out that $\operatorname{Re}\Omega \in C^\infty(\Lambda^n T^* M)$ is a calibration on the Riemannian manifold (M, g). Since the Levi-Civita connection ∇_g of g is torsion-free, the condition $\nabla_g \Omega = 0$ implies that Ω is closed, and therefore holomorphic. It follows that the canonical bundle K_M of (M, J) is trivial.

In the above situation (in particular with equations (2.27) and (2.28) holding) we shall say that (J, g, Ω) is a *Calabi-Yau structure* on M.

It is a well-known fact – see [38, Chapter IX, Theorem 4.6] for example – that a Kähler manifold (M, J, g) of dimension $2n$ is Ricci-flat precisely when the restricted holonomy group $\mathrm{Hol}^0(g) \leqslant \mathrm{Hol}(g)$ of (M, g) is contained inside $\mathrm{SU}(n)$. It follows that all Calabi-Yau manifolds are Ricci-flat. Compact Riemannian manifolds with holonomy in $\mathrm{SU}(n)$ were shown to exist in reasonable numbers by Yau's proof of the Calabi Conjecture [59], and hence the name Calabi-Yau as given above. Specifically, Yau's theorem implies that if (M, J) is a compact, complex manifold which admits Kähler metrics and has first Chern class $c_1(M) = 0$ then in each Kähler class there exists a unique metric g which is Ricci-flat, so that $\mathrm{Hol}^0(g) \leqslant \mathrm{SU}(n)$. If M is simply-connected we can then deduce that $\mathrm{Hol}(g) \leqslant \mathrm{SU}(n)$. Examples include non-singular hypersurfaces in $\mathbb{C}P^{n+1}$ defined by the vanishing of a homogeneous polynomial of degree $n + 2$. In particular when $n = 3$ we have the quintic hypersurface in $\mathbb{C}P^4$.

We should note here that in the definition of Calabi-Yau manifold some authors require M to be compact. One reason for this is that M being compact is specifically the case in which the Calabi Conjecture applies. Another reason is that the applications of Calabi-Yau geometry in physics – specifically string theory – all require compactness of the manifold M. However, we will not be too concerned about these issues: M being compact will not be needed in any of the theory we consider, and so we shall relax this condition.

Let (J, g, Ω) be a Calabi-Yau structure on a manifold M with $\dim M = 2n$. Let ω be the Kähler form. We say that a submanifold $f : X \to M$ of dimension n is *special Lagrangian* with respect to (J, g, Ω) if

$$f^*\omega = 0 \tag{2.29}$$
$$f^* \operatorname{Im}\Omega = 0 \tag{2.30}$$

in $C^\infty(\Lambda^* T^* X)$. When the ambient Calabi-Yau structure is clear we shall simply speak of "special Lagrangian submanifolds". The following proposition relates special Lagrangian submanifolds with the theory of calibrated submanifolds as given in Section 2.2.5, and is proved in the Harvey and Lawson paper [21, Chapter III, Corollary 1.11].

Proposition 2.21 *Let (J, g, Ω) be a Calabi-Yau structure on a manifold M and suppose $f : X \to M$ is a submanifold.*

1. *If $f : X \to M$ is special Lagrangian then X has a unique orientation such that $f : X \to M$ is calibrated with respect to* $\operatorname{Re}\Omega$.

2. *If X is oriented and $f : X \to M$ is calibrated with respect to* $\operatorname{Re}\Omega$ *then $f : X \to M$ is special Lagrangian.*

It follows by the theory given in Section 2.2.5 that special Lagrangian submanifolds will always be minimal submanifolds of the ambient Riemannian manifold (M, g).

We now give some examples of special Lagrangian submanifolds.

Example 2.22 *If we take $M = \mathbb{C}^n$ with its usual complex structure and metric then we have our most basic example of a Calabi-Yau manifold. The following examples of special Lagrangian submanifolds of $\mathbb{C}^n$ all have a large degree (cohomogeneity 1) of symmetry, which made them easy to find.*

1. *Given $a_1, \ldots, a_n, b \in \mathbb{R}$ with $b \neq 0$ the subset*

$$X_{a_1,\ldots,a_n,b} := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \begin{array}{l} |z_1|^2 - a_1 = \cdots = |z_n|^2 - a_n \\ \mathrm{Im}(i^{n+1} z_1 \ldots z_n) = b \end{array} \right\}$$

   is invariant under the group of diagonal matrices $\mathrm{U}(1)^{n-1} \leqslant \mathrm{SU}(n)$. Moreover, $X_{a_1,\ldots,a_n,b}$ is a special Lagrangian submanifold, with topology $\mathbb{R} \times \mathrm{U}(1)^{n-1}$. Note that this family of examples has n real parameters.

2. *Let $a \in \mathbb{R}$ be non-zero and let $\mathrm{SO}(n) \leqslant \mathrm{SU}(n)$ denote the subgroup of real matrices. Then*

$$X_a := \mathrm{SO}(n) \cdot \left\{ (z, 0, \ldots, 0) \in \mathbb{C}^n : \mathrm{Im}(z^n) = a \text{ and } 0 < \arg(z) < \tfrac{2\pi}{n} \right\}$$

   is an $\mathrm{SO}(n)$-invariant special Lagrangian submanifold of $\mathbb{C}^n$ with topology $\mathbb{R} \times S^{n-1}$. When $a = 0$ the above subset X_a becomes a union of special Lagrangian n-planes $V_j \leqslant \mathbb{C}^n$ with $V_i \cap V_j = \{0\}$ for $i \neq j$, so 0 is the only singular point of X_0. Note that this family of examples has 1 real parameter.

These were first examples of special Lagrangian submanifolds to be found, back in the Harvey and Lawson paper [21]. More examples of this kind are constructed in the author's dissertation [47] where other symmetry groups $G \leqslant \mathrm{SU}(n)$ are also considered.

The examples from both families above are non-compact. This is not just a coincidence: for if ϕ is a calibration on $\mathbb{R}^m$ and $f : X \to \mathbb{R}^m$ is a compact submanifold calibrated with respect to ϕ then noting $H^r(\mathbb{R}^m)$ is trivial we have by Stokes' Theorem

$$0 = \int_X f^* \phi = \int_X \mathrm{d}V_g = \mathrm{Vol}(X, g) > 0$$

a contradiction. Therefore no special Lagrangian submanifold $f : X \to \mathbb{C}^n$ can be compact. Another way of seeing this is to observe that no minimal submanifold of $\mathbb{R}^m$ can be compact (as performing dilations shows) and then we may appeal to Proposition 2.20 and Proposition 2.21.

Besides the examples given above, other more complicated special Lagrangian submanifolds of $\mathbb{C}^n$ have been constructed recently by Joyce: see the papers [26], [27], [28], [29], [30], [32], [34], [35], [36], [37] already cited in the introduction.

The remaining examples are taken from the paper [23] of Hitchin.

Example 2.23 *Suppose that (J, g, Ω) is a Calabi-Yau structure on M with Kähler form ω. Suppose further that $\sigma : M \to M$ is an anti-holomorphic involution (so $\sigma^2 = \mathrm{id}$ and σ is holomorphic as a map $M \to \overline{M}$ where $\overline{M}$ is the differentiable manifold M endowed with the complex structure $-J$) such that $\sigma^* \omega = -\omega$ and $\sigma^* \Omega = \overline{\Omega}$. Then the fixed point set $\{p \in M : \sigma(p) = p\}$ is a special Lagrangian submanifold of M.*

Example 2.24 *For each $n \geqslant 1$ the manifold $T^* S^n$ admits the structure of a Calabi-Yau manifold for which the zero section is a special Lagrangian submanifold. The complex structure on $T^* S^n$ comes from an embedding into $\mathbb{C}^{n+1}$ as an affine quadric*

$$Q = \left\{ (z_0, \ldots, z_n) \in \mathbb{C}^{n+1} : z_0^2 + \cdots + z_n^2 = 1 \right\}.$$

See the paper [57] of Stenzel for further details.

Example 2.25 *Let (M, g) be a hyperkähler manifold, of dimension $4k$. So g is Kähler with respect to three complex structures I, J, K on M and furthermore $IJK = -\mathrm{id}$. Let the Kähler forms of I, J, K be $\omega_I, \omega_J, \omega_K$ respectively. Put $\Omega := (\omega_I + i\omega_J)^k$, which is a nowhere vanishing form that is covariant constant, and of type $(2k, 0)$ with respect to the complex structure K on M. So when Ω is suitably normalised, both (K, g, Ω) and $(K, g, i\Omega)$ become Calabi-Yau structures on M.*

Let $f : X \to M$ be a submanifold of dimension $2k$. Then any two of the conditions

1. *$f : X \to M$ is a complex submanifold with respect to J*
2. *$f^*\omega_I = 0$*
3. *$f^*\omega_K = 0$*

holding implies the third, and when these conditions hold we shall say that $f : X \to M$ is complex Lagrangian. If $f : X \to M$ is complex Lagrangian, we obviously have $f^*\omega_K = 0$, and also

$$f^*\Omega = (f^*\omega_I + if^*\omega_J)^k = i^k(f^*\omega_J)^k$$

so that $f^* \operatorname{Im}\Omega = 0$ if k is even and $f^* \operatorname{Im}(i\Omega) = 0$ if k is odd. So in hyperkähler manifolds, complex Lagrangian submanifolds are special Lagrangian submanifolds with respect to some Calabi-Yau structure on M.

Conversely, when $k = 1$ we have $i\Omega = i\omega_I - \omega_J$ so that if $f : X \to M$ is special Lagrangian with respect to $(K, g, i\Omega)$ we have $f^*\omega_K = 0$ and $f^*\omega_I = f^* \operatorname{Im}(i\Omega) = 0$ so that $f : X \to M$ is complex Lagrangian as defined above.

2.3.2 Regularity of special Lagrangian submanifolds

In this section we bring together some of the material of Section 2.2 as applied to a special Lagrangian submanifold got from a tubular neighbourhood.

Suppose (J, g, Ω) is a Calabi-Yau structure on a manifold M and that $f : X \to M$ is a special Lagrangian submanifold with normal bundle $N \to X$. Suppose also that $f(X) \subseteq M$ is a closed subspace. Then using Theorem 2.15 we have a tubular neighbourhood $\tilde{U} \subseteq N$ such that the exponential map restricts to a diffeomorphism from $\tilde{U}$ onto an open subset of M. Let ξ be a section of N with $\xi_x \in \tilde{U}$ for all $x \in X$. Then ξ induces a submanifold $f_\xi : X \to M$ where

$$f_\xi(x) := \exp_{f(x)}(\xi_x)$$

for each $x \in X$.

However, now suppose that ξ is only of class $C^{l,a}$ for some integer $l \geqslant 3$. Then although we have a submanifold $f_\xi : X \to M$ the map f_ξ will a priori only be of class $C^{l,a}$ and not smooth. But if we suppose further that the submanifold $f_\xi : X \to M$ is special Lagrangian then Proposition 2.19 and the fact that special Lagrangian submanifolds are minimal implies that the subset $f_\xi(X) \subseteq M$ has a smooth structure coming from slices of charts for M, so that the inclusion $f_\xi(X) \to M$ is a smooth submanifold. We now use the fact that our submanifold f_ξ comes from a tubular neighbourhood: translating over to the normal bundle we have an inclusion $i : \xi(X) \to N$ that is smooth and a commuting diagram

$$\begin{tikzcd} \xi(X) \arrow[r, "i"] & N \arrow[d, "\pi_N"] \\ X \arrow[u, "\xi"] \arrow[r, "\mathrm{id}"'] & X. \end{tikzcd}$$

Since the projection $\pi_N : N \to X$ is smooth it follows that $\xi^{-1} = \pi_N \circ i : \xi(X) \to X$ is a smooth map which is a C^l-diffeomorphism. Hence by Corollary 2.8 we see that $\xi : X \to \xi(X)$ is a C^∞-diffeomorphism, so that $\xi : X \to N$ is a smooth submanifold. Consequently $f_\xi : X \to M$ is a smooth submanifold, too.

2.3.3 Some pointwise calculations

In this section we give some elementary results concerning the objects that live on Calabi-Yau manifolds. These results will be needed in Lemma 2.29.

Let $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}\right)$ be the standard basis for the real vector space $\mathbb{C}^n$ with dual basis $(\mathrm{d}x_1, \ldots, \mathrm{d}x_n, \mathrm{d}y_1, \ldots, \mathrm{d}y_n)$. Then we have the following relevant objects

$$\begin{aligned} g_0 &= \sum_{j=1}^{n} (\mathrm{d}x_j \otimes \mathrm{d}x_j + \mathrm{d}y_j \otimes \mathrm{d}y_j) \\ \omega_0 &= \sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}y_j \\ \Omega_0 &= (\mathrm{d}x_1 + i\mathrm{d}y_1) \wedge \ldots \wedge (\mathrm{d}x_n + i\mathrm{d}y_n) \end{aligned}$$

that are respectively the metric, Kähler form and the holomorphic volume form. The endomorphism $J_0 : \mathbb{C}^n \to \mathbb{C}^n$ defined as in equation (2.27) is then multiplication by $i \in \mathbb{C}$.

Lemma 2.26 *Let $V \leqslant \mathbb{C}^n$ be an n-plane with $\omega_0|_V = 0$ and $\operatorname{Im}\Omega_0|_V = 0$. Given $\xi \in V^\perp$ we have*

$$\begin{aligned} \big(\iota(\xi)\omega_0\big)\big|_V &= (\flat_0 J_0)\xi, & (2.31) \\ \big(\iota(\xi)\operatorname{Im}\Omega_0\big)\big|_V &= -*_0 (\flat_0 J_0)\xi & (2.32) \end{aligned}$$

where $\flat_0 : V \to V^*$ is the usual isomorphism induced by the restriction of g_0 to V and $*_0 : \Lambda^* V^* \to \Lambda^* V^*$ is the Hodge star isomorphism got from the restriction of g_0 to V and the orientation induced on V by $\operatorname{Re}\Omega_0|_V$. Moreover, for arbitrary $\xi \in \mathbb{C}^n$ we have

$$\big(\iota(\xi)\operatorname{Im}\Omega_0\big)\big|_V = -*_0 \Big(\big(\iota(\xi)\omega_0\big)\Big)\Big|_V. \qquad (2.33)$$

Proof: Because the first two equations are $\mathrm{SU}(n)$-invariant and $\mathrm{SU}(n)$ acts transitively on the set of all n-planes $V \leqslant \mathbb{C}^n$ with $\omega_0|_V = \operatorname{Im}\Omega_0|_V = 0$ we need only check (2.31) and (2.32) in the case that $V = \mathbb{R}^n$. Then we must have $\xi = \sum_{j=1}^n v^j \frac{\partial}{\partial y_j}$ for some $v^1, \ldots, v^n \in \mathbb{R}$. In fact, as both sides of each equation are linear in ξ we can assume that $\xi = \frac{\partial}{\partial y_k}$ for some $k = 1, \ldots, n$. Then for (2.31) we have

$$\begin{aligned} \Big(\iota(\tfrac{\partial}{\partial y_k})\omega_0\Big)\Big|_V &= \iota(\tfrac{\partial}{\partial y_k}) \left(\sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}y_j\right)\Bigg|_V \\ &= -\big(\mathrm{d}x_k\big)\big|_V \\ &= (\flat_0 J_0)\tfrac{\partial}{\partial y_k} \end{aligned}$$

as required, and for (2.32)

$$\begin{aligned} \Big(\iota(\tfrac{\partial}{\partial y_k})\operatorname{Im}\Omega_0\Big)\Big|_V &= \Big(\iota(\tfrac{\partial}{\partial y_k})\operatorname{Im}\big[(\mathrm{d}x_1 + i\mathrm{d}y_1) \wedge \ldots \wedge (\mathrm{d}x_n + i\mathrm{d}y_n)\big]\Big)\Big|_V \\ &= (-1)^{k+1} \cdot \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_{k-1} \wedge \mathrm{d}x_{k+1} \wedge \ldots \wedge \mathrm{d}x_n \\ &= *_0(\mathrm{d}x_k) \\ &= -*_0 (\flat_0 J_0)\tfrac{\partial}{\partial y_k} \end{aligned}$$

as required. To prove the third equation note that $\mathbb{C}^n = V \oplus V^\perp$ and clearly (2.33) holds for all $\xi \in V^\perp$: this follows from equations (2.31) and (2.32). Also (2.33) holds for all $\xi \in V$ because $\omega_0|_V = 0$ and $\operatorname{Im}\Omega_0|_V = 0$.

∎

We would like a result for Calabi-Yau manifolds that is analogous to Lemma 2.26. For this we need the following result.

Proposition 2.27 *Let (M, J, g, Ω) be a Calabi-Yau manifold, with Kähler form ω. Then given $x \in M$ there exists an open subset $U \subseteq M$ containing x and local orthonormal frames $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ for $TM|_U$ such that*

$$\begin{aligned} \omega &= \sum_{j=1}^{n} e^j \wedge f^j \\ \Omega &= (e^1 + if^1) \wedge \ldots \wedge (e^n + if^n) \end{aligned}$$

where $\dim M = 2n$ *and* $(e^1, \ldots, e^n, f^1, \ldots, f^n)$ *is the dual frame for* $T^*M|_U$.

Proof: By an inductive argument for $k = 1, 2, \ldots, n$ we can find local orthonormal vector fields $(e_1, \ldots, e_k, Je_1, \ldots, Je_k)$, since the metric g is Hermitian. So now fix $k = n$, and set $f_j := Je_j$ for $j = 1, \ldots, n$. Then we have g, ω and J in the form needed. Also, by equation (2.28) we see that

$$\Omega = \lambda \cdot (e^1 + if^1) \wedge \ldots \wedge (e^n + if^n)$$

for some $S^1 \subseteq \mathbb{C}$ valued function λ. Now by rotating (say) the pair of vectors $e_1 + if_1$ by λ throws Ω into the form required, whilst preserving g, ω and J.

∎

Corollary 2.28 *Let (M, J, g, Ω) be a Calabi-Yau manifold, with Kähler form ω. Let $f : X \to M$ be a special Lagrangian submanifold with normal bundle $N \to X$. If $\xi \in C^\infty(N)$ then we have*

$$f^*\big(\iota(\xi)\omega\big) = (\flat_g J)\xi \tag{2.34}$$

$$f^*\big(\iota(\xi)\operatorname{Im}\Omega\big) = - *_g (\flat_g J)\xi \tag{2.35}$$

where $\flat_g : TX \to T^*X$ is the usual isomorphism induced by the restriction of g to X and $*_g : \Lambda^*T^*X \to \Lambda^*T^*X$ is the Hodge star X induced by the restriction of g to X and the orientation on X induced by $f^*\operatorname{Re}\Omega$. Moreover, for arbitrary $\xi \in C^\infty(f^*TM)$ we have

$$f^*\big(\iota(\xi)\operatorname{Im}\Omega\big) = - *_g \Big(f^*\big(\iota(\xi)\omega\big)\Big). \tag{2.36}$$

Proof: The equations (2.34), (2.35) and (2.36) can be checked pointwise, and at each point $x \in M$ we have by Proposition 2.27 an isomorphism $T_xM \cong \mathbb{C}^m$ compatible with the relevant structure. Hence the result follows from Lemma 2.26.

∎

2.3.4 Infinitesimal deformations of special Lagrangian submanifolds

Suppose that (J, g, Ω) is a Calabi-Yau structure on a manifold M^{2n} with Kähler form ω and that $f : X \to M$ is a special Lagrangian submanifold. Since $f^*\omega = 0$ it follows from equation (2.27) that the complex structure J defines a vector bundle isomorphism

$$J : N \to TX$$

where $N \to X$ is the normal bundle of X in M. Also using the restriction of the metric g to X we have as usual the vector bundle isomorphism

$$\flat_g : TX \to T^*X$$

so that $\flat_g J$ identifies normal vector fields on X with 1-forms on X.

When $f(X)$ is a closed subset of M we may apply the Tubular Neighbourhood Theorem 2.15 to $f : X \to M$ and obtain an open neighbourhood $\tilde{U} \subseteq N$ of the zero section such that the exponential map defines a diffeomorphism

$$\exp|_{\tilde{U}} : \tilde{U} \to M$$

onto an open subset of M. By shrinking $\tilde{U}$ if necessary we may suppose further that for each $x \in X$ the subset $\tilde{U} \cap N_x \subseteq N_x$ is star-shaped with respect to 0. Let us define

$$U := (\flat_g J)\tilde{U}$$

an open subset of T^*X containing the zero section, and correspondingly

$$\tilde{U}^\infty := \{\xi \in C^\infty(N) : \xi_x \in \tilde{U} \text{ for all } x \in X\} \tag{2.37}$$

$$U^\infty := \{\eta \in C^\infty(T^*X) : \eta_x \in U \text{ for all } x \in X\}. \tag{2.38}$$

Then both $\tilde{U}^\infty$ and U^∞ are star-shaped with respect to 0.

Suppose that $\xi \in C^\infty(N)$ and $\varepsilon > 0$ are such that $t\xi \in \tilde{U}^\infty$ for all $|t| < \varepsilon$. It follows by the comment after Theorem 2.15 that we have a 1-parameter family of submanifolds $f_{t\xi} : X \to M$ parameterised by $t \in (-\varepsilon, \varepsilon)$ where

$$f_{t\xi}(x) = \exp_{f(x)}(t\xi_x)$$

for each $x \in X$. Note that $f_0 = f$. We would like to know whether or not deforming the special Lagrangian submanifold $f : X \to M$ in the direction $\xi \in C^\infty(N)$ keeps X special Lagrangian, at least infinitesimally. Now the condition that $f_{t\xi} : X \to M$ be special Lagrangian is

$$\begin{aligned} f_{t\xi}^*\omega &= 0 \\ f_{t\xi}^* \operatorname{Im}\Omega &= 0 \end{aligned}$$

in $C^\infty(\Lambda^* T^* X)$, so that in order to answer our question we perform the following pointwise computations on X.

Lemma 2.29 *Refer to the above notation. Let $\eta = (\flat_g J)\xi$ be the 1-form corresponding to ξ. Then*

$$\begin{aligned} \frac{\partial}{\partial t}\Big(f_{t\xi}^*\omega\Big)\Big|_{t=0} &= \mathrm{d}\eta \\ \frac{\partial}{\partial t}\Big(f_{t\xi}^* \operatorname{Im}\Omega\Big)\Big|_{t=0} &= -\mathrm{d}(*_g\eta) \end{aligned}$$

where the derivatives on the left hand sides of these equations are calculated pointwise on X, and we use the restriction of the metric g on X and the orientation on X induced by $f^*\operatorname{Re}\Omega$ to define the Hodge star operator $*_g$.

Proof: Define $F_\xi : (-\varepsilon, \varepsilon) \times X \to M$ by $F_\xi(t, x) = f_{t\xi}(x)$ for all $|t| < \varepsilon$ and $x \in X$. Then Corollary 2.28, the results of Section 2.2.2, and the fact ω is closed give

$$\begin{aligned} \frac{\partial}{\partial t}\Big(f_{t\xi}^*\omega\Big)\Big|_{t=0} &= \mathcal{L}_{\frac{\partial}{\partial t}}\Big(F_\xi^*\omega\Big)\Big|_{X_0} \\ &= f^*\big(\iota(\xi^0)\mathrm{d}\omega + \mathrm{d}(\iota(\xi^0)\omega)\big) \\ &= \mathrm{d}\big(f^*(\iota(\xi^0)\omega)\big) \\ &= \mathrm{d}\eta. \end{aligned}$$

In the above ξ^0 is any extension of ξ to a neighbourhood of $f(X)$ in M. Similarly we have

$$\begin{aligned} \frac{\partial}{\partial t}\Big(f_{t\xi}^* \operatorname{Im}\Omega\Big)\Big|_{t=0} &= \mathrm{d}\big(f^*(\iota(\xi^0)\operatorname{Im}\Omega)\big) \\ &= -\mathrm{d}(*_g\eta) \end{aligned}$$

and we are done. ∎

Chapter 3

Deformations of compact special Lagrangian submanifolds

3.1 Analysis on compact manifolds

In this section we give a brief description of some analytic tools for compact manifolds. Useful references for this section are the books of Adams [2], Aubin [4, Chapters 2,3,4], Besse [7, Appendix], Gilbarg and Trudinger [18] and Joyce [25, Chapter 1]. As for the notation we shall use, we refer the reader to Table 1.1, so that for instance, throughout Section 3.1 we have $p > 1$, $0 < a, b < 1$ and i, j, k, m non-negative integers.

3.1.1 Construction of suitable Banach spaces

Let E be a vector bundle over a compact manifold X, where $\dim X = n$. Our primary method of constructing Banach spaces of sections of E is via coordinate charts. To this end, pick any finite open covering $\mathcal{U} = \{U_1, \ldots, U_N\}$ of X such that both E and X are trivial over each U_ν and each U_ν is a domain when considered as a subset of $\mathbb{R}^n$. If ξ is a section of E we denote by $\xi_1^\nu, \ldots, \xi_{\operatorname{rank} E}^\nu$ the components of ξ in the open set U_ν. We also fix a partition of unity $\{\rho_1, \ldots, \rho_N\}$ subordinate to the open covering $\mathcal{U}$ of X.

Sobolev spaces

Considering each U_ν as being a subset of Euclidean space $\mathbb{R}^n$, we have the usual Euclidean measure $\mathrm{d}V_e$ defined on each U_ν. So given $u \in C^\infty(X)$ with $\operatorname{supp} u \subseteq U_\nu$ we may define

$$\|u\|_{L^p(U_\nu)} := \left(\int_{U_\nu} |u|^p \mathrm{d}V_e\right)^{\frac{1}{p}}$$

the usual L^p-*norm* of u. For $\xi \in C^\infty(E)$ we also have the *Sobolev norm* defined by

$$\|\xi\|_{W_k^p(E)} := \left(\sum_{\nu=1}^{N} \sum_{j=1}^{\operatorname{rank} E} \sum_{0 \leqslant |\lambda| \leqslant k} \left\|\rho_\nu\left(\partial^\lambda \xi_j^\nu\right)\right\|_{L^p(U_\nu)}^p\right)^{\frac{1}{p}}. \tag{3.1}$$

We define $W_k^p(E)$ to be the vector space completion of $C^\infty(E)$ with respect to the norm (3.1). We call the Banach space $W_k^p(E)$ a *Sobolev space*. Note that, additionally, each $W_k^2(E)$ is a Hilbert space.

As a topological vector space $W_k^p(E)$ is independent of all choices U_j, ρ_j. Additionally, we can view elements of $W_k^p(E)$ as genuine sections of E, whose components in the various trivialisations U_ν are k times weakly differentiable, with all derivatives L^p-integrable.

An alternative, coordinate free method for constructing the Sobolev spaces $W_k^p(E)$ is to endow the manifold X with a Riemannian metric g, and the vector bundle E with a fibre metric $(\ ,\ )_E$ and a compatible connection ∇_E. Then the norm (3.1) is equivalent to the norm on $C^\infty(E)$ defined by

$$\|\xi\| := \left(\sum_{j=0}^{k} \int_X \left|\nabla_E^j \xi\right|_E^p \mathrm{d}V_g\right)^{\frac{1}{p}}$$

for all $\xi \in C^\infty(E)$. Then $W_0^2(E)$ is simply the Hilbert space $L^2(E)$ defined as in Section 2.1.2.

Note that there is a constant $C > 0$ such that

$$\left|\langle \xi_1 | \xi_2 \rangle_{L^2(E)}\right| \leqslant C \|\xi_1\|_{W_0^p(E)} \|\xi_2\|_{W_0^{p'}(E)}$$

for all $\xi_1, \xi_2 \in C^\infty(E)$. It follows that the L^2-inner product defined in Section 2.1.2 extends to a continuous bilinear map

$$\langle\ |\ \rangle_{L^2(E)} : W_0^p(E) \times W_0^{p'}(E) \to \mathbb{R} \tag{3.2}$$

and in fact the pairing (3.2) induces a Banach space isomorphism

$$\Phi : W_0^p(E) \to W_0^{p'}(E)^* \tag{3.3}$$

defined by $\Phi(\xi)(\eta) := \langle \xi | \eta \rangle_{L^2(E)}$ for all $\xi \in W_0^p(E)$ and $\eta \in W_0^{p'}(E)$. See the book [2, Section 3.4] of Adams for further details, where the following useful result is also proved.

Proposition 3.1 *The Banach spaces $W_k^p(E)$ are reflexive.*

The important point in Proposition 3.1 is that $p > 1$.

Hölder spaces

Given $\xi \in C^k(E)$ we define

$$\|\xi\|_{C^k(E)} := \sum_{\nu=1}^{N} \sum_{j=1}^{\operatorname{rank} E} \sum_{0 \leqslant |\lambda| \leqslant k} \sup_{U_\nu} \left|\rho_\nu\left(\partial^\lambda \xi_j^\nu\right)\right|. \tag{3.4}$$

The norm (3.4) makes $C^k(E)$ into a Banach space, which we call a C^k*-space*. We also define

$$\|\xi\|_{C^{k,a}(E)} = \|\xi\|_{C^k(E)} + \sum_{\nu=1}^{N} \sum_{j=1}^{\operatorname{rank} E} \sum_{|\lambda|=k} \left[\rho_\nu\left(\partial^\lambda \xi_j^\nu\right)\right]_{a;U_\nu} \tag{3.5}$$

which may, or may not, be finite: refer to Section 2.2.4 for the definition of $[\cdot]_{a,A}$. We now put

$$C^{k,a}(E) := \left\{\xi \in C^k(E) : \|\xi\|_{C^{k,a}(E)} < \infty\right\}$$

which becomes a Banach space when equipped with the norm (3.5). This Banach space is called a *Hölder space*.

The C^k and Hölder spaces can also be constructed in a coordinate free manner, as we now describe. Suppose that X is equipped with a Riemannian metric g and E is equipped with a fibre metric and compatible connection. Then given $\xi \in C^k(E)$ and $0 \leqslant j \leqslant k$ we may form the jth covariant derivative $\nabla_E^j \xi \in C^0\left((\otimes^j T^*X) \otimes E\right)$ of ξ using the Levi-Civita connection of g and the connection on E. Furthermore, using g and the fibre metric on E we may compute the pointwise norm of $\nabla_E^j \xi$, which we write as $\left|\nabla_E^j \xi\right|_E \in C^0(X)$. It turns out that the norm (3.4) on $C^k(E)$ defined above is equivalent to the norm on $C^k(E)$ defined by

$$\|\xi\| \;:=\; \sum_{j=0}^{k} \sup_X \left|\nabla_E^j \xi\right|_E$$

for each $\xi \in C^k(E)$.

To construct the Hölder norm (3.5) in a coordinate free manner we first need a preliminary discussion. Suppose V is any vector bundle over a manifold Y which is endowed with a connection. Then given any piecewise-smooth curve which joins points $x, y \in Y$ we may define a linear isomorphism $V_x \to V_y$ using parallel transport along this curve. If further V is equipped with a fibre metric which is compatible with the connection, then the parallel transport map will be an isometry.

Suppose Y has a Riemannian metric h and the injectivity radius $\mathrm{inj}(Y, h)$ is positive. Then we may choose some $0 < \varepsilon \leqslant \mathrm{inj}(Y, h)$. Let $d_h(x, y)$ denote the distance between two points x, y in the Riemannian manifold (Y, h). Then given $x, y \in Y$ with $d_h(x, y) < \varepsilon$ there exists a unique geodesic in (Y, h) of length $d_h(x, y)$ which joins x to y.

Consequently, for any $A \subseteq Y$ and section v of V we can define

$$[v]^h_{a;A} := \sup \left\{ \frac{|v_x - v_y|_V}{d_h(x, y)^a} : x, y \in A \text{ with } 0 < d_h(x, y) < \varepsilon \right\} \tag{3.6}$$

which may or may not be finite. In equation (3.6) we make sense of $|v_x - v_y|_V$ by identifying $V_x \cong V_y$ isometrically using parallel transport along the unique geodesic in (Y, h) from x to y which has length $d_h(x, y)$, and then applying the fibre metric $|\cdot|_V$ on V.

If we return to our compact Riemannian manifold (X, g), then $\mathrm{inj}(X, g) > 0$ and it turns out that

$$C^{k,a}(E) = \left\{ \xi \in C^k(E) : \left[\nabla^k_E \xi\right]^g_{a;X} < \infty \right\}$$

and the norm (3.5) is equivalent to the norm on $C^{k,a}(E)$ defined by

$$\|\xi\| := \left(\sum_{j=0}^{k} \sup_X \left| \nabla^j_E \xi \right|_E \right) + \left[\nabla^k_E \xi\right]^g_{a;X} \tag{3.7}$$

for each $\xi \in C^{k,a}(E)$.

Embedding and Compactness Theorems

The Banach spaces defined above are actually closely related, as we see from the following very useful results.

Theorem 3.2 (Embedding Theorems) *Refer to Section* 2.1.1 *for the definition of a continuous embedding between Banach spaces.*

1. *If $k \geqslant l \geqslant 0$ and $k - \frac{n}{p} \geqslant l - \frac{n}{q}$ then there is a continuous embedding $W^p_k(E) \leqslant W^q_l(E)$.*
2. *If $k + a \geqslant l + b$ then there are continuous embeddings $C^{k+1}(E) \leqslant C^{k,a}(E) \leqslant C^{l,b}(E) \leqslant C^l(E)$ and $C^k(E) \leqslant C^l(E)$.*
3. *If $k - \frac{n}{p} \geqslant l + a$ then there are continuous embeddings $W^p_k(E) \leqslant C^{l,a}(E) \leqslant C^l(E) \leqslant W^q_l(E)$.*

A consequence of Theorem 3.2 is that

$$\bigcap_{k=0}^{\infty} W^p_k(E) = \bigcap_{k=0}^{\infty} C^{k,a}(E) = C^\infty(E). \tag{3.8}$$

Theorem 3.3 (Compactness Theorems) *Refer to Section* 2.1.1 *for the definition of a compact embedding between Banach spaces.*

1. *The embedding $W^p_k(E) \leqslant W^q_l(E)$ is compact when $k > l \geqslant 0$ and $k - \frac{n}{p} > l - \frac{n}{q}$.*
2. *The embedding $C^{k,a}(E) \leqslant C^k(E)$ is compact.*
3. *The embedding $W^p_k(E) \leqslant C^{l,a}(E)$ is compact whenever $k - \frac{n}{p} > l + a$.*

The real substance of Theorem 3.2 and Theorem 3.3 are the corresponding local results from PDE theory. See for example the book [18, Section 7.7, Section 7.10] of Gilbarg and Trudinger.

3.1.2 The theory of elliptic operators

We now describe the properties of differential operators when acting between the spaces introduced above. First of all, if $P : C^\infty(E) \to C^\infty(F)$ is a smooth, linear differential operator of order $l \geqslant 1$ then P extends to bounded linear maps

$$P : W^p_{k+l}(E) \quad \to \quad W^p_k(F) \tag{3.9}$$
$$P : C^{k+l,a}(E) \quad \to \quad C^{k,a}(F). \tag{3.10}$$

It follows from continuity that the defining identity (2.7) of the formal adjoint P^* of P extends to an identity

$$\langle \xi | P^* \eta \rangle_{L^2(E)} = \langle P\xi | \eta \rangle_{L^2(F)} \tag{3.11}$$

valid for all $\xi \in W^p_l(E)$, $\eta \in W^{p'}_l(F)$, and hence all $\xi \in C^{l,a}(E)$, $\eta \in C^{l,a}(F)$. Use of the extended identity (3.11) is called *integration by parts*.

Elliptic operators are very important in the theory of analysis on compact manifolds because of the usefulness of results such as the following: refer to Section 2.1.2 for the definition of weak solution.

Theorem 3.4 *Let X be a compact manifold and $E, F \to X$ vector bundles over X. Let $P : C^\infty(E) \to C^\infty(F)$ be an elliptic, smooth, linear differential operator of order $l \geqslant 1$. Suppose that $\eta \in L^1(F)$ and that $\xi \in L^1(E)$ is a weak solution of the equation $P\xi = \eta$.*

1. *If $\eta \in W^p_k(F)$ then $\xi \in W^p_{k+l}(E)$ with $P\xi = \eta$ and*

$$\|\xi\|_{W^p_{k+l}(E)} \leqslant C_1 \left(\|P\xi\|_{W^p_k(F)} + \|\xi\|_{L^1(E)} \right) \tag{3.12}$$

   where the constant $C_1 > 0$ does not depend on ξ.

2. *If $\xi \in C^0(E)$ with $\eta \in C^{k,a}(F)$ then $\xi \in C^{k+l,a}(E)$ with $P\xi = \eta$ and*

$$\|\xi\|_{C^{k+l,a}(E)} \leqslant C_2 \left(\|P\xi\|_{C^{k,a}(F)} + \|\xi\|_{C^0(E)} \right) \tag{3.13}$$

   where the constant $C_2 > 0$ does not depend on ξ.

The proof of Theorem 3.4 is best thought of as being in two parts: we firstly deduce that the given conditions imply that ξ is locally of class W^p_{k+l} in the first case or locally of class $C^{k+l,a}$ in the second case. Then one can give local estimates for ξ in terms of the relevant norms, and the passage to the whole of X is then entirely straightforward. The relevant theorems are the Morrey interior estimates as in [53, Theorem 6.4.8] or the Schauder interior estimates as in [16, Theorem 1]. (Actually the Morrey estimates are stronger as he proves that the $C^0(E)$ norm on the right hand side of the inequality (3.13) can in fact be replaced with the weaker $L^1(E)$ norm, but for our purposes the estimate (3.13) is sufficient.)

It turns out that results such as Theorem 3.4 will fail for the Banach spaces $C^k(E)$: that is why we have had to introduce the more complicated Sobolev and Hölder spaces.

We now have the following corollary.

Corollary 3.5 *Let X be a compact manifold and $E, F \to X$ vector bundles over X. Let $P : C^\infty(E) \to C^\infty(F)$ be an elliptic, smooth, linear differential operator of order $l \geqslant 1$. If $\xi \in L^1(E)$ with $P\xi = 0$ holding weakly then $\xi \in C^\infty(E)$ and $P\xi = 0$ holds in the usual sense.*

Proof: If $\xi \in L^1(E)$ with $P\xi = 0$ holding weakly then by Theorem 3.4 we have $\xi \in W^p_k(E)$ for all $k \geqslant 0$. Then $\xi \in C^\infty(E)$ follows from equation (3.8).

∎

It follows that when P is elliptic the kernels of the maps (3.9) and (3.10) coincide and are independent of p, k, a. Moreover, this kernel is a subspace of $C^\infty(E)$.

Theorem 3.6 *Let X be a compact manifold and E, F vector bundles over X. Let $P : C^\infty(E) \to C^\infty(F)$ be an elliptic, smooth, linear differential operator of order $l \geqslant 1$. Then the maps $P : W^p_{k+l}(E) \to W^p_k(F)$ and $P : C^{k+l,a}(E) \to C^{k,a}(F)$ both have finite-dimensional kernels and closed images.*

We give a proof of the $C^{k,a}$-half of this corollary to illustrate the various techniques that are used in the theory of elliptic operators, in particular the use of Theorem 3.3 and Theorem 3.4. The method of proof for is similar to that of Cantor [10], who considers the Sobolev case.

Proof: A Banach space is finite-dimensional precisely when its closed unit ball is compact. Now $\operatorname{Ker} P$ is a closed subspace of $C^{k+l,a}(E)$ and hence is a Banach space. Let $\mathcal{B}$ be the closed unit ball in $\operatorname{Ker} P$. Suppose that $(\xi_j) \subseteq \mathcal{B}$ is any sequence. Then (ξ_j) is bounded in the $C^{k+l,a}$-norm, and by Theorem 3.3 there exists a subsequence (ξ_{j_r}) that is C^{k+l}-Cauchy, and hence C^0-Cauchy. Now applying Theorem 3.4 gives a constant $C_2 > 0$ such that

$$\|\xi_{j_r} - \xi_{j_s}\|_{C^{k+l,a}(E)} \leqslant C_2 \cdot \|\xi_{j_r} - \xi_{j_s}\|_{C^0(E)}$$

for all $r, s \geqslant 1$ so that (ξ_{j_r}) is $C^{k+l,a}$-Cauchy. It follows that this sequence has a limit $\xi \in \mathcal{B}$. Hence $\mathcal{B}$ is compact, and $\operatorname{Ker} P$ is finite-dimensional.

To show that $\operatorname{Im} P \leqslant C^{k,a}(F)$ is closed we firstly define the closed subspace $\mathcal{A} := (\operatorname{Ker} P)^\perp \leqslant C^{k+l,a}(E)$. Here we use the L^2-inner product on $C^{k+l,a}(E)$ to form $\mathcal{A}$. Note that $C^{k+l,a}(E) = \operatorname{Ker} P \oplus \mathcal{A}$, as one can see by picking an L^2-orthonormal basis for $\operatorname{Ker} P$.

Suppose for a contradiction that there exists a sequence $(\xi_j) \subseteq \mathcal{A}$ with

$$\begin{aligned} \|\xi_j\|_{C^{k+l,a}(E)} &= 1 \quad \text{for all } j \geqslant 1 & (3.14)\\ \|P\xi_j\|_{C^{k,a}(F)} &\to 0 \quad \text{as } j \to \infty. & (3.15)\end{aligned}$$

By equation (3.14) and Theorem 3.3 there exists a subsequence (ξ_{j_r}) that is C^{k+l}-Cauchy, and therefore C^0-Cauchy. By equation (3.15) and Theorem 3.4 we deduce (ξ_{j_r}) is $C^{k+l,a}$-Cauchy and so converges to some $\xi \in \mathcal{A}$. Now by equation (3.15) we have $P\xi = 0$ so that necessarily $\xi = 0$. But this contradicts equation (3.14). It follows from this contradiction that there exists a constant $C_3 > 0$ such that

$$\|\xi\|_{C^{k+l,a}(E)} \leqslant C_3 \|P\xi\|_{C^{k,a}(E)} \tag{3.16}$$

for all $\xi \in \mathcal{A}$. We can finally show that $\operatorname{Im} P \leqslant C^{k,a}(F)$ is closed. Take a sequence $(\eta_j) \subseteq \operatorname{Im} P$ converging to $\eta \in C^{k,a}(F)$. Then put $\eta_j = P\xi_j$ for $j \geqslant 1$ with each $\xi_j \in \mathcal{A}$. By equation (3.16) the sequence $(\xi_j) \subseteq \mathcal{A}$ is $C^{k+l,a}$-Cauchy and thus converges to $\xi \in \mathcal{A}$, and moreover $P\xi = \eta$. This shows $\operatorname{Im} P$ is a closed subspace of $C^{k,a}(F)$.

∎

Theorem 3.6 allows us to prove the following result giving a characterisation of the image of an elliptic operator acting between Sobolev spaces. The method of proof for Theorem 3.7 is that of Cantor [10].

Theorem 3.7 *Let X be a compact manifold and E, F vector bundles over X. Let $P : C^\infty(E) \to C^\infty(F)$ be an elliptic, smooth, linear differential operator of order $l \geqslant 1$ with formal adjoint P^*. Then in the extension $P : W^p_{k+l}(E) \to W^p_k(F)$ we have*

$$\operatorname{Im} P = \Big\{\eta \in W^p_k(F) : \langle \eta | h \rangle_{L^2(F)} = 0 \textit{ for all } h \in \operatorname{Ker} P^*\Big\}. \tag{3.17}$$

Note from Corollary 3.5 that $\operatorname{Ker} P^*$ is a subspace of $C^\infty(F)$ and is therefore contained in $W^{p'}_0(F)$, so that the right hand side of equation (3.17) makes sense.

Proof: First note that

$$\operatorname{Im} P \leqslant \Big\{\eta \in W^p_k(F) : \langle \eta | h \rangle_{L^2(F)} = 0 \text{ for all } h \in \operatorname{Ker} P^*\Big\}$$

follows immediately from integration by parts.

Consider now the case $k = 0$. For the purposes of this proof, denote the Banach space adjoint of the map P by $P' : W^p_0(F)^* \to W^p_l(E)^*$, to distinguish from the formal adjoint P^* of P. Identify $W^p_0(F)^* \cong W^{p'}_0(F)$ as in (3.3). Then an integration by parts argument shows that

$$\operatorname{Ker} P^* \leqslant \operatorname{Ker} P'$$

in $W^p_0(F)^*$. Also, it is a consequence of Theorem 3.4 that

$$\operatorname{Ker} P' \leqslant \operatorname{Ker} P^* \tag{3.18}$$

because if $\eta \in W^{p'}_0(F)$ with $\langle P\phi|\eta\rangle_{L^2(F)} = 0$ for all $\phi \in W^p_l(E)$ then the equation $P^*\eta = 0$ holds weakly. (It is at the point of establishing the inclusion (3.18) that the corresponding proof for the Hölder spaces breaks down, because one does not have a good characterisation of their dual space.) Now take $\eta \in W^p_0(F)$ such that $\langle\eta|h\rangle_{L^2(F)} = 0$ for all $h \in \operatorname{Ker} P^*$. Then $\eta \in W^p_0(F)$ lies in

$$(\operatorname{Ker} P^*)^\circ = (\operatorname{Ker} P')^\circ = \operatorname{Im} P$$

as required. Here we are using Proposition 2.2 and Theorem 3.6. It follows that we have proved the result in the case $k = 0$.

Now suppose that $k \geqslant 1$ and that $\eta \in W^p_k(F)$ with $\langle\eta|h\rangle_{L^2(F)} = 0$ for all $h \in \operatorname{Ker} P^*$. A consequence of the case $k = 0$ proof is that there exists $\xi \in W^p_l(E)$ such that $P\xi = \eta$. But then Theorem 3.4 implies $\xi \in W^p_{k+l}(E)$ and we are done.

∎

Theorem 3.7 is important because it generalises very easily to the non-compact case we shall consider later. It also allows us to prove our next result, which is again very useful. The method is that of the author, we do not know if it is in the literature.

Theorem 3.8 *Let X be a compact manifold and E, F vector bundles over X. Let $P : C^\infty(E) \to C^\infty(F)$ be an elliptic, smooth, linear differential operator of order $l \geqslant 1$, with formal adjoint P^*. Then there are L^2-orthogonal decompositions*

$$W^p_k(F) = P\big(W^p_{k+l}(E)\big) \oplus \operatorname{Ker} P^* \tag{3.19}$$
$$C^{k,a}(F) = P\big(C^{k+l,a}(E)\big) \oplus \operatorname{Ker} P^*. \tag{3.20}$$

Proof: We first prove the Sobolev decomposition (3.19) which can then be used to prove the Hölder decomposition (3.20).

From Corollary 3.5 and Theorem 3.6 above we have that $\operatorname{Ker} P^*$ is a finite dimensional subspace of $W^p_k(F)$ contained inside $C^\infty(F)$. Choose an L^2-orthonormal basis $\{h_1, \ldots, h_N\}$ of $\operatorname{Ker} P^*$. Given $\eta \in W^p_k(F)$ we may write

$$\eta = \left(\eta - \sum_{j=1}^N \langle\eta|h_j\rangle_{L^2(F)} h_j\right) + \left(\sum_{j=1}^N \langle\eta|h_j\rangle_{L^2(F)} h_j\right)$$

and this shows that

$$W^p_k(F) = \Big\{\eta \in W^p_k(F) : \langle\eta|h\rangle_{L^2(F)} = 0 \text{ for all } h \in \operatorname{Ker} P^*\Big\} \oplus \operatorname{Ker} P^*$$

and the Sobolev decomposition now follows from Theorem 3.7.

Note now that $C^{k,a}(F) \leqslant W^p_k(F)$. If we intersect the decomposition (3.19) with $C^{k,a}(F)$ we obtain

$$C^{k,a}(F) = \Big\{\eta \in C^{k,a}(F) : \eta = P\xi \text{ for some } \xi \in W^p_{k+l}(E)\Big\} \oplus \operatorname{Ker} P^*.$$

Now choose $p > 1$ so that $k + l - a \geqslant \frac{n}{p}$ so that by Theorem 3.2 we have $W^p_{k+l}(E) \leqslant C^{0,a}(E)$. Then Theorem 3.4 implies that

$$\Big\{\eta \in C^{k,a}(F) : \eta = P\xi \text{ for some } \xi \in W^p_{k+l}(E)\Big\} = P\big(C^{k+l,a}(E)\big)$$

and we are done.

∎

Given the hypotheses of Theorem 3.8 one may deduce immediately that the linear map $P : C^\infty(E) \to C^\infty(F)$ admits an L^2-orthogonal decomposition

$$C^\infty(F) = P\big(C^\infty(E)\big) \oplus \operatorname{Ker} P^* \tag{3.21}$$

with a similar orthogonality property. This follows by intersecting both sides of equation (3.20) with $C^\infty(F)$ and using the elliptic regularity results of Theorem 3.4.

3.1.3 An application: Hodge theory

On a compact Riemannian manifold (X, g) we have the exterior derivative d, its formal adjoint d^*_g, and the elliptic, self-adjoint, smooth, linear differential operator of order 1

$$\mathrm{d}^*_g + \mathrm{d} : C^\infty(\Lambda^* T^* X) \to C^\infty(\Lambda^* T^* X).$$

Furthermore we have an elliptic, self-adjoint, smooth, linear differential operator $(\mathrm{d}^*_g + \mathrm{d})^2 = \Delta_g$ on $C^\infty(\Lambda^* T^* X)$. The analysis of the operators $\mathrm{d} + \mathrm{d}^*_g$ and Δ_g is what we shall call *Hodge theory* on (X, g). Although in Section 3.1.3 we shall work with Hölder spaces, the corresponding results for Sobolev spaces also hold.

For the rest of Section 3.1.3 we consider $\mathrm{d}^*_g + \mathrm{d}$ as a map

$$\mathrm{d}^*_g + \mathrm{d} : C^{k+1,a}(\Lambda^* T^* X) \to C^{k,a}(\Lambda^* T^* X) \tag{3.22}$$

which is a bounded linear map of Banach spaces. By Corollary 3.6 the subspace $\operatorname{Im}(\mathrm{d}^*_g + \mathrm{d}) \leqslant C^{k,a}(\Lambda^* T^* X)$ is closed and $\operatorname{Ker}(\mathrm{d}^*_g + \mathrm{d})$ is a finite-dimensional subspace of $C^\infty(\Lambda^* T^* X)$. Also, by Theorem 3.8 and the fact that $\mathrm{d}^*_g + \mathrm{d}$ is self-adjoint we have a direct sum decomposition

$$C^{k,a}(\Lambda^* T^* X) = \operatorname{Im}(\mathrm{d}^*_g + \mathrm{d}) \oplus \operatorname{Ker}(\mathrm{d}^*_g + \mathrm{d}) \tag{3.23}$$

which is L^2-orthogonal.

Proposition 3.9 *The bounded linear map of Banach spaces* (3.22) *has kernel*

$$\operatorname{Ker}(\mathrm{d}^*_g + \mathrm{d}) = \big\{\xi \in C^{k+1,a}(\Lambda^* T^* X) : \mathrm{d}^*_g \xi = 0 \text{ and } \mathrm{d}\xi = 0\big\}$$

and image

$$\begin{aligned} \operatorname{Im}(\mathrm{d}^*_g + \mathrm{d}) &= \big\{\mathrm{d}^*_g \theta_1 + \mathrm{d}\theta_2 : \theta_1, \theta_2 \in C^{k+1,a}(\Lambda^* T^* X)\big\} \\ &= \mathrm{d}^*_g\big(C^{k+1,a}(\Lambda^* T^* X)\big) \oplus \mathrm{d}\big(C^{k+1,a}(\Lambda^* T^* X)\big) \end{aligned}$$

a direct sum of vector spaces that is L^2-orthogonal.

Proof: It is clear that if $\xi \in C^{k+1,a}(\Lambda^* T^* X)$ with $\mathrm{d}^*_g \xi = 0$ and $\mathrm{d}\xi = 0$ then $\xi \in \operatorname{Ker}(\mathrm{d}^*_g + \mathrm{d})$. Suppose conversely that $\xi \in \operatorname{Ker}(\mathrm{d}^*_g + \mathrm{d})$. Then $\mathrm{d}^*_g \xi = -\mathrm{d}\xi$ and

$$\|\mathrm{d}^*_g \xi\|^2_{L^2} = \langle \mathrm{d}^*_g \xi | \mathrm{d}^*_g \xi \rangle_{L^2} = \langle \xi | \mathrm{d}\mathrm{d}^*_g \xi \rangle_{L^2} = -\langle \xi | \mathrm{d}\mathrm{d}\xi \rangle_{L^2} = 0$$

where by L^2 we mean $L^2(\Lambda^* T^* X)$ throughout. It follows that $\mathrm{d}^*_g \xi = \mathrm{d}\xi = 0$.

For the second set of equations we note that clearly $\mathrm{Im}(\mathrm{d}_g^* + \mathrm{d})$ is contained inside

$$\mathcal{A} := \left\{ \mathrm{d}_g^*\theta_1 + \mathrm{d}\theta_2 : \theta_1, \theta_2 \in C^{k+1,a}(\Lambda^* T^* X) \right\}$$

but the problem is to show the reverse inclusion $\mathcal{A} \leqslant \mathrm{Im}(\mathrm{d}_g^* + \mathrm{d})$. For this take any $\theta_1, \theta_2 \in C^{k+1,a}(\Lambda^* T^* X)$. Then we have $\mathrm{d}_g^*\theta_1 + \mathrm{d}\theta_2 \in C^{k,a}(\Lambda^* T^* X)$ so from the decomposition (3.23) there exist $h \in \mathrm{Ker}(\mathrm{d}_g^* + \mathrm{d})$ and $\theta_3 \in C^{k+1,a}(\Lambda^* T^* X)$ with

$$\mathrm{d}_g^*\theta_1 + \mathrm{d}\theta_2 = (\mathrm{d}_g^* + \mathrm{d})\theta_3 + h \tag{3.24}$$

From the description of $\mathrm{Ker}(\mathrm{d}_g^* + \mathrm{d})$ obtained above we have

$$\begin{aligned} \mathrm{d}_g^*\mathrm{d}(\theta_2 - \theta_3) &= 0 \\ \mathrm{d}\mathrm{d}_g^*(\theta_1 - \theta_3) &= 0 \end{aligned}$$

by respectively applying d_g^* and d to equation (3.24). Then

$$\left\| \mathrm{d}(\theta_2 - \theta_3) \right\|_{L^2}^2 = \left\langle \mathrm{d}(\theta_2 - \theta_3) \middle| \mathrm{d}(\theta_2 - \theta_3) \right\rangle_{L^2} = \left\langle \theta_2 - \theta_3 \middle| \mathrm{d}_g^*\mathrm{d}(\theta_2 - \theta_3) \right\rangle_{L^2} = 0$$

so that $\mathrm{d}\theta_2 = \mathrm{d}\theta_3$, and similarly $\mathrm{d}_g^*\theta_1 = \mathrm{d}_g^*\theta_3$. Hence $h = 0$ and $(\mathrm{d}_g^* + \mathrm{d})\theta_3 = \mathrm{d}_g^*\theta_1 + \mathrm{d}\theta_2$ so we have proved $\mathrm{Im}(\mathrm{d}_g^* + \mathrm{d}) = \mathcal{A}$. The last remaining equality

$$\mathcal{A} = \mathrm{d}_g^*\big(C^{k+1,a}(\Lambda^* T^* X)\big) \oplus \mathrm{d}\big(C^{k+1,a}(\Lambda^* T^* X)\big) \tag{3.25}$$

is obvious: the sum of vector spaces is direct because given any $\theta_1, \theta_2 \in C^{k+1,a}(\Lambda^* T^* X)$ we have

$$\langle \mathrm{d}_g^*\theta_1 | \mathrm{d}\theta_2 \rangle_{L^2} = \langle \theta_1 | \mathrm{d}\mathrm{d}\theta_2 \rangle_{L^2} = 0$$

and this also shows the splitting (3.25) is L^2-orthogonal, in that

$$\mathrm{d}_g^*\big(C^{k+1,a}(\Lambda^* T^* X)\big) \leqslant \big(\mathrm{d}(C^{k+1,a}(\Lambda^* T^* X))\big)^{\perp}.$$

∎

For the rest of Section 3.1.3 we consider Δ as a map

$$\Delta : C^{k+2,a}(\Lambda^* T^* X) \to C^{k,a}(\Lambda^* T^* X) \tag{3.26}$$

which is a bounded linear map of Banach spaces. By Corollary 3.6 the subspace $\mathrm{Im}\,\Delta \leqslant C^{k,a}(\Lambda^* T^* X)$ is closed and $\mathrm{Ker}\,\Delta$ is a finite-dimensional subspace of $C^\infty(\Lambda^* T^* X) \leqslant C^{k+2,a}(\Lambda^* T^* X)$. Also, by Theorem 3.8 and the fact that Δ is self-adjoint we have a direct sum decomposition

$$C^{k,a}(\Lambda^* T^* X) = \mathrm{Im}\,\Delta \oplus \mathrm{Ker}\,\Delta \tag{3.27}$$

which is L^2-orthogonal.

Proposition 3.10 *In equation* (3.27) *we have*

$$\begin{aligned} \mathrm{Ker}\,\Delta &= \mathrm{Ker}(\mathrm{d}_g^* + \mathrm{d}) \\ \mathrm{Im}\,\Delta &= \mathrm{Im}(\mathrm{d}_g^* + \mathrm{d}). \end{aligned}$$

Proof: Clearly $\mathrm{Ker}(\mathrm{d}_g^* + \mathrm{d}) \leqslant \mathrm{Ker}\,\Delta$ as both are subspaces of $C^\infty(\Lambda^* T^* X)$ and $(\mathrm{d}_g^* + \mathrm{d})^2 = \Delta$. To show the reverse inclusion, suppose that $\xi \in \mathrm{Ker}\,\Delta$. Then

$$\left\| (\mathrm{d}_g^* + \mathrm{d})\xi \right\|_{L^2}^2 = \left\langle (\mathrm{d}_g^* + \mathrm{d})\xi \middle| (\mathrm{d}_g^* + \mathrm{d})\xi \right\rangle_{L^2} = \left\langle \xi \middle| (\mathrm{d}_g^* + \mathrm{d})^2\xi \right\rangle_{L^2} = \left\langle \xi \middle| \Delta\xi \right\rangle_{L^2} = 0$$

as $\mathrm{d}_g^* + \mathrm{d}$ is self-adjoint. We conclude that $\xi \in \mathrm{Ker}(\mathrm{d}_g^* + \mathrm{d})$.

For the second equation, it is obvious that

$$\operatorname{Im}\Delta \leqslant \big\{\mathrm{d}_g^*\theta_1 + \mathrm{d}\theta_2 : \theta_1,\theta_2 \in C^{k+1,a}(\Lambda^* T^* X)\big\} = \operatorname{Im}(\mathrm{d}_g^* + \mathrm{d}).$$

Now suppose that $\xi \in \operatorname{Im}(\mathrm{d}_g^* + \mathrm{d}) \leqslant C^{k,a}(\Lambda^* T^* X)$. Using the splitting (3.27) we write $\xi = \xi_1 + \xi_2$ where $\xi_1 \in \operatorname{Im}\Delta$ and $\xi_2 \in \operatorname{Ker}\Delta$. But then $\xi - \xi_1 = \xi_2$ lies inside

$$\operatorname{Im}(\mathrm{d}_g^* + \mathrm{d}) \cap \operatorname{Ker}(\mathrm{d}_g^* + \mathrm{d}) = \{0\}$$

so that $\xi = \xi_1$ lies inside $\operatorname{Im}\Delta$.

∎

We usually refer to elements of $\mathcal{H} := \operatorname{Ker}\Delta$ as *harmonic* forms. If $\dim X =: n$ then there is a grading

$$C^{k,a}(\Lambda^* T^* X) = \bigoplus_{r=0}^{n} C^{k,a}(\Lambda^r T^* X) \tag{3.28}$$

and we define $\mathcal{H}^r := \mathcal{H} \cap C^\infty(\Lambda^r T^* X)$ for each $0 \leqslant r \leqslant n$.

Proposition 3.11 *There is a decomposition*

$$C^{k,a}(\Lambda^r T^* X) = \mathcal{H}^r \oplus \mathrm{d}_g^*\big(C^{k+1,a}(\Lambda^{r+1} T^* X)\big) \oplus \mathrm{d}\big(C^{k+1,a}(\Lambda^{r-1} T^* X)\big) \tag{3.29}$$

which is L^2-orthogonal.

Proof: From equation (3.23) and Proposition 3.9 we have an L^2-orthogonal decomposition

$$C^{k,a}(\Lambda^* T^* X) = \mathcal{H} \oplus \mathrm{d}_g^*\big(C^{k+1,a}(\Lambda^* T^* X)\big) \oplus \mathrm{d}\big(C^{k+1,a}(\Lambda^* T^* X)\big)$$

and now intersecting both sides of this equation with $C^{k,a}(\Lambda^r T^* X)$ gives the result.

∎

Corollary 3.12 *If $\eta \in C^1(\Lambda^{r-1} T^* X)$ with $\mathrm{d}\eta \in C^{k,a}(\Lambda^r T^* X)$ then there exists $\xi \in C^{k+1,a}(\Lambda^{r-1} T^* X)$ with $\mathrm{d}\eta = \mathrm{d}\xi$.*

Proof: In the direct sum decomposition (3.29) we may write $\mathrm{d}\eta = h + \mathrm{d}_g^*\theta + \mathrm{d}\xi$ where $\theta \in C^{k+1,a}(\Lambda^{r+1} T^* X)$ and $\xi \in C^{k+1,a}(\Lambda^{r-1} T^* X)$. Now an integration by parts argument as in the proof of Proposition 3.9 shows that $h = 0$ and $\mathrm{d}_g^*\theta = 0$ as required.

∎

There are versions of the grading (3.28) and decomposition (3.29) in the smooth situation:

$$C^\infty(\Lambda^* T^* X) = \bigoplus_{r=0}^{n} C^\infty(\Lambda^r T^* X) \tag{3.30}$$

$$C^\infty(\Lambda^r T^* X) = \mathcal{H}^r \oplus \mathrm{d}_g^*\big(C^\infty(\Lambda^{r+1} T^* X)\big) \oplus \mathrm{d}\big(C^\infty(\Lambda^{r-1} T^* X)\big). \tag{3.31}$$

It follows that for each $0 \leqslant r \leqslant n$ we have a canonical isomorphism of real vector spaces

$$\begin{array}{ccc} \mathcal{H}^r & \to & H^r(X) \\ h & \mapsto & [h] \end{array}$$

and this is because the smooth, closed r-forms split as $\mathcal{H}^r \oplus \mathrm{d}\big(C^\infty(\Lambda^{r-1} T^* X)\big)$, from equation (3.31).

3.2 The McLean Theorem

As a part of his doctoral thesis, published as [50], McLean showed that whenever X is a compact special Lagrangian submanifold of a Calabi-Yau manifold we may deform X via normal vector fields to nearby submanifolds, and the special Lagrangian deformations correspond to a finite-dimensional smooth submanifold of the total (infinite-dimensional) space of normal deformations. This smooth submanifold is a manifold modelled locally as an open subset of the affine space $H^1(X)$, as is explained in the paper [23] of Hitchin.

Using the background material we have already discussed, we give a rigorous proof of McLean's theorem, and also extend the theorem to the situation where the background Calabi-Yau structure is being deformed, and ask whether or not compact special Lagrangian submanifolds persist under deformations of the ambient Calabi-Yau structure. The main result of this section is Theorem 3.21.

3.2.1 Deformations of Calabi-Yau structures

We begin by explaining what we mean by deforming a Calabi-Yau structure.

Let Z be a manifold and let $E \to Z$ be a vector bundle. We would like to formalise the notion of a smoothly varying family of smooth sections of E. For this, let $D \subseteq \mathbb{R}^m$ be a domain containing 0 and $\pi_Z : D \times Z \to Z$ be the projection onto the second factor.

Given $\hat{e} \in C^\infty(\pi_Z^* E)$ and $p \in D$ define the section $e(p) \in C^\infty(E)$ by

$$e(p)_z := \hat{e}_{(p,z)}$$

for all $z \in Z$. Here we are identifying $E_z \cong (\pi_Z^* E)_{(p,z)}$ in the usual way, for each $z \in Z$. We shall say that $\hat{e}$ is a *(smooth, m-dimensional) deformation* of $e(0)$. We shall call D the *parameter space* of the deformation $\hat{e}$.

Given the above situation, we have for each $i = 1, \ldots, m$ a section $\widehat{\partial_i e} \in C^\infty(\pi_Z^* E)$ defined

$$(\widehat{\partial_i e})_{(p,z)} := \frac{\partial}{\partial r_i}\Big(\hat{e}_{(r,z)}\Big)\bigg|_{r=p}$$

for each $(p, z) \in D \times Z$. In other words, $\widehat{\partial_i e}$ is just the derivative of $\hat{e}$ in the ith direction in D, where we compute derivatives in each fibre of E separately. We then have sections $(\partial_i e)(p) \in C^\infty(E)$ for each $1 \leqslant i \leqslant m$ and $p \in D$, as described above.

Suppose now that M is a manifold with a Calabi-Yau structure (J, g, Ω). Then by a *deformation of Calabi-Yau structures* of (J, g, Ω) we mean a deformation $(\hat{J}, \hat{g}, \hat{\Omega})$ of (J, g, Ω), such that $\big(J(p), g(p), \Omega(p)\big)$ is a Calabi-Yau structure on M for each $p \in D$, the common parameter space of $(\hat{J}, \hat{g}, \hat{\Omega})$.

3.2.2 The deformation map F

For the rest of Section 3.2 let M be a manifold with $\dim M = 2n$ and (J, g, Ω) a Calabi-Yau structure on M with Kähler form ω. Let $f : X \to M$ be a compact submanifold which is special Lagrangian with respect to (J, g, Ω). We let $\flat_g : TX \to T^*X$ be the usual bundle isomorphism induced by the restriction of g to X and $*_g$ the Hodge star on X induced by the restriction of g to X and the orientation on X determined by $f^* \operatorname{Re}\Omega$. Also d_g^* denotes the formal adjoint of the exterior derivative d on X got using the restriction of the metric g to X. Note that $f(X) \subseteq M$ will be a closed subset as X is compact.

Let $N \to X$ be the normal bundle of $f : X \to M$ and $\tilde{U}, U$ be tubular neighbourhoods of X, as in Section 2.3.4, with corresponding smooth sections $\tilde{U}^\infty, U^\infty$ as in equations (2.37) and (2.38). For the rest of Section 3.2 fix some $k \geqslant 2$ and further define

$$\begin{aligned}
\tilde{U}^{k+1,a} \quad &:= \quad \Big\{\xi \in C^{k+1,a}(N) : \xi_x \in \tilde{U} \text{ for all } x \in X\Big\} \\
U^{k+1,a} \quad &:= \quad \Big\{\eta \in C^{k+1,a}(T^*X) : \eta_x \in U \text{ for all } x \in X\Big\}.
\end{aligned}$$

Then both $\tilde{U}^{k+1,a} \subseteq C^{k+1,a}(N)$ and $U^{k+1,a} \subseteq C^{k+1,a}(T^*X)$ are open subsets of Banach spaces, containing 0. Moreover both $\tilde{U}^{k+1,a}$ and $U^{k+1,a}$ are star-shaped with respect to 0.

Given $\xi \in \tilde{U}^{k+1,a}$ we have a map $f_\xi : X \to M$ defined as in Section 2.2.1 by

$$f_\xi(x) := \exp_{f(x)}(\xi_x)$$

for all $x \in X$. Here f_ξ will not necessarily be smooth, but instead has components of class $C^{k+1,a}$ in local trivialisations for X and M. However the map $f_\xi : X \to M$ still defines a submanifold of M in that f_ξ is an injective immersion that is a homeomorphism onto its image. Also, we may pull back forms $\theta \in C^\infty(\Lambda^r T^*M)$ to obtain forms $f_\xi^*\theta \in C^{k,a}(\Lambda^r T^*X)$.

Let us suppose that $(\hat{J}, \hat{g}, \hat{\Omega})$ is a deformation of Calabi-Yau structures of (J, g, Ω) which has a common parameter space $D \subseteq \mathbb{R}^m$. So $J = J(0)$, $g = g(0)$ and $\Omega = \Omega(0)$. Also, given $p \in D$ we denote the Kähler form of the Calabi-Yau structure $\big(J(p), g(p), \Omega(p)\big)$ by $\omega(p)$.

We now consider the problem of which $p \in D$ and $\xi \in \tilde{U}^{k+1,a}$ give rise to submanifolds $f_\xi : X \to M$ which are special Lagrangian with respect to the Calabi-Yau structure $\big(J(p), g(p), \Omega(p)\big)$. That is, which $(p, \xi) \in D \times \tilde{U}^{k+1,a}$ satisfy

$$f_\xi^*\omega(p) = f_\xi^* \operatorname{Im}\Omega(p) = 0$$

in $C^{k,a}(\Lambda^* T^*X)$. To this end we define a map

$$\begin{aligned} \tilde{F} : D \times \tilde{U}^{k+1,a} &\to C^{k,a}(\Lambda^* T^*X) \\ (p, \xi) &\mapsto *_g f_\xi^* \operatorname{Im}\Omega(p) + f_\xi^*\omega(p) \end{aligned}$$

which is a map between open subsets of Banach spaces. Then clearly we have $\tilde{F}(0,0) = 0$ since $f : X \to M$ is special Lagrangian with respect to (J, g, Ω) and more generally the $(p, \xi) \in \tilde{F}^{-1}(0)$ correspond precisely to the submanifolds $f_\xi : X \to M$ which are special Lagrangian with respect to $\big(J(p), g(p), \Omega(p)\big)$.

A priori the map $f_\xi : X \to M$ will be only have regularity $C^{k+1,a}$ but the material from Section 2.3.2 shows us that the $f_\xi : X \to M$ which are special Lagrangian with respect to some $\big(J(p), g(p), \Omega(p)\big)$ must in fact be smooth. It follows that we are interested in looking at the structure of the subset

$$\tilde{F}^{-1}(0) \subseteq D \times \tilde{U}^\infty \subseteq D \times \tilde{U}^{k+1,a}$$

which is precisely the set of $(p, \xi) \in D \times \tilde{U}^{k+1,a}$ such that $f_\xi : X \to M$ is a smooth submanifold which is special Lagrangian with respect to $\big(J(p), g(p), \Omega(p)\big)$. The right tool to study $\tilde{F}^{-1}(0)$ is the Implicit Function Theorem 2.10, and in order to invoke this theorem we shall need to establish some facts about the map $\tilde{F}$. The first such fact is given in the following theorem.

Theorem 3.13 *The map $\tilde{F} : D \times \tilde{U}^{k+1,a} \to C^{k,a}(\Lambda^* T^*X)$ is a smooth mapping between open subsets of Banach spaces.*

Theorem 3.13 is proved in the thesis [5, Theorem 2.2.15] of Baier. Essentially, for each $l \geqslant 1$ and $(p, \xi) \in D \times \tilde{U}^{k+1,a}$ one obtains a candidate: the so-called Gâteaux derivative [1, Corollary 2.4.10], for the lth order derivative

$$(D^l \tilde{F})_{(p,\xi)} : \underbrace{\big(\mathbb{R}^m \oplus C^{k+1,a}(N)\big) \times \cdots \times \big(\mathbb{R}^m \oplus C^{k+1,a}(N)\big)}_{l \text{ factors}} \to C^{k,a}(\Lambda^* T^*X) \tag{3.32}$$

where we use the identification (2.1). Straightforward estimates on the components of $\tilde{F}$ show that the multilinear map (3.32) is bounded for all $(p, \xi) \in D \times \tilde{U}^{k+1,a}$, and that the map $(p, \xi) \mapsto (D^l \tilde{F})_{(p,\xi)}$ is continuous. So $\tilde{F}$ is of class C^l for each $l \geqslant 1$ and therefore smooth.

For convenience we shall work mainly with the cotangent bundle T^*X rather that the normal bundle N: we can interchange the two pictures using the isomorphism $\flat_g J$. Define the mapping $F : D \times U^{k+1,a} \to C^{k,a}(\Lambda^* T^*X)$ by

$$F\big(p, (\flat_g J)\xi\big) := \tilde{F}(p, \xi) = *_g f_\xi^* \operatorname{Im}\Omega(p) + f_\xi^*\omega(p)$$

for all $\big(p, (\flat_g J)\xi\big) \in D \times U^{k+1,a}$. In other words, $\tilde{F} = F \circ \big(\mathrm{id} \times (\flat_g J)\big)$. Now $\flat_g J : N \to T^*X$ is a vector bundle isomorphism and the induced map $\flat_g J : C^{k+1,a}(N) \to C^{k+1,a}(T^*X)$ is a topological linear isomorphism. The Chain Rule 2.5 and the Product Rule 2.6 then imply that the map F is smooth and moreover

$$\tilde{F}'(0,0) = F'(0,0) \circ \big(\mathrm{id} \times (\flat J)\big)$$

as a bounded linear map $\mathbb{R}^m \oplus C^{k+1,a}(N) \to C^{k,a}(\Lambda^* T^* X)$.

Now, the derivative of F at $(0,0)$ is a bounded linear map

$$F'(0,0) : \mathbb{R}^m \oplus C^{k+1,a}(T^*X) \to C^{k,a}(\Lambda^* T^* X)$$

and the decomposition (3.29) of Section 3.1.3 allows us to write

$$C^{k+1,a}(T^*X) = \mathcal{H}^1 \oplus \mathrm{d}_g^*\big(C^{k+2,a}(\Lambda^2 T^*X)\big) \oplus \mathrm{d}\big(C^{k+2,a}(X)\big).$$

We define

$$\begin{aligned} \mathcal{X}_1 &:= \mathbb{R}^m \oplus \mathcal{H}^1 \\ \mathcal{X}_2 &:= \mathrm{d}_g^*\big(C^{k+2,a}(\Lambda^2 T^*X)\big) \oplus \mathrm{d}\big(C^{k+2,a}(X)\big) \end{aligned}$$

and pick open subsets $\mathcal{V}_1 \subseteq \mathcal{H}^1$, $\mathcal{V}_2 \subseteq \mathcal{X}_2$ containing 0 such that $\mathcal{V}_1 \times \mathcal{V}_2 \subseteq U^{k+1,a}$. For the purposes of applying Theorem 2.10, put $\mathcal{U}_1 := D \times \mathcal{V}_1$ and $\mathcal{U}_2 := \mathcal{V}_2$ so that we have a restriction

$$F : \mathcal{U}_1 \times \mathcal{U}_2 \to C^{k,a}(\Lambda^* T^* X). \tag{3.33}$$

For the rest of Section 3.2 we consider F as the smooth map (3.33).

Proposition 3.14 *The smooth map $F : \mathcal{U}_1 \times \mathcal{U}_2 \to C^{k,a}(\Lambda^* T^* X)$ has partial derivative*

$$F_2'(0,0) : \mathcal{X}_2 \to C^{k,a}(\Lambda^* T^* X) \tag{3.34}$$

at $(0,0)$ in the $\mathcal{X}_2$ direction which acts as $\mathrm{d}_g^* + \mathrm{d}$.

Proof: Given $x \in X$ let $\mathrm{ev}_x : C^{k,a}(\Lambda^* T^* X) \to \Lambda^* T_x^* X$ denote the bounded linear map which evaluates sections at $x \in X$. Also if $\xi \in C^{k+1,a}(N)$ define a bounded linear map

$$\begin{aligned} \mathrm{mult}_\xi : \mathbb{R} &\to C^{k+1,a}(N) \\ t &\mapsto t\xi. \end{aligned}$$

The partial derivative we have to calculate is the derivative at 0 of the map $T : \mathcal{U}_2 \to C^{k,a}(\Lambda^* T^* X)$ defined by

$$T\big((\flat_g J)\xi\big) = *_g f_\xi^* \operatorname{Im} \Omega + f_\xi^* \omega$$

for all $(\flat_g J)\xi \in \mathcal{U}_2$. Further define $\tilde{T} = T \circ (\flat_g J)$ and then if $\eta = (\flat_g J)\xi \in \mathcal{X}_2$ we have

$$\begin{aligned} (T'(0)\eta)_x &= \big(\tilde{T}'(0)\xi\big)_x \\ &= \big(\mathrm{ev}_x \circ \tilde{T}'(0)\big)\xi \\ &= \big(\mathrm{ev}_x \circ \tilde{T}\big)'(0)\xi \\ &= \big(\mathrm{ev}_x \circ \tilde{T}\big)'(0)\mathrm{mult}_\xi(1) \\ &= \big(\mathrm{ev}_x \circ \tilde{T} \circ \mathrm{mult}_\xi\big)'(0)(1) \end{aligned}$$

where we use the Chain Rule 2.5. It follows that

$$\begin{aligned} \big(T'(0)\eta\big)_x &= \frac{\partial}{\partial t}\Big(\big(*_g f_{t\xi}^* \operatorname{Im}\Omega\big)_x + \big(f_{t\xi}^*\omega\big)_x\Big)\Big|_{t=0} && (3.35) \\ &= *_g \frac{\partial}{\partial t}\Big((f_{t\xi}^* \operatorname{Im}\Omega)_x\Big)\Big|_{t=0} + \frac{\partial}{\partial t}\Big((f_{t\xi}^*\omega)_x\Big)\Big|_{t=0} && (3.36) \\ &= -(*_g \mathrm{d} *_g \eta)_x + (\mathrm{d}\eta)_x \\ &= (\mathrm{d}_g^*\eta + \mathrm{d}\eta)_x \end{aligned}$$

where the derivatives in equations (3.35) and (3.36) are the pointwise derivatives calculated in Lemma 2.29. The result now follows.

∎

Theorem 3.15 *The partial derivative $F_2'(0,0) : \mathcal{X}_2 \to C^{k,a}(\Lambda^* T^* X)$ is a topological linear isomorphism onto the closed subspace*

$$\mathcal{Y} := \mathrm{d}_g^*\big(C^{k+1,a}(T^*X)\big) \oplus \mathrm{d}\big(C^{k+1,a}(T^*X)\big) \tag{3.37}$$

of $C^{k,a}(\Lambda^* T^* X)$.

Proof: Recall from Proposition 3.11 that there is an L^2-orthogonal decomposition

$$C^{k+1,a}(T^*X) = \mathcal{H}^1 \oplus \mathcal{X}_2. \tag{3.38}$$

Now, any element $\eta \in \mathcal{X}_2$ which lies inside $\operatorname{Ker} F_2'(0,0)$ must, by Proposition 3.14, satisfy $\mathrm{d}_g^*\eta = \mathrm{d}\eta = 0$ and therefore lie in $\mathcal{H}^1$. It follows from the orthogonality of (3.38) that $\eta = 0$ and hence $\operatorname{Ker} F_2'(0,0) = \{0\}$.

From Proposition 3.11 with $r = 0$ and $r = 2$ we have that both

$$\begin{aligned} \mathrm{d}_g^*\big(C^{k+1,a}(T^*X)\big) &\leqslant C^{k,a}(X) \\ \mathrm{d}\big(C^{k+1,a}(T^*X)\big) &\leqslant C^{k,a}(\Lambda^2 T^*X) \end{aligned}$$

are closed subspaces. Therefore

$$\mathrm{d}_g^*\big(C^{k+1,a}(T^*X)\big) \oplus \mathrm{d}\big(C^{k+1,a}(T^*X)\big) \leqslant C^{k,a}(X) \oplus C^{k,a}(\Lambda^2 T^*X) \leqslant C^{k,a}(\Lambda^* T^*X)$$

is a sequence of closed inclusions, and $\mathcal{Y}$ is closed in $C^{k,a}(\Lambda^* T^* X)$.

It is obvious that $\operatorname{Im} F_2'(0,0) \leqslant \mathcal{Y}$. To see the reverse inclusion, take any $\theta_1, \theta_2 \in C^{k+1,a}(T^*X)$. Then by Proposition 3.9 we have

$$\mathrm{d}_g^*\theta_1 + \mathrm{d}\theta_2 \in \mathrm{d}_g^*\big(C^{k+1,a}(\Lambda^*T^*X)\big) \oplus \mathrm{d}\big(C^{k+1,a}(\Lambda^*T^*X)\big) = \operatorname{Im}(\mathrm{d}_g^* + \mathrm{d})$$

where we consider $\mathrm{d}_g^* + \mathrm{d}$ as a map on the whole exterior bundle $C^{k+1,a}(\Lambda^*T^*X)$ as in Section 3.1.3. It follows that there exists $\eta \in C^{k+1,a}(\Lambda^*T^*X)$ with $(\mathrm{d}_g^* + \mathrm{d})\eta = \mathrm{d}_g^*\theta_1 + \mathrm{d}\theta_2$. Now write $\eta = \eta_0 + \cdots + \eta_n$ where $\eta_r \in C^{k+1,a}(\Lambda^r T^*X)$ for each $0 \leqslant r \leqslant n$. Then it is easy to show that

$$(\mathrm{d}_g^* + \mathrm{d})\eta_1 = \mathrm{d}_g^*\theta_1 + \mathrm{d}\theta_2.$$

Using the decomposition $C^{k+1,a}(T^*X) = \mathcal{H}^1 \oplus \mathcal{X}_2$ to write $\eta_1 = h + \tilde{\eta}_1$, we deduce that $\mathrm{d}_g^*\theta_1 + \mathrm{d}\theta_2 \in \operatorname{Im} F_2'(0,0)$, and hence $\operatorname{Im} F_2'(0,0) = \mathcal{Y}$.

We have shown that $F_2'(0,0)$ is a continuous linear isomorphism from $\mathcal{X}_2$ onto $\mathcal{Y}$. But $\mathcal{Y}$ is closed in $C^{k,a}(\Lambda^*T^*X)$ and so must be a Banach space. We then deduce from the Open Mapping Theorem 2.9 that $F_2'(0,0)$ is a topological linear isomorphism from $\mathcal{X}_2$ onto $\mathcal{Y}$.

∎

3.2.3 The moduli space of compact special Lagrangians

We have nearly completed our task of fulfilling the requirements of the Implicit Function Theorem 2.10. However in the statement of Theorem 2.10 we require the partial derivative of our mapping F to be a topological linear isomorphism of Banach spaces. But

$$F : \mathcal{U}_1 \times \mathcal{U}_2 \to C^{k,a}(\Lambda^*T^*X) \tag{3.39}$$

has partial derivative $F_2'(0,0) : \mathcal{X}_1 \oplus \mathcal{X}_2 \to C^{k,a}(\Lambda^*T^*X)$ which is *not* surjective. In fact, by Theorem 3.15 we have

$$\operatorname{Im} F_2'(0,0) = \mathrm{d}_g^*\big(C^{k+1,a}(T^*X)\big) \oplus \mathrm{d}\big(C^{k+1,a}(T^*X)\big)$$

a proper subspace of $C^{k,a}(\Lambda^*T^*X)$. We must somehow get around this problem.

Lemma 3.16 *If $p \in D$ is such that there exists $\xi \in \tilde{U}^{k+1,a}$ with $f_\xi : X \to M$ special Lagrangian with respect to $\big(J(p), g(p), \Omega(p)\big)$ then $[f^*\omega(p)] = 0$ in $H^2(X)$ and $[f^* \operatorname{Im}\Omega(p)] = 0$ in $H^n(X)$.*

Proof: Firstly note that $\xi \in \tilde{U}^\infty$. This follows from the regularity of special Lagrangian submanifolds as described in Section 2.3.2. Then the smooth mappings $f, f_\xi : X \to M$ are homotopic, in the sense of Bott and Tu [8]. The homotopy $H : \mathbb{R} \times X \to M$ can be given as

$$H(t,x) = f_{\chi(t)\xi}(x)$$

where $\chi \in C^\infty(\mathbb{R})$ is non-negative with $\operatorname{supp}(\chi) \subseteq (0,\infty)$ and $\operatorname{supp}(1-\chi) \subseteq (-\infty, 1)$. From the material of [8] it follows that

$$[f^*\omega(p)] = [f_\xi^*\omega(p)] = 0$$

in $H^2(X)$. Similarly $[f^* \operatorname{Im}\Omega(p)] = 0$ in $H^n(X)$ and we are done.

As in the proof of Proposition 2.20, we can use the variations theory of Section 2.2.2 and see explicitly that

$$f_\xi^*\omega(p) - f^*\omega(p) = \mathrm{d}\left(\int_0^1 f_{s\xi}^*\big(\iota(\xi^s)\omega(p)\big)\mathrm{d}s\right).$$

∎

We are interested in the $(p,\xi) \in D \times \tilde{U}^{k+1,a}$ such that $f_\xi : X \to M$ is special Lagrangian with respect to $\big(J(p), g(p), \Omega(p)\big)$. Lemma 3.16 gives us necessary conditions on $p \in D$ for this to hold. In order to prove a kind of converse result to Lemma 3.16 we make the following assumption about our deformation of Calabi-Yau structures $(\hat{J}, \hat{g}, \hat{\Omega})$.

Assumption 3.17 *For each $p \in D$ we have*

$$[f^*\omega(p)] = 0 \tag{3.40}$$

in $H^2(X)$ and

$$[f^* \operatorname{Im}\Omega(p)] = 0 \tag{3.41}$$

in $H^n(X)$.

Note that condition (3.41) can always be arranged by rotating the holomorphic volume form $\Omega(p)$ by some S^1-valued function of p. An interesting point to consider is whether or not the conditions (3.40) and (3.41) cut out a subset $D' \subseteq D$ which is a submanifold in a neighbourhood of 0. For the rest of Section 3.2 we shall assume that Assumption 3.17 holds.

Proposition 3.18 *The mapping $\tilde{F} : D \times \tilde{U}^{k+1,a} \to C^{k,a}(\Lambda^* T^* X)$ has image*

$$\operatorname{Im}\tilde{F} \subseteq \mathrm{d}_g^*\big(C^{k+1,a}(T^*X)\big) \oplus \mathrm{d}\big(C^{k+1,a}(T^*X)\big).$$

Proof: Given $(p,\xi) \in D \times \tilde{U}^{k+1,a}$ we have that

$$\tilde{F}(p,\xi) = *_g f_\xi^* \operatorname{Im}\Omega(p) + f_\xi^*\omega(p).$$

Now, there exists an $\varepsilon > 0$ such that the map $f_{t\xi} : X \to M$ defines a submanifold of M for each $-\varepsilon < t < 1+\varepsilon$. These maps will have class $C^{k+1,a}$ coefficients. Define the map $H : (-\varepsilon, 1+\varepsilon) \times X \to M$ by

$$H(t,x) = f_{t\xi}(x)$$

for all $t \in (-\varepsilon, 1+\varepsilon)$ and $x \in X$. Also for $0 \leqslant t \leqslant 1$ define $X_t := \{t\} \times X \cong X$ and the vector field $\xi^t \in C^{k,a}(f_{t\xi}^* TM)$ by

$$\xi_x^t := \mathrm{d}H_{(t,x)}\left(\frac{\partial}{\partial t}\right)$$

for all $x \in X$. We then extend ξ^t to any neighbourhood of $f_{t\xi}(X)$ in M.

If θ is any closed form on M and $0 \leqslant s \leqslant 1$ then as in Section 2.2.2 and the proof of Proposition 2.20, we may compute

$$\begin{aligned}\frac{\partial}{\partial t}\Big(f_{t\xi}^{*}\theta\Big)\Big|_{t=s} &= \mathcal{L}_{\frac{\partial}{\partial t}}\big(H^{*}\theta\big)\Big|_{X_s}\\ &= f_{s\xi}^{*}\big(\iota(\xi^{s})\mathrm{d}\theta + \mathrm{d}(\iota(\xi^{s})\theta)\big)\\ &= f_{s\xi}^{*}\big(\mathrm{d}(\iota(\xi^{s})\theta)\big)\\ &= \mathrm{d}\big(f_{s\xi}^{*}(\iota(\xi^{s})\theta)\big)\end{aligned}$$

and then

$$\begin{aligned}f_{\xi}^{*}\theta - f^{*}\theta &= f_{\xi}^{*}\theta - f_{0}^{*}\theta\\ &= \int_0^1 \frac{\partial}{\partial t}\Big(f_{t\xi}^{*}\theta\Big)\Big|_{t=s}\mathrm{d}s\\ &= \int_0^1 \mathrm{d}\big(f_{s\xi}^{*}(\iota(\xi^{s})\theta)\big)\mathrm{d}s\\ &= \mathrm{d}\left(\int_0^1 f_{s\xi}^{*}\big(\iota(\xi^{s})\theta\big)\mathrm{d}s\right). \qquad (3.42)\end{aligned}$$

Now Assumption 3.17 implies that $f^{*}\omega(p) = \mathrm{d}\phi_1$ for some $\phi_1 \in C^{\infty}(T^{*}X)$, so that equation (3.42) applied with $\theta = \omega(p)$ yields $f_{\xi}^{*}\omega(p) = \mathrm{d}\phi_2$ where

$$\phi_2 := \phi_1 + \int_0^1 f_{s\xi}^{*}\big(\iota(\xi^{s})\omega(p)\big)\mathrm{d}s$$

is a 1-form with coefficients of class $C^{k,a}$. Unfortunately this is one less derivative than we need, but we can get around this problem using Lemma 3.12, which implies that there exists $\phi_3 \in C^{k+1,a}(T^{*}X)$ with $f_{\xi}^{*}\omega(p) = \mathrm{d}\phi_3$. Similarly, there exists $\phi_4 \in C^{k+1,a}(T^{*}X)$ with $*_g f_{\xi}^{*}\,\mathrm{Im}\,\Omega(p) = \mathrm{d}_g^{*}\phi_4$. Hence

$$\tilde{F}(p,\xi) \in \mathrm{d}_g^{*}\big(C^{k+1,a}(T^{*}X)\big) \oplus \mathrm{d}\big(C^{k+1,a}(T^{*}X)\big)$$

as required.

∎

The following result is a prelude to our main theorem.

Proposition 3.19 *There exist open subsets $W \subseteq D$, $W_1 \subseteq \mathcal{V}_1$, $\mathcal{W}_2 \subseteq \mathcal{V}_2$ each containing* 0, *and a unique map $\chi : W \times W_1 \to \mathcal{W}_2$ such that*

$$F^{-1}(0) \cap \big(W \times W_1 \times \mathcal{W}_2\big) = \Big\{\big(p, \xi_1, \chi(p,\xi_1)\big) : (p,\xi_1) \in W \times W_1\Big\} \qquad (3.43)$$

in $W \times W_1 \times \mathcal{W}_2$. Moreover the map χ is smooth.

Proof: By Proposition 3.18 we can consider F as a smooth map $\mathcal{U}_1 \times \mathcal{U}_2 \to \mathcal{Y}$, and then Theorem 3.15 tells us that this map has a partial derivative

$$F_2'(0,0) : \mathcal{X}_2 \to \mathcal{Y}$$

which is a topological linear isomorphism. Now invoking the Implicit Function Theorem 2.10 we see that there exist open subsets $\mathcal{W}_1 \subseteq \mathcal{U}_1$, $\mathcal{W}_2 \subseteq \mathcal{U}_2$ containing 0 and a unique map $\chi : \mathcal{W}_1 \to \mathcal{W}_2$ such that

$$F^{-1}(0) \cap (\mathcal{W}_1 \times \mathcal{W}_2) = \Big\{\big(w_1, \chi(w_1)\big) : w_1 \in \mathcal{W}_1\Big\}.$$

Moreover χ is smooth.

Recall now that $\mathcal{U}_1 = D \times \mathcal{V}_1 \subseteq \mathbb{R}^m \oplus \mathcal{H}^1$ so that we may choose open subsets $W \subseteq D$ and $W_1 \subseteq \mathcal{V}_1$ containing 0 such that $W \times W_1 \subseteq \mathcal{W}_1$ and then consider χ as a map defined on $W \times W_1$ to give us the required result: an inspection of the proof of Theorem 2.10 shows that equation (3.43) characterises χ uniquely as a function $\chi : W \times W_1 \to \mathcal{W}_2$.

∎

One can think of Proposition 3.19 as saying that in the cube $W \times W_1 \times \mathcal{W}_2$ the zero set $F^{-1}(0)$ of F is given as the graph of a uniquely determined function $\chi : W \times W_1 \to \mathcal{W}_2$ that is smooth. An important point is that W is an open subset of the finite-dimensional space $\mathbb{R}^m$ and W_1 is a subset of the finite-dimensional space $\mathcal{H}^1$. The open set $W \subseteq \mathbb{R}^m$ corresponds to deformations of the Calabi-Yau structure on M and the open set $W_1 \subseteq \mathcal{H}^1$ corresponds to deformations of the compact special Lagrangian submanifold $f : X \to M$.

It follows from Proposition 3.19 that there is a bijection

$$\begin{array}{rcl} W \times W_1 & \to & F^{-1}(0) \cap \big(W \times W_1 \times \mathcal{W}_2\big) \\ (p, \xi_1) & \mapsto & \big(p, \xi_1, \chi(p, \xi_1)\big) \end{array} \tag{3.44}$$

and we can put the structure of a smooth manifold onto

$$\mathcal{M} := F^{-1}(0) \cap \big(W \times W_1 \times \mathcal{W}_2\big) \tag{3.45}$$

by declaring that the map (3.44) be a chart in the maximal smooth atlas for $\mathcal{M}$.

Lemma 3.20 1. *The manifold $\mathcal{M}$ is diffeomorphic to the open subset $W \times W_1 \subseteq \mathbb{R}^m \oplus \mathcal{H}^1$ and consequently* $\dim \mathcal{M} = m + b^1(X)$.

2. *With the smooth structure on $\mathcal{M}$ defined above, the inclusion*

$$i : \mathcal{M} \to W \times W_1 \times \mathcal{W}_2 \tag{3.46}$$

is a smooth injective map that is an immersion, and a homeomorphism onto its image. In other words, the inclusion (3.46) *is a smooth submanifold of $W \times W_1 \times \mathcal{W}_2$.*

Of course, here we are extending the notions of immersion and submanifold to the Banach space situation, in the obvious manner.

Proof: The first part is obvious. For the second part, it is obvious that i is a smooth (as χ is smooth) injective immersion. Now if $G \subseteq W \times W_1$ is open, the subset $G \times \mathcal{W}_2$ is an open subset of $W \times W_1 \times \mathcal{W}_2$. Consequently $\mathcal{M} \cap \big(G \times \mathcal{W}_2\big)$ is open with respect to the subspace topology on $\mathcal{M}$ and i is a homeomorphism onto its image.

∎

Most of the above results are summarised in the following theorem, whose proof is straightforward given the above discussion.

Theorem 3.21 *Let (M, J, g, Ω) be a Calabi-Yau manifold and $(\hat{J}, \hat{g}, \hat{\Omega})$ a deformation of Calabi-Yau structures of (J, g, Ω), with common parameter space the open subset $D \subseteq \mathbb{R}^m$ containing* 0*. Suppose that $f : X \to M$ is a compact submanifold which is special Lagrangian with respect to (J, g, Ω), and that $(\hat{J}, \hat{g}, \hat{\Omega})$ satisfies Assumption* 3.17*. Let $N \to X$ be the normal bundle of $f : X \to M$ and identify $N \cong T^*X$ via the bundle isomorphism $\flat_g J$. If $k \geqslant 2$ then there exist open subsets*

$$\begin{array}{rcl} W & \subseteq & D \\ W_1 & \subseteq & \mathcal{H}^1 = \big\{\xi \in C^{k+1,a}(T^*X) : \Delta_g \xi = 0\big\} \\ \mathcal{W}_2 & \subseteq & \mathrm{d}_g^*\big(C^{k+2,a}(T^*X)\big) \oplus \mathrm{d}\big(C^{k+2,a}(T^*X)\big) \end{array}$$

all containing 0 *and a smooth map $\chi : W \times W_1 \to \mathcal{W}_2$ with $\chi(0) = 0$ such that the following holds:*

1. *Every*

$$\begin{array}{rcl} \xi = (\xi_1, \xi_2) & \in & W_1 \times \mathcal{W}_2 \\ & \subseteq & \mathcal{H}^1 \oplus \mathrm{d}_g^*\big(C^{k+2,a}(T^*X)\big) \oplus \mathrm{d}\big(C^{k+2,a}(T^*X)\big) \\ & = & C^{k+1,a}(T^*X) \\ & \cong & C^{k+1,a}(N) \end{array}$$

gives rise to a submanifold $f_\xi : X \to M$ *of class* $C^{k+1,a}$.

2. *For all* $\xi = (\xi_1, \xi_2) \in W_1 \times \mathcal{W}_2$ *and* $p \in W$ *we have*

$$\Big[f_\xi : X \to M \textit{ is special Lagrangian wrt } \big(J(p), g(p), \Omega(p)\big)\Big] \Longleftrightarrow \Big[\xi_2 = \chi(p, \xi_1)\Big]$$

and consequently $\chi(W \times W_1) \subseteq \mathrm{d}_g^*\big(C^\infty(T^*X)\big) \oplus \mathrm{d}\big(C^\infty(T^*X)\big)$.

3.

$$\begin{array}{rl} \mathcal{M} := \Big\{ & (p, \xi) = (p, \xi_1, \xi_2) \in W \times W_1 \times \mathcal{W}_2 : \\ & f_\xi : X \to M \textit{ is special Lagrangian wrt } \big(J(p), g(p), \Omega(p)\big)\Big\} \end{array}$$

is a smooth manifold with dimension $\dim \mathcal{M} = m + b^1(X)$. *Moreover,*

$$\begin{array}{rcl} W \times W_1 & \to & \mathcal{M} \\ (p, \xi_1) & \mapsto & \big(p, \xi_1, \chi(p, \xi_1)\big) \end{array}$$

is a diffeomorphism, and the inclusion $\mathcal{M} \to W \times W_1 \times \mathcal{W}_2$ *is a smooth submanifold.*

4. *Given* $p \in W$

$$\mathcal{M}_p := \Big\{\xi = (\xi_1, \xi_2) \in W_1 \times \mathcal{W}_2 : f_\xi : X \to M \textit{ is special Lagrangian wrt } \big(J(p), g(p), \Omega(p)\big)\Big\}$$

is a smooth manifold with dimension $\dim \mathcal{M}_p = b^1(X)$. *Moreover,*

$$\begin{array}{rcl} W_1 & \to & \mathcal{M}_p \\ \xi_1 & \mapsto & \big(\xi_1, \chi(p, \xi_1)\big) \end{array}$$

is a diffeomorphism, and the inclusion $\mathcal{M}_p \to \mathcal{M}$ *is a smooth submanifold.*

5. *Given* $\xi_1 \in W_1$

$$\mathcal{M}_{\xi_1} := \Big\{(p, \xi_2) \in W \times \mathcal{W}_2 : f_{\xi_1 + \xi_2} : X \to M \textit{ is special Lagrangian wrt } \big(J(p), g(p), \Omega(p)\big)\Big\}$$

is a smooth manifold with dimension $\dim \mathcal{M}_{\xi_1} = m$. *Moreover,*

$$\begin{array}{rcl} W & \to & \mathcal{M}_{\xi_1} \\ p & \mapsto & \big(\xi_1, \chi(p, \xi_1)\big) \end{array}$$

is a diffeomorphism, and the inclusion $\mathcal{M}_{\xi_1} \to \mathcal{M}$ *is a smooth submanifold.*

Chapter 4

Fredholm theory on non-compact manifolds

In this chapter we give a description of the analytic theory we shall be needing later. The relevant sources are the papers of Bartnik [6], Lockhart [45], and Lockhart and McOwen [46]. We begin by explaining some of the theory from these papers, and in later chapters go on to say how this theory can be applied in our situation.

The chapter is roughly split into three parts, which correspond to three different types of operators we shall consider on our non-compact manifold: translation invariant operators, asymptotically translation invariant operators, and asymptotically conical operators. Although the results in each section are closely related, the techniques required in the proofs are not, and so we give a separate exposition for each.

4.1 Manifolds with ends

We begin by describing the type of non-compact manifolds we shall consider. Throughout this chapter, suppose that X is a non-compact manifold of dimension $n \geqslant 3$ and that Σ is a compact manifold of dimension $n-1$ with L connected components

$$\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_L.$$

Also suppose that there exists a compact submanifold with boundary $X_0 \subseteq X$ and a diffeomorphism

$$X \setminus X_0 \to (T, \infty) \times \Sigma \tag{4.1}$$

for some $T \in \mathbb{R}$. We shall say that X is a *manifold with ends*. The sort of thing we have in mind, at least topologically, is indicated in Figure 4.1. We shall always consider $X_\infty := X \setminus X_0$ and $(T, \infty) \times \Sigma$ as being identified via the diffeomorphism (4.1). The canonical coordinate on (T, ∞) will be denoted by t and we denote a typical coordinate on Σ by $\sigma = (x_2, \ldots, x_n)$. Also $\pi : X_\infty \to \Sigma$ is the projection onto the link of the cylindrical part of X, got from the identification (4.1). If $S \geqslant 0$ we put

$$X_S := X_0 \cup \big((T, T+S] \times \Sigma\big)$$

which is a compact submanifold of X with boundary. If $\beta = (\beta_1, \ldots, \beta_L) \in \mathbb{R}^L$ then expressions such as βt refer to smooth functions $X \to \mathbb{R}$ which are equal to $\beta_j t$ on the jth end $(T, \infty) \times \Sigma_j$ of X.

We now describe the special types of coordinate charts on X we shall need in order to define Banach spaces of sections of bundles over X. Although there are coordinate independent methods for defining such spaces with a metric, with coordinate charts one obtains a very explicit description of the Banach spaces involved, with which one can prove easily results via the standard local results of PDE theory. Fix any covering $U_1, \ldots, U_N$ of Σ by coordinate charts and put $V_\nu := (T, \infty) \times U_\nu$ for

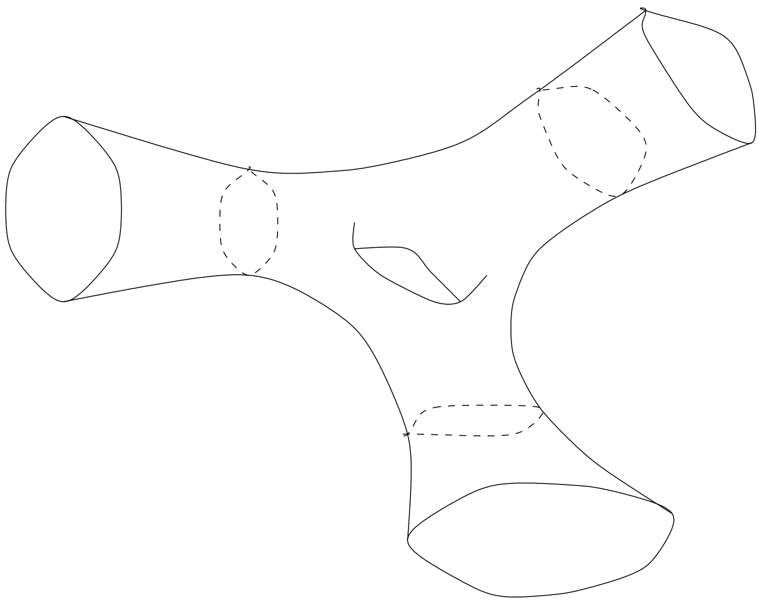

Figure 4.1: A portion of a manifold X with ends, divided into a compact piece X_0 and a non-compact piece $X \setminus X_0$

each $\nu = 1, \ldots, N$. Then $V_1, \ldots, V_N$ is an open cover of X_∞ consisting of coordinate charts. Fix any open covering $V_{N+1}, \ldots, V_{N+K}$ of X_0 by coordinate charts such that

$$\bigcup_{\nu=N+1}^{N+K} V_\nu \subseteq X_1.$$

The coordinates on V_ν will be denoted $x = (x_1, \ldots, x_n)$. In particular, if $1 \leqslant \nu \leqslant N$ we put $x_1 = t$ and $(x_2, \ldots, x_n) = \sigma$ the coordinates on U_ν. We also fix a partition of unity $\rho_1, \ldots, \rho_{N+K}$ subordinate to the open cover $V_1, \ldots, V_{N+K}$ of X, chosen so that ρ_ν is translation invariant on $(T+1, \infty) \times U_\nu$ for each $1 \leqslant \nu \leqslant N$. In other words,

$$\rho_\nu(s, \sigma) = \rho_\nu(t, \sigma)$$

for all $1 \leqslant \nu \leqslant N$, $s, t \geqslant T+1$ and $\sigma \in U_\nu$.

We now describe the vector bundles $E \to X$ we shall consider to build the Banach spaces mentioned above. The main requirement is the existence of suitable trivialisations for E over the infinite piece X_∞ of X, and this is certainly no significant restriction for our purposes. Let $E_\Sigma \to \Sigma$ be a vector bundle which is trivial over each U_ν. Then we have induced trivialisations for the vector bundle $\pi^* E_\Sigma \to X_\infty$ over $V_1, \ldots, V_N$. Suppose that $E \to X$ is a vector bundle, trivialised over each V_ν so that $E|_{X_\infty} = \pi^* E_\Sigma$ on $X \setminus X_S$ for some large $S \geqslant 0$. We shall call such a bundle E over X *admissible*, and the vector bundle $E_\Sigma \to \Sigma$ the *slice* of E over Σ. Actually, the terminology is a bit misleading, because being admissible is really a property of the vector bundle $E \to X$ *together with* an additional piece of data, namely the charts for $E|_{X_\infty}$ as described above. However, we shall abuse terminology and simply refer to the bundle $E \to X$ itself as being admissible.

Obviously we can direct sum, exterior product or tensor product admissible bundles to obtain new admissible bundles. If ξ is a section of an admissible bundle E we denote by $\xi_1^\nu, \ldots, \xi_{\operatorname{rank} E}^\nu$ the components of ξ in the given trivialisation of E over V_ν.

Suppose that E is an admissible vector bundle over X, with slice E_Σ. We shall say that a fibre metric $\widetilde{\langle \mid \rangle}_E$ on E is *translation invariant* if there is a fibre metric $\langle \mid \rangle_{E_\Sigma}$ on the vector bundle E_Σ such that

$$\pi^* \langle \mid \rangle_{E_\Sigma} = \widetilde{\langle \mid \rangle}_E$$

over $X \setminus X_S$ for some large $S \geqslant 0$. We shall say that a fibre metric $\langle\ |\ \rangle_E$ on E is *asymptotically translation invariant* if there is a translation invariant fibre metric $\widetilde{\langle\ |\ \rangle}_E$ on E such that

$$\sup_{\{t\}\times U_\nu} \left|\rho_\nu \partial^\lambda \big(e_{ij} - \tilde{e}_{ij}\big)\right| = o(1) \tag{4.2}$$

for each $1 \leqslant \nu \leqslant N$, $1 \leqslant i,j \leqslant \operatorname{rank} E$ and $|\lambda| \geqslant 0$. Here the e_{ij} and $\tilde{e}_{ij}$ are the components of $\langle\ |\ \rangle_E$ and $\widetilde{\langle\ |\ \rangle}_E$ respectively in the given trivialisations of E and X over each $V_\nu = (T,\infty)\times U_\nu$.

Examples of admissible bundles are:

1. The tensor bundles
$$E := \big(\otimes^r T^*X\big) \otimes \big(\otimes^s TX\big) \tag{4.3}$$
which have slices $\bigoplus_{i=r,r-1\ j=s,s-1}\big(\otimes^i T^*\Sigma\big)\otimes\big(\otimes^j T\Sigma\big)$.
2. The exterior bundles
$$E := \Lambda^r T^*X \tag{4.4}$$
which have slices $\Lambda^r T^*\Sigma \oplus \Lambda^{r-1}T^*\Sigma$.
3. The total exterior bundle
$$E := \Lambda^* T^*X \tag{4.5}$$
which has slice $\Lambda^*T^*\Sigma \oplus \Lambda^*T^*\Sigma$.

To see why the slices are as given, take for example the case $E = \Lambda^*T^*X$. Then given $x \in X_\infty$ and any $\xi \in \Lambda^*T^*_xX$ there are unique $\psi,\phi \in \Lambda^*T^*_\sigma\Sigma$ such that $\xi = \psi + \mathrm{d}t\wedge\phi$ where $x = (t,\sigma)$ in $X_\infty = (T,\infty)\times\Sigma$.

Whenever E is one of the three bundles (4.3), (4.4), (4.5) given above, we define a linear operator $e^{(s-r)t}$ acting on sections ξ of E as follows. The basic idea is that $e^{(s-r)t}$ scales ξ depending on what type of tensor or form ξ is. If ξ has r covariant (T^*X) parts and s contravariant (TX) parts we declare $e^{(s-r)t}\xi := f_{r,s}\xi$ where $f_{r,s} : X \to (0,\infty)$ is a smooth function which over X_∞ is equal to the function $e^{(s-r)t}$. We now extend the *operator* $e^{(s-r)t}$ by linearity to act on any section ξ of E. Obviously $e^{(s-r)t}$ is an invertible operator, and will be a topological linear isomorphism between certain Banach spaces we shall consider in the sequel.

4.2 Fredholm theory on manifolds with cylindrical ends

In order to consider the Fredholmness of various differential operators on X, we need to introduce suitable classes of Banach spaces between which such operators can act. It turns out that *weighted* versions of the usual Sobolev and Hölder spaces on X are the most appropriate. After defining these spaces we go on to describe the corresponding differential operators and Fredholm results.

4.2.1 Construction of suitable Banach spaces

Let g_Σ be a Riemannian metric on Σ. A metric $\tilde{h}$ on X which is of the form

$$\tilde{h} = \mathrm{d}t^2 + g_\Sigma$$

over $X\setminus X_S$ for some large $S \geqslant 0$ is called a *cylindrical* metric on X. We say that a metric h on X is *asymptotically cylindrical* if there exists a cylindrical metric $\tilde{h}$ such that

$$\sup_{\{t\}\times U_\nu} \left|\rho_\nu \partial^\lambda \big(h_{ij} - \tilde{h}_{ij}\big)\right| = o(1) \tag{4.6}$$

for each $1 \leqslant \nu \leqslant N$, $1 \leqslant i,j \leqslant n$ and $|\lambda| \geqslant 0$. In coordinate free terms, equation (4.6) is the same as requiring

$$\sup_{\{t\}\times\Sigma} \left|\nabla^j_{\tilde{h}}(h-\tilde{h})\right|_{\tilde{h}} = o(1)$$

for each $j \geqslant 0$. Any asymptotically cylindrical metric h on X will be complete. Note also that an (asymptotically) cylindrical metric induces an (asymptotically) translation invariant fibre metric on each of the admissible bundles (4.3), (4.4), (4.5).

Damped Sobolev spaces

Suppose now we have an asymptotically cylindrical metric h on X. Then we have an induced measure $\mathrm{d}V_h$ on the space X. It follows that if $u \in C_c^\infty(X)$ with $\mathrm{supp}(u) \subseteq V_\nu$ we may define

$$\|u\|_{L^p(V_\nu)} := \left(\int_{V_\nu} |u|^p \mathrm{d}V_h \right)^{\frac{1}{p}} \tag{4.7}$$

the usual L^p-norm of u. Given $\xi \in C_c^\infty(E)$ we define the *damped Sobolev norm*

$$\|\xi\|_{W^p_{k,\beta}(E)} := \left(\sum_{j=1}^{\mathrm{rank}\,E} \sum_{0 \leqslant |\lambda| \leqslant k} \left(\sum_{\nu=1}^{N} \left\| e^{-\beta t} \rho_\nu \left(\partial^\lambda \xi_j^\nu\right) \right\|^p_{L^p(V_\nu)} + \sum_{\nu=N+1}^{N+K} \left\| \rho_\nu \left(\partial^\lambda \xi_j^\nu\right) \right\|^p_{L^p(V_\nu)} \right) \right)^{\frac{1}{p}}. \tag{4.8}$$

Let $W^p_{k,\beta}(E)$ be the vector space completion of $C_c^\infty(E)$ with respect to the norm (4.8). We call the Banach space $W^p_{k,\beta}(E)$ a *damped Sobolev space*. Note that each $W^2_{k,\beta}(E)$ is a Hilbert space, equipped with the inner product

$$\langle \xi | \eta \rangle_{W^2_{k,\beta}(E)} := \sum_{j=1}^{\mathrm{rank}\,E} \sum_{0 \leqslant |\lambda| \leqslant k} \left(\sum_{\nu=1}^{N} \int_{V_\nu} e^{-2\beta t} \rho_\nu^2 (\partial^\lambda \xi_j^\nu)(\partial^\lambda \eta_j^\nu) \mathrm{d}V_h + \sum_{\nu=N+1}^{N+K} \int_{V_\nu} \rho_\nu^2 (\partial^\lambda \xi_j^\nu)(\partial^\lambda \eta_j^\nu) \mathrm{d}V_h \right).$$

As a topological vector space $W^p_{k,\beta}(E)$ is independent of all choices $U_j, V_j, \rho_j, h, \tilde{h}, g_\Sigma$. Additionally, we can view elements of $W^p_{k,\beta}(E)$ as genuine sections of E, whose components in the various trivialisations V_ν are k times weakly differentiable, and satisfy appropriate L^p-decay conditions as one goes to infinity on X_∞.

If E is equipped with an asymptotically translation invariant fibre metric then we have $L^2(E) = W^2_{0,0}(E)$ where we form the space $L^2(E)$ as in Section 2.1.2 using the asymptotically cylindrical metric h. Furthermore, in the case that E is a tensor bundle (4.3) or an exterior bundle (4.4), (4.5) we have the Levi-Civita connection ∇_h and fibre metric $|\cdot|_h$ on E induced by the asymptotically translation invariant metric h, and the norm (4.8) is equivalent to the norm on $C_c^\infty(E)$ defined by

$$\|\xi\| := \left(\sum_{j=0}^{k} \int_X \left| e^{-\beta t} \nabla_h^j \xi \right|_h^p \mathrm{d}V_h \right)^{\frac{1}{p}}$$

for all $\xi \in C_c^\infty(E)$.

Note that there is a constant $C > 0$ such that

$$\left| \langle \xi_1 | \xi_2 \rangle_{W^2_{0,\delta}(E)} \right| \leqslant C \|\xi_1\|_{W^p_{0,\delta+\beta}(E)} \|\xi_2\|_{W^{p'}_{0,\delta-\beta}(E)}$$

for all $\xi_1, \xi_2 \in C_c^\infty(E)$. It follows that the inner product $\langle\ |\ \rangle_{W^2_{0,\delta}(E)}$ extends to a continuous bilinear map

$$\langle\ |\ \rangle_{W^2_{0,\delta}} : W^p_{0,\delta+\beta}(E) \times W^{p'}_{0,\delta-\beta}(E) \to \mathbb{R} \tag{4.9}$$

and in fact the pairing (4.9) induces a Banach space isomorphism

$$\Phi : W^p_{0,\delta+\beta}(E) \to W^{p'}_{0,\delta-\beta}(E)^* \tag{4.10}$$

defined by $\Phi(\xi)(\eta) := \langle \xi | \eta \rangle_{W^2_{0,\delta}(E)}$ for all $\xi \in W^p_{0,\delta+\beta}(E)$ and $\eta \in W^{p'}_{0,\delta-\beta}(E)$. We now have the following useful result, which can be proved as in the book [2, Section 3.4] of Adams.

Proposition 4.1 *The Banach spaces $W^p_{k,\beta}(E)$ are reflexive.*

The important point in Proposition 4.1 is that $p > 1$.

Damped Hölder spaces

Besides the damped Sobolev spaces $W^p_{k,\beta}(E)$ defined above we shall also introduce a class of Banach spaces $B^{k,a}_\beta(E)$ whose elements are forced to decay at rates $O(e^{\beta t})$ (as measured, for example, by the asymptotically cylindrical metric h when E is a bundle of forms or tensors) on the infinite piece X_∞ of X.

Define $B^k_\beta(E)$ to be the vector space consisting of those $\xi \in C^k(E)$ such that

$$\sup_{\{t\}\times U_\nu} \left|\rho_\nu\left(\partial^\lambda \xi^\nu_j\right)\right| = O(e^{\beta t})$$

for all $1 \leqslant \nu \leqslant N$, $1 \leqslant j \leqslant \operatorname{rank} E$ and $0 \leqslant |\lambda| \leqslant k$. Given $\xi \in B^k_\beta(E)$ we define

$$\|\xi\|_{B^k_\beta(E)} := \sum_{j=1}^{\operatorname{rank} E} \sum_{0\leqslant|\lambda|\leqslant k} \left(\sum_{\nu=1}^{N} \sup_{V_\nu} \left|e^{-\beta t}\rho_\nu\left(\partial^\lambda \xi^\nu_j\right)\right| + \sum_{\nu=N+1}^{N+K} \sup_{V_\nu} \left|\rho_\nu\left(\partial^\lambda \xi^\nu_j\right)\right| \right). \tag{4.11}$$

Then the norm (4.11) makes $B^k_\beta(E)$ into a Banach space, which we call a *damped* B^k*-space*.

We also define $B^{k,a}_\beta(E)$ to be the vector space consisting of those $\xi \in B^k_\beta(E)$ such that

1. $\left[e^{-\beta t}\rho_\nu\left(\partial^\lambda \xi^\nu_j\right)\right]_{a,V_\nu} < \infty$ for all $1 \leqslant \nu \leqslant N$, $1 \leqslant j \leqslant \operatorname{rank} E$ and $|\lambda| = k$
2. $\left[\rho_\nu\left(\partial^\lambda \xi^\nu_j\right)\right]_{a,V_\nu} < \infty$ for all $N+1 \leqslant \nu \leqslant N+K$, $1 \leqslant j \leqslant \operatorname{rank} E$ and $|\lambda| = k$

where each $[\,\cdot\,]_{a,V_\nu}$ is defined as in Section 2.2.4. Given $\xi \in B^{k,a}_\beta(E)$ we now put

$$\|\xi\|_{B^{k,a}_\beta(E)} := \|\xi\|_{B^k_\beta(E)} + \sum_{j=1}^{\operatorname{rank} E} \sum_{|\lambda|=k} \left(\sum_{\nu=1}^{N} \left[e^{-\beta t}\rho_\nu\left(\partial^\lambda \xi^\nu_j\right)\right]_{a,V_\nu} + \sum_{\nu=N+1}^{N+K} \left[\rho_\nu\left(\partial^\lambda \xi^\nu_j\right)\right]_{a,V_\nu} \right). \tag{4.12}$$

Then the norm (4.12) makes $B^{k,a}_\beta(E)$ into a Banach space, which we call a *damped Hölder space*. Note that as topological vector spaces both $B^k_\beta(E)$ and $B^{k,a}_\beta(E)$ are independent of all choices of U_ν, V_ν, ρ_ν.

In the case that E is one of the bundles (4.3), (4.4), (4.5), we may give an equivalent description of the norms (4.11) and (4.12) in terms of the connection ∇_h and fibre metric $|\cdot|_h$ on E got from the asymptotically cylindrical metric h. Firstly, it is not hard to show that $B^k_\beta(E)$ consists of those $\xi \in C^k(E)$ such that

$$\sup_{\{t\}\times\Sigma} \left|\nabla^j_h \xi\right|_h = O(e^{\beta t})$$

for all $0 \leqslant j \leqslant k$. Furthermore the norm (4.11) is equivalent to the norm on $B^k_\beta(E)$ given by

$$\|\xi\| := \sum_{j=0}^{k} \sup_X \left|e^{-\beta t}\nabla^j_h \xi\right|_h$$

for all $\xi \in B^k_\beta(E)$.

We secondly deal with the Hölder norm (4.12). For this it turns out that

$$B^{k,a}_\beta(E) = \left\{\xi \in B^k_\beta(E) : \left[e^{-\beta t}\nabla^k_h \xi\right]^h_{a,X} < \infty\right\} \tag{4.13}$$

and the norm (4.12) is equivalent to the norm on $B^{k,a}_\beta(E)$ given by

$$\|\xi\| := \left(\sum_{j=0}^{k} \sup_X \left|e^{-\beta t}\nabla^j_h \xi\right|_h\right) + \left[e^{-\beta t}\nabla^k_h \xi\right]^h_{a,X}. \tag{4.14}$$

In equation (4.13) and equation (4.14) we make sense of the quantity $\left[e^{-\beta t}\nabla^k_h \xi\right]^h_{a,X}$ as in the discussion of Section 3.1.1.

Embedding and Compactness Theorems

We now state some embedding and compactness theorems for the spaces introduced above.

Theorem 4.2 (Damped Embedding Theorems) *Refer to Section* 2.1.1 *for the definition of a continuous embedding between Banach spaces.*

1. *If* $k \geqslant l \geqslant 0$ *and* $k - \frac{n}{p} \geqslant l - \frac{n}{q}$ *and one of the following two conditions holds:*

   (a) $p \leqslant q$ *and* $\beta \leqslant \delta$

   (b) $p > q$ *and* $\beta < \delta$

   then there is a continuous embedding $W^p_{k,\beta}(E) \leqslant W^q_{l,\delta}(E)$.

2. *If* $\beta \leqslant \delta$ *and* $k+a \geqslant l+b$ *then there are continuous embeddings* $B^{k+1}_\beta(E) \leqslant B^{k,a}_\beta(E) \leqslant B^{l,b}_\delta(E) \leqslant B^l_\delta(E)$ *and* $B^k_\beta(E) \leqslant B^l_\delta(E)$.

3. *If* $\beta < \delta$ *and* $k - \frac{n}{p} \geqslant l + a$ *then there are continuous embeddings* $W^p_{k,\beta}(E) \leqslant B^{l,a}_\beta(E) \leqslant W^q_{l,\delta}(E)$.

Part 1 of Theorem 4.2 is proved in the paper [46, Lemma 7.2] of Lockhart and McOwen, part 2 is proved in the paper [13, Lemma 2] of Chaljub-Simon and Choquet-Bruhat, and part 3 is proved in the paper [6, Theorem 1.2] of Bartnik. A consequence of Theorem 4.2 is that

$$\begin{aligned} W^p_{\infty,\beta}(E) &:= \bigcap_{k=0}^{\infty} W^p_{k,\beta}(E) \\ B^\infty_\beta(E) &:= \bigcap_{k=0}^{\infty} B^{k,a}_\beta(E) \end{aligned}$$

are both subspaces of $C^\infty(E)$ with the latter independent of a and

$$W^p_{\infty,\beta}(E) \leqslant B^\infty_\beta(E) \leqslant W^q_{\infty,\delta}(E)$$

for all $\beta < \delta$.

We also have results which tell us when the embeddings of Theorem 4.2 are compact.

Theorem 4.3 (Damped Compactness Theorems) *Refer to Section* 2.1.1 *for the definition of a compact embedding between Banach spaces.*

1. *The embedding* $W^p_{k,\beta}(E) \leqslant W^q_{l,\delta}(E)$ *is compact whenever* $k > l \geqslant 0$, $k - \frac{n}{p} > l - \frac{n}{q}$ *and* $\beta < \delta$.

2. *The embedding* $B^{k,a}_\beta(E) \leqslant B^k_\delta(E)$ *is compact whenever* $\beta < \delta$.

Part 1 of Theorem 4.3 is proved in the paper [45, Theorem 4.9] of Lockhart and part 2 is proved in the paper [13, Lemma 3] of Chaljub-Simon and Choquet-Bruhat.

4.2.2 Translation invariant operators

Suppose that $F \to X$ is a second admissible vector bundle over X, with slice $F_\Sigma \to \Sigma$ over Σ. Let $P_\infty : C^\infty_c(E) \to C^\infty_c(F)$ be a smooth, linear differential operator of order $l \geqslant 1$. Over each V_ν the operator P_∞ acts as a $\operatorname{rank} F \times \operatorname{rank} E$ matrix of operators of the form:

$$(P_\infty|_{X_\nu})_{ij} = \sum_{0 \leqslant |\lambda| \leqslant l} (P_\infty)^{\nu\lambda}_{ij} \partial^\lambda \tag{4.15}$$

where each $(P_\infty)_{ij}^{\nu\lambda} : V_\nu \to \mathbb{R}$ is smooth. We shall say that P_∞ is *translation invariant* if for each $1 \leqslant \nu \leqslant N$ the functions $(P_\infty)_{ij}^{\nu\lambda}$ got above are translation invariant on $(T+S,\infty) \times U_\nu$ for some large $S \geqslant 0$. In this case we may write

$$P_\infty|_{X \setminus X_S} = \sum_{j=0}^{l} \partial_1^{l-j} A_j \tag{4.16}$$

where each $A_j : C^\infty(E_\Sigma) \to C^\infty(F_\Sigma)$ is a smooth, linear differential operator of order $\leqslant j$ and $\partial_1 = \frac{\partial}{\partial t}$.

Whenever we write P_∞^* to denote the formal adjoint of a translation invariant differential operator P_∞, we always mean with respect to some translation invariant fibre metrics on E, F and the cylindrical metric $\tilde{h}$ on X. Having said this, we now give the following basic result:

Lemma 4.4 1. *The set of translation invariant differential operators is a subalgebra of the algebra of all differential operators.*

2. *For any $\beta \in \mathbb{R}^L$ and translation invariant differential operator P_∞ the differential operator $e^{-\beta t} P_\infty e^{\beta t}$ is also translation invariant.*

3. *If $P_\infty : C_c^\infty(E) \to C_c^\infty(F)$ is a translation invariant differential operator then the formal adjoint $P_\infty^* : C_c^\infty(F) \to C_c^\infty(E)$ is also translation invariant.*

Proof: Although this result is entirely straightforward, we give a few details as some of the formulae we obtain will be useful later.

The first assertion is quickly verified in local coordinates. For the second assertion, note that if P_∞ is as given in equation (4.16) and $\beta = (\beta_1, \ldots, \beta_L)$ then

$$e^{-\beta t} P_\infty e^{\beta t} = \sum_{j=0}^{l} (\partial_1 + \beta_i)^{l-j} A_j \tag{4.17}$$

on the ith end of X. For the third assertion, it is straightforward to check that there is a large $S \geqslant 0$ such that over $X \setminus X_S$ the operator P_∞^* acts as

$$P_\infty^* = \sum_{j=0}^{l} (-\partial_1)^{l-j} A_j^*$$

where the formal adjoint of each A_j is computed using the metric g_Σ on Σ and the given fibre metrics on E_Σ and F_Σ.

∎

Examples of translation invariant operators include the exterior derivative d, its formal adjoint $\mathrm{d}_{\tilde{h}}^*$ and the Laplacian $\Delta_{\tilde{h}}$ of any cylindrical metric $\tilde{h}$ on X.

Translation invariant differential operators always extend to bounded linear maps on the Banach spaces defined above.

Proposition 4.5 *Let $P_\infty : C_c^\infty(E) \to C_c^\infty(F)$ be a translation invariant differential operator of order $l \geqslant 1$. Then P_∞ extends to bounded linear maps*

$$P_\infty : W_{k+l,\beta}^p(E) \to W_{k,\beta}^p(F) \tag{4.18}$$

$$P_\infty : B_\beta^{k+l,a}(E) \to B_\beta^{k,a}(F) \tag{4.19}$$

Proof: This is straightforward estimation using the given definitions.

∎

Using Lemma 4.4 and Proposition 4.5 it is easy to show that if $P_\infty : C^\infty_c(E) \to C^\infty_c(F)$ is a translation invariant differential operator then the defining identity (2.7) of the formal adjoint P^*_∞ of P_∞ extends to an identity

$$\langle \xi | P^*_\infty \eta \rangle_{L^2(E)} = \langle P_\infty \xi | \eta \rangle_{L^2(F)}$$

valid for all $\xi \in W^p_{l,\beta}(E)$ and $\eta \in W^{p'}_{l,-\beta}(F)$.

We shall denote the kernel of the map (4.18) by $\mathrm{Ker}(P_\infty)^p_{k+l,\beta} \leqslant W^p_{k+l,\beta}(E)$, the image of (4.18) by $\mathrm{Im}(P_\infty)^p_{k+l,\beta} \leqslant W^p_{k,\beta}(F)$ and the cokernel by $\mathrm{Coker}(P_\infty)^p_{k+l,\beta}$. We also define the index of (4.18) to be

$$\mathrm{Ind}(P_\infty)^p_{k+l,\beta} := \dim \mathrm{Ker}(P_\infty)^p_{k+l,\beta} - \dim \mathrm{Coker}(P_\infty)^p_{k+l,\beta}$$

whenever this is finite. Similarly the kernel, image, cokernel, index of the map (4.19) are denoted $\mathrm{Ker}(P_\infty)^{k+l,a}_\beta$, $\mathrm{Im}(P_\infty)^{k+l,a}_\beta$, $\mathrm{Coker}(P_\infty)^{k+l,a}_\beta$, $\mathrm{Ind}(P_\infty)^{k+l,a}_\beta$ respectively.

Notice the Embedding Theorem 4.2 shows that

$$\begin{aligned} \mathrm{Ker}(P_\infty)^p_{k+l,\beta} &\leqslant \mathrm{Ker}(P_\infty)^p_{k+l,\delta} \\ \mathrm{Ker}(P_\infty)^{k+l,a}_\beta &\leqslant \mathrm{Ker}(P_\infty)^{k+l,a}_\delta \end{aligned}$$

whenever $\beta \leqslant \delta$. Also from Theorem 4.2 we have

$$\mathrm{Ker}(P_\infty)^p_{m+l,\beta} \leqslant \mathrm{Ker}(P_\infty)^{k+l,a}_\beta \leqslant \mathrm{Ker}(P_\infty)^q_{k+l,\delta} \tag{4.20}$$

whenever $\beta < \delta$ and $m - \frac{n}{p} \geqslant k + a$.

The following a priori estimates for translation invariant operators are very useful. The Sobolev estimates are given in the paper [46, inequality (2.4)] of Lockhart and McOwen, and the Hölder parts are proved in [48, Theorem 3.16].

Theorem 4.6 *Suppose that $P_\infty : C^\infty_c(E) \to C^\infty_c(F)$ is an elliptic, translation invariant differential operator of order $l \geqslant 1$. Suppose that $\eta \in L^1_{loc}(F)$ and that $\xi \in L^1_{loc}(E)$ is a weak solution of the equation $P\xi = \eta$.*

1. *If $\xi \in W^p_{0,\beta}(E)$ and $\eta \in W^p_{k,\beta}(F)$ then $\xi \in W^p_{k+l,\beta}(E)$ with $P_\infty \xi = \eta$ and*

$$\|\xi\|_{W^p_{k+l,\beta}(E)} \leqslant C_1 \Big(\|P_\infty \xi\|_{W^p_{k,\beta}(F)} + \|\xi\|_{W^p_{0,\beta}(E)} \Big)$$

where the constant $C_1 > 0$ is independent of ξ.

2. *If $\xi \in B^0_\beta(E)$ and $\eta \in B^{k,a}_\beta(F)$ then $\xi \in B^{k+l,a}_\beta(E)$ with $P_\infty \xi = \eta$ and*

$$\|\xi\|_{B^{k+l,a}_\beta(E)} \leqslant C_2 \Big(\|P_\infty \xi\|_{B^{k,a}_\beta(F)} + \|\xi\|_{B^0_\beta(E)} \Big)$$

where the constant $C_2 > 0$ is independent of ξ.

Results such as Theorem 4.6 rely on some kind of "uniform ellipticity" condition on the operator P_∞ as one moves off to infinity on the manifold X. Here this condition is provided by the translation invariance of P_∞ and the usual pointwise ellipticity. As in the estimates Theorem 3.4 for the compact case, the proof of Theorem 4.6 is best thought of as being in two parts. The first part is the local elliptic regularity and estimates, which follow just as in Theorem 3.4. The second part is then a passage to the global estimates, which involves piecing all the local estimates together in a straightforward manner. The asymptotic behaviour of P_∞ ensures the local estimates are kept under control as one goes off to infinity on X_∞.

It follows from Theorem 4.6 that when P_∞ is elliptic we have

$$\begin{aligned} \mathrm{Ker}(P_\infty)^p_{k+l,\beta} &\leqslant W^p_{\infty,\beta}(E) \\ \mathrm{Ker}(P_\infty)^{k+l,a}_\beta &\leqslant B^\infty_\beta(E). \end{aligned}$$

In particular, when P_∞ is elliptic we have $\operatorname{Ker}(P_\infty)^p_{k+l,\beta}$ independent of k and $\operatorname{Ker}(P_\infty)^{k+l,a}_\beta$ independent of k and a. Therefore, when P_∞ is elliptic we shall write

$$\begin{aligned} \operatorname{Ker}(P_\infty)^p_\beta &:= \operatorname{Ker}(P_\infty)^p_{k+l,\beta} \\ \operatorname{Ker}(P_\infty)_\beta &:= \operatorname{Ker}(P_\infty)^{k+l,a}_\beta. \end{aligned}$$

With this notation, equation (4.20) becomes

$$\operatorname{Ker}(P_\infty)^p_\beta \leqslant \operatorname{Ker}(P_\infty)_\beta \leqslant \operatorname{Ker}(P_\infty)^q_\delta \tag{4.21}$$

valid for all $\beta < \delta$ whenever P_∞ is elliptic.

One of the key questions for us is now: when is the map (4.18) Fredholm? The following theorem, proved in [46, Theorem 1.1], provides the answer.

Theorem 4.7 *Suppose that $P_\infty : C^\infty_c(E) \to C^\infty_c(F)$ is an elliptic, translation invariant differential operator. Then there exists a subset $\mathcal{D}(P_\infty) \subseteq \mathbb{R}^L$, independent of p, k, such that (4.18) is Fredholm if and only if $\beta \in \mathbb{R}^L \setminus \mathcal{D}(P_\infty)$. Moreover, the subset $\mathcal{D}(P_\infty)$ is of the form*

$$\mathcal{D}(P_\infty) = \big(\mathcal{D}(P_\infty,1) \times \mathbb{R}^{L-1}\big) \cup \big(\mathbb{R} \times \mathcal{D}(P_\infty,2) \times \mathbb{R}^{L-2}\big) \cup \cdots \cup \big(\mathbb{R}^{L-1} \times \mathcal{D}(P_\infty,L)\big)$$

where each $\mathcal{D}(P_\infty,i) \subseteq \mathbb{R}$ is countable and discrete.

In the situation of Theorem 4.7, if $\beta \in \mathbb{R}^L \setminus \mathcal{D}(P_\infty)$ we denote the connected component of $\mathbb{R}^L \setminus \mathcal{D}(P_\infty)$ containing β by $\big(\mathbb{R}^L \setminus \mathcal{D}(P_\infty)\big)_\beta$. Obviously, one would like to know as much as possible about the subsets $\mathcal{D}(P_\infty,i) \subseteq \mathbb{R}$. We now give some brief details: a fuller account can again be found in [46].

The idea is to formally substitute $w \in \mathbb{C}$ for the differential operator ∂_1 in the expression (4.16) for P_∞, as a Fourier transform in the t coordinate. Take $1 \leqslant i \leqslant L$ and put $E_{\Sigma_i} := E_\Sigma|_{\Sigma_i}$ and $F_{\Sigma_i} := F_\Sigma|_{\Sigma_i}$. For each $w \in \mathbb{C}$ we have a smooth, linear differential operator

$$P_\infty(w) := \sum_{j=0}^{l} w^{l-j} A_j \tag{4.22}$$

where $P_\infty(w) : C^\infty(E_{\Sigma_i} \otimes \mathbb{C}) \to C^\infty(F_{\Sigma_i} \otimes \mathbb{C})$ and this extends to a bounded linear map

$$P_\infty(w) : W^p_{k+l}(E_{\Sigma_i} \otimes \mathbb{C}) \to W^p_k(F_{\Sigma_i} \otimes \mathbb{C}). \tag{4.23}$$

Note that since P_∞ is elliptic we have that A_l is an elliptic differential operator of order l. Therefore $P_\infty(w)$ is an elliptic differential operator of order l for each $w \in \mathbb{C}$. Using this fact and the analyticity of the map

$$\begin{aligned} \mathbb{C} &\to \mathcal{B}\big(W^p_{k+l}(E_{\Sigma_i} \otimes \mathbb{C}), W^p_k(F_{\Sigma_i} \otimes \mathbb{C})\big) \\ w &\mapsto P_\infty(w) \end{aligned} \tag{4.24}$$

one can show that (4.23) is an isomorphism if and only if $w \in \mathbb{C} \setminus \mathcal{C}(P_\infty,i)$, where $\mathcal{C}(P_\infty,i) \subseteq \mathbb{C}$ is discrete, countable, and finite in any complex strip $\{w \in \mathbb{C} : \varepsilon_1 < \operatorname{Re} w < \varepsilon_2\}$: these results are all proved in the paper [3, Theorem 5.4] of Agmon and Nirenberg. A map of the form (4.24) is called an *operator pencil*, which is a much studied object in the theory of PDEs. In fact there is much to be said about the form of the subset $\mathcal{C}(P_\infty,i)$: see the books [40] and [41] of Kozlov, Maz'ya and Rossmann for example. However, we shall not go into this here, but instead merely state that the subsets $\mathcal{D}(P_\infty,i)$ of Theorem 4.7 are given by

$$\mathcal{D}(P_\infty,i) = \big\{\operatorname{Re} w : w \in \mathcal{C}(P_\infty,i)\big\}.$$

It follows from the proof of Lemma 4.4 that if $\beta = (\beta_1,\ldots,\beta_L) \in \mathbb{R}^L$ then

$$(e^{-\beta t} P_\infty e^{\beta t})(w) = \sum_{j=0}^{l} (w+\beta_i)^{l-j} A_j$$

so that $\mathcal{D}(e^{-\beta t}P_\infty e^{\beta t}, i) = \mathcal{D}(P_\infty, i) - \beta_i$ for each $1 \leqslant i \leqslant L$. Therefore

$$\mathcal{D}(e^{-\beta t}P_\infty e^{\beta t}) = \mathcal{D}(P_\infty) - \beta.$$

We also have from the proof of Lemma 4.4 that

$$\begin{aligned} P_\infty^*(w) &= \sum_{j=0}^{l}(-w)^{l-j}A_j^* \\ &= \big(P_\infty(-\overline{w})\big)^* \end{aligned}$$

where the formal adjoint of each $P_\infty(-\overline{w})$ is computed using the metric g_Σ on Σ and the induced hermitian fibre metrics on $E_\Sigma \otimes \mathbb{C}$ and $F_\Sigma \otimes \mathbb{C}$. It follows that $\mathcal{D}(P_\infty^*) = -\mathcal{D}(P_\infty)$.

The next result is a useful corollary of Theorem 4.7.

Corollary 4.8 *Suppose that $P_\infty : C_c^\infty(E) \to C_c^\infty(F)$ is an elliptic, translation invariant differential operator. Then both*

$$\begin{aligned} \mathrm{Ker}(P_\infty)_\beta^p &\leqslant W_{k+l,\beta}^p(E) \\ \mathrm{Ker}(P_\infty)_\beta &\leqslant B_\beta^{k+l,a}(E) \end{aligned}$$

are finite-dimensional.

Proof: Given any $\beta \in \mathbb{R}^L$ choose $\delta \in \mathbb{R}^L \setminus \mathcal{D}(P_\infty)$ with $\beta < \delta$, and then appeal to Theorem 4.7 and the inclusion (4.20).

∎

We now turn to the Fredholm index of the map (4.18). Given $1 \leqslant i \leqslant L$ consider the operator P_∞ over $(T, \infty) \times \Sigma_i$. If $w \in \mathcal{C}(P_\infty, i)$ let $d(i, w)$ be the dimension of the (complex) vector space of solutions of $P_\infty \xi = 0$ of the form

$$\xi(t, \sigma) = e^{wt}p(t, \sigma)$$

where $p(t, \sigma)$ is a polynomial in $t \in (T, \infty)$ with coefficients in $C^\infty(E_{\Sigma_i} \otimes \mathbb{C})$. Now given $\beta_i, \delta_i \in \mathbb{R} \setminus \mathcal{D}(P_\infty, i)$ with $\delta_i \leqslant \beta_i$ define

$$N(\beta_i, \delta_i, i) := \sum \big\{ d(i, w) : w \in \mathcal{C}(P_\infty, i) \text{ with } \delta_i < \mathrm{Re}\, w < \beta_i \big\}$$

and then if $\beta, \delta \in \mathbb{R}^L \setminus \mathcal{D}(P_\infty)$ with $\delta \leqslant \beta$ put

$$N(\beta, \delta) := \sum_{i=1}^{L} N(\beta_i, \delta_i, i).$$

We are now in a position to state a theorem regarding the index of the map (4.18) for $\beta \in \mathbb{R}^L \setminus \mathcal{D}(P_\infty)$.

Theorem 4.9 *Suppose $P_\infty : C_c^\infty(E) \to C_c^\infty(F)$ is an elliptic, translation invariant differential operator. If $\beta, \delta \in \mathbb{R}^L \setminus \mathcal{D}(P_\infty)$ with $\delta \leqslant \beta$ then* $\mathrm{Ind}(P_\infty)_{k+l,\beta}^p - \mathrm{Ind}(P_\infty)_{k+l,\delta}^p = N(\beta, \delta)$.

Theorem 4.9 is also proved in the paper [46, Theorem 1.2] of Lockhart and McOwen, but we shall not give any details of the proof, as they shall not be required by us in the sequel.

4.2.3 Asymptotically translation invariant operators

In this section we shall extend the theory of Section 4.2.2 to perturbations of translation invariant operators.

Let $P_1, P_2 : C_c^\infty(E) \to C_c^\infty(F)$ be smooth, linear differential operator of order $l \geqslant 1$. As in equation (4.15) the operators P_1, P_2 acts as a $\operatorname{rank} F \times \operatorname{rank} E$ matrix of operators

$$\begin{aligned}(P_1|_{V_\nu})_{ij} &= \sum_{0\leqslant|\lambda|\leqslant l} (P_1)_{ij}^{\nu\lambda}\partial^\lambda \\ (P_2|_{V_\nu})_{ij} &= \sum_{0\leqslant|\lambda|\leqslant l} (P_2)_{ij}^{\nu\lambda}\partial^\lambda\end{aligned}$$

on the open subsets V_ν of X. We shall say that P_1 and P_2 are *asymptotic* and write $P_1 \sim P_2$ if

$$\sup_{\{t\}\times U_\nu} \left|\rho_\nu \partial^{\lambda_1}\big((P_1)_{\nu,ij}^{\lambda_2} - (P_2)_{\nu,ij}^{\lambda_2}\big)\right| = o(1)$$

for all $1 \leqslant i \leqslant \operatorname{rank} F$, $1 \leqslant j \leqslant \operatorname{rank} E$, $1 \leqslant \nu \leqslant N$, $|\lambda_1| \geqslant 0$ and $0 \leqslant |\lambda_2| \leqslant l$. We shall say that a smooth, linear differential operator P is *asymptotically translation invariant* if there exists a translation invariant differential operator P_∞ such that $P \sim P_\infty$.

Whenever we write P^* to denote the formal adjoint of an asymptotically translation invariant differential operator P, we always mean with respect to some asymptotically translation invariant fibre metrics on E and F and the asymptotically cylindrical metric h on X. We now have the analogue of Lemma 4.4.

Lemma 4.10 1. *The set of asymptotically translation invariant differential operators is a subalgebra of the algebra of all differential operators.*

2. *For any $\beta \in \mathbb{R}^L$ and asymptotically translation invariant operator P the differential operator $e^{-\beta t}Pe^{\beta t}$ is asymptotically translation invariant.*

3. *If $P : C_c^\infty(E) \to C_c^\infty(F)$ is an asymptotically translation invariant differential operator then the formal adjoint $P^* : C_c^\infty(F) \to C_c^\infty(E)$ is asymptotically translation invariant.*

Proof: These assertions follows quickly from Lemma 4.4 and calculations in local coordinates: if $P_1 \sim P_{1,\infty}$ and $P_2 \sim P_{2,\infty}$ then $P_1P_2 \sim P_{1,\infty}P_{2,\infty}$. For the second assertion note that if $P \sim P_\infty$ then $e^{-\beta t}Pe^{\beta t} \sim e^{-\beta t}P_\infty e^{\beta t}$. For the third assertion, if $P \sim P_\infty$ then $P^* \sim P_\infty^*$ where P_∞^* is formed using the translation invariant metric $\tilde{h}$ on X and the translation invariant fibre metrics on E, F which the asymptotically translation invariant fibre metrics tend towards.

∎

Just as cylindrical metrics $\tilde{h}$ on X gave rise to translation invariant operators $\mathrm{d}_{\tilde{h}}^*$ and $\Delta_{\tilde{h}}$, metrics h on X which are asymptotically cylindrical give rise to asymptotically translation invariant operators d_h^* and Δ_h, where $\mathrm{d}_h^* \sim \mathrm{d}_{\tilde{h}}^*$ and $\Delta_h \sim \Delta_{\tilde{h}}$.

As above, asymptotically translation invariant operators always extend to bounded linear maps on the damped Sobolev and Hölder spaces.

Proposition 4.11 *Let $P : C_c^\infty(E) \to C_c^\infty(F)$ be an asymptotically translation invariant differential operator of order $l \geqslant 1$. Then P extends to bounded linear maps*

$$P : W_{k+l,\beta}^p(E) \to W_{k,\beta}^p(F) \tag{4.25}$$

$$P : B_\beta^{k+l,a}(E) \to B_\beta^{k,a}(F) \tag{4.26}$$

Proof: This is another straightforward estimation.

∎

It follows from Lemma 4.10 and Proposition 4.11 that if $P : C_c^\infty(E) \to C_c^\infty(F)$ is an asymptotically translation invariant differential operator then the defining identity (2.7) of the formal adjoint P^* of P extends to an identity

$$\langle\xi|P^*\eta\rangle_{L^2(E)} = \langle P\xi|\eta\rangle_{L^2(F)}$$

valid for all $\xi \in W^p_{l,\beta}(E)$ and $\eta \in W^{p'}_{l,-\beta}(F)$.

Again, we establish notation: the kernel, image, cokernel, index of the map (4.25) are denoted $\mathrm{Ker}(P)^p_{k+l,\beta}$, $\mathrm{Im}(P)^p_{k+l,\beta}$, $\mathrm{Coker}(P)^p_{k+l,\beta}$, $\mathrm{Ind}(P)^p_{k+l,\beta}$ respectively. The kernel, image, cokernel, index of the map (4.26) are denoted $\mathrm{Ker}(P)^{k+l,a}_\beta$, $\mathrm{Im}(P)^{k+l,a}_\beta$, $\mathrm{Coker}(P)^{k+l,a}_\beta$, $\mathrm{Ind}(P)^{k+l,a}_\beta$ respectively. As before, the Embedding Theorem 4.2 shows that

$$\begin{aligned}\mathrm{Ker}(P)^p_{k+l,\beta} &\leqslant \mathrm{Ker}(P)^p_{k+l,\delta}\\ \mathrm{Ker}(P)^{k+l,a}_\beta &\leqslant \mathrm{Ker}(P)^{k+l,a}_\delta\end{aligned}$$

whenever $\beta \leqslant \delta$. Also Theorem 4.2 gives

$$\mathrm{Ker}(P)^p_{m+l,\beta} \leqslant \mathrm{Ker}(P)^{k+l,a}_\beta \leqslant \mathrm{Ker}(P)^q_{k+l,\delta} \tag{4.27}$$

whenever $\beta < \delta$ and $m - \frac{n}{p} \geqslant k + a$.

If both P and P_∞ are elliptic then we say that the asymptotically translation invariant operator P is *uniformly elliptic*. In this case the regularity and Fredholm theory of the map (4.25) is very similar to that of the map (4.18) given in the theorems above. We now state the corresponding results.

Theorem 4.12 *Suppose that $P : C^\infty_c(E) \to C^\infty_c(F)$ is a uniformly elliptic, asymptotically translation invariant differential operator of order $l \geqslant 1$. Suppose that $\eta \in L^1_{loc}(F)$ and that $\xi \in L^1_{loc}(E)$ is a weak solution of the equation $P\xi = \eta$.*

1. *If $\xi \in W^p_{0,\beta}(E)$ and $\eta \in W^p_{k,\beta}(F)$ then $\xi \in W^p_{k+l,\beta}(E)$ with $P\xi = \eta$ and*

$$\|\xi\|_{W^p_{k+l,\beta}(E)} \leqslant C_1\Big(\|P\xi\|_{W^p_{k,\beta}(F)} + \|\xi\|_{W^p_{0,\beta}(E)}\Big)$$

   where the constant $C_1 > 0$ is independent of ξ.

2. *If $\xi \in B^0_\beta(E)$ and $\eta \in B^{k,a}_\beta(F)$ then $\xi \in B^{k+l,a}_\beta(E)$ with $P\xi = \eta$ and*

$$\|\xi\|_{B^{k+l,a}_\beta(E)} \leqslant C_2\Big(\|P\xi\|_{B^{k,a}_\beta(F)} + \|\xi\|_{B^0_\beta(E)}\Big)$$

   where the constant $C_2 > 0$ is independent of ξ.

The estimates of Theorem 4.12 are proved in a manner similar to those if Theorem 4.6. All of the comments following Theorem 4.6 apply here also. We note that the Hölder estimate of Theorem 4.12 is proved in [48, Theorem 3.16].

As before, Theorem 4.12 implies that when P is uniformly elliptic we have

$$\begin{aligned}\mathrm{Ker}(P)^p_{k+l,\beta} &\leqslant W^p_{\infty,\beta}(E)\\ \mathrm{Ker}(P)^{k+l,a}_\beta &\leqslant B^\infty_\beta(E)\end{aligned}$$

so that $\mathrm{Ker}(P)^p_{k+l,\beta}$ is independent of k and $\mathrm{Ker}(P)^{k+l,a}_\beta$ is independent of k and a. Therefore, whenever P is uniformly elliptic we shall write

$$\begin{aligned}\mathrm{Ker}(P)^p_\beta &:= \mathrm{Ker}(P)^p_{k+l,\beta}\\ \mathrm{Ker}(P)_\beta &:= \mathrm{Ker}(P)^{k+l,a}_\beta.\end{aligned}$$

With this notation, equation (4.27) becomes

$$\mathrm{Ker}(P)^p_\beta \leqslant \mathrm{Ker}(P)_\beta \leqslant \mathrm{Ker}(P)^q_\delta \tag{4.28}$$

valid for all $\beta < \delta$ whenever P is uniformly elliptic.

The following theorem tells us when a uniformly elliptic asymptotically translation invariant operator is Fredholm.

Theorem 4.13 *Suppose that P is a uniformly elliptic, asymptotically translation invariant operator with $P \sim P_\infty$ an elliptic, translation invariant differential operator. Then there exists a subset $\mathcal{D}(P) \subseteq \mathbb{R}^L$, independent of p, k, such that (4.25) is Fredholm if and only if $\beta \in \mathbb{R}^L \setminus \mathcal{D}(P)$. Moreover $\mathcal{D}(P) = \mathcal{D}(P_\infty)$.*

The proof of Theorem 4.13 can again be found in [46, Theorem 6.1]. It follows that if both P and P_∞ are elliptic then the bounded linear map (4.18) is Fredholm precisely when the bounded linear map (4.25) is Fredholm. In the situation Theorem 4.13, if $\beta \in \mathbb{R}^L \setminus \mathcal{D}(P)$ we denote the connected component of $\mathbb{R}^L \setminus \mathcal{D}(P)$ containing β by $\left(\mathbb{R}^L \setminus \mathcal{D}(P)\right)_\beta$.

Note that if P is asymptotically translation invariant with $P \sim P_\infty$ then $e^{-\beta t} P e^{\beta t}$ is uniformly elliptic precisely when P is uniformly elliptic, and in this situation

$$\mathcal{D}(e^{-\beta t} P e^{\beta t}) = \mathcal{D}(P) - \beta.$$

Also, if P^* is the formal adjoint of P as described above then P^* is uniformly elliptic precisely when P is uniformly elliptic and in this situation we have

$$\mathcal{D}(P^*) = -\mathcal{D}(P).$$

The following corollary to Theorem 4.13 is very useful.

Corollary 4.14 *Suppose that $P : C_c^\infty(E) \to C_c^\infty(F)$ is a uniformly elliptic, asymptotically translation invariant differential operator of order $l \geqslant 1$. Then both*

$$\begin{aligned} \mathrm{Ker}(P)^p_\beta &\leqslant W^p_{k+l,\beta}(E) \\ \mathrm{Ker}(P)_\beta &\leqslant B^{k+l,a}_\beta(E) \end{aligned}$$

are finite-dimensional.

Proof: Given any $\beta \in \mathbb{R}^L$ choose $\delta \in \mathbb{R}^L \setminus \mathcal{D}(P)$ with $\beta < \delta$, and then appeal to Theorem 4.13 and the inclusion (4.27).

∎

The following theorem tells us that as β crosses over the "bad" points $\mathcal{D}(P) = \mathcal{D}(P_\infty)$ the change in index for P is the same as for P_∞. Again, the proof is in the paper [46, Theorem 6.1] of Lockhart and McOwen.

Theorem 4.15 *Suppose that P is a uniformly elliptic, asymptotically translation invariant operator with $P \sim P_\infty$ an elliptic, translation invariant differential operator. If $\beta, \delta \in \mathbb{R}^L \setminus \mathcal{D}(P)$ with $\delta \leqslant \beta$ then*

$$\mathrm{Ind}(P)^p_{k+l,\beta} - \mathrm{Ind}(P)^p_{k+l,\delta} = N(\beta, \delta) \tag{4.29}$$

where the quantity $N(\beta, \delta)$ is defined at the end of Section 4.2.2*.*

It follows from Theorem 4.9 and Theorem 4.15 that if $\beta, \delta \in \mathbb{R}^L \setminus \mathcal{D}(P)$ with $\delta \leqslant \beta$ then

$$\mathrm{Ind}(P)^p_{k+l,\beta} - \mathrm{Ind}(P_\infty)^p_{k+l,\beta} = \mathrm{Ind}(P)^p_{k+l,\delta} - \mathrm{Ind}(P_\infty)^p_{k+l,\delta} \tag{4.30}$$

and hence the Fredholm indices of (4.18) and (4.25) differ only by a constant as $\beta \in \mathbb{R}^L \setminus \mathcal{D}(P)$ varies.

4.3 Asymptotically conical Riemannian manifolds

In Section 4.2 we considered a rather general Fredholm theory for asymptotically translation invariant operators acting between vector bundles over X which had some kind of product structure off a compact piece X_S of X. We shall now apply this theory to the specific situation where E is one of the admissible bundles

$$E := \begin{cases} \left(\otimes^r T^*X\right) \otimes \left(\otimes^s TX\right) \\ \Lambda^r T^*X \\ \Lambda^* T^*X \end{cases} \tag{4.31}$$

where $r, s \geqslant 0$ are integers. We shall also consider X as having a metric g which is approximately of *conical* form on the infinite piece X_∞, and consider Fredholm theory for operators Δ_g^r and $\mathrm{d} + \mathrm{d}_g^*$ derived from the metric g. Correspondingly, we define new Banach spaces on which these operators act, describe the relationship with the spaces of Section 4.2, and derive the corresponding Fredholm theory.

Although it may seem a little perverse to define new classes of Banach spaces in this section we believe that in the end it gives a cleaner exposition. The Fredholm theory of Section 4.2 is best described in terms of the cylindrical type spaces $W^p_{k,\beta}(E), B^{k,a}_\beta(E)$ and (asymptotically) translation invariant operators P_∞, P. However, for Riemannian manifolds with conical type metrics, introducing new types of spaces and operators is more appropriate.

4.3.1 Construction of suitable Banach spaces

Let us suppose that the manifold Σ has a Riemannian metric g_Σ. The *cone metric* on X_∞ is then defined to be

$$\tilde{g} := e^{2t}(\mathrm{d}t^2 + g_\Sigma).$$

The reason for this terminology is clear: for after putting $r := e^t$ we obtain $\tilde{g} = \mathrm{d}r^2 + r^2 g_\Sigma$, which is the usual cone metric on the manifold $(e^T, \infty) \times \Sigma$. We prefer the t coordinate rather than the r coordinate because the 1-forms $\mathrm{d}t$ and $\mathrm{d}\sigma_j$ have the same growth rate e^{-t} on X_∞ in the conical metric $\tilde{g}$, whereas $\mathrm{d}r$ has unit length over X_∞.

We say that a metric g on X is *asymptotically conical* if there exists a conical metric $\tilde{g}$ on X such that

$$\sup_{\{t\}\times U_\nu} \left|\rho_\nu \partial^\lambda \left(g_{ij} - \tilde{g}_{ij}\right)\right| = o(e^{2t}) \tag{4.32}$$

for each $1 \leqslant \nu \leqslant N$, $1 \leqslant i, j \leqslant n$ and $|\lambda| \geqslant 0$. In coordinate free terms, equation (4.32) is the same as requiring

$$\sup_{\{t\}\times\Sigma} \left|\nabla^j_{\tilde{g}}(g - \tilde{g})\right|_{\tilde{g}} = o(e^{-jt})$$

for each $j \geqslant 0$. An asymptotically conical metric on X will always be complete.

Conical damped Sobolev spaces

Let us suppose that X is endowed with some asymptotically conical metric g, asymptotic to the conical metric $\tilde{g}$ on X. It is easy to show that $h := e^{-2t}g$ is asymptotically cylindrical, asymptotic to the cylindrical metric $\tilde{h} := e^{-2t}\tilde{g}$. Therefore the L^p-norm (4.7) becomes

$$\|u\|_{L^p(V_\nu)} = \left(\int_{V_\nu} |u|^p e^{-nt} \mathrm{d}V_g\right)^{\frac{1}{p}} \tag{4.33}$$

for all $1 \leqslant \nu \leqslant N$ and $u \in C_c^\infty(X)$ with $\mathrm{supp}(u) \subseteq V_\nu$. Suppose now that E is one of the bundles (4.31). As in Section 4.2.1 we can use the L^p-norm (4.33) to construct the Banach space $W^p_{k,\beta}(E)$, which has a norm $\|\cdot\|_{W^p_{k,\beta}(E)}$. Given $\xi \in C_c^\infty(E)$ we now define

$$\|\xi\|_{L^p_{k,\beta}(E)} := \|e^{(s-r)t}\xi\|_{W^p_{k,\beta}(E)} \tag{4.34}$$

and let $L^p_{k,\beta}(E)$ be the vector space completion of $C_c^\infty(E)$ with respect to the norm (4.34). We shall call $L^p_{k,\beta}(E)$ a *conical damped Sobolev space*. Obviously the map $e^{(s-r)t} : C_c^\infty(E) \to C_c^\infty(E)$ lifts to an isometric isomorphism

$$e^{(s-r)t} : L^p_{k,\beta}(E) \to W^p_{k,\beta}(E). \tag{4.35}$$

The reason we have bothered to introduce a new class of Banach space is the following: the bundle E comes equipped with a natural connection ∇_g and fibre metric $|\cdot|_g$ got from the asymptotically conical metric g on X. Therefore we have a second norm on the vector space $C^\infty_c(E)$ given by

$$\|\xi\| := \left(\sum_{j=0}^{k} \int_X \left| e^{(j-\beta)t} \nabla^j_g \xi \right|^p_g e^{-nt} \mathrm{d}V_g \right)^{\frac{1}{p}}. \tag{4.36}$$

The norms (4.34) and (4.36) on $C^\infty_c(E)$ are equivalent: this is because we have included the correction factor $e^{(s-r)t}$ into the $W^p_{k,\beta}(E)$ norm (4.8).

Each $L^2_{k,\beta}(E)$ is a Hilbert space, and the norm (4.36) is induced by the inner product

$$\langle \xi_1 | \xi_2 \rangle := \sum_{j=0}^{k} \int_X e^{2(j-\beta)t} \langle \nabla^j_g \xi_1 | \nabla^j_g \xi_2 \rangle_g e^{-nt} \mathrm{d}V_g. \tag{4.37}$$

Note that $L^2_{0,-\frac{n}{2}}(E) = L^2(E)$ where we use the asymptotically conical metric g to define the space $L^2(E)$ as in Section 2.1.2. Note also that there is a constant $C > 0$ such that

$$\left| \langle \xi_1 | \xi_2 \rangle_{L^2_{0,\delta}(E)} \right| \leqslant C \|\xi_1\|_{L^p_{0,\delta+\beta}(E)} \|\xi_2\|_{L^{p'}_{0,\delta-\beta}(E)}$$

for all $\xi_1, \xi_2 \in C^\infty_c(E)$. Therefore

$$\left| \langle \xi_1 | \xi_2 \rangle_{L^2(E)} \right| \leqslant C \|\xi_1\|_{L^p_{0,\beta}(E)} \|\xi_2\|_{L^{p'}_{0,-\beta-n}(E)}$$

for all $\xi_1, \xi_2 \in C^\infty_c(E)$. It follows that the L^2-inner product defined in Section 2.1.2 extends to a continuous bilinear map

$$\langle\ |\ \rangle_{L^2(E)} : L^p_{0,\beta}(E) \times L^{p'}_{0,-\beta-n}(E) \to \mathbb{R} \tag{4.38}$$

and in fact the pairing (4.38) induces a Banach space isomorphism

$$\Phi : L^p_{0,\beta}(E) \to L^{p'}_{0,-\beta-n}(E)^* \tag{4.39}$$

defined by $\Phi(\xi)(\eta) := \langle \xi | \eta \rangle_{L^2(E)}$ for all $\xi \in L^p_{0,\beta}(E)$ and $\eta \in L^{p'}_{0,-\beta-n}(E)$. The following useful result can now be proved as in the book [2, Section 3.4] of Adams.

Proposition 4.16 *The Banach spaces $L^p_{k,\beta}(E)$ are reflexive.*

The important point in Proposition 4.16 is that $p > 1$.

Conical damped Hölder spaces

Besides the damped Sobolev spaces $L^p_{k,\beta}(E)$ defined above we shall also introduce a class of Banach spaces $C^{k,a}_\beta(E)$ whose elements are forced to decay at rates $O(e^{\beta t})$ on the infinite piece X_∞ of X, as measured using the asymptotically *conical* metric g on X. Recall that the spaces $B^{k,a}_\beta(E)$ had a similar decay property, but instead using the asymptotically *cylindrical* metric h.

We shall call the $C^{k,a}_\beta(E)$ spaces *conical damped Hölder spaces*. Given what we have already said about the $B^{k,a}_\beta(E)$ spaces, their definition is very straightforward. First of all, declare a section ξ of E to lie in $C^k_\beta(E)$ precisely when $e^{(s-r)t}\xi \in B^k_\beta(E)$, so as vector spaces we have $C^k_\beta(E) := e^{(r-s)t} B^k_\beta(E)$. Now given $\xi \in C^k_\beta(E)$ define the norm

$$\|\xi\|_{C^k_\beta(E)} := \|e^{(s-r)t}\xi\|_{B^k_\beta(E)} \tag{4.40}$$

which makes $C^k_\beta(E)$ into a Banach space, because $B^k_\beta(E)$ is a Banach space and

$$e^{(s-r)t} : C^k_\beta(E) \to B^k_\beta(E)$$

is an isometric isomorphism.

Similarly, as a vector space, we define $C^{k,a}_\beta(E) := e^{(r-s)t}B^{k,a}_\beta(E)$, endowed with the norm

$$\|\xi\|_{C^{k,a}_\beta(E)} := \|e^{(s-r)t}\xi\|_{B^{k,a}_\beta(E)} \tag{4.41}$$

which makes $C^{k,a}_\beta(E)$ a Banach space too.

In fact, we may give an equivalent description of the norms (4.40) and (4.41) in terms of the connection ∇_g and fibre metric $|\cdot|_g$ on E got from the asymptotically conical metric g. Firstly, it is not hard to show that $\xi \in C^k(E)$ lies in $C^k_\beta(E)$ precisely when

$$\sup_{\{t\}\times\Sigma} \left|\nabla^j_g \xi\right|_g = O(e^{(\beta-j)t})$$

for all $0 \leqslant j \leqslant k$, and that the norm (4.40) is equivalent to the norm on $C^k_\beta(E)$ given by

$$\|\xi\| := \sum_{j=0}^{k} \sup_X \left|e^{(j-\beta)t}\nabla^j_g\xi\right|_g .$$

For the Hölder norm (4.41) it turns out that

$$C^{k,a}_\beta(E) = \left\{\xi \in C^k_\beta(E) : \left[e^{(k+a-\beta)t}\nabla^k_g\xi\right]^g_{a,X} < \infty\right\}$$

and the norm (4.41) is equivalent to the norm on $C^{k,a}_\beta(E)$ given by

$$\|\xi\| := \left(\sum_{j=0}^{k} \sup_X \left|e^{(j-\beta)t}\nabla^j_g\xi\right|_g\right) + \left[e^{(k+a-\beta)t}\nabla^k_g\xi\right]^g_{a,X} . \tag{4.42}$$

The fact that the spaces $C^k_\beta(E)$ and $C^{k,a}_\beta(E)$ are canonically got from the asymptotically conical metric g is another reason for introducing them. In (4.42) we make sense of the quantity $\left[e^{(k+a-\beta)t}\nabla^k_g\xi\right]^g_{a,X}$ using the arguments of Section 3.1.1.

To see that the isomorphisms $e^{(s-r)t} : C^{k,a}_\beta(E) \to B^{k,a}_\beta(E)$ do what we expect, consider for example the case $s = 0$, $r = 1$. The 1-form $e^t\mathrm{d}t$ lies inside $C^0_0(T^*X)$ because $\mathrm{d}t$ grows like e^{-t} in the asymptotically conical metric g and as expected we have $e^{-t}e^t\mathrm{d}t = \mathrm{d}t \in B^0_0(T^*X)$ because $\mathrm{d}t$ grows like 1 in the asymptotically cylindrical metric h.

Embedding and Compactness Theorems

We now state the Embedding and Compactness Theorems for the spaces $L^p_{k,\beta}(E)$ and $C^{k,a}_\beta(E)$ defined above. They all follow immediately after applying the isometric isomorphism $e^{(s-r)t}$ to the corresponding theorems of Section 4.2.1.

Theorem 4.17 (Conical Damped Embedding Theorems) *Refer to Section* 2.1.1 *for the definition of a continuous embedding between Banach spaces.*

1. *If $k \geqslant l \geqslant 0$ and $k - \frac{n}{p} \geqslant l - \frac{n}{q}$ and one of the following two conditions holds*

   (a) *$p \leqslant q$ and $\beta \leqslant \delta$*

   (b) *$p > q$ and $\beta < \delta$*

   then there is a continuous embedding $L^p_{k,\beta}(E) \leqslant L^q_{l,\delta}(E)$.

2. *If $\beta \leqslant \delta$ and $k+a \geqslant l+b$ then there are continuous embeddings $C^{k+1}_\beta(E) \leqslant C^{k,a}_\beta(E) \leqslant C^{l,b}_\delta(E) \leqslant C^l_\delta(E)$ and $C^k_\beta(E) \leqslant C^l_\delta(E)$.*

3. *If $\beta < \delta$ and $k - \frac{n}{p} \geqslant l + a$ then there are continuous embeddings $L^p_{k,\beta}(E) \leqslant C^{l,a}_\beta(E) \leqslant L^q_{l,\delta}(E)$.*

A consequence of Theorem 4.17 is that

$$L^p_{\infty,\beta}(E) := \bigcap_{k=0}^{\infty} L^p_{k,\beta}(E) \qquad C^\infty_\beta(E) := \bigcap_{k=0}^{\infty} C^{k,a}_\beta(E)$$

are both subspaces of $C^\infty(E)$ with the latter independent of a and

$$L^p_{\infty,\beta}(E) \leqslant C^\infty_\beta(E) \leqslant L^q_{\infty,\delta}(E)$$

for all $\beta < \delta$.

Theorem 4.18 (Conical Damped Compactness Theorems) *Refer to Section* 2.1.1 *for the definition of a compact embedding between Banach spaces.*

1. *The embedding $L^p_{k,\beta}(E) \leqslant L^q_{l,\delta}(E)$ is compact whenever $k > l \geqslant 0$ and $k - \frac{n}{p} > l - \frac{n}{q}$ and $\beta < \delta$.*

2. *The embedding $C^{k,a}_\beta(E) \leqslant C^k_\delta(E)$ is compact whenever $\beta < \delta$.*

Note that Theorem 4.17 explains the way we have chosen to define our $L^p_{k,\beta}(E)$ spaces: the index β genuinely indicates the rate of growth of a section in terms of the asymptotically conical metric.

4.3.2 Asymptotically conical operators

Suppose now that F is a second vector bundle over X, of the form (4.31). Then we have linear isomorphisms

$$\begin{aligned} e^{(s-r)t} : C^\infty_c(E) &\rightarrow C^\infty_c(E) \\ e^{(s-r)t} : C^\infty_c(F) &\rightarrow C^\infty_c(F). \end{aligned}$$

We shall say that a smooth, linear differential operator $Q : C^\infty_c(E) \to C^\infty_c(F)$ of order $l \geqslant 1$ is an *asymptotically conical operator* of *rate* $\gamma \in \mathbb{R}^L$ if

$$P : C^\infty_c(E) \xrightarrow{e^{(r-s)t}} C^\infty_c(E) \xrightarrow{Q} C^\infty_c(F) \xrightarrow{e^{(\gamma+s-r)t}} C^\infty_c(F) \tag{4.43}$$

is an asymptotically translation invariant operator. Whenever we take the formal adjoint Q^* of an asymptotically conical operator Q we always mean with respect to the asymptotically conical metric g, which induces fibre metrics on each of the bundles E, F.

Here are some basic properties of asymptotically conical operators:

Lemma 4.19 1. *The set of asymptotically conical operators is a subalgebra of the algebra of all differential operators. Moreover, if Q_1, Q_2 are asymptotically conical operators of rates $\gamma_1, \gamma_2 \in \mathbb{R}^L$ then $Q_1 Q_2$ is an asymptotically conical operator of rate $\gamma_1 + \gamma_2$.*

2. *For any $\beta \in \mathbb{R}^L$ and asymptotically conical operator Q of rate γ the differential operator $e^{-\beta t} Q e^{\beta t}$ is an asymptotically conical operator of rate γ.*

3. *If $Q : C^\infty_c(E) \to C^\infty_c(F)$ is an asymptotically conical differential operator of rate $\gamma \in \mathbb{R}^L$ then the formal adjoint $Q^* : C^\infty_c(F) \to C^\infty_c(E)$ is asymptotically conical of rate γ.*

Proof: For the first assertion, suppose that we have asymptotically translation invariant operators

$$\begin{aligned} P_1 : C^\infty_c(E) &\xrightarrow{e^{(r-s)t}} C^\infty_c(E) \xrightarrow{Q_1} C^\infty_c(F) \xrightarrow{e^{(\gamma_1+s-r)t}} C^\infty_c(F) \\ P_2 : C^\infty_c(F) &\xrightarrow{e^{(r-s)t}} C^\infty_c(F) \xrightarrow{Q_2} C^\infty_c(G) \xrightarrow{e^{(\gamma_2+s-r)t}} C^\infty_c(G). \end{aligned}$$

It follows that

$$e^{\gamma_1 t} P_2 e^{-\gamma_1 t} P_1 : C_c^\infty(E) \stackrel{e^{(r-s)t}}{\longrightarrow} C_c^\infty(E) \stackrel{Q_2 Q_1}{\longrightarrow} C_c^\infty(G) \stackrel{e^{(\gamma_1+\gamma_2+s-r)t}}{\longrightarrow} C_c^\infty(G)$$

and we are done, after appealing to Lemma 4.10. The second assertion is now a special case of the first.

To prove the third assertion, we note that if P, Q are as in (4.43) then

$$e^{(\gamma-n)t} P^* e^{-(\gamma-n)t} = e^{\gamma t} e^{(s-r)t} Q^* e^{(r-s)t}$$

where on the left hand side P^* denotes the formal adjoint of P with respect to the asymptotically cylindrical metric h. Appealing to Lemma 4.10 now completes the proof of the third assertion.

∎

Table 4.1 gives some examples of asymptotically conical differential operators $Q : C_c^\infty(E) \to C_c^\infty(F)$, along with the corresponding asymptotically translation invariant operators P and rates $\gamma \in \mathbb{R}^L$.

Q	P	γ	E	F
d	$e^{-rt}\mathrm{d}e^{rt}$	1	$\Lambda^r T^* X$	$\Lambda^{r+1} T^* X$
d_g^*	$e^{(2-r)t}\mathrm{d}_g^* e^{rt}$	1	$\Lambda^r T^* X$	$\Lambda^{r-1} T^* X$
Δ_g^r	$e^{(2-r)t}\Delta_g^r e^{rt}$	2	$\Lambda^r T^* X$	$\Lambda^r T^* X$
$\mathrm{d}_g^* + \mathrm{d}$	$e^{-rt}(e^{2t}\mathrm{d}_g^* + \mathrm{d})e^{rt}$	1	$\Lambda^* T^* X$	$\Lambda^* T^* X$

Table 4.1: Examples of asymptotically conical operators: the row for $\mathrm{d}_g^* + \mathrm{d}$ gives P in terms of the action on r-forms

Here are some comments on Table 4.1.

1. Clearly the exterior derivative d is asymptotically conical of rate 1, because

$$\mathrm{d} : C_c^\infty(\Lambda^r T^* X) \to C_c^\infty(\Lambda^{r+1} T^* X)$$

is translation invariant. Now we just conjugate by e^{-rt} and apply Lemma 4.10.

2. To see that d_g^* is asymptotically conical of rate 1, we calculate

$$e^{2t}\mathrm{d}_g^* \xi = \mathrm{d}_h^* \xi + (-1)^{nr+n+1}(n-2r) *_h \big(\mathrm{d}t \wedge (*_h \xi)\big) \tag{4.44}$$

for all r-forms ξ. Here $h = e^{-2t} g$ is the asymptotically cylindrical metric on X. It follows that $e^{2t}\mathrm{d}_g^*$ is asymptotically translation invariant: now we can again conjugate by e^{-rt} and apply Lemma 4.10.

3. Lemma 4.19 then implies that both $\mathrm{d}_g^* + \mathrm{d}$ and Δ_g^r are asymptotically conical, with the required rates. An alternative way of seeing the Laplacian is asymptotically conical of rate 2 is to consider

$$e^{2t}\Delta_g = (e^{2t}\mathrm{d}_g^*)\mathrm{d} + (e^{2t}\mathrm{d}e^{-2t})(e^{2t}\mathrm{d}_g^*)$$

and then appeal to equation (4.44) and Lemma 4.10.

Asymptotically conical operators extend to bounded linear maps between the conical damped Sobolev and Hölder spaces.

Proposition 4.20 *Let $Q : C_c^\infty(E) \to C_c^\infty(F)$ be an asymptotically conical differential operator of order $l \geqslant 1$ and rate $\gamma \in \mathbb{R}^L$. Then Q extends to bounded linear maps*

$$Q : L^p_{k+l,\beta+\gamma}(E) \to L^p_{k,\beta}(F) \tag{4.45}$$

$$Q : C^{k+l,a}_{\beta+\gamma}(E) \to C^{k,a}_{\beta}(F). \tag{4.46}$$

Proof: This is immediate from the definition of asymptotically conical and Proposition 4.11. ∎

It follows from Lemma 4.19 and Proposition 4.20 that if $Q : C_c^\infty(E) \to C_c^\infty(F)$ is an asymptotically conical differential operator of order $l \geqslant 1$ and rate $\gamma \in \mathbb{R}^L$ then the defining identity (2.7) of the formal adjoint Q^* of Q extends to an identity

$$\langle \xi | Q^* \eta \rangle_{L^2(E)} = \langle Q\xi | \eta \rangle_{L^2(F)}$$

valid for all $\xi \in L^p_{l,\beta+\gamma}(E)$ and $\eta \in L^{p'}_{l,-\beta-n}(F)$.

If $Q : C_c^\infty(E) \to C_c^\infty(F)$ is an asymptotically conical operator of rate $\gamma \in \mathbb{R}^L$ then Proposition 4.20 is really saying we have a commutative diagram

$$\begin{array}{ccc} W^p_{k+l,\beta+\gamma}(E) & \xrightarrow{P} & W^p_{k,\beta+\gamma}(F) \\ {\scriptstyle e^{(s-r)t}}\big\uparrow & & \big\uparrow{\scriptstyle e^{(\gamma+s-r)t}} \\ L^p_{k+l,\beta+\gamma}(E) & \xrightarrow[Q]{} & L^p_{k,\beta}(F) \end{array} \tag{4.47}$$

where the vertical maps are topological linear isomorphisms. In the situation of the diagram (4.47) we shall always identify the top row with the bottom row, via the vertical isomorphisms. In particular we then have $\mathrm{Ker}(Q)^p_{k+l,\beta+\gamma} := \mathrm{Ker}(P)^p_{k+l,\beta+\gamma}$, $\mathrm{Im}(Q)^p_{k+l,\beta+\gamma} := \mathrm{Im}(P)^p_{k+l,\beta+\gamma}$, $\mathrm{Coker}(Q)^p_{k+l,\beta+\gamma} := \mathrm{Coker}(P)^p_{k+l,\beta+\gamma}$ and when the horizontal maps of (4.47) are Fredholm

$$\mathrm{Ind}(Q)^p_{k+l,\beta+\gamma} := \mathrm{Ind}(P)^p_{k+l,\beta+\gamma}.$$

From Theorem 4.17 we see

$$\begin{aligned} \mathrm{Ker}(Q)^p_{k+l,\beta+\gamma} &\leqslant \mathrm{Ker}(Q)^p_{k+l,\delta+\gamma} \\ \mathrm{Ker}(Q)^{k+l,a}_{\beta+\gamma} &\leqslant \mathrm{Ker}(Q)^{k+l,a}_{\delta+\gamma} \end{aligned}$$

whenever $\beta \leqslant \delta$. Also Theorem 4.17 gives

$$\mathrm{Ker}(Q)^p_{m+l,\beta+\gamma} \leqslant \mathrm{Ker}(Q)^{k+l,a}_{\beta+\gamma} \leqslant \mathrm{Ker}(Q)^q_{k+l,\delta+\gamma} \tag{4.48}$$

whenever $\beta < \delta$ and $m - \frac{n}{p} \geqslant k + a$.

In the situation of diagram (4.47) we shall say that Q is *uniformly elliptic* if the corresponding asymptotically translation invariant operator P is uniformly elliptic. The following theorem is merely the version of Theorem 4.12 got from identifying the top row of diagram (4.47) with the bottom row.

Theorem 4.21 *Suppose that $Q : C_c^\infty(E) \to C_c^\infty(F)$ is a uniformly elliptic, asymptotically conical operator of order $l \geqslant 1$ and rate $\gamma \in \mathbb{R}^L$. Suppose that $\eta \in L^1_{loc}(F)$ and that $\xi \in L^1_{loc}(E)$ is a weak solution of the equation $P\xi = \eta$.*

1. *If $\xi \in L^p_{0,\beta+\gamma}(E)$ and $\eta \in L^p_{k,\beta}(F)$ then $\xi \in L^p_{k+l,\beta+\gamma}(E)$ with $Q\xi = \eta$ and*

$$\|\xi\|_{L^p_{k+l,\beta+\gamma}(E)} \leqslant C_1\Big(\|Q\xi\|_{L^p_{k,\beta}(F)} + \|\xi\|_{L^p_{0,\beta+\gamma}(E)}\Big)$$

   where the constant $C_1 > 0$ is independent of ξ.

2. *If $\xi \in C^0_{\beta+\gamma}(E)$ and $\eta \in C^{k,a}_\beta(F)$ then $\xi \in C^{k+l,a}_{\beta+\gamma}(E)$ with $Q\xi = \eta$ and*

$$\|\xi\|_{C^{k+l,a}_{\beta+\gamma}(E)} \leqslant C_2\Big(\|Q\xi\|_{C^{k,a}_\beta(F)} + \|\xi\|_{C^0_{\beta+\gamma}(E)}\Big)$$

   where the constant $C_2 > 0$ is independent of ξ.

Proof: This follows immediately from Theorem 4.12. ∎

As before, Theorem 4.21 implies that when Q is uniformly elliptic we have

$$\begin{aligned} \mathrm{Ker}(Q)^p_{k+l,\beta+\gamma} &\leqslant L^p_{\infty,\beta+\gamma}(E) \\ \mathrm{Ker}(Q)^{k+l,a}_{\beta+\gamma} &\leqslant C^\infty_{\beta+\gamma}(E) \end{aligned}$$

so that $\mathrm{Ker}(Q)^p_{k+l,\beta+\gamma}$ is independent of k and $\mathrm{Ker}(Q)^{k+l,a}_{\beta+\gamma}$ is independent of k and a. We therefore write

$$\begin{aligned} \mathrm{Ker}(Q)^p_{\beta+\gamma} &:= \mathrm{Ker}(Q)^p_{k+l,\beta+\gamma} \\ \mathrm{Ker}(Q)_{\beta+\gamma} &:= \mathrm{Ker}(Q)^{k+l,a}_{\beta+\gamma} \end{aligned}$$

whenever Q is uniformly elliptic. With this notation, equation (4.48) becomes

$$\mathrm{Ker}(Q)^p_{\beta+\gamma} \leqslant \mathrm{Ker}(Q)_{\beta+\gamma} \leqslant \mathrm{Ker}(Q)^q_{\delta+\gamma} \tag{4.49}$$

valid for all $\beta < \delta$.

We now have the first main theorem on Fredholm theory.

Theorem 4.22 *Suppose that $Q : C^\infty_c(E) \to C^\infty_c(F)$ is a uniformly elliptic, asymptotically conical operator of rate $\gamma \in \mathbb{R}^L$. Then there exists a subset $\mathcal{D}(Q) \subseteq \mathbb{R}^L$ independent of p, k such that* (4.45) *is Fredholm if and only if $\beta + \gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$. Moreover*

$$\mathcal{D}(Q) = \mathcal{D}(P) = \mathcal{D}(P_\infty)$$

where P corresponds to Q as in diagram (4.47) *and $P \sim P_\infty$.*

Proof: This follows straight from Theorem 4.13. ∎

In the situation of the Theorem 4.22, if $\beta + \gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$ we denote the connected component of $\mathbb{R}^L \setminus \mathcal{D}(Q)$ containing $\beta + \gamma$ by $\left(\mathbb{R}^L \setminus \mathcal{D}(Q)\right)_{\beta+\gamma}$.

Notice from Lemma 4.19 that if Q is asymptotically conical then Q is uniformly elliptic precisely when $e^{-\beta t} Q e^{\beta t}$ is uniformly elliptic, and in this situation

$$\mathcal{D}(e^{-\beta t} Q e^{\beta t}) = \mathcal{D}(Q) - \beta.$$

Furthermore, Q is uniformly elliptic precisely when Q^* is uniformly elliptic, and in this situation

$$\mathcal{D}(Q^*) = \gamma - n - \mathcal{D}(Q)$$

so that $\beta + \gamma \in \mathcal{D}(Q)$ precisely when $-\beta - n \in \mathcal{D}(Q^*)$.

Another useful corollary:

Corollary 4.23 *Suppose that $Q : C^\infty_c(E) \to C^\infty_c(F)$ is a uniformly elliptic, asymptotically conical differential operator of order $l \geqslant 1$ and rate $\gamma \in \mathbb{R}^L$. Then both*

$$\begin{aligned} \mathrm{Ker}(Q)^p_{\beta+\gamma} &\leqslant L^p_{k+l,\beta+\gamma}(E) \\ \mathrm{Ker}(Q)_{\beta+\gamma} &\leqslant C^{k+l,a}_{\beta+\gamma}(E) \end{aligned}$$

are finite-dimensional.

Proof: This follows straight from the corresponding Corollary 4.14 and the commutative diagram (4.47). ∎

The second main theorem on Fredholm theory is now:

Theorem 4.24 *Suppose that $Q : C_c^\infty(E) \to C_c^\infty(F)$ is a uniformly elliptic, asymptotically conical operator of rate $\gamma \in \mathbb{R}^L$. If $\beta+\gamma, \delta+\gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$ and $\delta+\gamma \leqslant \beta+\gamma$ then*

$$\mathrm{Ind}(Q)^p_{k+l,\beta+\gamma} - \mathrm{Ind}(Q)^p_{k+l,\delta+\gamma} = N(\beta,\delta).$$

Proof: This is immediate from Theorem 4.15. ∎

4.3.3 The images of asymptotically conical operators

In Section 3.1 we stated that the useful characterisation Theorem 3.7 of the image of an elliptic operator on a compact manifold has an extension to the non-compact case. We now give that extension.

Theorem 4.25 *Let $Q : C_c^\infty(E) \to C_c^\infty(F)$ be a uniformly elliptic, asymptotically conical operator of order $l \geqslant 1$ and rate $\gamma \in \mathbb{R}^L$. Suppose that $\beta+\gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$ so that the bounded linear map*

$$Q : L^p_{k+l,\beta+\gamma}(E) \to L^p_{k,\beta}(F) \tag{4.50}$$

is Fredholm. Then the image of the map (4.50) *is given by*

$$\mathrm{Im}(Q)^p_{k+l,\beta+\gamma} = \Big\{\eta \in L^p_{k,\beta}(F) : \langle \eta | h \rangle_{L^2(F)} = 0 \text{ for all } h \in \mathrm{Ker}(Q^*)^{p'}_{-\beta-n}\Big\}. \tag{4.51}$$

Proof: First note that

$$\mathrm{Im}(Q)^p_{k+l,\beta+\gamma} \leqslant \Big\{\eta \in L^p_{k,\beta}(F) : \langle \eta | h \rangle_{L^2(F)} = 0 \text{ for all } h \in \mathrm{Ker}(Q^*)^{p'}_{-\beta-n}\Big\}$$

follows immediately from integration by parts.

Consider now the case $k=0$. For the purposes of this proof, denote the Banach space adjoint of the map (4.50) by $Q' : L^p_{0,\beta}(F)^* \to L^p_{l,\beta+\gamma}(E)^*$ to distinguish from the formal adjoint Q^* of Q. If we identify $L^p_{0,\beta}(F)^* \cong L^{p'}_{0,-\beta-n}(F)$ as in (4.39) then it is a consequence of Theorem 4.21 that $\mathrm{Ker}\, Q' = \mathrm{Ker}\, Q^*$ in $L^p_{0,\beta}(F)^*$. To see this, note that if $\eta \in L^{p'}_{0,-\beta-n}(F)$ with $\langle Q\phi|\eta\rangle_{L^2(F)} = 0$ for all $\phi \in L^p_{l,\beta+\gamma}(E)$ then the equation $Q^*\eta = 0$ holds weakly.

Now take $\eta \in L^p_{0,\beta}(F)$ such that $\langle \eta|h\rangle_{L^2(F)} = 0$ for all $h \in \mathrm{Ker}(Q^*)^{p'}_{-\beta-n}$. Then we have that $\eta \in L^p_{0,\beta}(F)$ lies in

$$(\mathrm{Ker}\, Q^*)^\circ = (\mathrm{Ker}\, Q')^\circ = \mathrm{Im}\, Q$$

as required. Here we are using the fact that (4.50) has closed image, together with Proposition 2.2. We have now proved the result in the case $k=0$.

Now suppose that $k \geqslant 1$ and that $\eta \in L^p_{k,\beta}(F)$ with $\langle \eta|h\rangle_{L^2(F)} = 0$ for all $h \in \mathrm{Ker}(Q^*)^{p'}_{-\beta-n}$. A consequence of the case $k=0$ proof is that there exists $\xi \in L^p_{l,\beta+\gamma}(E)$ such that $Q\xi = \eta$. But then Theorem 4.21 implies $\xi \in L^p_{k+l,\beta+\gamma}(E)$ and we are done.

∎

Corollary 4.26 *Let $Q : C_c^\infty(E) \to C_c^\infty(F)$ be a uniformly elliptic, asymptotically conical operator of order $l \geqslant 1$ and rate $\gamma \in \mathbb{R}^L$. Suppose that $\beta+\gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$. Then we may write*

$$L^p_{k,\beta}(F) = \mathrm{Im}(Q)^p_{k+l,\beta+\gamma} \oplus V$$

where $V \leqslant L^p_{k,\beta}(F)$ is a subspace of finite dimension $\dim V = \dim \mathrm{Ker}(Q^*)^{p'}_{-\beta-n}$, and in particular, $\dim \mathrm{Coker}(Q)^p_{k+l,\beta+\gamma} = \dim \mathrm{Ker}(Q^*)^{p'}_{-\beta-n}$. If $\mathrm{Ker}(Q^*)^{p'}_{-\beta-n} \leqslant L^p_{k,\beta}(F)$ then we may take V to be equal to $\mathrm{Ker}(Q^*)^{p'}_{-\beta-n}$.

Proof: The first part is straightforward: using Theorem 4.25 we have

$$\dim \mathrm{Coker}(Q)^p_{k+l,\beta+\gamma} := \mathrm{codim}\,\mathrm{Im}(Q)^p_{k+l,\beta+\gamma} = \dim \mathrm{Ker}(Q^*)^{p'}_{-\beta-n}$$

as required, and the existence of the subspace V is trivial to establish.

For the second part, suppose $\mathrm{Ker}(Q^*)^{p'}_{-\beta-n} \leqslant L^p_{k,\beta}(F)$ and pick an $L^2(F)$-orthonormal basis $\{e_1, \dots, e_K\}$ for $\mathrm{Ker}(Q^*)^{p'}_{-\beta-n}$. Given $\eta \in L^p_{k,\beta}(F)$ we may observe

$$\eta - \sum_{k=1}^{K} \langle \eta | e_k \rangle_{L^2(F)} e_k \in \mathrm{Im}(Q)^p_{k+l,\beta+\gamma}$$

and we are done.

∎

It follows that if $Q : C^\infty_c(E) \to C^\infty_c(F)$ is a uniformly elliptic, asymptotically conical operator of order $l \geqslant 1$ and rate $\gamma \in \mathbb{R}^L$ and $\beta, \delta \in \mathbb{R}^L$ with $\beta + \gamma, \delta + \gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$ and $\delta \leqslant \beta$ then

$$\dim \mathrm{Coker}(Q)^p_{k+l,\beta+\gamma} \leqslant \dim \mathrm{Coker}(Q)^p_{k+l,\delta+\gamma}. \tag{4.52}$$

But now if $\beta + \gamma$ and $\delta + \gamma$ lie in the same connected component of $\mathbb{R}^L \setminus \mathcal{D}(Q)$ then by Theorem 4.24 we have $\mathrm{Ind}(Q)^p_{k+l,\beta+\gamma} = \mathrm{Ind}(Q)^p_{k+l,\delta+\gamma}$ so that by equation (4.49) and equation (4.52)

$$\begin{aligned} \dim \mathrm{Ker}(Q)^p_{\beta+\gamma} &= \dim \mathrm{Ker}(Q)^p_{\delta+\gamma} \\ \dim \mathrm{Coker}(Q)^p_{k+l,\beta+\gamma} &= \dim \mathrm{Coker}(Q)^p_{k+l,\delta+\gamma} \end{aligned}$$

hold. Since $\mathrm{Ker}(Q)^p_{\delta+\gamma} \leqslant \mathrm{Ker}(Q)^p_{\beta+\gamma}$ we deduce that

$$\mathrm{Ker}(Q)^p_{\delta+\gamma} = \mathrm{Ker}(Q)^p_{\beta+\gamma}. \tag{4.53}$$

Corollary 4.27 *Let $Q : C^\infty_c(E) \to C^\infty_c(F)$ be a uniformly elliptic, asymptotically conical operator of order $l \geqslant 1$ and rate $\gamma \in \mathbb{R}^L$. If $\beta \in \mathbb{R}^L$ with $\beta + \gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$ then*

$$\mathrm{Ker}(Q)_{\beta+\gamma} = \mathrm{Ker}(Q)^p_{\beta+\gamma}.$$

Proof: Choose some small $\varepsilon > 0$ so that $\delta + \gamma := \beta + \gamma + \varepsilon$ lies in $\mathbb{R}^L \setminus \mathcal{D}(Q)$. The result now follows from the inclusions (4.49) and equation (4.53).

∎

It follows from Corollary 4.26 and Corollary 4.27 that $\dim \mathrm{Coker}(Q)^p_{k+l,\beta+\gamma}$ and hence $\mathrm{Ind}(Q)^p_{k+l,\beta+\gamma}$ are independent of p and k, provided that $\beta+\gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$. This is because $-\beta-n \in \mathcal{D}(Q^*)$ precisely when $\beta + \gamma \in \mathcal{D}(Q)$. Similar remarks hold for asymptotically translation invariant operators P.

Chapter 5

Infinitesimal deformations of AC special Lagrangians

In this chapter we give applications of the material of Chapter 4.

5.1 A study of Δ_g^0

5.1.1 Analytic properties of Δ_g^0

In this section we apply some of the results of Chapter 4 to the Laplacian Δ_g^0 of an asymptotically conical metric g on X, acting on functions. The resulting theory will give many of the analytic results we shall need later.

Recall that $\Delta_g^0 : C_c^\infty(X) \to C_c^\infty(X)$ is an asymptotically conical operator of order 2 and rate 2. Our first task is to determine the set of $\beta + 2 \in \mathbb{R}^L$ such that

$$\Delta_g^0 : L^p_{k+2,\beta+2}(X) \to L^p_{k,\beta}(X) \tag{5.1}$$

fails to be Fredholm. In other words, we are computing the subset $\mathcal{D}(\Delta_g^0) \subseteq \mathbb{R}^L$ and then (5.1) fails to be Fredholm precisely when $\beta + 2 \in \mathcal{D}(\Delta_g^0)$. Recall from Theorem 4.22 that this subset is of the form $\mathcal{D}(\Delta_g^0) = \mathcal{D}(P) = \mathcal{D}(P_\infty)$ where

$$\mathcal{D}(P_\infty) = \big(\mathcal{D}(P_\infty, 1) \times \mathbb{R}^{L-1}\big) \cup \big(\mathbb{R} \times \mathcal{D}(P_\infty, 2) \times \mathbb{R}^{L-2}\big) \cup \cdots \cup \big(\mathbb{R}^{L-1} \times \mathcal{D}(P_\infty, L)\big)$$

and $P = e^{2t}\Delta_g^0$ is the asymptotically translation invariant operator corresponding to Δ_g^0, and P_∞ is the translation invariant operator with $P \sim P_\infty$. Here we have $P_\infty = e^{2t}\Delta_{\tilde{g}}^0$.

If $u : X_\infty \to \mathbb{R}$ is a twice differentiable function then a brief calculation in local coordinates shows that

$$e^{2t}\Delta_{\tilde{g}}^0 u = \Delta_{\tilde{h}}^0 u - (n-2)(\mathrm{d}t, \mathrm{d}u)_{\tilde{h}} = \Delta_{g_\Sigma}^0 u - \frac{\partial^2 u}{\partial t^2} - (n-2)\frac{\partial u}{\partial t}$$

where $\Delta_{g_\Sigma}^0$ is the Laplacian of the metric g_Σ on the manifold Σ, acting on functions. Referring back to Section 4.2.2 and in particular the discussion after Theorem 4.7, we replace each $\frac{\partial}{\partial t}$ in the operator P_∞ by $w \in \mathbb{C}$, and then for each $1 \leqslant j \leqslant L$ we have an operator

$$\begin{aligned} P_\infty(w) : W^p_{k+2}(\Sigma_j \otimes \mathbb{C}) &\to W^p_k(\Sigma_j \otimes \mathbb{C}) \\ \xi &\mapsto \Delta_{g_\Sigma}^0 \xi - w(w+n-2)\xi. \end{aligned} \tag{5.2}$$

We now have:

Lemma 5.1 *If $w \in \mathbb{C}$ then (5.2) is not an isomorphism precisely when*

$$w(w+n-2) \in \mathrm{Spec}(\Sigma_j, g_\Sigma, 0) \subseteq [0, \infty)$$

and the set of such $w \in \mathbb{C}$ is a subset of $\mathbb{R}$.

Proof: Firstly, suppose that $w \in \mathbb{C}$ is a complex number such that $w(w+n-2) = \mu$ is real and non-negative. Then since

$$\left(w + \frac{n-2}{2}\right)^2 = \left(\frac{n-2}{2}\right)^2 + \mu$$

we see immediately that w must in fact be real.

Now suppose that $w \in \mathbb{C}$. Obviously $w(w+n-2) \in \operatorname{Spec}(\Sigma_j, g_\Sigma, 0)$ precisely when (5.2) fails to be injective. If (5.2) is not surjective then by standard Hodge Theory the adjoint operator

$$P_\infty(w)^* = (\Delta^0_{g_\Sigma})^* - \overline{w(w+n-2)} = P_\infty(\overline{w})$$

fails to be injective. Thus $\overline{w}(\overline{w}+n-2) \in \operatorname{Spec}(\Sigma_j, g, 0)$, and $\overline{w}$ is real, completing the proof. ∎

Following Section 4.2.2 we define

$$\mathcal{C}(P_\infty, j) := \big\{ w \in \mathbb{C} : \text{(5.2) is not an isomorphism} \big\}$$

and then $\mathcal{D}(P_\infty, j) := \big\{ \operatorname{Re} w : w \in \mathcal{C}(P_\infty, j) \big\}$. It follows from Lemma 5.1 we have that

$$\mathcal{D}(P_\infty, j) = \left\{ -\left(\frac{n-2}{2}\right) \pm \left(\left(\frac{n-2}{2}\right)^2 + \mu_{j,i} \right)^{\frac{1}{2}} : i \geqslant 0 \right\}$$

where $0 = \mu_{j,0} < \mu_{j,1} < \mu_{j,2} < \dots$ are the points of $\operatorname{Spec}(\Sigma_j, g_\Sigma, 0)$. In other words, $\beta_j + 2 \in \mathcal{D}(P_\infty, j)$ precisely when $(\beta_j+2)(\beta_j+n) = \mu_{j,i}$ for some $i \geqslant 0$.

As in the introduction, we put $\mu_j = \mu_{j,1}$ the first positive element of $\operatorname{Spec}(\Sigma_j, g_\Sigma, 0)$, and then define $\lambda_j > 0$ to be such that $\lambda_j(\lambda_j + n - 2) = \mu_j$. We finally put $\lambda = (\lambda_1, \dots, \lambda_L)$. Figure 5.1 shows the quadratic equation $(\beta_j+2)(\beta_j+n) = \mu$. The horizontal axis corresponds to the growth rates $\beta_j + 2$ of the harmonic functions on X, and the vertical axis corresponds to the elements of $\operatorname{Spec}(\Sigma_j, g_\Sigma, 0)$. The first positive eigenvalue μ_j of the Laplacian on Σ_j is marked, together with the growth rates $\lambda_j > 0$ and $2 - n - \lambda_j$. By Theorem 4.22 the map $\Delta^0_g : L^p_{k+2,\beta+2}(X) \to L^p_{k,\beta}(X)$ fails to be Fredholm precisely when $\beta + 2 \in \mathcal{D}(P_\infty)$, and these points can now just be read off the quadratic equation given in Figure 5.1.

The operator Δ^0_g is an example of a second order elliptic operator acting on functions. We can exploit these features of Δ^0_g to obtain strong control over the kernels $\operatorname{Ker}(\Delta^0_g)^p_{\beta+2}$ and $\operatorname{Ker}(\Delta^0_g)_{\beta+2}$. The required ingredient is the *Maximum Principle*, which does not hold for the operators Δ^r_g when $r \geqslant 1$.

Theorem 5.2 (Maximum Principle) *Let $G \subseteq \mathbb{R}^n$ be a domain. Define the operator $L : C^2(G) \to C^0(G)$ by*

$$Lu := a^{ij}\partial_i\partial_j u + b^i \partial_i u$$

where $a^{ij} = a^{ji}, b^i : G \to \mathbb{R}$ are functions. For each $x \in G$ let $m(x)$ be the least eigenvalue of $\big(a^{ij}(x)\big)$, so that we have a function $m : G \to \mathbb{R}$. Assume that

1. *$m > 0$ on G*
2. *The function $\frac{|b^i|}{m} : G \to \mathbb{R}$ is bounded for each $1 \leqslant i \leqslant n$.*

If $u \in C^2(G) \cap C^0(\overline{G})$ with $Lu = 0$ in G then the maximum and the minimum of u on $\overline{G}$ are both attained on $\partial G := \overline{G} \setminus G$.

Theorem 5.2 is proved in the book [18, Theorem 3.1] of Gilbarg and Trudinger. We have an immediate corollary for the operator Δ^0_g.

Corollary 5.3 *If $\beta + 2 < 0$ then* $\operatorname{Ker}(\Delta^0_g)^p_{\beta+2} = \operatorname{Ker}(\Delta^0_g)_{\beta+2} = \{0\}$.

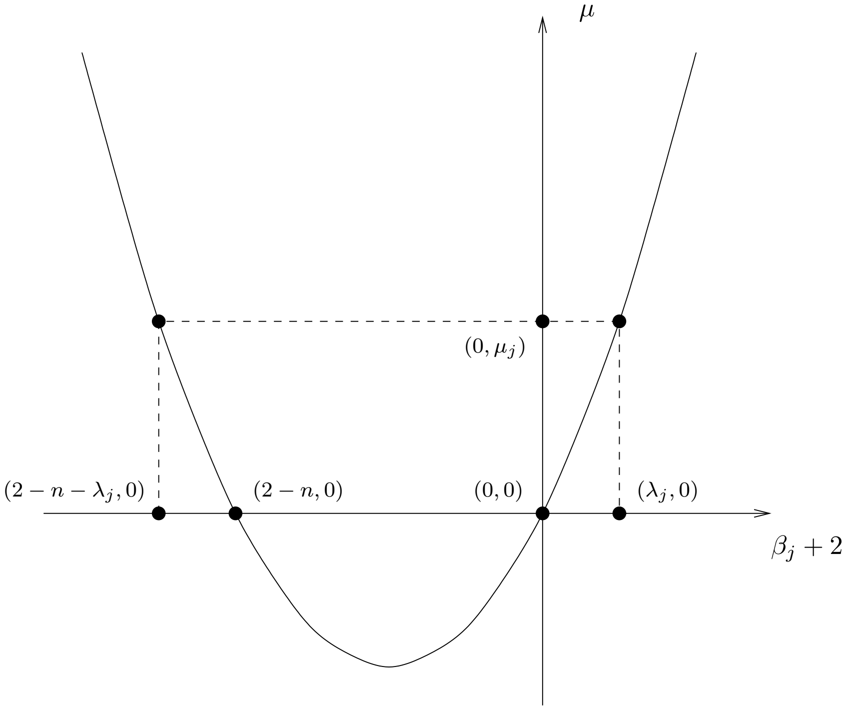

Figure 5.1: The quadratic equation $(\beta_j + 2)(\beta_j + n) = \mu$

Proof: From the inclusions (4.49) we need only show that $\mathrm{Ker}(\Delta_g^0)_{\beta+2} = \{0\}$. So suppose for a contradiction that $u \in \mathrm{Ker}(\Delta_g^0)_{\beta+2}$ and there exists $x \in X$ with $u(x) \neq 0$. Since $u \in C^0_{\beta+2}(X)$ we have that $|u|$ is bounded by some multiple $C > 0$ of the function $e^{(\beta+2)t}$ on X. Also we may choose a large $S \geqslant 0$ so that $x \in X_S$ and

$$e^{(\beta_j+2)(S+T)}C \quad < \quad |u(x)| \tag{5.3}$$

for each $1 \leqslant j \leqslant L$. It follows that the function $u : X_S \to \mathbb{R}$ cannot attain both its maximum and minimum on ∂X_S, which contradicts the Maximum Principle 5.2.

∎

Corollary 5.4 *If $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$ and $\beta + 2 > 2 - n$ then the map $\Delta_g^0 : L^p_{k+2,\beta+2}(X) \to L^p_{k,\beta}(X)$ is surjective, and therefore $\mathrm{d}_g^* : L^p_{k+1,\beta+1}(T^*X) \to L^p_{k,\beta}(X)$ is surjective also.*

Proof: If $\beta + 2 > 2 - n$ then $-\beta - n < 0$ and the assertion now follows from Theorem 4.25 and Corollary 5.3.

∎

We now have a fairly explicit description of the kernel and cokernel of the map

$$\Delta_g^0 : L^p_{k+2,\beta+2}(X) \to L^p_{k,\beta}(X) \tag{5.4}$$

for various $\beta + 2 \in \mathbb{R}^L$. Firstly, the map (5.4) always has finite-dimensional kernel, and is injective for $\beta + 2 < 0$. Therefore the map (5.4) has finite-dimensional cokernel precisely when it is Fredholm, which is precisely when $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$: a set of points we have a very explicit description of. Furthermore, the map (5.4) is surjective when $\beta + 2 > 2 - n$ and $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$.

When $2 - n < \beta + 2 < 0$ the map (5.4) is an isomorphism, and so

$$\mathrm{Ind}(\Delta_g^0)^p_{k+2,\beta+2} = 0. \tag{5.5}$$

Exactly the same arguments as above work for the conical metric $\tilde{g}$ so that $\operatorname{Ind}(\Delta^0_{\tilde{g}})^p_{k+2,\beta+2} = 0$ for $2-n < \beta+2 < 0$, and therefore by equation (4.30) we have

$$\operatorname{Ind}(\Delta^0_g)^p_{k+2,\beta+2} = \operatorname{Ind}(\Delta^0_{\tilde{g}})^p_{k+2,\beta+2} \tag{5.6}$$

for all $\beta+2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$.

We now turn to evaluating the indices (5.6) for all $\beta+2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$. To do this we use the "jumping" formula (4.29) and equation (5.5). Take $1 \leqslant j \leqslant L$ and $w \in \mathcal{C}(P_\infty, j)$, where $P_\infty = e^{2t}\Delta^0_{\tilde{g}}$. Then w is a real number such that

$$w(w+n-2) =: \mu \in \operatorname{Spec}(\Sigma_j, g_\Sigma, 0).$$

We are interested in the dimension $d(j,w)$ of the space of solutions of the equation $P_\infty u = 0$ which have the form

$$u(t,\sigma) = e^{wt}p(t,\sigma)$$

for some polynomial $p(t,\sigma)$ in t with coefficients in $C^\infty(\Sigma_j)$. We now appeal to the following lemma.

Lemma 5.5 *Let $m \geqslant 0$ and*

$$p(t,\sigma) := t^m f_m + \cdots + t f_1 + f_0 \tag{5.7}$$

be a polynomial in t with coefficients $f_k \in C^\infty(\Sigma_j)$, where $f_m \neq 0$. Then

$$P_\infty(e^{wt}p) = 0 \tag{5.8}$$

precisely when $m = 0$ and $(\Delta^0_{g_\Sigma} - \mu)f_0 = 0$.

Proof: Note that

$$P_\infty(e^{wt}p) = \left(\Delta^0_{g_\Sigma} - (n-2)\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2}\right)(e^{wt}p) = e^{wt}\left(\left(\Delta^0_{g_\Sigma} - \mu\right)p - \big(2w+(n-2)\big)\frac{\partial p}{\partial t} - \frac{\partial^2 p}{\partial t^2}\right).$$

Therefore equation (5.8) is equivalent to

$$\left(\Delta^0_{g_\Sigma} - \mu\right)p - \big(2w+(n-2)\big)\frac{\partial p}{\partial t} - \frac{\partial^2 p}{\partial t^2} = 0. \tag{5.9}$$

Clearly when $m = 0$ and $(\Delta^0_{g_\Sigma} - \mu)f_0 = 0$ we see that p satisfies equation (5.9). For the converse, suppose that p is as given in equation (5.7) and satisfies (5.9). Then comparing coefficients of t^m in (5.9) gives $(\Delta^0_{g_\Sigma} - \mu)f_m = 0$. Suppose for a contradiction that $m \geqslant 1$. Then comparing coefficients of t^{m-1} in (5.9) gives

$$(\Delta^0_{g_\Sigma} - \mu)f_{m-1} = m\big(2w+(n-2)\big)f_m. \tag{5.10}$$

Now the right hand side of (5.10) is non-zero, but lies in the μ-eigenspace of $\Delta^0_{g_\Sigma}$. The operator $(\Delta^0_{g_\Sigma} - \mu)$ preserves the splitting of $C^\infty(\Sigma_j)$ into eigenspaces of $\Delta^0_{g_\Sigma}$ and annihilates the μ-eigenspace. This is a contradiction, as required.

∎

It follows from Lemma 5.5 that

$$d(j,w) = \dim\big(\operatorname{Ker}(\Delta^0_{g_\Sigma} - \mu) \cap C^\infty(\Sigma_j)\big) \tag{5.11}$$

whenever $w \in \mathcal{C}(P_\infty, j)$ with $w(w+n-2) = \mu \in \operatorname{Spec}(\Sigma_j, g_\Sigma, 0)$. In equation (5.11) we are thinking of $C^\infty(\Sigma_j) \leqslant C^\infty(\Sigma)$ as the functions which vanish on Σ_k for $k \neq j$. We can now easily obtain expressions for $\dim\operatorname{Ker}(\Delta^0_g)_{\beta+2} = \dim\operatorname{Ker}(\Delta^0_g)^p_{\beta+2}$ and $\dim\operatorname{Coker}(\Delta^0_g)^p_{k+2,\beta+2}$ for all $\beta+2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$. For example, if $\beta+2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$ with $\beta+2 > 0$ then

$$\dim\operatorname{Ker}(\Delta^0_g)^p_{\beta+2} = L + \chi(\beta+2) \tag{5.12}$$

where for $\beta + 2 \in \mathbb{R}^L$ with $\beta + 2 \geqslant 0$ we define

$$\chi(\beta+2) := \sum_{j=1}^{L} \sum \left\{ \dim\bigl(\mathrm{Ker}(\Delta^0_{g_\Sigma} - \mu) \cap C^\infty(\Sigma_j)\bigr) : \begin{array}{l} 0 < \mu \leqslant (\beta_j + 2)(\beta_j + n) \\ \mu \in \mathrm{Spec}(\Sigma_j, \Delta^0_{g_\Sigma}, 0) \end{array} \right\} \tag{5.13}$$

which is an analytic piece of data got from the Riemannian manifold (Σ, g_Σ). We also define, for future reference

$$\hat{\chi}(\beta+2) := \sum_{j=1}^{L} \sum \left\{ \dim\bigl(\mathrm{Ker}(\Delta^0_{g_\Sigma} - \mu) \cap C^\infty(\Sigma_j)\bigr) : \begin{array}{l} 0 < \mu < (\beta_j + 2)(\beta_j + n) \\ \mu \in \mathrm{Spec}(\Sigma_j, \Delta^0_{g_\Sigma}, 0) \end{array} \right\} \tag{5.14}$$

for all $\beta + 2 \in \mathbb{R}^L$ with $\beta + 2 \geqslant 0$.

5.1.2 Cohomology and homology

For each $t > T$ define a submanifold $i_t : \Sigma \to X$ by $i_t(\sigma) := (t, \sigma)$ for each $\sigma \in \Sigma$. We shall say that a form ξ defined over X is *translation invariant* if there exists an $S \geqslant 0$ such that

$$\begin{aligned} i_s^* \xi &= i_t^* \xi \\ i_s^* \left(\iota(\tfrac{\partial}{\partial t})\xi\right) &= i_t^* \left(\iota(\tfrac{\partial}{\partial t})\xi\right) \end{aligned}$$

for all $s, t > S + T$. We shall say that a translation invariant form ξ defined over X is a *lift* if there exists an $S \geqslant 0$ such that

$$i_t^* \left(\iota(\tfrac{\partial}{\partial t})\xi\right) = 0$$

for all $t > S + T$. A form ξ on X is a lift precisely when there exists a form η on Σ such that $\xi = \pi^* \eta$ over $X \setminus X_S$ for some $S \geqslant 0$. We shall say that a function $f : X \to \mathbb{R}$ is *constant on the ends of* X if there exists $S \geqslant 0$ and $c = (c_1, \ldots, c_L) \in \mathbb{R}^L$ such that $f(t, \sigma) = c_j$ for all $t > S + T$ and $\sigma \in \Sigma_j$. The notation we use in this situation is $f_c := f$. So given $c \in \mathbb{R}^L$ we have $f_c \in C^\infty(X)$ well-defined up to elements of $C^\infty_c(X)$.

On our manifold X we have the usual de Rham cohomology groups $H^r(X)$ and the compactly supported de Rham cohomology groups $H^r_c(X)$. Moreover, if X is oriented there is a pairing $H^r_c(X) \times H^{n-r}(X) \to \mathbb{R}$ defined

$$[\xi] \cdot [\eta] := \int_X \xi \wedge \eta \tag{5.15}$$

which induces isomorphisms

$$H^r_c(X) \cong H^{n-r}(X)^* \tag{5.16}$$
$$H^r_c(X)^* \cong H^{n-r}(X). \tag{5.17}$$

We also have the usual real singular homology groups $H_r(X)$, and a pairing $H_{n-r}(X) \times H^{n-r}(X) \to \mathbb{R}$ defined

$$[\tau] \cdot [\eta] := \int_\tau \xi \tag{5.18}$$

which induces isomorphisms

$$\begin{aligned} H_{n-r}(X) &\cong H^{n-r}(X)^* \\ H_{n-r}(X)^* &\cong H^{n-r}(X). \end{aligned} \tag{5.19}$$

Now the isomorphisms (5.16) and (5.19) imply

$$H_{n-r}(X) \cong H^r_c(X). \tag{5.20}$$

In order to obtain homological objects which are isomorphic to the usual de Rham cohomology groups $H^r(X)$ we need to consider X as the interior of a compact manifold $\overline{X}$ with boundary $\partial\overline{X} \subseteq \overline{X}$. This is certainly no problem because we can pick a homeomorphism $(T,\infty)\times\Sigma \cong (T,T+1)\times\Sigma$ so that $\partial\overline{X} \cong \Sigma$. Then we have the relative homology groups $H_r(\overline{X},\partial\overline{X})$ and a pairing $H^r_c(X)\times H_r(\overline{X},\partial\overline{X}) \to \mathbb{R}$ defined

$$[\xi]\cdot[\tau] := \int_\tau \xi \tag{5.21}$$

which induces isomorphisms

$$\begin{array}{rcl} H^r_c(X) & \cong & H_r(\overline{X},\partial\overline{X})^* \\ H^r_c(X)^* & \cong & H_r(\overline{X},\partial\overline{X}) \end{array} \tag{5.22}$$

which combined with (5.17) show that

$$H_r(\overline{X},\partial\overline{X}) \cong H^{n-r}(X). \tag{5.23}$$

Note further that $H_r(\overline{X}) \cong H_r(X)$ for all $r \geqslant 0$.

We now describe how to build a long exact sequence of cohomology groups which shall be useful later. First of all, we have the natural map

$$\begin{array}{rcl} \phi_r : H^r_c(X) & \to & H^r(X) \\ {[\xi]} & \mapsto & [\xi]. \end{array} \tag{5.24}$$

Also, given $t > T$ the embedding $i_t : \Sigma \to X$ induces a pull-back homomorphism

$$\begin{array}{rcl} p_r : H^r(X) & \to & H^r(\Sigma) \\ {[\xi]} & \mapsto & [i_t^*\xi]. \end{array} \tag{5.25}$$

Note that by Stokes' Theorem the map (5.25) is independent of our choice of $t > T$. We also have a boundary map

$$\partial_r : H^r(\Sigma) \to H^{r+1}_c(X) \tag{5.26}$$

which we define as follows. Fix any $\rho \in C^\infty(X)$ such that

$$\begin{array}{rcll} \rho(x) & = & 0 & \text{for all } x \in X_0 \\ \rho(x) & = & 1 & \text{for all } x \in X\setminus X_1. \end{array}$$

Given $\xi \in C^\infty(\Lambda^r T^*\Sigma)$ with $\mathrm{d}\xi = 0$ we lift to a translation invariant form $\pi^*\xi \in C^\infty(\Lambda^r T^* X_\infty)$, which in turn extends to a form $\rho\pi^*\xi \in C^\infty(\Lambda^r T^*X)$. We now put $\partial_r[\xi] := [\mathrm{d}(\rho\pi^*\xi)]$ and this gives us a well-defined map (5.26) as required.

Proposition 5.6 *The sequence*

$$\cdots \longrightarrow H^r_c(X) \xrightarrow{\phi_r} H^r(X) \xrightarrow{p_r} H^r(\Sigma) \xrightarrow{\partial_r} H^{r+1}_c(X) \longrightarrow \cdots \tag{5.27}$$

is exact.

Proof: Under the Poincaré Duality isomorphisms (5.20) and (5.23) defined above the sequence (5.27) of cohomology groups is isomorphic to the usual long exact sequence for relative homology

$$\cdots \longrightarrow H_{n-r}(\overline{X}) \longrightarrow H_{n-r}(\overline{X},\partial\overline{X}) \longrightarrow H_{n-r-1}(\partial\overline{X}) \longrightarrow H_{n-r-1}(\overline{X}) \longrightarrow \cdots. \tag{5.28}$$

∎

Lemma 5.7 *In the long exact sequence* (5.27) *we have* $\operatorname{Ker}\phi_1 = \{[\mathrm{d}f_c] : c \in \mathbb{R}^L\}$, *and furthermore* $\dim\operatorname{Ker}\phi_1 = L-1$.

Proof: Obviously
$$\{[\mathrm{d}f_c] : c \in \mathbb{R}^L\} \leqslant \operatorname{Ker}\phi_1.$$
Now suppose that $f \in C^\infty(X)$ with $\mathrm{d}f \in C_c^\infty(T^*X)$. Then f is locally constant off some compact subset of X, and it follows that f must be constant on the ends of X. Hence
$$\operatorname{Ker}\phi_1 \leqslant \{[\mathrm{d}f_c] : c \in \mathbb{R}^L\},$$
as required, and it is easy to show this vector space has dimension $L-1$.

An alternative approach is to use the exactness of the sequence
$$0 \longrightarrow H^0_c(X) \xrightarrow{\phi_0} H^0(X) \xrightarrow{p_0} H^0(\Sigma) \xrightarrow{\partial_0} H^1_c(X) \xrightarrow{\phi_1} H^1(X) \longrightarrow \cdots$$
and the fact that $H^0_c(X) = 0$, $H^0(X) \cong \mathbb{R}$, $H^0(\Sigma) \cong \mathbb{R}^L$.

∎

If we denote the standard basis of $\mathbb{R}^L$ by $\{e_1, \dots, e_L\}$ then clearly
$$\operatorname{Ker}\phi_1 = \operatorname{Span}\{[\mathrm{d}f_{e_1}], \dots, [\mathrm{d}f_{e_L}]\}$$
and moreover given $(c_1, \dots, c_L) \in \mathbb{R}^L$ we have
$$c_1[\mathrm{d}f_{e_1}] + \cdots + c_L[\mathrm{d}f_{e_L}] = 0$$
in $H^1_c(X)$ precisely when $c_1 = \cdots = c_L$. This is because the map p_0 sends $c \in \mathbb{R}$ to $(c, \dots, c) \in \mathbb{R}^L$ and the map ∂_0 sends $c = (c_1, \dots, c_L) \in \mathbb{R}^L \cong H^0(\Sigma)$ to the element $[\mathrm{d}f_c] \in H^1_c(X)$.

Proposition 5.8 *Let $0 \leqslant r \leqslant n$ and $[\eta] \in H^r(X)$ be a cohomology class, so that $\eta \in C^\infty(\Lambda^r T^*X)$ with $\mathrm{d}\eta = 0$. Then there exists $\xi \in C^\infty(\Lambda^r T^*X)$ which is a lift such that $\mathrm{d}\xi = 0$ and $[\eta] = [\xi]$.*

Proof: We give two proofs. The first method relies on the exactness of the sequence (5.27) and the second method gives an explicit construction for ξ.

Method 1

Since $[\eta] \in H^r(X)$ we have $p_r[\eta] = [i_t^*\eta] \in H^r(\Sigma)$ for any $t > T$. Define $\theta := i_t^*\eta$ so that $\theta \in C^\infty(\Lambda^r T^*\Sigma)$ with $\mathrm{d}\theta = 0$. By the exactness of (5.27) we have $[\theta] \in \operatorname{Ker}\partial_r$ and there exists $\hat{\theta} \in C_c^\infty(\Lambda^r T^*X)$ with
$$\mathrm{d}(\rho\pi^*\theta) = \mathrm{d}\hat{\theta}$$
as $\partial_r[\theta] = 0$ in $H_c^{r+1}(X)$. Consider now the cohomology class $[\rho\pi^*\theta - \hat{\theta}] \in H^r(X)$. We have
$$i_t^*(\rho\pi^*\theta - \hat{\theta}) = i_t^*(\pi^*\theta) = \theta = i_t^*\eta.$$
It follows by the exactness of (5.27) that
$$[\rho\pi^*\theta - \hat{\theta} - \eta] \in \operatorname{Ker} p_r = \operatorname{Im}\phi_r \subseteq H^r(X)$$
and there exists $h \in C^\infty(\Lambda^{r-1}T^*X)$ and $\phi \in C_c^\infty(\Lambda^r T^*X)$ with $\mathrm{d}\phi = 0$ such that
$$\phi + \mathrm{d}h = \rho\pi^*\theta - \hat{\theta} - \eta,$$
and then $[\eta] = [\rho\pi^*\theta - \hat{\theta} - \phi]$ in $H^r(X)$ with $\rho\pi^*\theta - \hat{\theta} - \phi$ a lift as required.

Method 2

First of all, work on $X_\infty = (T,\infty)\times\Sigma$. Then given $t > T$ we have the embedding

$$\begin{array}{rcl} i_t : \Sigma & \to & X_\infty \\ \sigma & \mapsto & (t,\sigma). \end{array}$$

Suppose that $\eta \in C^\infty(\Lambda^r T^* X_\infty)$ with $\mathrm{d}\eta = 0$. Write $\eta = \eta_0 + \mathrm{d}t\wedge\eta_1$ where $\eta_0 \in C^\infty(\Lambda^r T^* X_\infty)$ and $\eta_1 \in C^\infty(\Lambda^{r-1} T^* X_\infty)$ with $\iota(\frac{\partial}{\partial t})\eta_0 = \iota(\frac{\partial}{\partial t})\eta_1 = 0$, so that η_0, η_1 have no $\mathrm{d}t$ component. Since $\mathrm{d}\eta = 0$ we have $\mathrm{d}\eta_0 - \mathrm{d}t\wedge \mathrm{d}\eta_1 = 0$ and then

$$\iota(\tfrac{\partial}{\partial t})(\mathrm{d}\eta_0) = \iota(\tfrac{\partial}{\partial t})(\mathrm{d}t\wedge\mathrm{d}\eta_1) = \mathrm{d}\eta_1$$

and also $i_t^*\mathrm{d}\eta_0 = i_t^*\mathrm{d}\eta = 0$ for all $t > T$. Now given $t > T$ define

$$\sigma_t = \int_0^t (i_s^*\eta_1)\mathrm{d}s \in C^\infty(\Lambda^{r-1}T^*\Sigma),$$

and then define further $\sigma \in C^\infty(\Lambda^{r-1}T^*X_\infty)$ by the equations

$$\begin{array}{rcll} \iota(\frac{\partial}{\partial t})\sigma & = & 0 & \\ i_t^*\sigma & = & \sigma_t & \quad \text{for all } t > T. \end{array}$$

We now show that $\xi := \eta - \mathrm{d}\sigma$ is a lift, by looking at the component parts. Firstly

$$\begin{array}{rcl} \frac{\partial}{\partial t}\big(i_t^*(\eta - \mathrm{d}\sigma)\big) & = & \frac{\partial}{\partial t}\big(i_t^*\eta_0 - \mathrm{d}\sigma_t\big) \\ & = & \frac{\partial}{\partial t}\left(i_t^*\eta_0 - \int_0^t i_s^*(\mathrm{d}\eta_1)\mathrm{d}s\right) \\ & = & \frac{\partial}{\partial t}\left(i_t^*\eta_0\right) - i_t^*\mathrm{d}\eta_1 \\ & = & \frac{\partial}{\partial t}\left(i_t^*\eta_0\right) - i_t^*\left(\iota(\frac{\partial}{\partial t})\mathrm{d}\eta_0\right) \\ & = & i_t^*\left(\mathcal{L}_{\frac{\partial}{\partial t}}\eta_0 - \iota(\frac{\partial}{\partial t})\mathrm{d}\eta_0\right) \\ & = & i_t^*\left(\mathrm{d}\left(\iota(\frac{\partial}{\partial t})\eta_0\right)\right) \\ & = & 0 \end{array}$$

and therefore $i_s^*(\eta - \mathrm{d}\sigma) = i_t^*(\eta - \mathrm{d}\sigma)$ for all $s, t > T$. Secondly, we have

$$\begin{array}{rcl} i_t^*\left(\iota(\frac{\partial}{\partial t})(\eta - \mathrm{d}\sigma)\right) & = & i_t^*\left(\eta_1 - \iota(\frac{\partial}{\partial t})\mathrm{d}\sigma\right) \\ & = & i_t^*\eta_1 - \left(i_t^*\left(\mathcal{L}_{\frac{\partial}{\partial t}}\sigma - \mathrm{d}\left(\iota(\frac{\partial}{\partial t})\sigma\right)\right)\right) \\ & = & i_t^*\eta_1 - i_t^*\left(\mathcal{L}_{\frac{\partial}{\partial t}}\sigma\right) \\ & = & i_t^*\eta_1 - \frac{\partial}{\partial s}\left(i_s^*\sigma\right)\Big|_{s=t} \\ & = & i_t^*\eta_1 - \frac{\partial}{\partial s}\left(\int_0^s i_u^*\eta_1\mathrm{d}u\right)\Big|_{s=t} \\ & = & 0 \end{array}$$

and hence ξ is a lift, as required.

Now suppose we are working on the whole manifold X. Given $\eta \in C^\infty(\Lambda^r T^*X)$ with $\mathrm{d}\eta = 0$ put $\hat\eta := \eta|_{X_\infty}$. Then from the above there exists $\hat\sigma \in C^\infty(\Lambda^{r-1}T^*X_\infty)$ such that $\hat\xi := \hat\eta - \mathrm{d}\hat\sigma$ is a lift on X_∞. Now define $\sigma := \rho\hat\sigma \in C^\infty(\Lambda^{r-1}T^*X)$ and then

$$\mathrm{d}\sigma = \mathrm{d}\rho\wedge\hat\sigma + \rho\mathrm{d}\hat\sigma$$

so that $\eta - \mathrm{d}\sigma = (\eta - \rho\mathrm{d}\hat{\sigma}) - (\mathrm{d}\rho \wedge \hat{\sigma})$, which is a lift.

∎

Corollary 5.9 *If $0 \leqslant r \leqslant n$ then any cohomology class in $H^r(X)$ can be represented by some closed form $\xi \in C^\infty_{-r}(\Lambda^r T^* X)$, and moreover ξ can be chosen to be a lift.*

Proof: Using Proposition 5.8 we may pick a lift $\xi \in C^\infty(\Lambda^r T^* X)$ representing the cohomology class. This form has the required decay properties.

∎

5.1.3 Calculations with functions

Lemma 5.10 *Let $\xi \in C^{k+1,a}_{\beta+1}(T^*X)$ and $f \in C^1(X)$ with $\mathrm{d}f = \xi$.*

1. *If $\beta + 2 > 0$ then $f \in C^{k+2,a}_{\beta+2}(X)$.*
2. *If $\beta + 2 = 0$ then $f \in C^{k+2,a}_{\gamma+2}(X)$ for all $\gamma + 2 > 0$.*
3. *If $\beta+2 < 0$ then there exists $f_c \in C^\infty(X)$ constant on the ends of X such that $f - f_c \in C^{k+2,a}_{\beta+2}(X)$.*

Obviously the condition $\mathrm{d}f = \xi$ determines f uniquely up to constants. Also when $\beta + 2 < 0$ the function f tends to the constants $c_j \in \mathbb{R}$ on the jth end of X, where $c = (c_1, \ldots, c_L) \in \mathbb{R}^L$.

Proof: We begin by considering the first part. Using the elliptic estimates of Theorem 4.21 applied to the operator $Q = \mathrm{d}^*_g + \mathrm{d}$ on $C^\infty_c(\Lambda^* T^* X)$ we observe that we need only show $f \in C^0_{\beta+2}(X)$.

Fix $1 \leqslant j \leqslant L$ and some $\tilde{\sigma} \in \Sigma_j$, $S > 0$. Then for any $t > T + S$ and $\sigma \in \Sigma_j$ we have by Stokes' Theorem

$$f(t,\sigma) - f(T+S,\tilde{\sigma}) = \int_{\gamma_1} \xi + \int_{\gamma_2} \xi$$

where γ_1 is the straight line path in $(T,\infty) \times \Sigma_j$ going from $(T+S,\tilde{\sigma})$ to $(t,\tilde{\sigma})$ and γ_2 is a geodesic of minimum length in $\{t\} \times \Sigma_j \subseteq (T+S,\infty) \times \Sigma_j$ going from $(t,\tilde{\sigma})$ to (t,σ). Suppose that in coordinates (s,τ) over $(T,\infty) \times \Sigma$ we have $\xi(s,\tau) = a(s,\tau)\mathrm{d}s + b(s,\tau)\mathrm{d}\tau$. Then since $\xi \in C^0_{\beta+1}(T^*X)$ we have $a, b \in C^0_{\beta+2}(X)$, and furthermore

$$\begin{aligned} \int_{\gamma_1} \xi &= \int_{T+S}^t a(s,\tilde{\sigma})\mathrm{d}s \\ \int_{\gamma_2} \xi &= \int_{\gamma_3} b(t,\tau)\mathrm{d}\tau \end{aligned}$$

where γ_3 is the geodesic in Σ_j with $\gamma_2 = \{t\} \times \gamma_3$. Working on the jth end $(T,\infty) \times \Sigma_j$ of X we have

$$\begin{aligned} \left| \int_{\gamma_1} \xi \right| &\leqslant \int_{T+S}^t |a(s,\hat{\sigma})|\mathrm{d}s \\ &\leqslant \|a\|_{C^0_{\beta+2}(X)} \int_{T+S}^t e^{(\beta_j+2)s}\mathrm{d}s \\ &= \frac{\|a\|_{C^0_{\beta+2}(X)}}{\beta_j + 2} \left(e^{(\beta_j+2)t} - e^{(\beta_j+2)(T+S)} \right) \\ &\leqslant \frac{\|a\|_{C^0_{\beta+2}(X)}}{\beta_j + 2} e^{(\beta_j+2)t} \end{aligned}$$

and

$$\left| \int_{\gamma_2} \xi \right| \leqslant \mathrm{diam}(\Sigma_j, g_\Sigma) \|b\|_{C^0_{\beta+2}(X)} e^{(\beta_j+2)t}$$

so that

$$\begin{aligned}|f(t,\sigma)| &\leqslant |f(T+S,\tilde{\sigma})| + |f(t,\sigma) - f(T+S,\tilde{\sigma})| \\ &\leqslant |f(T+S,\tilde{\sigma})| + \left(\frac{\|a\|_{C^0_{\beta+2}(X)}}{\beta_j+2} + \operatorname{diam}(\Sigma_j, g_\Sigma)\|b\|_{C^0_{\beta+2}(X)}\right)e^{(\beta_j+2)t}.\end{aligned}$$

It follows that $f \in C^0_{\beta+2}(X)$, as required.

For the second part we need only show $f \in C^0_{\gamma+2}(X)$ for all $\gamma + 2 > 0$. The proof is now very similar to part 1. Keeping the same notation we compute

$$\left|\int_{\gamma_1} \xi\right| \leqslant (t-T-S)\|a\|_{C^0_{\beta+2}(X)}$$

from which it follows

$$|f(t,\sigma)| \leqslant |f(T+S,\tilde{\sigma})| + (t-T-S)\|a\|_{C^0_{\beta+2}(X)} + \operatorname{diam}(\Sigma_j, g_\Sigma)\|b\|_{C^0_{\beta+2}(X)}$$

and we are done.

In case 3 we need only show that there exists $f_c \in C^\infty(X)$ constant on the ends of X such that $f - f_c \in C^0_{\beta+2}(X)$, since then $(\mathrm{d}^*_g + \mathrm{d})(f - f_c) = \xi - \mathrm{d}f_c \in C^{k+1,a}_{\beta+1}(T^*X)$. With notation as in part 1 we have

$$f(t,\sigma) - f(T+S,\tilde{\sigma}) - \int_{T+S}^\infty a(s,\tilde{\sigma})\mathrm{d}s \quad = \quad -\int_t^\infty a(s,\tilde{\sigma})\mathrm{d}s + \int_{\gamma_3} b(t,\sigma)\mathrm{d}\sigma$$

and then

$$\begin{aligned}\left|\int_t^\infty a(t,\sigma)\mathrm{d}t\right| &\leqslant \int_t^\infty |a(s,\sigma)|\mathrm{d}s \\ &\leqslant \|a\|_{C^0_{\beta+2}(X)}\int_t^\infty e^{(\beta_j+2)s}\mathrm{d}s \\ &= -\frac{\|a\|_{C^0_{\beta+2}(X)}}{\beta_j+2}e^{(\beta_j+2)t}.\end{aligned}$$

It follows that

$$\left|f(t,\sigma) - f(T+S,\tilde{\sigma}) - \int_{T+S}^\infty a(s,\tilde{\sigma})\mathrm{d}s\right| \leqslant \left(\operatorname{diam}(\Sigma_j, g_\Sigma)\|b\|_{C^0_{\beta+2}(X)} - \frac{\|a\|_{C^0_{\beta+2}(X)}}{\beta_j+2}\right)e^{(\beta_j+2)t}$$

and we are done, with f_c constant on the ends of X chosen so that

$$c_j := f(T+S,\tilde{\sigma}) + \int_{T+S}^\infty a(s,\tilde{\sigma})\mathrm{d}s. \tag{5.29}$$

A straightforward application of Stokes' Theorem shows the right hand side of equation (5.29) is independent of $S > 0$ and a similar application together with a convergence argument shows the right hand side of equation (5.29) is independent of $\tilde{\sigma} \in \Sigma_j$, as we expect.

∎

In the second part of Lemma 5.10 we can never hope to have $f \in C^{k+1,a}_0(X)$, as the example $f(t,\sigma) := \log t$ shows. For here we have

$$\xi := \mathrm{d}f = \frac{\mathrm{d}t}{t}$$

which lies in $C^\infty_{-1}(X)$, but $f \notin C^0_0(X)$.

Lemma 5.11 *Let $h \in C^\infty(X)$ be harmonic and $f_c \in C^\infty(X)$ be constant on the ends of X. Then*

$$\int_X (\Delta_g^0 f_c) h \mathrm{d}V_g = \sum_{j=1}^{L} c_j [\Sigma_j] \cdot [*_g \mathrm{d}h]$$

where in the right hand side we use the pairing (5.18).

Proof: Choosing some large $S \geqslant 0$ and noting $\Delta_g^0 f_c$ is compactly supported we have

$$\int_X (\Delta_g^0 f_c) h \mathrm{d}V_g = -\int_{X_S} h(\mathrm{d} *_g \mathrm{d}f_c)$$

from the definition of Δ_g^0. Then Stokes' Theorem and the fact that

$$\mathrm{d}(h *_g \mathrm{d}f_c) = h(\mathrm{d} *_g \mathrm{d}f_c) + \mathrm{d}h \wedge *_g \mathrm{d}f_c$$

gives

$$\int_X (\Delta_g^0 f_c) h \mathrm{d}V_g = \int_{X_S} \mathrm{d}h \wedge *_g \mathrm{d}f_c - \int_{\partial X_S} h *_g \mathrm{d}f_c = \int_{X_S} \mathrm{d}f_c \wedge *_g \mathrm{d}h = \int_{\partial X_S} f_c *_g \mathrm{d}h,$$

again using Stokes' Theorem and the fact that $\mathrm{d}(f_c *_g \mathrm{d}h) = \mathrm{d}f_c \wedge *_g \mathrm{d}h$. We now have

$$\int_X (\Delta_g^0 f_c) h \mathrm{d}V_g = \sum_{j=1}^{L} c_j \int_{\Sigma_j} *_g \mathrm{d}h = \sum_{j=1}^{L} c_j [\Sigma_j] \cdot [*_g \mathrm{d}h]$$

as required. ∎

Recall the definition of the L-tuple $\lambda = (\lambda_1, \ldots, \lambda_L)$ in Section 5.1.1 above. If $\gamma + 2 \in \mathbb{R}^L$ with

$$\begin{array}{ll} 0 < \gamma_k + 2 < \lambda_j & \text{if } k = j \\ 2 - n < \gamma_k + 2 < 0 & \text{if } k \neq j \end{array}$$

for each $1 \leqslant k \leqslant L$, then a consequence of Theorem 4.24, Corollary 5.4 and Lemma 5.5 is that

$$\dim \operatorname{Ker}(\Delta_g^0)_{\gamma+2} = 1.$$

Let h_j be a non-zero element of $\operatorname{Ker}(\Delta_g^0)_{\gamma+2}$. Then by equation (4.53) we have $h_j \in \operatorname{Ker}(\Delta_g^0)_{\beta+2}$ for all $\beta + 2 \in \left(\mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)\right)_{\gamma+2}$, and as $h_j \notin \operatorname{Ker}(\Delta_g^0)_{\delta+2} = \{0\}$ for any $\delta + 2 < 0$, it follows that $\{h_1, \ldots, h_L\}$ is a linearly independent set.

Now suppose that $\gamma + 2 \in \mathbb{R}^L$ with $0 < \gamma + 2 < \lambda$. Then again using Theorem 4.24, Corollary 5.4 and Lemma 5.5, we see that

$$\dim \operatorname{Ker}(\Delta_g^0)_{\gamma+2} = L$$

so that $\{h_1, \ldots, h_L\}$ is necessarily a basis for $\operatorname{Ker}(\Delta_g^0)_{\gamma+2}$. Since $1 \in \operatorname{Ker}(\Delta_g^0)_{\gamma+2}$ there exist $a_1, \ldots, a_L \in \mathbb{R}$ such that

$$a_1 h_1 + \cdots + a_L h_L = 1 \tag{5.30}$$

and each $a_j \neq 0$, as otherwise the left hand side of equation (5.30) tends to zero on some end of X. We now rescale the h_j so that

$$h_1 + \cdots + h_L = 1 \tag{5.31}$$

and fix this preferred basis $\{h_1, \ldots, h_L\}$ for the rest of Chapter 5.

For each $1 \leqslant j \leqslant L$ define $h_j^1 \in C^\infty(X)$ to be a function constant on the ends of X, constantly equal to 1 on the jth end of X and 0 on the other ends of X. So in the notation of Section 5.1.2, $h_j = f_{e_j}$. Then from equation (5.31) we see that

$$h_j - h_j^1 \in C^\infty_{\gamma+2}(X) \tag{5.32}$$

for all $\gamma+2>2-n$ and $1\leqslant j\leqslant L$. Note that a consequence of equation (5.32) is that $h_j\in C^\infty_0(X)$ for each $1\leqslant j\leqslant L$, and therefore

$$\mathrm{Ker}(\Delta^0_g)_0=\mathrm{Ker}(\Delta^0_g)_{\gamma+2}=\mathrm{Span}\{h_1,\dots,h_L\} \tag{5.33}$$

for all $0\leqslant\gamma+2<\lambda$.

Note that $\mathrm{d}h_j\in C^\infty_{\gamma+1}(T^*X)$ for all $\gamma+2>2-n$ and $1\leqslant j\leqslant L$. Also the vector space

$$\mathrm{Span}\{\mathrm{d}h_1,\dots,\mathrm{d}h_L\}\leqslant C^\infty_{\gamma+1}(T^*X) \tag{5.34}$$

has dimension $L-1$, because the linear map

$$\mathrm{d}:\mathrm{Ker}(\Delta^0_g)_0\to\mathrm{d}\,\mathrm{Ker}(\Delta^0_g)_0 \tag{5.35}$$

is surjective and has a 1-dimensional kernel $\{c_1h_1+\cdots+c_Lh_L:c_1=\cdots=c_L\}$. Note that although

$$\mathrm{d}\,\mathrm{Ker}(\Delta^0_g)_{\gamma+2}=\mathrm{Span}\{\mathrm{d}h_1,\dots,\mathrm{d}h_L\} \tag{5.36}$$

for all $0\leqslant\gamma+2<\lambda$ it is not the case (unless $L=1$) that

$$\mathrm{d}\,\mathrm{Ker}(\Delta^0_g)_{\gamma+2}=\mathrm{Span}\{\mathrm{d}h_1,\dots,\mathrm{d}h_L\} \tag{5.37}$$

for *any* $2-n<\gamma+2<0$, because for such $\gamma+2$ the left hand side of equation (5.37) is zero, whereas the right hand side has dimension $L-1$. All we assert is that the $\mathrm{d}h_j$ lie in $C^\infty_{\gamma+1}(T^*X)$ for all $2-n<\gamma+2<0$.

Lemma 5.12 *For each $1\leqslant j,k\leqslant L$ we have*

$$\int_X \mathrm{d}h_j\wedge *_g\mathrm{d}h_k=[\Sigma_j]\cdot[*_g\mathrm{d}h_k]$$

where in the right hand side we use the pairing (5.18).

Proof: This follows from Lemma 5.11 and an integration by parts once we note that, for all $S\geqslant 0$:

$$\int_{X_S}\mathrm{d}(h_j-h^1_j)\wedge *_g\mathrm{d}h_k=\int_{X_S}\mathrm{d}\big((h_j-h^1_j)*_g\,\mathrm{d}h_k\big)=\int_{\partial X_S}(h_j-h^1_j)*_g\,\mathrm{d}h_k\to 0$$

since $(h_j-h^1_j)\in C^0_{\gamma+2}(X)$, $\mathrm{d}h_k\in C^0_{\gamma+1}(T^*X)$ for all $\gamma+2>2-n$, and furthermore $\mathrm{Vol}(\partial X_S,g)=O(e^{(n-1)S})$.

∎

It follows from Lemma 5.12 that the vector space $\mathrm{Ker}(\Delta^0_g)_0=\mathrm{Span}\{h_1,\dots,h_L\}$ is endowed with a positive, semi-definite bilinear form $\langle\ ,\ \rangle$ defined by

$$\langle h,h\rangle:=\int_X|\mathrm{d}h|^2_g\mathrm{d}V_g=\sum_{j,k=1}^{L}c_jc_k[\Sigma_j]\cdot[*_g\mathrm{d}h_k]$$

for all $h=c_1h_1+\cdots+c_Lh_L\in\mathrm{Ker}(\Delta^0_g)_0$. Since $\mathrm{d}h=0$ precisely when $c_1=\cdots=c_L$ we deduce that $\langle\ ,\ \rangle$ becomes the usual L^2-inner product on $\mathrm{Span}\{\mathrm{d}h_1,\dots,\mathrm{d}h_L\}$ after we push down using the map (5.35). If we let A be the symmetric $L\times L$ real matrix with entries

$$a_{jk}:=[\Sigma_j]\cdot[*_g\mathrm{d}h_k] \tag{5.38}$$

then we have

$$\begin{aligned}\mathrm{Ker}\,A&=\big\{(c_1,\dots,c_L)\in\mathbb{R}^L:c_1=\cdots=c_L\big\}\\ \mathrm{Im}\,A&=\big\{(a_1,\dots,a_L)\in\mathbb{R}^L:a_1+\cdots+a_L=0\big\},\end{aligned}$$

since A is self-adjoint. These facts will be useful later. In the sequel we shall always let $(a_{jk})\subseteq\mathbb{R}$ denote the real numbers defined as in equation (5.38) above.

We conclude this section with some integration by parts formulae. For each $1\leqslant j\leqslant L$ we define $h^2_j\in C^\infty(X)$ to be any function equal to $e^{(2-n)t}$ on the jth end of X and 0 on the other ends of X.

Lemma 5.13 *For each $1 \leqslant j, k \leqslant L$ we have*

$$\int_X (\Delta_g^0 h_j^1) h_k \mathrm{d}V_g = [\Sigma_j] \cdot [*_g \mathrm{d}h_k] \tag{5.39}$$

$$\int_X (\Delta_g^0 h_j^2) h_k \mathrm{d}V_g = \begin{cases} (n-2)\operatorname{Vol}(\Sigma_j, g_\Sigma) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{5.40}$$

where in the right hand side of equation (5.39) *we use the pairing* (5.18)*.*

Proof: Equation (5.39) follows straight from Lemma 5.11 as each h_j^1 is constant on the ends of X. For equation (5.40) we compute, for $S \geqslant 0$:

$$\begin{aligned} \int_{X_S} (\Delta_g^0 h_j^2) h_k \mathrm{d}V_g &= -\int_{X_S} h_k (\mathrm{d} *_g \mathrm{d}h_j^2) \\ &= -\int_{X_S} \big(\mathrm{d}(h_k *_g \mathrm{d}h_j^2) - \mathrm{d}h_k \wedge *_g \mathrm{d}h_j^2\big) \\ &= \int_{X_S} \mathrm{d}h_k \wedge *_g \mathrm{d}h_j^2 - \int_{\partial X_S} h_k (*_g \mathrm{d}h_j^2). \end{aligned} \tag{5.41}$$

The first term of (5.41) is

$$\int_{X_S} \mathrm{d}h_k \wedge *_g \mathrm{d}h_j^2 = \int_{X_S} \mathrm{d}h_j^2 \wedge *_g \mathrm{d}h_k = \int_{X_S} \mathrm{d}(h_j^2 *_g \mathrm{d}h_k) = \int_{\partial X_S} h_j^2 *_g \mathrm{d}h_k$$

and this tends to 0 as $S \to \infty$, because $h_j^2 \in C^0_{2-n}(X)$, $\mathrm{d}h_k \in C^0_{\gamma+1}(T^*X)$ for all $\gamma + 2 > 2 - n$, and $\operatorname{Vol}(\partial X_S, g) = O(e^{(n-1)S})$.

The second term of (5.41) is

$$\int_{\partial X_S} h_k (*_g \mathrm{d}h_j^2) = \int_{\partial X_S} h_k^1 (*_g \mathrm{d}h_j^2) + \int_{\partial X_S} (h_k - h_k^1) *_g \mathrm{d}h_j^2$$

and since $\mathrm{d}h_j^2 \in C^0_{1-n}(T^*X)$, $h_k - h_k^1 \in C^\infty_{\gamma+2}(X)$ for all $\gamma + 2 > 2 - n$, we deduce that

$$\int_{\partial X_S} h_k (*_g \mathrm{d}h_j^2) \to \begin{cases} (2-n)\operatorname{Vol}(\Sigma_j, g_\Sigma) & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

as $S \to \infty$. The Dominated Convergence Theorem now implies that the integral (5.40) exists and is as given.

∎

5.1.4 Strongly asymptotically conical metrics

In order to derive further properties of the harmonic functions $\operatorname{Ker}(\Delta_g^0)_{\beta+2}$ we shall have to assume stronger decay conditions on our asymptotically conical metric g on X. Let $\alpha \in \mathbb{R}^L$ with $\alpha < 0$. We shall say that the metric g on X is *strongly asymptotically conical with rate* α if

$$\sup_{\{t\} \times U_\nu} \left|\rho_\nu \partial^\lambda \big(g_{ij} - \tilde{g}_{ij}\big)\right| = O(e^{(\alpha+2)t}) \tag{5.42}$$

for each $1 \leqslant \nu \leqslant N$, $1 \leqslant i, j \leqslant n$ and $|\lambda| \geqslant 0$. In coordinate-free terms, equation (5.42) is the same as requiring

$$\sup_{\{t\} \times \Sigma} \left|\nabla^j_{\tilde{g}} (g - \tilde{g})\right|_{\tilde{g}} = O(e^{(\alpha - j)t})$$

for each $j \geqslant 0$. Of course, any strongly asymptotically conical metric is asymptotically conical. For the rest of this chapter (except for Section 5.2.4) we assume that g is a strongly asymptotically conical metric on X which has rate $\alpha < 0$.

Lemma 5.14 1. *The operator* $\mathrm{d}_g^* - \mathrm{d}_{\tilde{g}}^* : C_c^\infty(\Lambda^r T^*X) \to C_c^\infty(\Lambda^{r-1}T^*X)$ *is an asymptotically conical operator of rate* $1-\alpha$.

2. *The operator* $(\mathrm{d}_g^* + \mathrm{d}) - (\mathrm{d}_{\tilde{g}}^* + \mathrm{d}) : C_c^\infty(\Lambda T^*X) \to C_c^\infty(\Lambda T^*X)$ *is an asymptotically conical operator of rate* $1-\alpha$.

3. *The operator* $\Delta_g^r - \Delta_{\tilde{g}}^r : C_c^\infty(\Lambda^r T^*X) \to C_c^\infty(\Lambda^r T^*X)$ *is an asymptotically conical operator of rate* $2-\alpha$.

Proof: The first assertion can be established via a local coordinate calculation. The second and third assertions then follow from the equations

$$\begin{aligned}(\mathrm{d}_g^* + \mathrm{d}) - (\mathrm{d}_{\tilde{g}}^* + \mathrm{d}) &= \mathrm{d}_g^* - \mathrm{d}_{\tilde{g}}^* \\ \Delta_g^r - \Delta_{\tilde{g}}^r &= \mathrm{d}(\mathrm{d}_g^* - \mathrm{d}_{\tilde{g}}^*) + (\mathrm{d}_g^* - \mathrm{d}_{\tilde{g}}^*)\mathrm{d}\end{aligned}$$

and Lemma 4.19.

∎

One can consider (5.32) as the first in a series of equations giving an *asymptotic expansion* of the harmonic functions $h_1, \ldots, h_L$, in terms of functions which are harmonic on the *exactly* conical Riemannian manifold $(X_\infty, \tilde{g})$. Then the first such approximation to h_j is given in equation (5.32) by h_j^1. When the metric g is strongly asymptotically conical we can give the second order terms in the asymptotic expansion for the h_j. For the purposes of the following lemma, if $\beta, \delta \in \mathbb{R}^L$ then we denote the L-tuple with jth entry $\max\{\beta_j, \delta_j\}$ by $\max\{\beta, \delta\}$.

Lemma 5.15 *For each* $1 \leqslant j, k \leqslant L$ *define*

$$b_{jk} := \frac{-a_{jk}}{(n-2)\operatorname{Vol}(\Sigma_k, g_\Sigma)} = \frac{-[\Sigma_j] \cdot [*_g \mathrm{d}h_k]}{(n-2)\operatorname{Vol}(\Sigma_k, g_\Sigma)}. \tag{5.43}$$

Then the functions

$$f_j := h_j - h_j^1 - \sum_{k=1}^L b_{jk} h_k^2 \tag{5.44}$$

lie in $C_{\beta+2}^\infty(X)$ *for all* $\max\{2-n+\alpha, 2-n-\lambda\} < \beta + 2 < 2-n$ *and* $1 \leqslant j \leqslant L$.

Proof: Let $1 \leqslant j \leqslant L$. For arbitrary $(b_{jk}) \subseteq \mathbb{R}$ consider

$$\Delta_g^0 \left(h_j - h_j^1 - \sum_{k=1}^L b_{jk} h_k^2 \right) = \left(\Delta_{\tilde{g}}^0 - \Delta_g^0\right) \left(\sum_{k=1}^L b_{jk} h_k^2 \right) \tag{5.45}$$

modulo elements of $C_c^\infty(X)$. It follows from Lemma 5.14 that

$$\Delta_g^0 \left(h_j - h_j^1 - \sum_{k=1}^L b_{jk} h_k^2 \right) \in C_{-n+\alpha}^\infty(X). \tag{5.46}$$

Now for each $1 \leqslant l \leqslant L$ we have:

$$\begin{aligned}\int_X \Delta_g^0 \left(h_j - h_j^1 - \sum_{k=1}^L b_{jk} h_k^2 \right) h_l \mathrm{d}V_g &= -\int_X (\Delta_g^0 h_j^1) h_l \mathrm{d}V_g - \sum_{k=1}^L b_{jk} \int_X (\Delta_g^0 h_k^2) h_l \mathrm{d}V_g \\ &= -[\Sigma_j] \cdot [*_g \mathrm{d}h_l] - \sum_{k=1}^L b_{jk}(n-2)\operatorname{Vol}(\Sigma_k, g_\Sigma)\delta_{kl} \\ &= -[\Sigma_j] \cdot [*_g \mathrm{d}h_l] - b_{jl}(n-2)\operatorname{Vol}(\Sigma_l, g_\Sigma)\end{aligned}$$

where we use Lemma 5.13. Now put

$$b_{jk} := \frac{-[\Sigma_j]\cdot[*_g \mathrm{d}h_k]}{(n-2)\operatorname{Vol}(\Sigma_k, g_\Sigma)} \tag{5.47}$$

and pick any $\max\{2-n+\alpha, 2-n-\lambda\} < \beta+2 < 2-n$. Then from equation (5.46) we have

$$\Delta_g^0\left(h_j - h_j^1 - \sum_{k=1}^L b_{jk}h_k^2\right) \in L^p_{\infty,\beta}(X) \tag{5.48}$$

and $2-n-\lambda < \beta+2 < 2-n$ so by Theorem 4.25 there exists $f_j \in L^p_{k+2,\beta+2}(X)$ such that

$$\Delta_g^0 f_j = \Delta_g^0\left(h_j - h_j^1 - \sum_{k=1}^L b_{jk}h_k^2\right).$$

Now the elliptic regularity Theorem 4.21 together with equation (5.48) tells us that

$$f_j \in L^p_{\infty,\beta+2}(X) \subseteq C^\infty_{\beta+2}(X)$$

and the Maximum Principle 5.2 then gives

$$f_j = h_j - h_j^1 - \sum_{k=1}^L b_{jk}h_k^2$$

so we are done.

■

In the sequel we shall always let $(b_{jk}) \subseteq \mathbb{R}$ denote the real numbers defined as in equation (5.43) above. Also, $(f_j) \subseteq C^\infty_{\beta+2}(X)$ will always denote the functions defined in equation (5.44) above. Using analogues of Lemma 5.13 for $\tilde{g}$-harmonic functions with even stronger decay we could compute further terms of the asymptotic expansion for the h_j in terms of the $\tilde{g}$-harmonic functions on X_∞. A particular consequence of (5.44) is the fact that

$$h_j - h_j^1 \in C^\infty_{2-n}(X) \tag{5.49}$$

with even stronger decay when the b_{jk} given in equation (5.43) vanish. It follows from (5.49) that $\mathrm{d}h_j \in C^\infty_{1-n}(T^*X)$ for each $1 \leqslant j \leqslant L$.

We can use Lemma 5.15 to deduce further useful information about the Laplacian Δ_g^0 acting on functions with low growth rate.

Corollary 5.16 *If* $\max\{2-n+\alpha, 2-n-\lambda\} < \beta+2 < 2-n$ *and* $\xi \in L^p_{k+1,\beta+1}(T^*X)$ *then:*

1. *There exists* $f \in C^\infty_{2-n}(X)$ *and* $\tilde{f} \in L^p_{k+2,\beta+2}(X)$ *such that*

$$\Delta_g^0(\tilde{f} + f) = \mathrm{d}_g^*\xi. \tag{5.50}$$

2. *There exists* $f_b \in C^\infty(X)$ *constant on the ends of* X *and* $F \in L^p_{k+2,\beta+2}(X)$ *such that*

$$\Delta_g^0(F + f_b) = \mathrm{d}_g^*\xi. \tag{5.51}$$

Proof: First of all, for arbitrary $(a_j) \subseteq \mathbb{R}$ and $1 \leqslant l \leqslant L$ consider:

$$\left\langle \mathrm{d}_g^*\xi - \sum_{j=1}^L a_j\Delta_g^0 h_j^2 \middle| h_l\right\rangle_{L^2(X)} = \langle \mathrm{d}_g^*\xi | h_l\rangle_{L^2(X)} - a_l(n-2)\operatorname{Vol}(\Sigma_l, g_\Sigma)$$

by Lemma 5.13. Therefore defining

$$a_j := \frac{\langle \mathrm{d}_g^*\xi | h_j \rangle_{L^2(X)}}{(n-2)\operatorname{Vol}(\Sigma_j, g_\Sigma)}$$

and using the fact that

$$\mathrm{d}_g^*\xi - \sum_{j=1}^{L} a_j \Delta_g^0 h_j^2 \in C^\infty_{-n+\alpha}(X) \subseteq L^p_{k,\beta}(X),$$

where $2-n-\lambda < \beta+2 < 2-n$, we deduce from Theorem 4.25 that there exists $\tilde{f} \in L^p_{k+2,\beta+2}(X)$ such that

$$\Delta_g^0 \left(\tilde{f} + \sum_{j=1}^{L} a_j h_j^2 \right) = \mathrm{d}_g^*\xi$$

and this proves the first assertion, putting $f := \sum_{j=1}^{L} a_j h_j^2$.

For the second assertion, observe that since

$$\sum_{j=1}^{L} (n-2)\operatorname{Vol}(\Sigma_j, g_\Sigma) a_j = \sum_{j=1}^{L} \langle \mathrm{d}_g^*\xi | h_j \rangle_{L^2(X)} = \langle \mathrm{d}_g^*\xi | 1 \rangle_{L^2(X)} = 0$$

there exists $(b_j) \subseteq \mathbb{R}$ such that

$$\sum_{j=1}^{L} a_{jk} b_j = (n-2)\operatorname{Vol}(\Sigma_k, g_\Sigma) a_k$$

for each $1 \leqslant k \leqslant L$, so that $\sum_{j=1}^{L} b_{jk} b_j = -a_k$ for each $1 \leqslant k \leqslant L$. Recall that the real numbers $(a_{jk}) \subseteq \mathbb{R}$ are as defined in Section 5.1.3. Now we have

$$\begin{aligned}
\mathrm{d}_g^*\xi &= \Delta_g^0 \left(\tilde{f} + \sum_{j=1}^{L} a_j h_j^2 \right) \\
&= \Delta_g^0 \left(\tilde{f} + \sum_{j=1}^{L} \left(a_j h_j^2 + b_j h_j \right) \right) \\
&= \Delta_g^0 \left(\tilde{f} + \sum_{j=1}^{L} \left(a_j h_j^2 + b_j \left(f_j + h_j^1 + \sum_{k=1}^{L} b_{jk} h_k^2 \right) \right) \right) \\
&= \Delta_g^0 \left(\tilde{f} + \sum_{j=1}^{L} b_j (f_j + h_j^1) \right)
\end{aligned}$$

and we are done, putting $F := \tilde{f} + \sum_{j=1}^{L} b_j f_j$, which lies in $L^p_{k+2,\beta+2}(X)$ by Lemma 5.15. ∎

5.2 The infinitesimal deformation space

The main reason we have developed the theory above is so that the vector space

$$K_{\beta+1} := \left\{ \xi \in C^\infty_{\beta+1}(T^*X) : \mathrm{d}\xi = \mathrm{d}_g^*\xi = 0 \right\} \tag{5.52}$$

may be examined more closely. Because of Lemma 2.29 the infinitesimal deformations of a special Lagrangian submanifold X may be thought of as the space of closed and coclosed 1-forms on the manifold X. When X is strongly asymptotically conical, the infinitesimal deformations which preserve being strongly asymptotically conical and special Lagrangian are precisely those in the vector space $K_{\beta+1}$ defined above, for a certain value of $\beta+1 \in \mathbb{R}^L$ which corresponds to the decay rate α of the metric on X (in fact, as we shall see in Chapter 6, the relationship is $\beta = \alpha$). This is the reason we are interested in the $K_{\beta+1}$ spaces. In particular we are interested in the dimension of the vector space $K_{\beta+1}$: we know it has finite dimension because $K_{\beta+1} \leqslant \operatorname{Ker}(\mathrm{d}_g^* + \mathrm{d})_{\beta+1}$ and we may then appeal to Corollary 4.23.

Define a map

$$\begin{array}{rcl} \psi_{\beta+1} : K_{\beta+1} & \to & H^1(X) \\ \xi & \mapsto & [\xi] \end{array} \tag{5.53}$$

for each $\beta+1 \in \mathbb{R}^L$. Then

$$\dim K_{\beta+1} = \dim \operatorname{Ker} \psi_{\beta+1} + \dim \operatorname{Im} \psi_{\beta+1} \tag{5.54}$$

where $\operatorname{Ker} \psi_{\beta+1}$ measures the failure of elements of $K_{\beta+1}$ to represent cohomology classes in $H^1(X)$ uniquely and $\operatorname{Im} \psi_{\beta+1}$ measures the extent to which $K_{\beta+1}$ represents the whole cohomology group $H^1(X)$. A proper understanding of the kernel and image of the map $\psi_{\beta+1}$ requires the use of the long exact sequence (5.27). Before we begin our calculations we make the trivial observation that $K_{\beta+1} \leqslant K_{\gamma+1}$, $\operatorname{Ker} \psi_{\beta+1} \leqslant \operatorname{Ker} \psi_{\gamma+1}$ and $\operatorname{Im} \psi_{\beta+1} \leqslant \operatorname{Im} \psi_{\gamma+1}$, whenever $\beta+2 \leqslant \gamma+2$.

5.2.1 Calculating $\operatorname{Ker} \psi_{\beta+1}$

Lemma 5.17 *If $\beta+2 \in \mathbb{R}^L$ and $\beta+2 > 0$ then $\operatorname{Ker} \psi_{\beta+1} = \mathrm{d}\operatorname{Ker}(\Delta_g^0)_{\beta+2}$.*

Proof: Clearly applying the exterior derivative to any harmonic function gives a closed and coclosed 1-form. Therefore $\mathrm{d}\operatorname{Ker}(\Delta_g^0)_{\beta+2} \subseteq \operatorname{Ker} \psi_{\beta+1}$. To see the reverse inclusion, take any $\xi \in \operatorname{Ker} \psi_{\beta+1}$. Then there exists $f \in C^\infty(X)$ such that $\mathrm{d}f = \xi$ and from Lemma 5.10 we deduce that $f \in C^\infty_{\beta+2}(X)$. Since ξ is coclosed we have $\Delta_g^0 f = 0$ so that $f \in \operatorname{Ker}(\Delta_g^0)_{\beta+2}$ as required.

∎

It follows from Lemma 5.17 and equation (5.36) that

$$\operatorname{Ker} \psi_{\beta+1} = \operatorname{Span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_L\}$$

for all $0 \leqslant \beta+2 < \lambda$. Since each $\mathrm{d}h_j$ is a closed and coclosed 1-form lying in $C^\infty_{1-n}(T^*X)$ we deduce that $\mathrm{d}h_j \in K_{1-n} \leqslant K_{\beta+1}$ for all $\beta+2 \geqslant 2-n$. Therefore

$$\operatorname{Ker} \psi_{\beta+1} = \operatorname{Span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_L\}$$

for all $2-n \leqslant \beta+2 < \lambda$. There is more to say about this space when $2-n < \beta+2 < 0$, as we see in the next lemma.

Lemma 5.18 *Recall the definition* (5.24) *of the map $\phi_1 : H^1_c(X) \to H^1(X)$. If $\beta+2 < 0$ then there exists an injective map*

$$\theta_{\beta+1} : \operatorname{Ker} \psi_{\beta+1} \to \operatorname{Ker} \phi_1. \tag{5.55}$$

When $2-n < \beta+2 < 0$ the map (5.55) *is onto and acts as*

$$\theta_{\beta+1}(c_1\mathrm{d}h_1 + \cdots + c_L\mathrm{d}h_L) = [\mathrm{d}f_c] \tag{5.56}$$

for all $c = (c_1, \ldots, c_L) \in \mathbb{R}^L$.

Proof: Let $\beta+2 < 0$ and suppose that $\xi \in \operatorname{Ker} \psi_{\beta+1}$. Then there exists $f \in C^\infty(X)$ such that $\mathrm{d}f = \xi$ and then from Lemma 5.10 we see that there exists $f_c \in C^\infty(X)$ constant on the ends of X such that $\hat{f} := f - f_c \in C^\infty_{\beta+2}(X)$. Define

$$\theta_{\beta+1}(\xi) := [\mathrm{d}f_c]$$

a class in $H^1_c(X)$ lying in $\operatorname{Ker}\phi_1$. Clearly f is uniquely determined by ξ up to constants and f_c is uniquely determined by f up to elements of $C^\infty_c(X)$, so that $\theta_{\beta+1}$ is well-defined as a map into $H^1_c(X)$. To see that the map (5.55) is injective, suppose that there exists $\tilde{f} \in C^\infty_c(X)$ such that $\mathrm{d}\tilde{f} = \mathrm{d}f_c$. Then $\tilde{f} - f_c$ is constant and the entries of the L-tuple c are all equal. Abusing notation we write $c = (c, \dots, c)$. Then $f - c \in C^\infty_{\beta+2}(X)$ and moreover

$$\Delta^0_g(f-c) = \Delta^0_g f = \mathrm{d}^*_g\xi = 0.$$

The Maximum Principle 5.2 now shows that $f = c$ so that $\xi = 0$ as required.

Suppose now that $2 - n < \beta + 2 < 0$. Let $[\mathrm{d}f_c] \in \operatorname{Ker}\phi_1$ where $c \in \mathbb{R}^L$. Then $\Delta^0_g f_c \in C^\infty_c(X)$ and by Corollary 5.4 we see that there exists $\hat{f} \in L^p_{k+2,\beta+2}(X)$ such that

$$\Delta^0_g \hat{f} = -\Delta^0_g f_c. \tag{5.57}$$

Put $f := \hat{f} + f_c$ and consider now $\xi := \mathrm{d}f$. Elliptic regularity for equation (5.57) shows that $\hat{f}$ and hence ξ are smooth, and since $\xi \in L^p_{k+1,\beta+1}(X)$ is closed and coclosed we may invoke Theorem 4.21 to see that $\xi \in \operatorname{Ker}\psi_{\beta+1}$. It is clear that $\theta_{\beta+1}(\xi) = [\mathrm{d}f_c]$. Also the properties of the harmonic functions h_j established in Section 5.1.3 show that the action of the map (5.55) is as given in (5.56). ∎

Note that Lemma 5.17 provides a good way of evaluating $\dim\operatorname{Ker}\psi_{\beta+1}$ when $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$ and $\beta + 2 > 0$, because in this situation we have

$$\dim\operatorname{Ker}\psi_{\beta+1} = \dim\bigl(\mathrm{d}\operatorname{Ker}(\Delta^0_g)_{\beta+2}\bigr) = \dim\operatorname{Ker}(\Delta^0_g)_{\beta+2} - 1 = \dim\operatorname{Ker}(\Delta^0_g)^p_{\beta+2} - 1$$

which we compute using the material of Section 5.1.1. The cases $\beta + 2 > 0$ and $\beta + 2 \in \mathcal{D}(\Delta^0_g)$ can be similarly dealt with once we prove the following lemma.

Lemma 5.19 *Let $\beta + 2 \geqslant 0$ with $\beta + 2 \in \mathcal{D}(\Delta^0_g)$. Then for suitably small $\varepsilon > 0$*

$$\begin{aligned}
\operatorname{Ker}(\Delta^0_g)^p_{\beta+2} &= \operatorname{Ker}(\Delta^0_g)^p_{\beta+2-\varepsilon}\\
\operatorname{Ker}(\Delta^0_g)_{\beta+2} &= \operatorname{Ker}(\Delta^0_g)_{\beta+2+\varepsilon}.
\end{aligned}$$

Proof: Take any small $\varepsilon > 0$ such that $\beta + 2 + t\varepsilon \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$ for each $0 < |t| \leqslant 1$. Then we have

$$\begin{aligned}
\operatorname{Ker}(\Delta^0_g)^p_{\beta+2-\varepsilon} &= \operatorname{Ker}(\Delta^0_g)_{\beta+2-\varepsilon}\\
&\leqslant \operatorname{Ker}(\Delta^0_g)^p_{\beta+2} && (5.58)\\
&\leqslant \operatorname{Ker}(\Delta^0_g)_{\beta+2} && (5.59)\\
&\leqslant \operatorname{Ker}(\Delta^0_g)^p_{\beta+2+\varepsilon} && (5.60)\\
&= \operatorname{Ker}(\Delta^0_g)_{\beta+2+\varepsilon}
\end{aligned}$$

and we wish to show that equality holds in inclusion (5.58) and inclusion (5.60). Since

$$\mathcal{D}(\Delta^0_g) = \bigl(\mathcal{D}(\Delta^0_g, 1) \times \mathbb{R}^{L-1}\bigr) \cup \bigl(\mathbb{R} \times \mathcal{D}(\Delta^0_g, 2) \times \mathbb{R}^{L-2}\bigr) \cup \dots \cup \bigl(\mathbb{R}^{L-1} \times \mathcal{D}(\Delta^0_g, L)\bigr)$$

there exists an $1 \leqslant r \leqslant L$ and $1 \leqslant j_1 < \dots < j_r \leqslant L$ such that for each $1 \leqslant j \leqslant L$ we have $\beta_j + 2 \in \mathcal{D}(\Delta^0_g, j)$ precisely when $j = j_l$ for some $1 \leqslant l \leqslant r$. Without loss of generality suppose that $j_l = l$ for each $1 \leqslant l \leqslant r$. Then given $1 \leqslant j \leqslant r$ we have

$$\mu^j := (\beta_j + 2)(\beta_j + n) \in \operatorname{Spec}(\Sigma_j, \Delta^0_g, 0).$$

For each $1 \leqslant j \leqslant r$ put

$$V_j := \operatorname{Ker}(\Delta^0_{g_\Sigma} - \mu^j) \cap C^\infty(\Sigma_j)$$

and choose a basis $\{k_j^1, \ldots, k_j^{n_j}\}$ for each V_j. Now for each $1 \leqslant j \leqslant r$ and $1 \leqslant i \leqslant n_j$ define functions $h_j^i \in C^\infty(X)$ such that $h_j^i = e^{(\beta_j+2)t} k_j^i$ on the jth end of X and $h_j^i = 0$ on the other ends of X. Then since each $\Delta_g^0 h_j^i$ is compactly supported we deduce

$$\Delta_g^0 h_j^i \in C^\infty_{\beta+\alpha}(X)$$

for each $1 \leqslant j \leqslant r$ and $1 \leqslant i \leqslant n_j$. Now choose some $\delta \in \mathbb{R}^L$ such that $0 < \delta < -\alpha$, $\beta + 2 + \alpha + \delta \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$ and $\beta + 2 + \alpha + \delta > 2 - n$. Then since

$$\Delta_g^0 : L^p_{k+2,\beta+2+\alpha+\delta}(X) \to L^p_{k,\beta+\alpha+\delta}(X)$$

is onto and each $\Delta_g^0 h_j^i \in C^\infty_{\beta+\alpha}(X) \leqslant L^p_{k,\beta+\alpha+\delta}(X)$ we deduce that there exist $f_j^i \in L^p_{k+2,\beta+2+\alpha+\delta}(X)$ such that

$$\Delta_g^0 f_j^i = \Delta_g^0 h_j^i$$

for each $1 \leqslant j \leqslant r$ and $1 \leqslant i \leqslant n_j$. Therefore we have

$$\mathrm{Span}\big\{f_j^i - h_j^i : 1 \leqslant j \leqslant r \text{ and } 1 \leqslant i \leqslant n_j\big\} \leqslant \mathrm{Ker}(\Delta_g^0)_{\beta+2} \tag{5.61}$$

and the left hand side of the inclusion (5.61) is a vector space of dimension $n_1 + \cdots + n_r$. We also have

$$\mathrm{Span}\big\{f_j^i - h_j^i : 1 \leqslant j \leqslant r \text{ and } 1 \leqslant i \leqslant n_j\big\} \cap \mathrm{Ker}(\Delta_g^0)^p_{\beta+2} = \{0\}$$

and

$$\dim \mathrm{Ker}(\Delta_g^0)_{\beta+2+\varepsilon} - \dim \mathrm{Ker}(\Delta_g^0)_{\beta+2-\varepsilon} = n_1 + \cdots + n_r$$

so that $0 \leqslant \dim \mathrm{Ker}(\Delta_g^0)_{\beta+2} - \dim \mathrm{Ker}(\Delta_g^0)^p_{\beta+2} \leqslant n_1 + \cdots + n_r$. It now follows that

$$\mathrm{Ker}(\Delta_g^0)_{\beta+2} = \mathrm{Span}\big\{f_j^i - h_j^i : 1 \leqslant j \leqslant r \text{ and } 1 \leqslant i \leqslant n_j\big\} \oplus \mathrm{Ker}(\Delta_g^0)^p_{\beta+2} \tag{5.62}$$

and we are done: equality in (5.58) and in (5.60) comes from equation (5.62), because the dimension jump at (5.59) is as big as it can be.

∎

So Lemma 5.19 is saying that for fixed $\beta + 2 \in \mathbb{R}^L$ the function $t \mapsto \dim \mathrm{Ker}(\Delta_g^0)_{\beta+2+t\varepsilon}$ is upper semi-continuous, and that $t \mapsto \dim \mathrm{Ker}(\Delta_g^0)^p_{\beta+2+t\varepsilon}$ lower semi-continuous.

The previous results show that for all $\beta + 2 > 0$ can choose a suitably small $\varepsilon > 0$ so that $\beta + 2 + \varepsilon \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$ and then we have

$$\begin{aligned}
\dim \mathrm{Ker}\, \psi_{\beta+1} &= \dim \mathrm{d}\, \mathrm{Ker}(\Delta_g^0)_{\beta+2} \\
&= \dim \mathrm{Ker}(\Delta_g^0)_{\beta+2} - 1 \\
&= \dim \mathrm{Ker}(\Delta_g^0)_{\beta+2+\varepsilon} - 1 \\
&= \dim \mathrm{Ker}(\Delta_g^0)^p_{\beta+2+\varepsilon} - 1 \\
&= L - 1 + \chi(\beta+2)
\end{aligned}$$

where the analytic piece of data $\chi(\beta + 2)$ is defined in equation (5.13).

We end this section on $\mathrm{Ker}\, \psi_{\beta+1}$ by showing that this space must vanish for small enough growth rates $\beta + 2$.

Lemma 5.20 *If* $\beta + 2 < 2 - n$ *then* $\mathrm{Ker}\, \psi_{\beta+1} = \{0\}$.

Proof: Suppose that $\beta + 2 < 2 - n$ and $\xi \in \mathrm{Ker}\, \psi_{\beta+1}$. Take some harmonic $h \in C^\infty(X)$ such that $\mathrm{d}h = \xi$. From Lemma 5.10 we deduce that h must be bounded, and then for $S \geqslant 0$ we may compute

$$\int_{X_S} |\xi|_g^2 \mathrm{d}V_g = \int_{X_S} \mathrm{d}h \wedge *_g \mathrm{d}h = \int_{\partial X_S} h(*_g \mathrm{d}h) = \int_{\partial X_S} h(*_g \xi) \tag{5.63}$$

since $\mathrm{d}(h *_g \mathrm{d}h) = \mathrm{d}h \wedge *_g \mathrm{d}h + h(\mathrm{d} *_g \mathrm{d}h)$. But we now observe that $\mathrm{Vol}(\partial X_S, g) = O(e^{(n-1)S})$ so that the right hand side of equation (5.63) tends to 0 as $S \to \infty$. Therefore $\xi = 0$ and we are done.

∎

5.2.2 Calculating $\operatorname{Im}\psi_{\beta+1}$

Lemma 5.21 *If $\beta+2<0$ then $\operatorname{Im}\psi_{\beta+1}\leqslant\operatorname{Im}\phi_1$.*

Proof: Suppose that $\xi\in K_{\beta+1}$ and consider the cohomology class $[\xi]\in H^1(X)$. Refer to the long exact sequence (5.27). To show that $[\xi]\in\operatorname{Im}\phi_1$ we show that $[\xi]\in\operatorname{Ker}p_1$ or equivalently that

$$[i_t^*\xi]\cdot[\tau]=\int_\tau i_t^*\xi=0$$

for all homology classes $[\tau]\in H_1(\Sigma)$. Without loss of generality suppose that $\tau\subseteq\Sigma_j$. Since $\xi\in C^\infty_{\beta+1}(T^*X)$ we have that $|i_t^*\xi|_{g_\Sigma}=O(e^{(\beta_j+2)t})$ on the component Σ_j of Σ. Then there exists a constant $A>0$ such that for suitably large $t>T$

$$\left|\int_\tau i_t^*\xi\right|\leqslant A\cdot e^{(\beta_j+2)t}$$

and hence we are done.

∎

Lemma 5.22 *If $\beta+2>2-n$ then $\operatorname{Im}\phi_1\leqslant\operatorname{Im}\psi_{\beta+1}$.*

Proof: Since $\operatorname{Im}\psi_{\beta+1}$ is decreasing with $\beta+1$ we may without loss of generality suppose that $\beta+2\in\mathbb{R}^L\setminus\mathcal{D}(\Delta_g^0)$. Suppose that $\xi\in C_c^\infty(T^*X)$ with $\mathrm{d}\xi=0$ and consider $[\xi]\in H^1(X)$. Then as $-\mathrm{d}_g^*\xi\in C_c^\infty(X)\leqslant L^p_{0,\beta}(X)$ we deduce from Corollary 5.4 that there exists $f\in L^p_{2,\beta+2}(X)$ with $\Delta_g^0 f=-\mathrm{d}_g^*\xi$. Since $-\mathrm{d}_g^*\xi\in L^p_{\infty,\beta}(X)$, elliptic regularity as in Theorem 4.21 shows that $f\in L^p_{\infty,\beta+2}(X)\leqslant C^\infty_{\beta+2}(X)$, so that $\xi+\mathrm{d}f$ is an element of $K_{\beta+1}$ and we are done.

∎

In actual fact, we have $\operatorname{Im}\phi_1\leqslant\operatorname{Im}\psi_{\beta+1}$ for $\beta+2$ slightly smaller than $2-n$, as we now go on to prove. Clearly this implies the previous result, but we have included the proof of Lemma 5.22 above because it is simpler than the proof of Lemma 5.23 below, and Lemma 5.22 does not rely on the strong decay properties of the asymptotically conical metric g.

Lemma 5.23 *If $\max\{2-n+\alpha,2-n-\lambda\}<\beta+2<2-n$ then $\operatorname{Im}\phi_1\leqslant\operatorname{Im}\psi_{\beta+1}$.*

Proof: Suppose that $\beta+2\in\mathbb{R}^L$ with $\max\{2-n+\alpha,2-n-\lambda\}<\beta+2<2-n$. Take any $\xi\in C_c^\infty(X)$ with $\mathrm{d}\xi=0$. We must find an $f\in C^\infty(X)$ such that $\mathrm{d}f+\xi\in K_{\beta+1}$, or equivalently $\mathrm{d}f\in C^\infty_{\beta+1}(T^*X)$ and $\Delta_g^0 f=-\mathrm{d}_g^*\xi$. For this we simply invoke Corollary 5.16: since $\xi\in L^p_{1,\beta+1}(T^*X)$ there exists $F\in L^p_{2,\beta+2}(X)$ and $f_b\in C^\infty(X)$ constant on the ends of X such that $\Delta_g^0(F+f_b)=-\mathrm{d}_g^*\xi$. Elliptic regularity tells us that since $-\Delta_g^0 f_b-\mathrm{d}_g^*\xi\in C_c^\infty(X)$ we have

$$F\in L^p_{\infty,\beta+2}(X)\subseteq C^\infty_{\beta+2}(X)$$

and then we are done after putting $f=F+f_b$.

∎

It follows from Lemma 5.21 and Lemma 5.23 that $\operatorname{Im}\psi_{\beta+1}=\operatorname{Im}\phi_1$ for all $\max\{2-n+\alpha,2-n-\lambda\}<\beta+2<0$. We now deal with the cases $\beta+2\geqslant 0$.

Lemma 5.24 *If $\beta+2>0$ then $\operatorname{Im}\psi_{\beta+1}=H^1(X)$.*

Proof: Since $\operatorname{Im}\psi_{\beta+1}$ is increasing with $\beta+1$ we may without loss of generality suppose that $\beta+2\in\mathbb{R}^L\setminus\mathcal{D}(\Delta_g^0)$. Now, Corollary 5.9 tells us that any class in $H^1(X)$ can be represented by a closed form $\xi\in C^\infty_{-1}(T^*X)$. Since

$$-\mathrm{d}_g^*\xi\in C^\infty_{-2}(X)\leqslant L^p_{0,\beta}(X)$$

we know from Corollary 5.4 that there exists $f \in L^p_{2,\beta+2}(X)$ with $\Delta^0_g f = -\mathrm{d}^*_g \xi$. Now the weighted regularity Theorem 4.21 implies $f \in L^p_{\infty,\beta+2}(X)$ and then the Embedding Theorem 4.17 shows that $f \in C^\infty_{\beta+2}(X)$ so that $\xi + \mathrm{d}f \in K_{\beta+1}$ as required.

∎

In actual fact, we have $\operatorname{Im}\psi_{-1} = H^1(X)$, as we now go on to prove. Clearly this implies the previous result, but we have included the proof of Lemma 5.24 above because it is simpler than the proof of Lemma 5.25 below, and Lemma 5.24 does not rely on the strong decay properties of the asymptotically conical metric g.

Lemma 5.25 $\operatorname{Im}\psi_{-1} = H^1(X)$.

Proof: By Corollary 5.9 any cohomology class in $H^1(X)$ can be represented by a closed form $\eta \in C^\infty_{-1}(T^*X)$, and moreover we may suppose that there exists an $S \geqslant 0$ and $\sigma \in C^\infty(T^*\Sigma)$ such that $\eta = \pi^*\sigma$ over $X \setminus X_S$. By perturbing σ by an exact 1-form on Σ we may further suppose that σ is g_Σ-harmonic.

We wish to find an $f \in C^\infty(X)$ such that $\mathrm{d}f \in C^\infty_{-1}(T^*X)$ and $\Delta^0_g f = -\mathrm{d}^*_g\eta$. Now, a priori $-\mathrm{d}^*_g\eta \in C^\infty_{-2}(X)$, but can we do any better? In fact, we can: working over $X \setminus X_S$ we have

$$*_{\tilde{g}}\eta = -e^{(n-2)t}\mathrm{d}t \wedge \big(\pi^*(*_{g_\Sigma}\sigma)\big)$$

so that

$$\mathrm{d}(*_{\tilde{g}}\eta) = e^{(n-2)t}\mathrm{d}t \wedge \mathrm{d}\big(\pi^*(*_{g_\Sigma}\sigma)\big) = e^{(n-2)t}\mathrm{d}t \wedge \pi^*\big(\mathrm{d}(*_{g_\Sigma}\sigma)\big).$$

It follows that $\mathrm{d}^*_{\tilde{g}}\eta = 0$ precisely when $\mathrm{d}^*_{g_\Sigma}\sigma = 0$, which is evidently the case as σ is g_Σ-harmonic and Σ is compact. Now over X we have $-\mathrm{d}^*_g\eta = -(\mathrm{d}^*_g - \mathrm{d}^*_{\tilde{g}})\eta$, modulo elements of $C^\infty_c(X)$ and from Lemma 5.14 we deduce that $-\mathrm{d}^*_g\eta \in C^\infty_{-2+\alpha}(X)$. Now choose any $\varepsilon \in \mathbb{R}^L$ so that $2-n < \varepsilon + \alpha < 0$ and $\varepsilon > 0$ and then

$$\Delta^0_g : L^p_{k+2,\alpha+\varepsilon}(X) \to L^p_{k,-2+\alpha+\varepsilon}(X)$$

is an isomorphism, by Corollary 5.3 and Corollary 5.4. Since $-\mathrm{d}^*_g\eta \in C^\infty_{-2+\alpha}(X) \leqslant L^p_{k,-2+\alpha+\varepsilon}(X)$ we deduce that there exists $f \in L^p_{k+2,\alpha+\varepsilon}(X)$ such that $\Delta^0_g f = -\mathrm{d}^*_g\eta$, with elliptic regularity showing

$$f \in L^p_{\infty,\alpha+\varepsilon}(X) \leqslant C^\infty_0(X)$$

and we are done.

∎

We finish this section on $\operatorname{Im}\psi_{\beta+1}$ by considering the remaining case $\beta + 2 < 2 - n$, so that $\operatorname{Im}\psi_{\beta+1} \leqslant \operatorname{Im}\phi_1$. In the following lemma we give a condition for an element of $\operatorname{Im}\phi_1$ to lie in $\operatorname{Im}\psi_{\beta+1}$. In general, it is quite hard to check explicitly if this condition holds. An exception is the case $L = 1$ and $\beta+2$ is only just smaller that $2-n$: in this situation there are no harmonic functions on X with small positive growth rate other than constants so that $\mathrm{d}\operatorname{Ker}(\Delta^0_g)_{-\beta-n} = 0$.

Lemma 5.26 *Let $\beta + 2 < 2 - n$ with $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$. Let $\tilde{\sigma} \in C^\infty_c(T^*X)$ with $\mathrm{d}\tilde{\sigma} = 0$ so that $[\tilde{\sigma}] \in \operatorname{Im}\phi_1$. Then $[\tilde{\sigma}] \in \operatorname{Im}\psi_{\beta+1}$ precisely when there exists $f_c \in C^\infty(X)$ constant on the ends of X such that*

$$\langle \mathrm{d}h | \tilde{\sigma} - \mathrm{d}f_c \rangle_{L^2(T^*X)} = 0 \tag{5.64}$$

for all $h \in \operatorname{Ker}(\Delta^0_g)_{-\beta-n}$.

Proof: Clearly $[\tilde{\sigma}] \in \operatorname{Im}\psi_{\beta+1}$ precisely when there exists an $\hat{f} \in C^\infty(X)$ such that $\tilde{\sigma} + \mathrm{d}\hat{f} \in K_{\beta+1}$, that is, $\mathrm{d}\hat{f} \in C^\infty_{\beta+1}(T^*X)$ and $\Delta^0_g \hat{f} = -\mathrm{d}^*_g\tilde{\sigma}$.

Suppose that there exists $f_c \in C^\infty(X)$ constant on the ends of X such that equation (5.64) holds for all $h \in \operatorname{Ker}(\Delta^0_g)_{-\beta-n}$. Then there exists $f \in L^p_{2,\beta}(X)$ such that $\Delta^0_g f = \Delta^0_g f_c - \mathrm{d}^*\tilde{\sigma}$ and by elliptic regularity and the embedding theorems $f \in C^\infty_{\beta+2}(X)$. Taking $\hat{f} := f - f_c$ shows $[\tilde{\sigma}] \in \operatorname{Im}\psi_{\beta+1}$.

Suppose conversely that there exists an $\hat{f} \in C^\infty(X)$ such that $\mathrm{d}\hat{f} \in C^\infty_{\beta+1}(T^*X)$ and $\Delta^0_g \hat{f} = -\mathrm{d}^*_g \tilde{\sigma}$. Then by Lemma 5.10 there exists $f_c \in C^\infty(X)$ constant on the ends of X such that $f := \hat{f} + f_c$ lies in $C^\infty_{\beta+2}(X)$ and therefore

$$\langle \mathrm{d}h | \tilde{\sigma} - \mathrm{d}f_c \rangle_{L^2(T^*X)} = \langle h | \mathrm{d}^*_g \tilde{\sigma} - \Delta^0_g (f - \hat{f}) \rangle_{L^2(X)} = -\langle \Delta^0_g h | f \rangle_{L^2(X)} = 0$$

for all $h \in \mathrm{Ker}(\Delta^0_g)_{-\beta-n}$. Hence we are done.

∎

5.2.3 Results for $\dim K_{\beta+1}$

The purpose of this section is merely to summarise the results of Section 5.2.1 and Section 5.2.2 above. We observe that for $\beta + 2 > 0$ we have $\dim \mathrm{d}\,\mathrm{Ker}(\Delta^0_g)_{\beta+2} = \dim \mathrm{Ker}(\Delta^0_g)_{\beta+2} - 1$ since the linear map $\mathrm{d} : \mathrm{Ker}(\Delta^0_g)_{\beta+2} \to \mathrm{d}\,\mathrm{Ker}(\Delta^0_g)_{\beta+2}$ is surjective and has kernel the 1-dimensional subspace of $\mathrm{Ker}(\Delta^0_g)_{\beta+2}$ consisting of the constant functions. In particular, for small positive $\beta + 2$ we have

$$\dim \mathrm{d}\,\mathrm{Ker}(\Delta^0_g)_{\beta+2} = \dim \mathrm{Span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_L\} = L - 1.$$

Also we have

$$\dim \mathrm{Im}\,\phi_1 = \dim H^1_c(X) - \dim \mathrm{Ker}\,\phi_1 = b^1_c(X) - L + 1.$$

The various results of Section 5.2 are now summarised in Table 5.1. The definition of the analytic piece of data $\chi(\beta + 2)$ is given in Section 5.1.1.

Growth rate	$\mathrm{Ker}\,\psi_{\beta+1}$	$\mathrm{Im}\,\psi_{\beta+1}$	$\dim K_{\beta+1}$
$\beta + 2 > 0$	$\mathrm{d}\,\mathrm{Ker}(\Delta^0_g)_{\beta+2}$	$H^1(X)$	$b^1(X) + L - 1 + \chi(\beta + 2)$
$\beta + 2 = 0$	$\mathrm{Span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_L\}$	$H^1(X)$	$b^1(X) + L - 1$
$2 - n < \beta + 2 < 0$	$\mathrm{Span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_L\}$	$\mathrm{Im}\,\phi_1$	$b^1_c(X)$
$\beta + 2 = 2 - n$	$\mathrm{Span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_L\}$	$\mathrm{Im}\,\phi_1$	$b^1_c(X)$
$\max\{2 - n + \alpha, 2 - n - \lambda\} < \beta + 2 < 2 - n$	0	$\mathrm{Im}\,\phi_1$	$b^1_c(X) - L + 1$
$\beta + 2 \leqslant \max\{2 - n + \alpha, 2 - n - \lambda\}$	0	$\leqslant \mathrm{Im}\,\phi_1$	$\leqslant b^1_c(X) - L + 1$

Table 5.1: The kernels and images of the representation map $\psi_{\beta+1}$, together with the dimensions of the infinitesimal deformation space $K_{\beta+1}$, for strongly asymptotically conical Riemannian manifolds (X, g) with rate $\alpha < 0$

Suppose that we are interested in the infinitesimal deformations of a strongly asymptotically conical special Lagrangian submanifold *as an asymptotically conical* special Lagrangian submanifold. To this end, define $\hat{C}^k_{\beta+1}(T^*X)$ to be the vector space of 1-forms ξ on X which are of class C^k such that

$$\sup_{\{t\}\times\Sigma} \left| \nabla^j_g \xi \right|_g = o(e^{(\beta+1-j)t})$$

for all $0 \leqslant j \leqslant k$. Then $\hat{C}^k_{\beta+1}(T^*X) \leqslant C^k_{\beta+1}(T^*X)$ is a closed subspace: in fact $\hat{C}^k_{\beta+1}(T^*X)$ is the closure of $C^\infty_c(T^*X)$ in $C^k_{\beta+1}(T^*X)$. We now put

$$\hat{C}^\infty_{\beta+1}(T^*X) := \bigcap_{k=0}^{\infty} \hat{C}^k_{\beta+1}(T^*X)$$

and then $\hat{K}_{\beta+1} := \left\{ \xi \in \hat{C}^\infty_{\beta+1}(T^*X) : \mathrm{d}\xi = \mathrm{d}^*_g \xi = 0 \right\} \leqslant K_{\beta+1}$, with a projection $\hat{\psi}_{\beta+1} := \psi_{\beta+1}|_{\hat{K}_{\beta+1}}$. From the results given in Table 5.1, together with small modifications of the proofs in Section 5.2.1 and Section 5.2.2 above, we can compile a Table 5.2 which gives the dimensions of the spaces $\hat{K}_{\beta+1}$ for various $\beta + 2 \in \mathbb{R}^L$. The definition of the analytic piece of data $\hat{\chi}(\beta + 2)$ is given in Section 5.1.1.

Growth rate	$\operatorname{Ker}\hat{\psi}_{\beta+1}$	$\operatorname{Im}\hat{\psi}_{\beta+1}$	$\dim \hat{K}_{\beta+1}$
$\beta+2>0$	$\mathrm{d}\operatorname{Ker}(\Delta^0_g)^p_{\beta+2}$	$H^1(X)$	$b^1(X)+L-1+\hat{\chi}(\beta+2)$
$\beta+2=0$	$\operatorname{Span}\{\mathrm{d}h_1,\dots,\mathrm{d}h_L\}$	$\operatorname{Im}\phi_1$	$b^1_c(X)$
$2-n<\beta+2<0$	$\operatorname{Span}\{\mathrm{d}h_1,\dots,\mathrm{d}h_L\}$	$\operatorname{Im}\phi_1$	$b^1_c(X)$
$\beta+2=2-n$	0	$\operatorname{Im}\phi_1$	$b^1_c(X)-L+1$
$\max\{2-n+\alpha,2-n-\lambda\}<\beta+2<2-n$	0	$\operatorname{Im}\phi_1$	$b^1_c(X)-L+1$
$\beta+2\leqslant\max\{2-n+\alpha,2-n-\lambda\}$	0	$\leqslant\operatorname{Im}\phi_1$	$\leqslant b^1_c(X)-L+1$

Table 5.2: The kernels and images of the representation map $\hat{\psi}_{\beta+1}$, together with the dimensions of the infinitesimal deformation space $\hat{K}_{\beta+1}$, for strongly asymptotically conical Riemannian manifolds (X,g) with rate $\alpha<0$

If we are interested in the space $\hat{K}_1$ of infinitesimal deformations of X as an asymptotically conical special Lagrangian submanifold, then we have $\beta+2=2$ and

$$\dim\hat{K}_1=b^1(X)+L-1+\hat{\chi}(2)$$

where

$$\hat{\chi}(2)=\sum\left\{\dim\operatorname{Ker}(\Delta^0_{g_\Sigma}-\mu):\begin{array}{l}0<\mu<2n\\ \mu\in\operatorname{Spec}(\Sigma,\Delta^0_{g_\Sigma},0)\end{array}\right\}.$$

5.2.4 Results for non-strong decay

Although it is not our primary concern, we shall note here what happens in the situation that the metric g on X is merely asymptotically conical, rather than strongly asymptotically conical. In this case we have Lemmas 5.17, 5.18, 5.20, 5.21, 5.22, 5.24, 5.26 holding and we can go on to fill in a version of Table 5.2 for asymptotically conical metrics. This is Table 5.3: we have only given proofs in the case $\beta+2\in\mathbb{R}^L\setminus\mathcal{D}(\Delta^0_g)$.

Growth rate	$\operatorname{Ker}\hat{\psi}_{\beta+1}$	$\operatorname{Im}\hat{\psi}_{\beta+1}$	$\dim \hat{K}_{\beta+1}$
$\beta+2>0$ and $\beta+2\in\mathbb{R}^L\setminus\mathcal{D}(\Delta^0_g)$	$\mathrm{d}\operatorname{Ker}(\Delta^0_g)^p_{\beta+2}$	$H^1(X)$	$b^1(X)+L-1+\hat{\chi}(\beta+2)$
$2-n<\beta+2<0$	$\operatorname{Span}\{\mathrm{d}h_1,\dots,\mathrm{d}h_L\}$	$\operatorname{Im}\phi_1$	$b^1_c(X)$
$\beta+2<2-n$	0	$\leqslant\operatorname{Im}\phi_1$	$\leqslant b^1_c(X)-L+1$

Table 5.3: The kernels and images of the representation map $\hat{\psi}_{\beta+1}$, together with the dimensions of the infinitesimal deformation space $\hat{K}_{\beta+1}$, for asymptotically conical Riemannian manifolds (X,g)

Chapter 6

Deformations of AC special Lagrangian submanifolds of $\mathbb{C}^n$

In chapter we consider the deformation problem for a certain class of special Lagrangian submanifolds of $\mathbb{C}^n$, in a manner analogous to the deformation problem for compact special Lagrangian submanifolds of a general Calabi-Yau manifold, as described in Section 3.2. The main result of this chapter is Theorem 6.45.

6.1 Further analytic results

6.1.1 Fredholm theory for Hölder spaces on non-compact manifolds

Most of the theory in the literature concerning the Fredholm theory of operators on non-compact manifolds is given in terms of Sobolev spaces of some type, as in the theory of Lockhart, McOwen and others presented in Section 4.2. However we shall be phrasing our deformation problem for AC special Lagrangian submanifolds $f : X \to \mathbb{C}^n$ in terms of conical damped Hölder spaces $C^{k,a}_\beta(E)$ for bundles $E \to X$. In this section we bridge the gap by deducing Fredholm theory for Hölder spaces analogous to the material for Sobolev spaces as given in Section 4.2.

For the rest of Section 6.1 we return to the mind-set of Chapter 4 and assume that X is a manifold with ends, as in Section 4.1. In fact, we shall assume all notation from Chapter 4.

Green's operators on the full cylinder

For the time being, let's suppose Σ is connected, so that $L = 1$, and define $\tilde{X} := \mathbb{R}\times\Sigma$ the full cylinder on Σ. We further put $\tilde{E} := \pi^* E_\Sigma$ which is a vector bundle over $\tilde{X}$. In order to define Banach spaces of sections of $\tilde{E}$ we work as in Section 4.2.1, except only consider charts of the form $V_\nu = \mathbb{R} \times U_\nu$ for $1 \leqslant \nu \leqslant N$, so that $V_1, \ldots, V_N$ is an open cover of $\tilde{X}$. All weight functions $e^{\beta t}$ are then extended to the full cylinder $\tilde{X}$, so that we obtain Banach spaces $W^p_{k,\beta}(\tilde{E})$ and $B^{k,a}_\beta(\tilde{E})$ as in Section 4.2.1 by omitting the terms with $N + 1 \leqslant \nu \leqslant N + K$ in the norms (4.8), (4.11), (4.12).

Notice that there is an obvious failure for many of the embeddings of Theorem 4.2, the problem being that for $\beta < \delta$ we have $t\beta > t\delta$ for $t < 0$. However, this aside, the Banach spaces $W^p_{k,\beta}(\tilde{E})$ and $B^{k,a}_\beta(\tilde{E})$ are very similar to the original spaces $W^p_{k,\beta}(E)$ and $B^{k,a}_\beta(E)$ we defined in Section 4.2.1. In particular, a translation invariant differential operator $P_\infty : C^\infty_c(\tilde{E}) \to C^\infty_c(\tilde{F})$ of order $l \geqslant 1$ will extend to bounded linear maps

$$P_\infty : W^p_{k+l,\beta}(\tilde{E}) \;\to\; W^p_{k,\beta}(\tilde{F}) \tag{6.1}$$

$$P_\infty : B^{k+l,a}_\beta(\tilde{E}) \;\to\; B^{k,a}_\beta(\tilde{F}). \tag{6.2}$$

We now have a very important result. For the rest of Section 6.1.1 we assume the (uniform) ellipticity of the operator P_∞ (respectively, P).

Theorem 6.1 (Green's function for $W^p_{k,\beta}(\tilde{E})$ spaces) *If $\beta \in \mathbb{R} \setminus \mathcal{D}(P_\infty)$ then the bounded linear map* (6.1) *is a topological linear isomorphism.*

Although we did not mention it in Chapter 4, Theorem 6.1 is the key result for proving the Fredholm theory of that chapter. For a proof of Theorem 6.1 see the paper [49, Theorem 4.1] of Maz'ya and Plamenevskiĭ. The main idea is that given $\xi \in C^\infty_c(\tilde{F})$ we can define, for $w \in \mathbb{C}$, the Fourier transform

$$\hat{\xi}(w, \sigma) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-wt} \xi(t, \sigma) \mathrm{d}t$$

so that $\hat{\xi}(w, \cdot)$ is a section of the bundle $F_\Sigma \otimes \mathbb{C}$. Now if $\beta \in \mathbb{R} \setminus \mathcal{D}(P_\infty)$ we have $P_\infty(w)$ invertible for all $w \in \mathbb{C}$ with $\operatorname{Re} w = \beta$ so we can define

$$(A_\beta \xi)(t, \sigma) := \frac{1}{\sqrt{2\pi}} \int_{\operatorname{Re} w = \beta} e^{wt} P_\infty(w)^{-1} \hat{\xi}(w, \sigma) \mathrm{d}w.$$

It turns out that A_β extends to a bounded linear operator $W^p_{k,\beta}(\tilde{F}) \to W^p_{k+l,\beta}(\tilde{E})$ which inverts the bounded linear map (6.1).

In order to deduce Fredholm theory for asymptotically translation invariant operators P on the $B^{k,a}_\beta(E)$ spaces, we shall need a result analogous to Theorem 6.1, valid for the damped Hölder spaces. Unfortunately, the results given in the literature do not deal with the Banach spaces $B^{k,a}_\beta(\tilde{E})$ directly, but only a closed subspace: define $\hat{B}^{k,a}_\beta(\tilde{E})$ to be the closure of $C^\infty_c(\tilde{E})$ in the space $B^{k,a}_\beta(\tilde{E})$. In more explicit terms, the space $\hat{B}^{k,a}_\beta(\tilde{E})$ can be defined as those sections ξ of $\tilde{E}$ which satisfy the decay conditions

$$\sup_{\{t\} \times U_\nu} \left| \rho_\nu \big(\partial^\lambda \xi^\nu_j \big) \right| = o(e^{\beta t})$$

for all $1 \leqslant \nu \leqslant N$, $1 \leqslant j \leqslant \operatorname{rank} E$ and $0 \leqslant |\lambda| \leqslant k$, together with a corresponding $o(e^{\beta t})$ decay condition on the Hölder norm of the kth derivatives of ξ.

The following result is proved in the paper [49, Theorem 5.1] of Maz'ya and Plamenevskiĭ.

Theorem 6.2 (Green's function for $\hat{B}^{k,a}_\beta(\tilde{E})$ spaces) *If $\beta \in \mathbb{R} \setminus \mathcal{D}(P_\infty)$ then the bounded linear map*

$$P_\infty : \hat{B}^{k+l,a}_\beta(\tilde{E}) \to \hat{B}^{k,a}_\beta(\tilde{F}) \tag{6.3}$$

is a topological linear isomorphism. We denote the inverse of the map (6.3) *by R_β.*

We can use Theorem 6.2 to prove various desirable properties of the map (6.2) for $\beta \in \mathbb{R} \setminus \mathcal{D}(P_\infty)$. We begin with the following lemma:

Lemma 6.3 *Let $K_1 \subseteq \mathbb{R} \setminus \mathcal{D}(P_\infty)$ be compact. Then there exists an $M > 0$ such that $\|R_\beta\| \leqslant M$ for all $\beta \in K_1$. Here $\|R_\beta\|$ denotes the operator norm of the map $R_\beta : \hat{B}^{k,a}_\beta(\tilde{F}) \to \hat{B}^{k+l,a}_\beta(\tilde{E})$.*

Proof: Fix any $\delta \in \mathbb{R}$. Then for all $\varepsilon \in \mathbb{R}$ we have a topological linear isomorphisms

$$e^{\varepsilon t} : \hat{B}^{k,a}_\delta(\tilde{E}) \to \hat{B}^{k,a}_{\delta+\varepsilon}(\tilde{E}) \tag{6.4}$$

Moreover, it is easy to check that one can obtain a bound on the operator norm $\|e^{\varepsilon t}\|$ of the map (6.4) which is independent of δ and polynomial in ε.

For the purposes of this proof, let $(P_\infty)_\delta$ denote P_∞ acting on the space $\hat{B}^{k+l,a}_\delta(\tilde{E})$. Then it is easy to see explicitly from equation (4.17) the map $\varepsilon \mapsto e^{-\varepsilon t}(P_\infty)_{\delta+\varepsilon} e^{\varepsilon t}$ is a continuous map

$$\mathbb{R} \to \mathcal{B}\big(\hat{B}^{k+l,a}_\delta(\tilde{E}), \hat{B}^{k,a}_\delta(\tilde{F})\big).$$

Since inversion is a continuous map we deduce that $\varepsilon \mapsto e^{-\varepsilon t} R_{\delta+\varepsilon} e^{\varepsilon t}$ is a continuous map

$$\big\{\varepsilon \in \mathbb{R} : \delta + \varepsilon \in \mathbb{R} \setminus \mathcal{D}(P_\infty)\big\} \to \mathcal{B}\big(\hat{B}^{k,a}_\delta(\tilde{F}), \hat{B}^{k+l,a}_\delta(\tilde{E})\big)$$

and so there is some $M_1 > 0$ such that $\|e^{-\varepsilon t} R_{\delta+\varepsilon} e^{\varepsilon t}\| \leqslant M_1$ for all $\varepsilon \in \mathbb{R}$ with $\delta + \varepsilon \in K_1$. Therefore for such $\varepsilon \in \mathbb{R}$ we have

$$\|R_{\delta+\varepsilon}\| = \|e^{\varepsilon t} e^{-\varepsilon t} R_{\delta+\varepsilon} e^{\varepsilon t} e^{-\varepsilon t}\| \leqslant M_1 \|e^{\varepsilon t}\| \|e^{-\varepsilon t}\|$$

and we are done because of the polynomial bound we can obtain on $\|e^{\pm\varepsilon t}\|$ in terms of ε.

∎

Corollary 6.4 *Suppose that $K_2 \subseteq \mathbb{R} \setminus \mathcal{D}(P_\infty)$ is compact. Then there exists a $C > 0$ such that if*

1. $\beta \in K_2$ *and* $\xi \in B^{k+l,a}_{\beta+\varepsilon}(\tilde{E})$ *for all* $\varepsilon > 0$
2. $\operatorname{supp}(\xi) \subseteq (T, \infty) \times \Sigma$ *for some* $T \in \mathbb{R}$
3. $P_\infty \xi \in B^{k,a}_{\beta}(\tilde{F})$

then $\xi \in B^{k+l,a}_{\beta}(\tilde{E})$ *with*

$$\|\xi\|_{B^{k+l,a}_{\beta}(\tilde{E})} \leqslant C \|P_\infty \xi\|_{B^{k,a}_{\beta}(\tilde{F})}. \tag{6.5}$$

Proof: Take a compact subset $K_1 \subseteq \mathbb{R} \setminus \mathcal{D}(P_\infty)$ such that $K_2 \subseteq V \subseteq K_1$ for some open subset $V \subseteq \mathbb{R}$, and then let $M > 0$ be as in Lemma 6.3.

Suppose that conditions 1, 2, 3 above hold. Then clearly $\xi \in \hat{B}^{k+l,a}_{\beta+\varepsilon}(\tilde{E})$ for all $\varepsilon > 0$ and furthermore we have

$$\|\xi\|_{B^{k+l,a}_{\beta+\varepsilon}(\tilde{E})} = \|R_{\beta+\varepsilon} P_\infty \xi\|_{B^{k+l,a}_{\beta+\varepsilon}(\tilde{E})} \leqslant \|R_{\beta+\varepsilon}\| \|P_\infty \xi\|_{B^{k,a}_{\beta+\varepsilon}(\tilde{F})} \leqslant M \|P_\infty \xi\|_{B^{k,a}_{\beta+\varepsilon}(\tilde{F})}$$

for all $\varepsilon > 0$ such that $\beta+\varepsilon \in V$. But now observe that if $\eta \in B^{k,a}_{\beta}(\tilde{F})$ is such that $\operatorname{supp}(\eta) \subseteq (T, \infty) \times \Sigma$ then $\eta \in B^{k,a}_{\beta+\varepsilon}(\tilde{F})$ for all $\varepsilon > 0$ and

$$\|\eta\|_{B^{k,a}_{\beta+\varepsilon}(\tilde{F})} \leqslant C' e^{-\varepsilon T} \|\eta\|_{B^{k,a}_{\beta}(\tilde{F})}$$

for some constant $C' > 0$ independent of T, η, β and ε. We now have

$$\|\xi\|_{B^{k+l,a}_{\beta+\varepsilon}(\tilde{E})} \leqslant M C' e^{-\varepsilon T} \|P_\infty \xi\|_{B^{k,a}_{\beta}(\tilde{F})}$$

for all $\varepsilon > 0$ with $\beta + \varepsilon \in V$. It follows quickly that $\xi \in B^{k+l,a}_{\beta}(\tilde{E})$ and that the inequality (6.5) holds for some $C > 0$ independent of T, ξ and $\beta \in K_2$. To see this, take for example $1 \leqslant \nu \leqslant N$, $1 \leqslant j \leqslant \operatorname{rank} \tilde{E}$ and $0 \leqslant |\lambda| \leqslant k$. Then for $x = (t, \sigma) \in (T, \infty) \times U_\nu$ we have

$$\left| e^{-(\beta+\varepsilon)t} \big(\rho_\nu \partial^\lambda \xi^\nu_j\big)_x \right| \leqslant \|\xi\|_{B^{k+l,a}_{\beta+\varepsilon}(\tilde{E})} \leqslant M C' e^{-\varepsilon T} \|P_\infty \xi\|_{B^{k,a}_{\beta}(\tilde{F})}$$

and letting $\varepsilon \to 0$ shows

$$\left| e^{-\beta t} \big(\rho_\nu \partial^\lambda \xi^\nu_j\big)_x \right| \leqslant M C' \|P_\infty \xi\|_{B^{k,a}_{\beta}(\tilde{F})}. \tag{6.6}$$

Taking a supremum in the inequality (6.6) over all $x = (t, \sigma) \in (T, \infty) \times U_\nu$, followed by summing $1 \leqslant \nu \leqslant N$, $1 \leqslant j \leqslant \operatorname{rank} \tilde{E}$ and $0 \leqslant |\lambda| \leqslant k$, gives the required bound: the Hölder part of the $B^{k+l,a}_{\beta}(\tilde{E})$ norm is handled just as for the $B^{k+l}_{\beta}(\tilde{E})$ part. Alternatively, one could just show that $\xi \in B^0_\beta(\tilde{E})$ and then appeal to the version of Theorem 4.6 which holds for the bundle $\tilde{E}$: it is easy to show that in results such as Theorem 4.6 the estimating constant C_2 can be taken to be independent of β provided we restrict β to some compact subset of $\mathbb{R}^L$.

∎

Scale broken estimates and Fredholm theory

We now apply the results above to the usual situation of bundles E, F over our manifold X, which we now assume to have an arbitrary number $L \geqslant 1$ of ends.

Corollary 6.5 *Suppose that $K_2 \subseteq \mathbb{R}^L \setminus \mathcal{D}(P_\infty)$ is compact. Then there exists a $C > 0$ such that if*

1. $\beta \in K_2$ *and* $\xi \in B^{k+l,a}_{\beta+\varepsilon}(E)$ *for all* $\varepsilon > 0$
2. $\operatorname{supp}(\xi) \subseteq X_\infty$
3. $P_\infty \xi \in B^{k,a}_\beta(F)$

then $\xi \in B^{k+l,a}_\beta(E)$ *and*

$$\|\xi\|_{B^{k+l,a}_\beta(E)} \leqslant C\|P_\infty\xi\|_{B^{k,a}_\beta(F)}. \tag{6.7}$$

Proof: This is immediate if we apply Corollary 6.4 on each end of X.

∎

Note that one would expect the inequality (6.7) to fail for non-zero elements $\xi \in \operatorname{Ker}(P_\infty)_\beta$ and that is what the second condition is meant to rule out. We now give a lemma which we shall need for the main result of this section, Theorem 6.7.

Lemma 6.6 (Interpolation Inequality) *Let $S \geqslant 0$. Then for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that*

$$\|\xi\|_{C^{k+1,a}(E|_{X_S})} \leqslant \varepsilon\|\xi\|_{C^{k+2,a}(E|_{X_S})} + C(\varepsilon)\|\xi\|_{C^0(E|_{X_S})} \tag{6.8}$$

for all $\xi \in C^{k+2,a}(E|_{X_S})$.

Proof: Define $B := \left\{\xi \in C^{k+2,a}(E|_{X_S}) : \|\xi\|_{C^{k+2,a}(E|_{X_S})} = 1 \text{ and } \|\xi\|_{C^{k+1,a}(E|_{X_S})} \geqslant \varepsilon\right\}$. Since the embedding $C^{k+2,a}(E|_{X_S}) \to C^{k+1,a}(E|_{X_S})$ is compact we deduce that the closure $\overline{B}$ of B in $C^{k+1,a}(E|_{X_S})$ is compact. It follows that the function

$$\begin{aligned} \overline{B} &\to \mathbb{R} \\ \xi &\mapsto \|\xi\|_{C^0(E|_{X_S})} \end{aligned}$$

attains its minimum, which must be strictly positive as $\|\xi\|_{C^{k+1,a}(E|_{X_S})} \geqslant \varepsilon$ for all $\xi \in B$. Therefore there exists a $C(\varepsilon) > 0$ such that

$$\|\xi\|_{C^0(E|_{X_S})} \geqslant \frac{1}{C(\varepsilon)}$$

for all $\xi \in B$, and the result follows: given $\xi \in C^{k+2,a}(E|_{X_S})$ with $\|\xi\|_{C^{k+2,a}(E|_{X_S})} = 1$ we have either $\xi \in B$ or $\|\xi\|_{C^{k+1,a}(E|_{X_S})} < \varepsilon$, and in both cases the inequality (6.8) holds.

∎

The main use of interpolation inequalities is to replace norms such as $\|\cdot\|_{C^{k+1,a}(E|_{X_S})}$ on the right hand side of an estimate with weaker norms such as $\|\cdot\|_{C^0(E|_{X_S})}$, when the left hand side of the estimate is a strong norm such as $\|\cdot\|_{C^{k+2,a}(E|_{X_S})}$. We shall do this in Theorem 6.7 below.

We can now move on to prove the following scale-broken estimate for asymptotically translation invariant differential operators P. Theorem 6.7 is the key result for proving Fredholmness for the asymptotically translation invariant operator $P : B^{k+l,a}_\beta(E) \to B^{k,a}_\beta(F)$.

Theorem 6.7 (Scale Broken Estimate) *Suppose that $K_2 \subseteq \mathbb{R}^L \setminus \mathcal{D}(P)$ is compact. Then there exist $C_1, S > 0$ such that if $\beta \in K_2$ then*

$$\|\xi\|_{B^{k+l,a}_\beta(E)} \leqslant C_1\Big(\|P\xi\|_{B^{k,a}_\beta(F)} + \|\xi\|_{C^0(E|_{X_{2S}})}\Big)$$

for all $\xi \in B^{k+l,a}_\beta(E)$.

Proof: Since $K_2 \subseteq \mathbb{R}^L \setminus \mathcal{D}(P) = \mathbb{R}^L \setminus \mathcal{D}(P_\infty)$ is compact we may take the $C > 0$ as given in Corollary 6.5. Now, given $S \geqslant 0$ define

$$\|P - P_\infty\|_S := \sup\left\{ \big\|(P - P_\infty)\xi\big\|_{B^{k,a}_\beta(F)} : \begin{array}{l} \xi \in B^{k+l,a}_\beta(E) \text{ with } \operatorname{supp}(\xi) \subseteq X \setminus X_S \\ \|\xi\|_{B^{k+l,a}_\beta(E)} \leqslant 1 \end{array} \right\}$$

the operator norm of $P - P_\infty$ restricted to $X \setminus X_S$. It is easy to show, using the asymptotic conditions on $P - P_\infty$, that $\|P - P_\infty\|_S \to 0$ as $S \to \infty$. So fix some $S > 0$ such that

$$\|P - P_\infty\|_S < \frac{1}{C}.$$

Also, fix some $\phi_S \in C^\infty(X)$ such that $\operatorname{supp}(\phi_S) \subseteq X_{2S-\varepsilon}$ and $\operatorname{supp}(1 - \phi_S) \subseteq X \setminus X_{S+\varepsilon}$ where $\varepsilon > 0$ is small.

Now let $\beta \in K_2$. Then given $\xi \in B^{k+l,a}_\beta(E)$ we can write $\xi = \xi_c + \xi_\infty$ where $\xi_c = \phi_S \xi$ and then

$$\|\xi_\infty\|_{B^{k+l,a}_\beta(E)} \leqslant C\|P_\infty \xi_\infty\|_{B^{k,a}_\beta(F)}$$

by Corollary 6.5. It follows that

$$\|\xi_\infty\|_{B^{k+l,a}_\beta(E)} \leqslant C\big(\|P\xi_\infty\|_{B^{k,a}_\beta(F)} + \|P - P_\infty\|_S \|\xi_\infty\|_{B^{k+l,a}_\beta(E)}\big)$$

so that

$$\|\xi_\infty\|_{B^{k+l,a}_\beta(E)} \leqslant C_2 \|P\xi_\infty\|_{B^{k,a}_\beta(F)} \tag{6.9}$$

where $C_2 = \frac{C}{1 - C\|P - P_\infty\|_S}$. We may now estimate

$$\begin{aligned} \|P\xi_\infty\|_{B^{k,a}_\beta(F)} &= \|P(1 - \phi_S)\xi\|_{B^{k,a}_\beta(F)} \\ &\leqslant \|P\xi\|_{B^{k,a}_\beta(F)} + \|\phi_S P\xi\|_{B^{k,a}_\beta(F)} + \|[P, \phi_S]\xi\|_{B^{k,a}_\beta(F)} \\ &\leqslant C_3\Big(\|P\xi\|_{B^{k,a}_\beta(F)} + \|\xi\|_{C^{k+l-1,a}(E|_{X_{2S}})}\Big) \end{aligned} \tag{6.10}$$

for some $C_3 > 0$ independent of β and ξ. This is because the commutator $[P, \phi_S]$ is differential operator of order $l - 1$ which is supported on X_{2S}. Now a standard interpolation inequality, such as in Lemma 6.6, together with inequalities (6.9) and (6.10), allow us to write

$$\|\xi_\infty\|_{B^{k+l,a}_\beta(E)} \leqslant C_4\Big(\|P\xi\|_{B^{k,a}_\beta(F)} + \|\xi\|_{C^0(E|_{X_{2S}})}\Big)$$

for some $C_4 > 0$ independent of β and ξ. The usual Schauder interior estimates [16, Theorem 1] for ξ_c on $X_{2S-\varepsilon} \subseteq X_{2S}$ now finish the proof of the theorem.

∎

Corollary 6.8 *Suppose that $\beta \in \mathbb{R}^L \setminus \mathcal{D}(P)$. If $\mathcal{A} \leqslant B^{k+l,a}_\beta(E)$ is any closed subspace such that*

$$B^{k+l,a}_\beta(E) = \mathcal{A} \oplus \operatorname{Ker}(P)_\beta \tag{6.11}$$

then there exists $C > 0$ such that

$$\|\xi\|_{B^{k+l,a}_\beta(E)} \leqslant C\|P\xi\|_{B^{k,a}_\beta(F)} \tag{6.12}$$

for all $\xi \in \mathcal{A}$.

Proof: Armed with Theorem 6.7 we can proceed as in the compact case of Theorem 3.6.

Take $S > 0$ as in Theorem 6.7 and consider the composite

$$B^{k+l,a}_\beta(E) \to C^{0,a}(E|_{X_{2S}}) \to C^0(E|_{X_{2S}}) \tag{6.13}$$

where the first map is restriction, which is a continuous map. By the Arzelà-Ascoli Theorem, the second map of (6.13) is compact. Therefore the composite (6.13) is compact.

Let $\mathcal{A} \leqslant B^{k+l,a}_\beta(E)$ be a closed subspace and suppose for a contradiction that (6.12) fails. Then there exists a sequence $(\xi_j) \subseteq \mathcal{A}$ such that $\|\xi_j\|_{B^{k+l,a}_\beta(E)} = 1$ for all $j \geqslant 1$ and $\|P\xi_j\|_{B^{k,a}_\beta(F)} \to 0$ as $j \to \infty$. Then by the compactness of (6.13) there is subsequence $(\xi_{j_r}) \subseteq (\xi_j)$ which is convergent, and hence Cauchy, in the $C^0(E|_{X_{2S}})$-norm. It follows from Theorem 6.7 that (ξ_{j_r}) is Cauchy in the $B^{k+l,a}_\beta(E)$-norm, so there exists $\xi \in \mathcal{A}$ with $\|\xi_{j_r} - \xi\|_{B^{k+l,a}_\beta(E)} \to 0$ as $r \to \infty$. But now $\|\xi\|_{B^{k+l,a}_\beta(E)} = 1$ and $P\xi = 0$ force the required contradiction.

∎

Note that, since $\mathrm{Ker}(P)_\beta \leqslant B^{k+l,a}_\beta(E)$ is finite-dimensional we can always find closed subspaces $\mathcal{A} \leqslant B^{k+l,a}_\beta(E)$ such that the decomposition (6.11) holds. It follows from Corollary 6.8 that the image $\mathrm{Im}(P)^{k+l,a}_\beta$ of the bounded linear map $P : B^{k+l,a}_\beta(E) \to B^{k,a}_\beta(F)$ must be closed if $\beta \in \mathbb{R}^L \setminus \mathcal{D}(P)$. In fact, we can do even better, as the next theorem shows.

Theorem 6.9 *If $\beta \in \mathbb{R}^L \setminus \mathcal{D}(P)$ then the bounded linear map*

$$P : B^{k+l,a}_\beta(E) \to B^{k,a}_\beta(F)$$

is Fredholm and has closed image

$$\mathrm{Im}(P)^{k+l,a}_\beta = \Big\{\eta \in B^{k,a}_\beta(F) : \langle \eta | h \rangle_{L^2(F)} = 0 \textit{ for all } h \in \mathrm{Ker}(P^*)_{-\beta}\Big\}.$$

Proof: An integration by parts argument shows that

$$\mathrm{Im}(P)^{k+l,a}_\beta \leqslant \Big\{\eta \in B^{k,a}_\beta(F) : \langle \eta | h \rangle_{L^2(F)} = 0 \text{ for all } h \in \mathrm{Ker}(P^*)_{-\beta}\Big\}.$$

but the problem is to show the reverse inclusion. For this, suppose that $\eta \in B^{k,a}_\beta(F)$ is such that $\langle \eta | h \rangle_{L^2(F)} = 0$ for all $h \in \mathrm{Ker}(P^*)_{-\beta}$. Then, since $\eta \in W^p_{k,\beta+\varepsilon}(F)$ for all $\varepsilon > 0$ we may use the version of Theorem 4.25 for asymptotically translation invariant operators

$$P : W^p_{k+l,\beta+\varepsilon}(E) \to W^p_{k,\beta+\varepsilon}(F)$$

to deduce that there exists $\xi \in \cap_{\varepsilon>0} W^p_{k+l,\beta+\varepsilon}(E)$ such that $P\xi = \eta$. We prove the result by showing that $\xi \in B^{k+l,a}_\beta(E)$. For this, pick $p > 1$ so that $k + l - \frac{n}{p} \geqslant a$. Then from the Embedding Theorem 4.2 we have that $\xi \in B^{0,a}_{\beta+\varepsilon}(E)$ for all $\varepsilon > 0$ and then elliptic regularity as in Theorem 4.12 shows $\xi \in B^{k+l,a}_{\beta+\varepsilon}(E)$ for all $\varepsilon > 0$. To show that in fact $\xi \in B^{k+l,a}_\beta(E)$ we appeal to Theorem 6.7: for then we see that there exist $C_1, S > 0$ independent of $\varepsilon > 0$ such that

$$\|\xi\|_{B^{k+l,a}_{\beta+\varepsilon}(E)} \leqslant C_1\big(\|\eta\|_{B^{k,a}_{\beta+\varepsilon}(F)} + \|\xi\|_{C^0(E|_{X_{2S}})}\big)$$

for all $\varepsilon > 0$. But now as in the proof of Corollary 6.4 we may write $\|\eta\|_{B^{k,a}_{\beta+\varepsilon}(F)} \leqslant C' e^{-\varepsilon T}\|\eta\|_{B^{k,a}_\beta(\tilde{F})}$ for some $C' > 0$ independent of T, η, β and ε. Then we have

$$\|\xi\|_{B^{k+l,a}_{\beta+\varepsilon}(E)} \leqslant C_1\big(C' e^{-\varepsilon T}\|\eta\|_{B^{k,a}_\beta(\tilde{F})} + \|\xi\|_{C^0(E|_{X_{2S}})}\big)$$

for all $\varepsilon > 0$ and we may show that $\xi \in B^{k+l,a}_\beta(E)$ using the method of letting $\varepsilon \to 0$ at each $x = (t, \sigma) \in X_\infty$ as in the proof of Corollary 6.4. Hence we are done: the Fredholmness of P follows immediately because the subspace

$$\Big\{\eta \in B^{k,a}_\beta(F) : \langle \eta | h \rangle_{L^2(F)} = 0 \text{ for all } h \in \mathrm{Ker}(P^*)_{-\beta}\Big\} \leqslant B^{k,a}_\beta(F)$$

has finite codimension $\dim \mathrm{Ker}(P^*)_{-\beta}$.

∎

We now state the immediate corollary for asymptotically conical operators acting on the conical damped Hölder spaces.

Theorem 6.10 *Let $Q : C_c^\infty(E) \to C_c^\infty(F)$ be a uniformly elliptic, asymptotically conical operator of order $l \geqslant 1$ and rate $\gamma \in \mathbb{R}^L$. If $\beta + \gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$ then the map*

$$Q : C^{k+l,a}_{\beta+\gamma}(E) \to C^{k,a}_{\beta}(F)$$

is Fredholm with image

$$\mathrm{Im}(Q)^{k+l,a}_{\beta+\gamma} = \Big\{\eta \in C^{k,a}_{\beta}(F) : \langle \eta | h \rangle_{L^2(F)} = 0 \textit{ for all } h \in \mathrm{Ker}(Q^*)_{-\beta-n}\Big\}$$

which is a subspace of finite codimension $\dim \mathrm{Ker}(Q^*)_{-\beta-n}$.

The following corollary of Theorem 6.10 is immediate.

Corollary 6.11 *Let $Q : C_c^\infty(E) \to C_c^\infty(F)$ be a uniformly elliptic, asymptotically conical operator of order $l \geqslant 1$ and rate $\gamma \in \mathbb{R}^L$. If $\beta + \gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$ then we may write*

$$C^{k,a}_{\beta}(F) = \mathrm{Im}(Q)^{k+l,a}_{\beta+\gamma} \oplus V$$

where $V \leqslant C^{k,a}_{\beta}(F)$ is a subspace of finite dimension $\dim V = \dim \mathrm{Ker}(Q^*)_{-\beta-n}$. *In particular,* $\dim \mathrm{Coker}(Q)^{k+l,a}_{\beta+\gamma} = \dim \mathrm{Ker}(Q^*)_{-\beta-n}$, *and if* $\mathrm{Ker}(Q^*)_{-\beta-n} \leqslant C^{k,a}_{\beta}(F)$ *then we may take V to be equal to* $\mathrm{Ker}(Q^*)_{-\beta-n}$.

It follows from Corollary 6.11 that for all $\beta + \gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$ we have

$$\mathrm{Ind}(Q)^{k+l,a}_{\beta+\gamma} = \dim \mathrm{Ker}(Q)_{\beta+\gamma} - \dim \mathrm{Ker}(Q^*)_{-\beta-n} = \mathrm{Ind}(Q)^{p}_{k+l,\beta+\gamma}. \tag{6.14}$$

It is now easy to see that if $\beta + \gamma \in \mathcal{D}(Q)$ then the bounded linear map $Q : C^{k+l,a}_{\beta+\gamma}(E) \to C^{k,a}_{\beta}(F)$ cannot be Fredholm: for otherwise Theorem 4.24 and equation (6.14) would imply the existence of a continuous family of Fredholm maps

$$\begin{array}{rcl} (-\tilde{\varepsilon}, \tilde{\varepsilon}) & \to & \mathcal{B}\big(C^{k+l,a}_{\beta+\gamma}(E), C^{k,a}_{\beta}(F)\big) \\ \varepsilon & \mapsto & e^{-\varepsilon t} Q e^{\varepsilon t} \end{array}$$

whose Fredholm index jumps as ε crosses $0 \in (-\tilde{\varepsilon}, \tilde{\varepsilon})$. This is a contradiction, because, as in [54, Theorem 1.4.17] and elsewhere, the index is an integer valued continuous function on the set of Fredholm maps $C^{k+l,a}_{\beta+\gamma}(E) \to C^{k,a}_{\beta}(F)$.

As in the case of the (conical) damped Sobolev spaces, we now see from Corollary 6.11 that $\dim \mathrm{Coker}(Q)^{k+l,a}_{\beta+\gamma}$ and $\mathrm{Ind}(Q)^{k+l,a}_{\beta+\gamma}$ are independent of k, a provided that $\beta + \gamma \in \mathbb{R}^L \setminus \mathcal{D}(Q)$, and similarly for asymptotically translation invariant operators on the $B^{k,a}_{\beta}(E)$ spaces.

6.1.2 Exceptional sets for operators derived from AC metrics

Recall that an asymptotically conical metric g on the manifold with ends X induces uniformly elliptic, asymptotically conical operators Δ^r_g and $\mathrm{d}^*_g + \mathrm{d}$, acting on the bundles $\Lambda^r T^*X$ and $\Lambda^* T^*X$ respectively. It is the purpose of this section to give an explicit description of the sets $\mathcal{D}(\Delta^r_g), \mathcal{D}(\mathrm{d}^*_g + \mathrm{d}) \subseteq \mathbb{R}^L$, so that we know when the operators Δ^r_g and $\mathrm{d}^*_g + \mathrm{d}$ are Fredholm when acting between conical damped Sobolev or Hölder spaces. As usual, $\tilde{g}$ denotes the metric on X which is exactly conical on the infinite piece X_∞ of X.

The spectra of compact Riemannian manifolds

We begin with some facts about the spectra $\mathrm{Spec}(\Sigma, g_\Sigma, r)$ of a compact Riemannian manifold (Σ, g_Σ) of dimension $\dim \Sigma = n - 1$. For each $0 \leqslant r \leqslant n-1$ we have $\mathrm{Spec}(\Sigma, g_\Sigma, r)$ a discrete, countable subset of $[0, \infty)$. Also, $0 \in \mathrm{Spec}(\Sigma, g_\Sigma, r)$ precisely when $H^r(\Sigma) \neq 0$. Given $\mu > 0$ we write

$$V^\mu_r := \mathrm{Ker}(\Delta^r_g - \mu)$$

which is a finite-dimensional subspace of $C^\infty(\Lambda^r T^*\Sigma)$. Moreover, $V_r^\mu \neq 0$ precisely when $\mu \in \mathrm{Spec}(\Sigma, g_\Sigma, r)$. We further put

$$\begin{aligned} U_r^\mu &:= V_r^\mu \cap \mathrm{d}C^\infty(\Lambda^{r-1}T^*\Sigma) \\ W_r^\mu &:= V_r^\mu \cap \mathrm{d}_g^* C^\infty(\Lambda^{r+1}T^*\Sigma) \end{aligned}$$

so that given $0 \leqslant r \leqslant n-1$ and $\mu > 0$ we have $U_r^\mu \neq 0$ ($W_r^\mu \neq 0$) precisely when $\mu \in \mathrm{Spec}(\Sigma, g_\Sigma, r)$ and μ has a non-zero (co)exact eigenform. We put

$$\begin{aligned} U_r &:= \{\mu > 0 : U_r^\mu \neq 0\} \\ W_r &:= \{\mu > 0 : W_r^\mu \neq 0\} \end{aligned}$$

so that each U_r, W_r is a discrete, countable subset of $[0, \infty)$ not containing 0. We call U_r the *rth exact spectrum* of (Σ, g_Σ) and W_r the *rth coexact spectrum* of (Σ, g_Σ). Note that $U_0 = \emptyset$ and that $W_0 \cup \{0\} = \mathrm{Spec}(\Sigma, g_\Sigma, 0)$.

It is easy to show that for all $0 \leqslant r \leqslant n-1$ and $\mu > 0$ we have

$$V_r^\mu = U_r^\mu \oplus W_r^\mu$$

and that the maps

$$\mathrm{d}_{g_\Sigma}^* : U_r^\mu \to W_{r-1}^\mu \tag{6.15}$$
$$\mathrm{d} : W_r^\mu \to U_{r+1}^\mu \tag{6.16}$$

are linear isomorphisms. Moreover, if Σ is oriented we have a Hodge star operator $*_{g_\Sigma}$ and linear isomorphisms

$$*_{g_\Sigma} : U_r^\mu \to W_{n-r-1}^\mu \tag{6.17}$$
$$*_{g_\Sigma} : W_r^\mu \to U_{n-r-1}^\mu \tag{6.18}$$
$$*_{g_\Sigma} : H^r(\Sigma) \to H^{n-r-1}(\Sigma). \tag{6.19}$$

From now on we suppose that Σ is oriented. It follows that $H^{n-1}(\Sigma) \cong H^0(\Sigma) \neq 0$, and the set of spectra $\{\mathrm{Spec}(\Sigma, g_\Sigma, r) : 0 \leqslant r \leqslant n-1\}$ is completely determined by:

1. The cohomology groups $H^1(\Sigma), H^2(\Sigma), \ldots, H^{r_1}(\Sigma)$ where $r_1 = \frac{n-2}{2}$ if n is even and $r_1 = \frac{n-1}{2}$ if n is odd.
2. The coexact spectra $W_0, W_1, \ldots, W_{r_2}$ where $r_2 = \frac{n-2}{2}$ if n is even and $r_2 = \frac{n-3}{2}$ if n is odd.

Note that both $H^1(\Sigma), H^2(\Sigma), \ldots, H^{r_1}(\Sigma)$ and $W_0, W_1, \ldots, W_{r_2}$ are "independent" sets of data in that no further identifications can be made using the isomorphisms (6.15), (6.16), (6.17), (6.18), (6.19).

Exceptional sets for Δ_g^r

Recall that differential operator $\Delta_g^r : C_c^\infty(\Lambda^r T^*X) \to C_c^\infty(\Lambda^r T^*X)$ is uniformly elliptic, order 2 and asymptotically conical with rate 2. Moreover the bounded linear maps

$$\Delta_g^r : L^p_{k+2,\beta+2}(\Lambda^r T^*X) \to L^p_{k,\beta}(\Lambda^r T^*X) \tag{6.20}$$
$$\Delta_g^r : C^{k+2,a}_{\beta+2}(\Lambda^r T^*X) \to C^{k,a}_{\beta}(\Lambda^r T^*X) \tag{6.21}$$

are Fredholm precisely when $\beta + 2 \in \mathbb{R}^L \backslash \mathcal{D}(\Delta_g^r)$ where $\mathcal{D}(\Delta_g^r) = \mathcal{D}(P_\infty)$ is computed as in Section 4.2.2. As in Table 4.1, the translation invariant operator $P_\infty : C_c^\infty(\Lambda^r T^*X) \to C_c^\infty(\Lambda^r T^*X)$ corresponding to the Laplacian Δ_g^r acts as

$$P_\infty = e^{(2-r)t} \Delta_{\tilde{g}}^r e^{rt}. \tag{6.22}$$

We view the bundle $\Lambda^r T^*X$ as being admissible with slice $\Lambda^r T^*\Sigma \oplus \Lambda^{r-1}T^*\Sigma$ got via the vector bundle isomorphism

$$\begin{aligned} \pi^*(\Lambda^r T^*\Sigma) \oplus \pi^*(\Lambda^{r-1}T^*\Sigma) &\cong \Lambda^r T^* X_\infty \\ (\psi, \phi) &\leftrightarrow \psi + \mathrm{d}t \wedge \phi \end{aligned} \tag{6.23}$$

and then we have the following result.

Lemma 6.12 *If we work with the identification* (6.23) *then the translation invariant operator* (6.22) *corresponding to* Δ_g^r *acts as*

$$P_\infty = \begin{pmatrix} \Delta_{g_\Sigma}^r - \left(\frac{\partial}{\partial t} + r\right)\left(\frac{\partial}{\partial t} + n - r - 2\right) & -2\mathrm{d} \\ -2\mathrm{d}_{g_\Sigma}^* & \Delta_{g_\Sigma}^{r-1} - \left(\frac{\partial}{\partial t} + r - 2\right)\left(\frac{\partial}{\partial t} + n - r\right) \end{pmatrix}$$

on the bundle $\pi^*(\Lambda^r T^*\Sigma) \oplus \pi^*(\Lambda^{r-1}T^*\Sigma)$.

The proof of Lemma 6.12 is a long, but entirely straightforward calculation: we omit the details. It follows from Lemma 6.12 that for each $w \in \mathbb{C}$ and $1 \leqslant j \leqslant L$ the operator $P_\infty(w)$ acts as

$$P_\infty(w) = \begin{pmatrix} \Delta_{g_\Sigma}^r - (w + r)(w + n - r - 2) & -2\mathrm{d} \\ -2\mathrm{d}_{g_\Sigma}^* & \Delta_{g_\Sigma}^{r-1} - (w + r - 2)(w + n - r) \end{pmatrix} \tag{6.24}$$

on the complexified bundle $(\Lambda^r T^*\Sigma_j \otimes \mathbb{C}) \oplus (\Lambda^{r-1}T^*\Sigma_j \otimes \mathbb{C})$. Given $w \in \mathbb{C}$ we define

$$\begin{aligned} a(w) &:= (w + r)(w + n - r - 2) & b(w) &:= (w + r - 2)(w + n - r) \\ c(w) &:= (w + r)(w + n - r) & d(w) &:= (w + r - 2)(w + n - r - 2) = c(w - 2). \end{aligned}$$

Note that by completing the square in w it is easy to show that if any one of $a(w), b(w), c(w), d(w)$ is real and non-negative then $w \in \mathbb{R}$. We can now state for which $w \in \mathbb{C}$ the map

$$P_\infty(w) : W_{k+2}^p\big((\Lambda^r T^*\Sigma_j \otimes \mathbb{C}) \oplus (\Lambda^{r-1}T^*\Sigma_j \otimes \mathbb{C})\big) \to W_k^p\big((\Lambda^r T^*\Sigma_j \otimes \mathbb{C}) \oplus (\Lambda^{r-1}T^*\Sigma_j \otimes \mathbb{C})\big)$$

is an isomorphism of Banach spaces: recall the definition of the subsets $\mathcal{C}(P_\infty, j) \subseteq \mathbb{C}$ from Section 4.2.2.

Lemma 6.13 *Let* $w \in \mathbb{C}$. *Then* $w \in \mathcal{C}(P_\infty, j)$ *precisely when at least one of the following holds:*

1. $a(w) = 0$ *and* $H^r(\Sigma_j) \neq 0$	2. $b(w) = 0$ *and* $H^{r-1}(\Sigma_j) \neq 0$
3. $a(w) \in W_r$	4. $b(w) \in W_{r-2}$
5. $c(w) \in W_{r-1}$	6. $d(w) \in W_{r-1}$

where the coexact spectra are those of the Riemannian manifold (Σ_j, g_Σ).

Note that cases 5 and 6 of are just translates of each other. The proof of Lemma 6.13 is a messy case by case analysis, whose details we omit. Since each of the cases of Lemma 6.13 give rise to a real, non-negative w it follows that $\mathcal{D}(P_\infty, j) = \mathcal{C}(P_\infty, j)$ for each $1 \leqslant j \leqslant L$ and we now have a very explicit picture for when the bounded linear maps (6.20) and (6.21) are Fredholm.

Corollary 6.14 *Let* $\beta + 2 \in \mathbb{R}^L$. *Then* $\beta + 2 \in \mathcal{D}(\Delta_g^r)$ *precisely when there exists a* $1 \leqslant j \leqslant L$ *such that* $\beta_j + 2 = w$ *for some* w *satisfying at least one of* 1, 2, 3, 4, 5, 6 *of Lemma* 6.13.

Using Lemma 6.13 we can also deduce the existence of open subsets

$$\underbrace{I_r \times \cdots \times I_r}_{L \text{ factors}} \subseteq \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^r) \tag{6.25}$$

where each $I_r \subseteq \mathbb{R}$ is a "good interval" for the growth parameters $\beta_j + 2$.

Corollary 6.15 *If* $0 \leqslant r < \frac{n}{2} - 1$ *define* $I_r := (r + 2 - n, -r)$, *an open interval of length* $n - 2 - 2r > 0$ *and if* $\frac{n}{2} + 1 < r \leqslant n$, *define* $I_r := (2 - r, r - n)$ *an open interval of length* $2r - n - 2 > 0$. *Then the inclusion* (6.25) *holds.*

Corollary 6.15 is easy to deduce from Lemma 6.13 and the definitions of the polynomials $a(w)$, $b(w)$, $c(w)$, $d(w)$ given above: we omit the details. Note that for r close to 0 or n the "good interval" I_r has length close to $n - 2$, and the length of I_r decreases to 0 as r approaches $\frac{n}{2}$.

The following two examples will be particularly useful for us: note we have already calculated $\mathcal{D}(\Delta_g^0)$ in Section 5.1.1.

Example 6.16 *The Laplacian Δ^1_g on 1-forms. After some brief calculations we find that if $1 \leqslant j \leqslant L$ then $\beta_j + 2 \in \mathcal{D}(\Delta^1_g, j)$ precisely when at least one of the following hold:*

1. $(\beta_j + 3)(\beta_j + n + 1) \in \mathrm{Spec}(\Sigma_j, g_\Sigma, 0)$
2. $(\beta_j + 1)(\beta_j + n - 1) \in W_0$,

where the coexact spectrum W_0 is that of the Riemannian manifold (Σ_j, g_Σ).

Example 6.17 *The Laplacian Δ^2_g on 2-forms. After some brief calculations we find that if $1 \leqslant j \leqslant L$ then $\beta_j + 2 \in \mathcal{D}(\Delta^2_g, j)$ precisely when at least one of the following hold:*

1. *($\beta_j + 2 = -2$ or $\beta_j + 2 = -n$) and $H^2(\Sigma_j) \neq 0$*
2. *($\beta_j + 2 = 0$ or $\beta_j + 2 = 2 - n$) and $H^1(\Sigma_j) \neq 0$*
3. $(\beta_j + 4)(\beta_j + n - 2) \in W_2$
4. $(\beta_j + 2)(\beta_j + n) \in W_0$
5. $(\beta_j + 4)(\beta_j + n) \in W_1$
6. $(\beta_j + 2)(\beta_j + n - 2) \in W_1$,

where the coexact spectra are those of the Riemannian manifold (Σ_j, g_Σ). Note that case 4 consists precisely of the points in $\mathcal{D}(\Delta^0_g, j) \setminus \{0, 2 - n\}$.

Exceptional sets for $\mathrm{d}^*_g + \mathrm{d}$

Recall that differential operator $\mathrm{d}^*_g + \mathrm{d} : C^\infty_c(\Lambda^* T^* X) \to C^\infty_c(\Lambda^* T^* X)$ is uniformly elliptic, order 1 and asymptotically conical with rate 1. Moreover the bounded linear maps

$$\mathrm{d}^*_g + \mathrm{d} : L^p_{k+1,\beta+1}(\Lambda^* T^* X) \;\to\; L^p_{k,\beta}(\Lambda^* T^* X) \tag{6.26}$$

$$\mathrm{d}^*_g + \mathrm{d} : C^{k+1,a}_{\beta+1}(\Lambda^* T^* X) \;\to\; C^{k,a}_{\beta}(\Lambda^* T^* X) \tag{6.27}$$

are Fredholm precisely when $\beta + 1 \in \mathbb{R}^L \setminus \mathcal{D}(\mathrm{d}^*_g + \mathrm{d})$ where $\mathcal{D}(\mathrm{d}^*_g + \mathrm{d}) = \mathcal{D}(P_\infty)$ is computed as in Section 4.2.2. As in Table 4.1 the translation invariant operator $P_\infty : C^\infty_c(\Lambda^* T^* X) \to C^\infty_c(\Lambda^* T^* X)$ corresponding to $\mathrm{d}^*_g + \mathrm{d}$ acts as

$$P_\infty = e^{-rt}(e^{2t}\mathrm{d}^*_g + \mathrm{d})e^{rt} \tag{6.28}$$

on r-forms. We view the bundle $\Lambda^* T^* X$ as being admissible with slice $\Lambda^* T^* \Sigma \oplus \Lambda^* T^* \Sigma$ got via the vector bundle isomorphism

$$\begin{aligned} \pi^*(\Lambda^* T^* \Sigma) \oplus \pi^*(\Lambda^* T^* \Sigma) &\;\cong\; \Lambda^* T^* X_\infty \\ (\psi, \phi) &\;\leftrightarrow\; \psi + \mathrm{d}t \wedge \phi. \end{aligned} \tag{6.29}$$

We now have the following result.

Lemma 6.18 *If we work with the identification* (6.29) *then the translation invariant operator* (6.28) *corresponding to $\mathrm{d}^*_g + \mathrm{d}$ acts as*

$$P_\infty = \begin{pmatrix} \mathrm{d}^*_{g_\Sigma} + \mathrm{d} & -(\frac{\partial}{\partial t} + n - r - 1) \\ \frac{\partial}{\partial t} + r & -(\mathrm{d}^*_{g_\Sigma} + \mathrm{d}) \end{pmatrix} \tag{6.30}$$

on the bundle $\pi^*(\Lambda^* T^* \Sigma) \oplus \pi^*(\Lambda^* T^* \Sigma)$. In equation (6.30) *$r$ denotes the operator which multiplies r-forms by r.*

The proof of Lemma 6.18 is a long, but entirely straightforward calculation: we omit the details. It follows from Lemma 6.18 that for each $w \in \mathbb{C}$ and $1 \leqslant j \leqslant L$ the operator $P_\infty(w)$ acts as

$$P_\infty(w) = \begin{pmatrix} \mathrm{d}^*_{g_\Sigma} + \mathrm{d} & -(w + n - r - 1) \\ w + r & -(\mathrm{d}^*_{g_\Sigma} + \mathrm{d}) \end{pmatrix} \tag{6.31}$$

on the complexified bundle $(\Lambda^* T^* \Sigma_j \otimes \mathbb{C}) \oplus (\Lambda^* T^* \Sigma_j \otimes \mathbb{C})$. Recall the definitions of $a(w)$, $b(w)$, $c(w)$, $d(w)$ given above. We can now state for which $w \in \mathbb{C}$ the map

$$P_\infty(w) : W^p_{k+1}\big((\Lambda^* T^* \Sigma_j \otimes \mathbb{C}) \oplus (\Lambda^* T^* \Sigma_j \otimes \mathbb{C})\big) \to W^p_k\big((\Lambda^* T^* \Sigma_j \otimes \mathbb{C}) \oplus (\Lambda^* T^* \Sigma_j \otimes \mathbb{C})\big)$$

is an isomorphism of Banach spaces.

Lemma 6.19 *Let $w \in \mathbb{C}$. Then $w \in \mathcal{C}(P_\infty, j)$ precisely when there exists an $0 \leqslant r \leqslant n$ such that at least one of the following holds:*

1. $w + r = 0$ *and* $H^r(\Sigma_j) \neq 0$
2. $w + n - r = 0$ *and* $H^{r-1}(\Sigma_j) \neq 0$
3. $a(w) \in W_r$
4. $b(w) \in W_{r-2}$
5. $c(w) \in W_{r-1}$

where the coexact spectra are those of the Riemannian manifold (Σ_j, g_Σ).

Again, we omit the details. Since each of the cases of Lemma 6.19 give rise to a real, non-negative w it follows that $\mathcal{D}(P_\infty, j) = \mathcal{C}(P_\infty, j)$ for each $1 \leqslant j \leqslant L$ and hence we have a very explicit picture for when the bounded linear maps (6.26) and (6.27) are Fredholm.

Corollary 6.20 *Let $\beta + 1 \in \mathbb{R}^L$. Then $\beta + 1 \in \mathcal{D}(\mathrm{d}_g^* + \mathrm{d})$ precisely when there exist $1 \leqslant j \leqslant L$ and $0 \leqslant r \leqslant n$ such that $\beta_j + 1 = w$ for some w satisfying at least one of* 1, 2, 3, 4, 5 *of Lemma* 6.19.

Note also from Lemma 6.13 and Lemma 6.19 that

$$\mathcal{D}(\mathrm{d}_g^* + \mathrm{d}) \subseteq \bigcup_{r=0}^{n} \mathcal{D}(\Delta_g^r),$$

but this is exactly what we should expect, because, for example, the composite map

$$\Delta_g : L^p_{k+2,\beta+2}(\Lambda^* T^* X) \xrightarrow{\mathrm{d}_g^*+\mathrm{d}} L^p_{k+1,\beta+1}(\Lambda^* T^* X) \xrightarrow{\mathrm{d}_g^*+\mathrm{d}} L^p_{k,\beta}(\Lambda^* T^* X) \tag{6.32}$$

is Fredholm precisely when

$$\beta + 2 \in \mathbb{R}^L \setminus \bigcup_{r=0}^{n} \mathcal{D}(\Delta_g^r),$$

and if (6.32) has finite dimensional cokernel then so has

$$\mathrm{d}_g^* + \mathrm{d} : L^p_{k+1,\beta+1}(\Lambda^* T^* X) \to L^p_{k,\beta}(\Lambda^* T^* X).$$

Note that in order to refine some of the above results one could also consider the bundles $\Lambda^{odd/even} T^* X$ over X, which are admissible with slice $\Lambda^{odd/even} T^*\Sigma \oplus \Lambda^{even/odd} T^*\Sigma$. Then the differential operator

$$(\mathrm{d}_g^* + \mathrm{d})^{odd/even} : C_c^\infty(\Lambda^{odd/even} T^* X) \to C_c^\infty(\Lambda^{even/odd} T^* X) \tag{6.33}$$

is uniformly elliptic, order 1 and asymptotically conical with rate 1. Using the techniques described above, it is easy to work out the exceptional sets of the operators (6.33), and it turns out that in building these subsets $\mathcal{D}\big((\mathrm{d}_g^* + \mathrm{d})^{odd/even}\big) \subseteq \mathbb{R}^L$ one only takes the $0 \leqslant r \leqslant n$ in Lemma 6.19 which are odd or even accordingly. As an application, we give two examples in low dimensions.

Example 6.21 $\dim X = 3$. *Note that the only "independent" pieces of data are $H^1(\Sigma)$ and W_0. After some brief calculations we find that if $1 \leqslant j \leqslant L$ then $\beta_j + 2 \in \mathcal{D}\big((\mathrm{d}_g^* + \mathrm{d})^{odd}, j\big)$ precisely when at least one of the following hold:*

1. $\beta_j + 1 = -2$ *or* $\beta_j + 1 = 0$
2. $\beta_j + 1 = -1$ *and* $H^1(\Sigma_j) \neq 0$
3. $(\beta_j + 1)(\beta_j + 2) \in W_0$
4. $(\beta_j + 2)(\beta_j + 3) \in W_0$,

where the coexact spectrum W_0 is that of the Riemannian manifold (Σ_j, g_Σ).

Example 6.22 $\dim X = 4$. *Note that the only "independent" pieces of data are $H^1(\Sigma)$, W_0 and W_1. After some brief calculations we find that if $1 \leqslant j \leqslant L$ then $\beta_j + 2 \in \mathcal{D}\big((\mathrm{d}_g^* + \mathrm{d})^{odd}, j\big)$ precisely when at least one of the following hold:*

1. $\beta_j + 1 = -3$
2. $\beta_j + 1 = -1$ *and* $H^1(\Sigma_j) \neq 0$
3. $(\beta_j + 2)(\beta_j + 4) \in W_0$
4. $(\beta_j + 2)^2 \in W_1$,

where the coexact spectra are those of the Riemannian manifold (Σ_j, g_Σ).

6.1.3 "Hodge theory" on non-compact manifolds

Although we shall be working with the conical damped Hölder spaces in this section, the material can be easily converted to the setting of the conical damped Sobolev spaces.

For non-compact Riemannian manifolds (X, g), many of the results of Section 3.1.3 fail to hold. In particular, if we suppose that X is a manifold with ends equipped with an asymptotically conical metric g then we have

$$\left\{\xi \in C^{1,a}_{\beta+2}(\Lambda^r T^*X) : \mathrm{d}^*_g\xi = 0 \text{ and } \mathrm{d}\xi = 0\right\} \leqslant \operatorname{Ker}(\Delta^r_g)_{\beta+2} \tag{6.34}$$

but in general the reverse inclusion in (6.34) fails to hold. Indeed, when $\beta + 2$ is too large the usual integration by parts argument is not valid. We can, however, give some useful results, and begin with the following:

Lemma 6.23 *If $(\beta + 1) + (\delta + 1) < 2 - n$ then*

$$\mathrm{d}^*_g\big(C^{k+1,a}_{\beta+1}(\Lambda^{r+1}T^*X)\big) \cap \mathrm{d}\big(C^{k+1,a}_{\delta+1}(\Lambda^{r-1}T^*X)\big) = 0.$$

Proof: Put $2\varepsilon := (2 - n) - (\beta + 1) - (\delta + 1) > 0$. If

$$\begin{aligned} \xi &\in C^{k+1,a}_{\beta+1}(\Lambda^{r+1}T^*X) \\ \eta &\in C^{k+1,a}_{\delta+1}(\Lambda^{r-1}T^*X) \end{aligned}$$

we use the fact that

$$\begin{aligned} \xi &\in L^p_{1,(\beta+\varepsilon)+1}(\Lambda^{r+1}T^*X) \\ \mathrm{d}\eta &\in L^{p'}_{0,-(\beta+\varepsilon)-n}(\Lambda^r T^*X) \end{aligned}$$

to write $\langle \mathrm{d}^*_g\xi|\mathrm{d}\eta\rangle_{L^2(\Lambda^r T^*X)} = \langle \xi|\mathrm{d}\mathrm{d}\eta\rangle_{L^2(\Lambda^{r+1}T^*X)} = 0$, as required. ∎

The following result gives conditions which ensure that the reverse inclusion of (6.34) holds.

Lemma 6.24 *If $\xi \in \operatorname{Ker}(\Delta^r_g)_{\beta+2}$ then $\mathrm{d}^*_g\xi = 0$ and $\mathrm{d}\xi = 0$ whenever either of the following conditions hold:*

1. $\beta + 2 < 1 - \frac{n}{2}$
2. $\beta + 2 < -r$ *or* $\beta + 2 < r - n$.

Proof: Let $\xi \in \operatorname{Ker}(\Delta^r_g)_{\beta+2}$. When condition 1 holds the integration by parts

$$0 = \langle \xi|\Delta^r_g\xi\rangle_{L^2(\Lambda^r T^*X)} = \langle \mathrm{d}\xi|\mathrm{d}\xi\rangle_{L^2(\Lambda^{r+1}T^*X)} + \langle \mathrm{d}^*_g\xi|\mathrm{d}^*_g\xi\rangle_{L^2(\Lambda^{r-1}T^*X)}$$

is valid, as in the proof of Lemma 6.23. We now introduce

3. $0 \leqslant r < \frac{n}{2} - 1$ and $\beta + 2 < -r$
4. $\frac{n}{2} + 1 < r \leqslant n$ and $\beta + 2 < r - n$.

If either condition 3 or condition 4 holds then from Corollary 6.15 and equation (4.53) we deduce that $\xi \in \operatorname{Ker}(\Delta^r_g)_{\delta+2}$ for some $\delta + 2 < 1 - \frac{n}{2}$, since in both cases we have $1 - \frac{n}{2} \in I_r$ and $\beta_j + 2 < \sup I_r$ for each $1 \leqslant j \leqslant L$. Therefore the conclusions of the lemma hold under either condition 3 or condition 4.

Supposing that $r \geqslant \frac{n}{2} - 1$ and $\beta + 2 < -r$, we have

$$\beta + 2 < -r \leqslant 1 - \frac{n}{2}$$

so that $\mathrm{d}^*_g\xi = \mathrm{d}\xi = 0$ by the first case. Similarly, if $r \leqslant \frac{n}{2} + 1$ and $\beta + 2 < r - n$ then

$$\beta + 2 < r - n \leqslant \frac{n}{2} + 1 - n = 1 - \frac{n}{2}$$

so that, again, $\mathrm{d}^*_g\xi = \mathrm{d}\xi = 0$. The result now follows. ∎

Corollary 6.25 *If $\beta+2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^r)$ then*

$$\Delta_g^r\big(C^{k+2,a}_{\beta+2}(\Lambda^r T^*X)\big) = \mathrm{d}_g^*\big(C^{k+1,a}_{\beta+1}(\Lambda^{r+1}T^*X)\big) + \mathrm{d}\big(C^{k+1,a}_{\beta+1}(\Lambda^{r-1}T^*X)\big)$$

whenever either of the following conditions hold:

1. $\beta+2 > 1-\frac{n}{2}$
2. $\beta+2 > r+2-n$ *or* $\beta+2 > 2-r$.

Proof: It is obvious that

$$\Delta_g^r\big(C^{k+2,a}_{\beta+2}(\Lambda^r T^*X)\big) \leqslant \mathrm{d}_g^*\big(C^{k+1,a}_{\beta+1}(\Lambda^{r+1}T^*X)\big) + \mathrm{d}\big(C^{k+1,a}_{\beta+1}(\Lambda^{r-1}T^*X)\big)$$

always. In order to prove the reverse inclusion, take $\xi \in C^{k+1,a}_{\beta+1}(\Lambda^{r+1}T^*X)$ and $\eta \in C^{k+1,a}_{\beta+1}(\Lambda^{r-1}T^*X)$. Then $\mathrm{d}_g^*\xi+\mathrm{d}\eta \in \Delta_g^r C^{k+2,a}_{\beta+2}(\Lambda^r T^*X)$ precisely when $\langle \mathrm{d}_g^*\xi+\mathrm{d}\eta | h\rangle_{L^2(\Lambda^r T^*X)} = 0$ for all $h \in \mathrm{Ker}(\Delta_g^r)_{-\beta-n}$. But from Lemma 6.24 this is clearly the case whenever one of

3. $-\beta-n < 1-\frac{n}{2}$
4. $-\beta-n < -r$ or $-\beta-n < r-n$

hold, and these conditions are exactly those given.

∎

Lemma 6.26 *If either of the following conditions hold:*

1. $\beta+2 \geqslant 2-\frac{n}{2}$
2. $\beta+2 > r+4-n$ *or* $\beta+2 > 4-r$

then $\mathrm{Ker}(\Delta_g^r)_{-\beta-n} \leqslant C^{k,a}_{\beta}(\Lambda^r T^*X)$.

Proof: If the first condition holds then $-\beta-n \leqslant \beta$ and we are done. Suppose now, for example, that $\beta+2 > r+4-n$. If $r < \frac{n}{2}-1$ then we have $I_r = (r+2-n, -r)$ and we are done since $-\beta-n < -r$ and $\beta > r+2-n$. If, on the other hand $r \geqslant \frac{n}{2}-1$ then

$$\beta+2 > r+4-n \geqslant \tfrac{n}{2}-1+4-n = 3-\tfrac{n}{2} > 2-\tfrac{n}{2}$$

so that the first condition holds, and we are also done. The case $\beta+2 > 4-r$ is handled similarly, proving the lemma.

∎

Using Corollary 6.11, Lemma 6.23, Lemma 6.24, Corollary 6.25 and Lemma 6.26 one can prove various Hodge decomposition theorems analogous to those of Section 3.1.3 which hold in the compact case, for example: Proposition 3.11.

Another typical failure in the asymptotically conical setting is that although we always have

$$(\mathrm{d}_g^*+\mathrm{d})\big(C^{k+1,a}_{\beta+1}(\Lambda^r T^*X)\big) \leqslant \mathrm{d}_g^*\big(C^{k+1,a}_{\beta+1}(\Lambda^r T^*X)\big) \oplus \mathrm{d}\big(C^{k+1,a}_{\beta+1}(\Lambda^r T^*X)\big)$$

the reverse inclusion will in general not hold. However, in the case of most interest to us, namely $r=1$, we can employ a device to get round this. The point in the proof of the following result is that the Laplacian on functions is only $\mathrm{d}_g^*\mathrm{d}$, rather than $\mathrm{d}_g^*\mathrm{d}+\mathrm{d}\mathrm{d}_g^*$.

Lemma 6.27 *If $\beta+2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$ and $\beta+2 > 2-n$ then*

$$(\mathrm{d}_g^*+\mathrm{d})\big(C^{k+1,a}_{\beta+1}(T^*X)\big) = \mathrm{d}_g^*\big(C^{k+1,a}_{\beta+1}(T^*X)\big) \oplus \mathrm{d}\big(C^{k+1,a}_{\beta+1}(T^*X)\big) \tag{6.35}$$

and furthermore $\mathrm{d}_g^*\big(C^{k+1,a}_{\beta+1}(T^*X)\big) = C^{k,a}_{\beta}(X)$.

Proof: As noted above, one inclusion of equation (6.35) is obvious. To prove the other inclusion, take $\xi_1, \xi_2 \in C^{k+1,a}_{\beta+1}(T^*X)$. Since the map

$$\Delta^0_g : C^{k+2,a}_{\beta+2}(X) \to C^{k,a}_{\beta}(X) \tag{6.36}$$

is surjective there exists $f \in C^{k+2,a}_{\beta+2}(X)$ such that $\Delta^0_g f = \mathrm{d}^*_g\xi_1 - \mathrm{d}^*_g\xi_2$. Putting $\xi = \xi_2 + \mathrm{d}f$ we see that $(\mathrm{d}^*_g + \mathrm{d})\xi = \mathrm{d}^*_g\xi_1 + \mathrm{d}\xi_2$ and we have proved (6.35). The final equation follows from the fact that (6.36) is surjective.

∎

When the metric g has a stronger decay rate than merely being asymptotically conical we can actually say more than Lemma 6.27. But we first give an intermediate result.

Lemma 6.28 *Suppose that the metric g is strongly asymptotically conical with rate $\alpha < 0$. If* $\max\{2-n+\alpha, 2-n-\lambda\} < \beta + 2 < 2 - n$ *and* $\xi \in C^{k+1,a}_{\beta+1}(T^*X)$ *then:*

1. *There exists $f \in C^{\infty}_{2-n}(X)$ and $\tilde{f} \in C^{k+2,a}_{\beta+2}(X)$ such that*
$$\Delta^0_g(\tilde{f} + f) = \mathrm{d}^*_g\xi. \tag{6.37}$$
2. *There exists $f_b \in C^{\infty}(X)$ constant on the ends of X and $F \in C^{k+2,a}_{\beta+2}(X)$ such that*
$$\Delta^0_g(F + f_b) = \mathrm{d}^*_g\xi. \tag{6.38}$$

Proof: The proof of this result is just as for its Sobolev counterpart Corollary 5.16: the important point is that we now have Theorem 6.10 at our disposal.

∎

Corollary 6.29 *Suppose that the metric g is strongly asymptotically conical with rate $\alpha < 0$. If* $\max\{2-n+\alpha, 2-n-\lambda\} < \beta + 2 < 2 - n$ *then*

$$(\mathrm{d}^*_g + \mathrm{d})\big(C^{k+1,a}_{\beta+1}(T^*X)\big) = \mathrm{d}^*_g\big(C^{k+1,a}_{\beta+1}(T^*X)\big) \oplus \mathrm{d}\big(C^{k+1,a}_{\beta+1}(T^*X)\big). \tag{6.39}$$

Proof: Take $\beta + 2 \in \mathbb{R}^L$ as given. It is clear that the left hand side of (6.39) is contained inside the right hand side. To prove the reverse inclusion, take $\xi_1, \xi_2 \in C^{k+1,a}_{\beta+1}(T^*X)$. By Lemma 6.28 there exists $f_b \in C^{\infty}(X)$ constant on the ends of X and $F \in C^{k+2,a}_{\beta+2}(X)$ such that $\Delta^0_g(F + f_b) = \mathrm{d}^*_g\xi_1 - \mathrm{d}^*_g\xi_2$. Putting $\xi := \xi_2 + \mathrm{d}F + \mathrm{d}f_b \in C^{k+1,a}_{\beta+1}(T^*X)$ we have $(\mathrm{d}^*_g + \mathrm{d})\xi = \mathrm{d}^*_g\xi_1 + \mathrm{d}\xi_2$, so that we are done.

∎

For applications in the sequel we now consider when the linear subspace

$$(\mathrm{d}^*_g + \mathrm{d})\big(C^{k+1,a}_{\beta+1}(T^*X)\big) \leqslant C^{k,a}_{\beta}(X) \oplus C^{k,a}_{\beta}(\Lambda^2 T^*X) \tag{6.40}$$

is closed. Note that since $\mathrm{d}^*_g\big(C^{k+1,a}_{\beta+1}(T^*X)\big) \leqslant C^{k,a}_{\beta}(X)$ is closed whenever $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$ it follows from Lemma 6.27 that for $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$ with $\beta + 2 > 2 - n$ we have (6.40) a closed subspace whenever

$$\mathrm{d}\big(C^{k+1,a}_{\beta+1}(T^*X)\big) \leqslant C^{k,a}_{\beta}(\Lambda^2 T^*X) \tag{6.41}$$

is closed. Using Corollary 6.39 we see a similar remark applies for $\max\{2-n+\alpha, 2-n-\lambda\} < \beta+2 < 2-n$ when the metric g on X is strongly asymptotically conical with rate $\alpha < 0$. Unfortunately the closure of the subspace (6.41) is not something that can be decided using the Laplacian on functions alone. Instead the operators Δ^1_g, Δ^2_g or $\mathrm{d}^*_g + \mathrm{d}$ must be introduced. This is, for example, one use of the computations of the relevant exceptional subsets in Section 6.1.2.

Proposition 6.30 *The bounded linear map*

$$\mathrm{d}_g^* + \mathrm{d} : C^{k+1,a}_{\beta+1}(T^*X) \to C^{k,a}_\beta(X) \oplus C^{k,a}_\beta(\Lambda^2 T^*X) \tag{6.42}$$

has closed image whenever any of the following conditions hold:

1. $\beta + 1 \in \mathbb{R}^L \setminus \mathcal{D}\big((\mathrm{d}_g^* + \mathrm{d})^{odd}\big)$
2. $k \geqslant 1$, $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^2)$, $\beta + 2 > 2 - n$ *and no coordinate of* $\beta + 2$ *is* 0
3. $k \geqslant 1$, $\beta + 1 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^1)$ *and* $\beta + 2 < 0$.

Proof: If condition 1 holds pick any closed linear subspace $\mathcal{A}_1 \leqslant C^{k+1,a}_{\beta+1}(\Lambda^{odd} T^*X)$ such that

$$C^{k+1,a}_{\beta+1}(\Lambda^{odd} T^*X) = \mathcal{A}_1 \oplus \mathrm{Ker}\big((\mathrm{d}_g^* + \mathrm{d})^{odd}\big)_{\beta+1}$$

and then by conical damped version of Corollary 6.8 there exists a $C > 0$ such that

$$\|\eta\|_{C^{k+1,a}_{\beta+1}(\Lambda^* T^*X)} \leqslant C \|\mathrm{d}_g^*\eta + \mathrm{d}\eta\|_{C^{k,a}_\beta(\Lambda^* T^*X)}$$

for all $\eta \in \mathcal{A}_1$. Now define $\tilde{\mathcal{A}}_1 := \mathcal{A}_1 \cap C^{k+1,a}_{\beta+1}(T^*X)$ and we have

$$C^{k+1,a}_{\beta+1}(T^*X) = \tilde{\mathcal{A}}_1 \oplus \big\{\xi \in C^{k+1,a}_{\beta+1}(T^*X) : \mathrm{d}_g^*\eta = 0 \text{ and } \mathrm{d}\eta = 0\big\}$$

with

$$\|\eta\|_{C^{k+1,a}_{\beta+1}(T^*X)} \leqslant C \|\mathrm{d}_g^*\eta + \mathrm{d}\eta\|_{C^{k,a}_\beta(\Lambda^* T^*X)} \tag{6.43}$$

for all $\eta \in \tilde{\mathcal{A}}_1$. Suppose now that $f \in C^{k,a}_\beta(X)$ and $\xi \in C^{k,a}_\beta(\Lambda^2 T^*X)$ are such that $f + \xi$ lies in the closure of the image of the map (6.42). Then there exist $(\eta_j) \subseteq \tilde{\mathcal{A}}_1$ such that $\mathrm{d}_g^*\eta_j \to f$ in $C^{k,a}_\beta(X)$ and $\mathrm{d}\eta_j \to \xi$ in $C^{k,a}_\beta(\Lambda^2 T^*X)$. It follows that $(\mathrm{d}_g^*\eta_j + \mathrm{d}\eta_j) \subseteq C^{k,a}_\beta(\Lambda^* T^*X)$ is Cauchy and hence from the inequality (6.43) we see that $(\eta_j) \subseteq C^{k+1,a}_{\beta+1}(T^*X)$ is Cauchy, and hence convergent to some $\eta \in \tilde{\mathcal{A}}_1$. Clearly $\mathrm{d}_g^*\eta = f$ and $\mathrm{d}\eta = \xi$ so that $f + \xi$ lies in the image of the map (6.42). Therefore (6.42) has closed image.

If condition 2 holds, then note from Example 6.17 that $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$. Let

$$W_1 := \Big\{\xi \in C^{k,a}_\beta(\Lambda^2 T^*X) : \mathrm{d}_g^*\xi = 0 \text{ and } \mathrm{d}\xi = 0\Big\}$$

which is a finite-dimensional vector space. Also let W_2 denote the image of the map (6.42). We begin by showing that $W_1 + W_2$ is closed in $C^{k,a}_\beta(X) \oplus C^{k,a}_\beta(\Lambda^2 T^*X)$. For this take any $(\eta_j) \subseteq C^{k+1,a}_{\beta+1}(T^*X)$ and $(w_j) \subseteq W_1$ such that $\mathrm{d}_g^*\eta_j \to f$ in $C^{k,a}_\beta(X)$ and $\mathrm{d}\eta_j + w_j \to \xi$ in $C^{k,a}_\beta(\Lambda^2 T^*X)$. Now since the map $\Delta_g^2 : C^{k+2,a}_{\beta+2}(\Lambda^2 T^*X) \to C^{k,a}_\beta(\Lambda^2 T^*X)$ is Fredholm we see

$$V := \Big\{\mathrm{d}_g^*\theta + \mathrm{d}\eta : \theta \in C^{k+1,a}_{\beta+1}(\Lambda^3 T^*X) \text{ and } \eta \in C^{k+1,a}_{\beta+1}(T^*X)\Big\}$$

has finite codimension in $C^{k,a}_\beta(\Lambda^2 T^*X)$, and so from Proposition 2.4 we deduce that V is closed in $C^{k,a}_\beta(\Lambda^2 T^*X)$. As W_1 is finite-dimensional, $V + W_1$ must be closed in $C^{k,a}_\beta(\Lambda^2 T^*X)$, and since $\mathrm{d}\eta_j + w_j \in V + W_1$ for all $j \geqslant 1$ there exist $\eta \in C^{k+1,a}_{\beta+1}(T^*X)$, $\theta \in C^{k+1,a}_{\beta+1}(\Lambda^3 T^*X)$ and $\tilde{\xi} \in W_1$ such that $\xi = \mathrm{d}_g^*\theta + \mathrm{d}\eta + \tilde{\xi}$. Now, it is easy to show that $\xi - \mathrm{d}\eta = \mathrm{d}_g^*\theta + \tilde{\xi}$ is closed and coclosed using the fact that $k \geqslant 1$, and therefore $\xi - \mathrm{d}\eta \in W_1$. Since $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$ and $\beta + 2 > 2 - n$ we also know that there exists $u \in C^{k+2,a}_{\beta+2}(X)$ such that $\Delta_g^0 u = f - \mathrm{d}_g^*\eta$ and then

$$f + \xi = (\xi - \mathrm{d}\eta) + \mathrm{d}_g^*(\mathrm{d}u + \eta) + \mathrm{d}(\mathrm{d}u + \eta)$$

shows that $f + \xi \in W_1 + W_2$ so that $W_1 + W_2$ is closed in $C^{k,a}_\beta(X) \oplus C^{k,a}_\beta(\Lambda^2 T^*X)$. We can now quickly deduce that the image W_2 of the map (6.42) must be closed. For this simply observe that W_2 has finite codimension in $W_1 + W_2$ and so by Proposition 2.4 must be closed in $W_1 + W_2$. It follows that W_2 is closed in $C^{k,a}_\beta(X) \oplus C^{k,a}_\beta(\Lambda^2 T^*X)$.

Finally, suppose that condition 3 holds. Pick any closed subspace $\mathcal{A}_2 \leqslant C^{k+1,a}_{\beta+1}(T^*X)$ such that

$$C^{k+1,a}_{\beta+1}(T^*X) = \mathcal{A}_2 \oplus \mathrm{Ker}(\Delta^1_g)_{\beta+1}$$

and then by the conical damped version of Corollary 6.8 there exists a $C > 0$ such that

$$\|\eta\|_{C^{k+1,a}_{\beta+1}(T^*X)} \leqslant C\|\Delta^1_g \eta\|_{C^{k-1,1}_{\beta-1}(T^*X)} \tag{6.44}$$

for all $\eta \in \mathcal{A}_2$. Now if $f \in C^{k,a}_\beta(X)$ and $\xi \in C^{k,a}_\beta(\Lambda^2 T^*X)$ are such that $f + \xi$ lie in the closure of the image of the map (6.42) then there exist $(\eta_j) \subseteq C^{k+1,a}_{\beta+1}(T^*X)$ such that $\mathrm{d}^*_g \eta_j \to f$ in $C^{k,a}_\beta(X)$ and $\mathrm{d}\eta_j \to \xi$ in $C^{k,a}_\beta(\Lambda^2 T^*X)$. Therefore

$$\Delta^1_g \eta_j \to \mathrm{d}f + \mathrm{d}^*_g \xi$$

in $C^{k-1,a}_{\beta-1}(T^*X)$. Using Lemma 6.24 together with the fact that $\beta + 2 < 0$ it is easy to deduce that every element of $\mathrm{Ker}(\Delta^1_g)_{\beta+1}$ is closed and coclosed, and therefore we may without loss of generality take the (η_j) above to lie in $\mathcal{A}_2$. Then the inequality (6.44) above implies that (η_j) is Cauchy in $C^{k+1,a}_{\beta+1}(T^*X)$ so that there exists an $\eta \in \mathcal{A}_2$ with $\eta_j \to \eta$ in $C^{k+1,a}_{\beta+1}(T^*X)$. It follows that $\mathrm{d}^*_g \eta = f$ and $\mathrm{d}\eta = \xi$ and we have shown that (6.42) has closed image, as required.

∎

There are various good and bad features of the three conditions given in Proposition 6.30: the first condition is good because it holds for almost all $\beta + 1 \in \mathbb{R}^L$, but is bad because on many of the non-generic points $\beta + 1 \in \mathcal{D}\big((\mathrm{d}^*_g + \mathrm{d})^{odd}\big)$ the map (6.42) will in fact have a closed image. The degree of this badness will increase with n, but for small n the set $\mathcal{D}\big((\mathrm{d}^*_g + \mathrm{d})^{odd}\big)$ is a close approximation to the set of points $\beta + 1$ for which the map (6.42) does not have a closed image. The good and bad features of conditions 2 and 3 are: there are fewer non-generic points, but we must restrict the values of $\beta + 1$ with some inequality.

6.2 The Deformation Theorem

6.2.1 Asymptotically conical submanifolds of $\mathbb{R}^m$

A *cone* in $\mathbb{R}^m$ is a non-empty closed subset $C \subseteq \mathbb{R}^m$ such that the inclusion $C \setminus \{0\} \to \mathbb{R}^m$ is a smooth submanifold and $e^t \cdot C = C$ for all $t \in \mathbb{R}$. It follows that $0 \in C$ always. If S^{m-1} denotes the unit sphere in $\mathbb{R}^m$ then we define $\Sigma := C \cap S^{m-1}$ which is a smooth, compact submanifold of $C \setminus \{0\}$, S^{m-1} and $\mathbb{R}^m$. If $C \setminus \{0\}$ has dimension n then Σ has dimension $n - 1$. We call Σ the *link* of the cone C, and sometimes say that C is a *cone on* Σ. Note that C itself is smooth precisely when C is a linear subspace of $\mathbb{R}^m$. When $m = 2n$ and we identify $\mathbb{R}^m \cong \mathbb{C}^n$ we abuse notation and say that a cone C is *(special) Lagrangian* when the submanifold $C \setminus \{0\} \to \mathbb{C}^n$ is (special) Lagrangian.

The Euclidean metric e on $\mathbb{R}^m$ endows the manifold $C \setminus \{0\}$ with a metric we denote $\tilde{g}$ and the manifold Σ with a metric we denote g_Σ. There is a diffeomorphism

$$\begin{aligned} i : \mathbb{R} \times \Sigma &\to C \setminus \{0\} \subseteq \mathbb{R}^m \\ (t, \sigma) &\mapsto e^t \sigma \end{aligned} \tag{6.45}$$

and we identify $\tilde{g}$ with the pulled-back metric

$$i^*\tilde{g} = e^{2t}(\mathrm{d}t^2 + g_\Sigma) \tag{6.46}$$

on $\mathbb{R} \times \Sigma$. The metric (6.46) is called the *cone metric* on $\mathbb{R} \times \Sigma$.

Suppose for the rest of Section 6.2 that $C \subseteq \mathbb{R}^m$ is a fixed cone with link $\Sigma \subseteq S^{m-1}$ and that the metrics g_Σ, $\tilde{g}$ are as described above. We also fix X a manifold with ends, as in Section 4.1, built using the link Σ of C, so that there exists a compact submanifold with boundary $X_0 \subseteq X$, and a fixed diffeomorphism

$$X \setminus X_0 \to (T, \infty) \times \Sigma \tag{6.47}$$

for some $T \in \mathbb{R}$. Now, using the metric g_Σ (or indeed any metric) on Σ we can build the Banach spaces $L^p_{k,\beta}(E)$ and $C^{k,a}_\beta(E)$ when E is a bundle of the form (4.3), (4.4), (4.5) on X. Moreover, as topological vector spaces, these Banach spaces are independent of g_Σ, asymptotically conical metric, open cover, and partition of unity satisfying the conditions given in Section 4.1. Define $\hat{C}^k_\beta(E)$ to be the closure of $C^\infty_c(E)$ in $C^k_\beta(E)$ and similarly define $\hat{C}^{k,a}_\beta(E)$ to be the closure of $C^\infty_c(E)$ in $C^{k,a}_\beta(E)$, so that the "hat" spaces replace $O(\cdot)$ decay conditions with the stronger $o(\cdot)$ decay conditions. We also put

$$\hat{C}^\infty_\beta(E) := \bigcap_{k \geqslant 0} \hat{C}^k_\beta(E).$$

If $f : X \to \mathbb{R}^m$ is a map we write $f \in C^k_\beta(X, \mathbb{R}^m)$ whenever the components $f_1, \ldots, f_m : X \to \mathbb{R}$ all lie in $C^k_\beta(X)$, and adopt similar conventions for the other vector spaces mentioned above.

Recall the definition (6.45) of the map i. Using the identification (6.47) we can extend the restricted map $i : (T, \infty) \times \Sigma \to \mathbb{R}^m$ to a smooth map $i : X \to \mathbb{R}^m$. In other words, the image of $X \setminus X_0$ under i is an infinite portion of the cone $C \subseteq \mathbb{R}^m$. It is easy to see that $i \in C^\infty_1(X, \mathbb{R}^m)$. In a similar way, we can consider the cone metric $\tilde{g}$ as being a metric on X, and indeed build our Banach spaces of sections using this metric. The fact that $i \in C^\infty_1(X, \mathbb{R}^m)$ is equivalent to

$$\sup_{\{t\} \times U_\nu} \left| \rho_\nu (\partial^\lambda i_k) \right| = O(e^t)$$

for all $1 \leqslant \nu \leqslant N$, multi-indices $|\lambda| \geqslant 0$ and $1 \leqslant k \leqslant m$, and this in turn is equivalent to the fact that

$$\sup_{\{t\} \times \Sigma} \left| \nabla^j_{\tilde{g}} i_k \right|_{\tilde{g}} = O(e^{(1-j)t})$$

for all $j \geqslant 0$ and $1 \leqslant k \leqslant m$.

We shall say that a submanifold $f : X \to \mathbb{R}^m$ is *asymptotically conical with cone* C if $f - i \in \hat{C}^\infty_1(X, \mathbb{R}^m)$. This condition is equivalent to either of the following:

1. $\sup_{\{t\} \times U_\nu} \left| \rho_\nu \partial^\lambda (f_k - i_k) \right| = o(e^t)$ for all $1 \leqslant \nu \leqslant N$, $|\lambda| \geqslant 0$, $1 \leqslant k \leqslant m$
2. $\sup_{\{t\} \times \Sigma} \left| \nabla^j_{\tilde{g}} (f_k - i_k) \right|_{\tilde{g}} = o(e^{(1-j)t})$ for all $j \geqslant 0$, $1 \leqslant k \leqslant m$.

Obviously if $f : X \to \mathbb{R}^m$ is asymptotically conical with cone C then $f \in C^\infty_1(X, \mathbb{R}^m)$, which is again equivalent to either of

1. $\sup_{\{t\} \times U_\nu} \left| \rho_\nu (\partial^\lambda f_k) \right| = O(e^t)$ for all $1 \leqslant \nu \leqslant N$, $|\lambda| \geqslant 0$, $1 \leqslant k \leqslant m$
2. $\sup_{\{t\} \times \Sigma} \left| \nabla^j_{\tilde{g}} f_k \right|_{\tilde{g}} = O(e^{(1-j)t})$ for all $j \geqslant 0$, $1 \leqslant k \leqslant m$.

If X is a manifold with ends then there are various rates of decay at which a submanifold $f : X \to \mathbb{R}^m$ might tend towards a cone $i : C \to \mathbb{R}^m$, and we shall now go on to consider rates which are stronger than the $o(e^t)$ decay given by the condition $f - i \in \hat{C}^\infty_1(X, \mathbb{R}^m)$ above. However, $f : X \to \mathbb{R}^m$ being asymptotically conical with cone C is the weakest useful rate at which $f : X \to \mathbb{R}^m$ could decay towards C: any weaker decay rates for $f - i$ mean a loss of control which makes the analysis of Chapter 4 and Chapter 5 impossible to implement: see Corollary 6.33 below. Also, if $f - i \in \hat{C}^\infty_1(X, \mathbb{R}^m)$ then the cone C to which $f : X \to \mathbb{R}^m$ is asymptotic is uniquely determined, but this is not the case for growth rates $f - i \in C^\infty_1(X, \mathbb{R}^m)$ or higher.

Let $\tilde{\alpha} \in \mathbb{R}^L$ with $\tilde{\alpha} < 1$. We shall say that a submanifold $f : X \to \mathbb{R}^m$ is *strongly asymptotically conical with cone* C *and rate* $\tilde{\alpha}$ if $f - i \in C^\infty_{\tilde{\alpha}}(X, \mathbb{R}^m)$. This is equivalent to either of the conditions

1. $\sup_{\{t\}\times U_\nu} \left|\rho_\nu \partial^\lambda (f_k - i_k)\right| = O(e^{\tilde{\alpha}t})$ for all $1 \leqslant \nu \leqslant N$, $|\lambda| \geqslant 0$, $1 \leqslant k \leqslant m$
2. $\sup_{\{t\}\times\Sigma} \left|\nabla^j_{\tilde{g}}(f_k - i_k)\right|_{\tilde{g}} = O(e^{(\tilde{\alpha}-j)t})$ for all $j \geqslant 0$, $1 \leqslant k \leqslant m$

Clearly if a submanifold $f : X \to \mathbb{R}^m$ is strongly asymptotically conical with cone C and rate $\tilde{\alpha} < 1$ then $f : X \to \mathbb{R}^m$ is asymptotically conical with cone C, since $C^\infty_{\tilde{\alpha}}(X,\mathbb{R}^m) \leqslant \hat{C}^\infty_1(X,\mathbb{R}^m)$.

We now give a result which shall be useful later, and demonstrates a typical application of the above asymptotic decay conditions.

Proposition 6.31 *Let $(e_1,\ldots,e_m)$ be the usual coframe on $\mathbb{R}^m$ and suppose that*

$$\theta = \sum_{j_1,\ldots,j_r=1}^{m} \theta_{j_1\ldots j_r} e_{j_1} \otimes \cdots \otimes e_{j_r}$$

is a covariant tensor of degree $r \geqslant 0$ on $\mathbb{R}^m$ such that each $\theta_{j_1\ldots j_r} : \mathbb{R}^m \to \mathbb{R}$ is a homogeneous polynomial of degree $\delta \geqslant 0$.

1. *If $f, \tilde{f} : X \to \mathbb{R}^m$ lie in $C^{k+1}_\gamma(X,\mathbb{R}^m)$ and $f - \tilde{f} \in C^{k+1}_\beta(X,\mathbb{R}^m)$ (respectively $f - \tilde{f} \in \hat{C}^{k+1}_\beta(X,\mathbb{R}^m)$) then $f^*\theta - \tilde{f}^*\theta \in C^k_{(r+\delta-1)\gamma+\beta-r}(\otimes^r T^*X)$ (respectively $f^*\theta - \tilde{f}^*\theta \in \hat{C}^k_{(r+\delta-1)\gamma+\beta-r}(\otimes^r T^*X)$).*
2. *If $f, \tilde{f} : X \to \mathbb{R}^m$ lie in $C^{k+1,a}_\gamma(X,\mathbb{R}^m)$ and $f - \tilde{f} \in C^{k+1,a}_\beta(X,\mathbb{R}^m)$ (respectively $f - \tilde{f} \in \hat{C}^{k+1,a}_\beta(X,\mathbb{R}^m)$) then $f^*\theta - \tilde{f}^*\theta \in C^{k,a}_{(r+\delta-1)\gamma+\beta-r}(\otimes^r T^*X)$ (respectively $f^*\theta - \tilde{f}^*\theta \in \hat{C}^{k,a}_{(r+\delta-1)\gamma+\beta-r}(\otimes^r T^*X)$).*

Proof: We assume that $\delta + r \geqslant 1$ because in the case $\delta = r = 0$ we just have a constant function θ on $\mathbb{R}^m$ and $f^*\theta = \tilde{f}^*\theta$.

Suppose firstly that $f, \tilde{f} \in C^{k+1}_\gamma(X,\mathbb{R}^m)$ and that $f - \tilde{f} \in C^{k+1}_\beta(X,\mathbb{R}^m)$. Then we have, in our usual coordinates $x = (t,\sigma)$ over each $(T,\infty) \times U_\nu$:

$$\begin{aligned}
f^*\theta &= \sum_{k_1,\ldots,k_r=1}^{n} \left(\sum_{j_1,\ldots,j_r=1}^{m} (\theta_{j_1\ldots j_r} \circ f) \frac{\partial f_{j_1}}{\partial x_{k_1}} \cdots \frac{\partial f_{j_r}}{\partial x_{k_r}} \right) \mathrm{d}x_{k_1} \otimes \cdots \otimes \mathrm{d}x_{k_r} \\
\tilde{f}^*\theta &= \sum_{k_1,\ldots,k_r=1}^{n} \left(\sum_{j_1,\ldots,j_r=1}^{m} (\theta_{j_1\ldots j_r} \circ \tilde{f}) \frac{\partial \tilde{f}_{j_1}}{\partial x_{k_1}} \cdots \frac{\partial \tilde{f}_{j_r}}{\partial x_{k_r}} \right) \mathrm{d}x_{k_1} \otimes \cdots \otimes \mathrm{d}x_{k_r}
\end{aligned}$$

and for each $1 \leqslant j_1,\ldots,j_r \leqslant m$ we may write:

$$\begin{aligned}
\theta_{j_1\ldots j_r} \circ f &= \sum_{1\leqslant i_1,\ldots,i_\delta\leqslant m} c_{i_1\ldots i_\delta} f_{i_1} \ldots f_{i_\delta} \\
\theta_{j_1\ldots j_r} \circ \tilde{f} &= \sum_{1\leqslant i_1,\ldots,i_\delta\leqslant m} c_{i_1\ldots i_\delta} \tilde{f}_{i_1} \ldots \tilde{f}_{i_\delta}
\end{aligned}$$

for some $c_{i_1\ldots i_\delta} \in \mathbb{R}$. Using the given decay conditions it is easy to see that $f^*\theta - \tilde{f}^*\theta \in C^k_{(r+\delta-1)\gamma+\beta-r}(\otimes^r T^*X)$: each $f_i, \tilde{f}_i$ and $\partial_k f_j, \partial_k \tilde{f}_j$ has decay $O(e^{\gamma t})$ in the first k derivatives, each $f_i - \tilde{f}_i$ and $\partial_k f_j - \partial_k \tilde{f}_j$ has decay $O(e^{\beta t})$ in the first k derivatives and each $\mathrm{d}x_{k_1} \otimes \cdots \otimes \mathrm{d}x_{k_r}$ has decay $O(e^{-rt})$ in the cone metric. Using a telescoping argument we see that the total decay rate of $f^*\theta - \tilde{f}^*\theta$ is therefore $O(e^{((r+\delta-1)\gamma+\beta-r)t})$ in the first k derivatives, as required. When $f - \tilde{f} \in \hat{C}^{k+1}_\beta(X,\mathbb{R}^m)$ each $f_i - \tilde{f}_i$ and $\partial_k f_j - \partial_k \tilde{f}_j$ have decay $o(e^{\beta t})$ in the first k derivatives so that the total decay rate of $f^*\theta - \tilde{f}^*\theta$ is $o(e^{((r+\delta-1)\gamma+\beta-r)t})$ in the first k derivatives, and $f^*\theta - \tilde{f}^*\theta \in \hat{C}^k_{(r+\delta-1)\gamma+\beta-r}(\otimes^r T^*X)$ as required.

The case of Hölder decay follows similarly: we omit the details.

∎

Proposition 6.31 has many useful consequences:

Corollary 6.32 *Suppose $f : X \to \mathbb{R}^m$ is an asymptotically conical submanifold with cone C. If θ is a covariant tensor on $\mathbb{R}^m$ which is constant in the usual coordinate system and $f^*\theta = 0$ then $i^*\theta = 0$ viewing i as a map $C \to \mathbb{R}^m$. Consequently, if $f : X \to \mathbb{C}^n$ is a (special) Lagrangian submanifold which is asymptotically conical with cone C then C is (special) Lagrangian.*

Proof: Suppose that $\theta \in C^\infty(\otimes^r T^*\mathbb{R}^m)$. Since $i, f \in C_1^\infty(X, \mathbb{R}^m)$ and $i - f \in \hat{C}_1^\infty(X, \mathbb{R}^m)$ we have, applying Proposition 6.31 with $\beta = \gamma = 1$ and $\delta = 0$:

$$i^*\theta = i^*\theta - f^*\theta \tag{6.48}$$

lying in $\hat{C}_0^\infty(\otimes^r T^*X)$. In equation (6.48) we view i as a map $i : X \to \mathbb{R}^m$, but it follows immediately that if we view i as a map $i : C \to \mathbb{R}^m$ we have

$$\sup_{\{t\}\times\Sigma} |i^*\theta|_{\tilde{g}} = o(1). \tag{6.49}$$

Now pick any $\sigma \in \Sigma$ and any g_Σ-orthonormal frame $(f_2, \ldots, f_n)$ for $T_\sigma^*\Sigma$. Then

$$(e^t \mathrm{d}t, e^t f_2, \ldots, e^t f_m)$$

is a $\tilde{g}$-orthonormal frame for the cotangent space of $\mathbb{R} \times \Sigma$ at each (t, σ) in which the form $i^*\theta$ is independent of t. It follows from equation (6.49) that we must have $i^*\theta = 0$ on $\mathbb{R} \times \Sigma$.

The second assertion now follows from the first because the forms ω and $\operatorname{Im}\Omega$ are constant in the usual coframe for $\mathbb{C}^n$.

∎

If $f : X \to \mathbb{R}^m$ is a submanifold we shall always denote the metric induced on X by $g := f^*e$. The following result relates the decay of f towards i with the decay of g towards the cone metric $\tilde{g}$.

Corollary 6.33 *If $f : X \to \mathbb{R}^m$ is a (strongly) asymptotically conical submanifold with cone C (and rate $\tilde{\alpha} < 1$) then the metric g on X is (strongly) asymptotically conical (with rate $\alpha = \tilde{\alpha} - 1 < 0$).*

Proof: Suppose that $f : X \to \mathbb{R}^m$ is asymptotically conical with cone C. Applying Proposition 6.31 with $r = 2$, $\delta = 0$ and $\beta = \gamma = 1$ yields $g - \tilde{g} \in \hat{C}_0^\infty(\otimes^2 T^*X)$ and that is the same thing as saying

$$\sup_{\{t\}\times\Sigma} \left|\nabla_{\tilde{g}}^j (g - \tilde{g})\right|_{\tilde{g}} = o(e^{-jt})$$

for all $j \geqslant 0$, which means that g is asymptotically conical. The case of strong decay follows similarly.

∎

6.2.2 Deforming AC Lagrangian submanifolds of $\mathbb{C}^n$

Suppose now for the rest of Section 6.2 that $f : X \to \mathbb{R}^m$ is a (strongly) asymptotically conical submanifold with cone $C \subseteq \mathbb{R}^m$ (and rate $\tilde{\alpha} < 1$), and let $N \to X$ be the normal bundle of X in $\mathbb{R}^m$. We would like to deform $f : X \to \mathbb{R}^m$ to "nearby" submanifolds $f_\xi : X \to \mathbb{R}^m$. By the Hopf-Rinow Theorem [24, Theorem 1.4.8] the subset $f(X) \subseteq \mathbb{R}^m$ is complete as a metric space and therefore closed in $\mathbb{R}^m$, so by the Tubular Neighbourhood Theorem 2.15 there exists an open subset $\tilde{U} \subseteq N$ containing the zero section such that the exponential map

$$\exp|_{\tilde{U}} : \tilde{U} \to \mathbb{R}^m \tag{6.50}$$

is a diffeomorphism onto an open subset of $\mathbb{R}^m$. Recall that for each

$$\xi \in \tilde{U}^\infty := \left\{\xi \in C^\infty(N) : \xi_x \in \tilde{U} \text{ for all } x \in X\right\}$$

we have a submanifold $f_\xi : X \to \mathbb{R}^m$ got by identifying $\tilde{U} \cong \exp(\tilde{U})$ via the diffeomorphism (6.50), and then the normal vector field ξ becomes identified with the map f_ξ.

One can think of a normal vector field $\xi \in C^\infty(N)$ as a function $\xi : X \to \mathbb{R}^m$ such that $\xi_x \in \mathbb{R}^m$ lies in $(T_xX)^\perp \leqslant T_x\mathbb{R}^m \cong \mathbb{R}^m$ for each $x \in X$, and then since the metric e on $\mathbb{R}^m$ is flat we have

$$f_\xi := f + \xi \tag{6.51}$$

for all $\xi \in \tilde{U}^\infty$.

If $\xi \in \tilde{U}^\infty$ then there is obviously a relationship between the growth rate of ξ and its derivatives and whether or not the submanifold f_ξ is (strongly) asymptotically conical with cone C (and rate $\tilde{\alpha} < 1$). However, we have not defined the notion of conical damped C^k-spaces or conical damped Hölder spaces for the bundle N. Also, we shall primarily be concerned with the case $m = 2n$ with a fixed identification $\mathbb{R}^m \cong \mathbb{C}^n$, and then only Lagrangian submanifolds of $\mathbb{C}^n$. Therefore, for the rest of Section 6.2 we assume that $m = 2n$, and we have a fixed identification $\mathbb{R}^m \cong \mathbb{C}^n$, and (in addition to the previous assumptions on f) that $f : X \to \mathbb{C}^n$ is a Lagrangian submanifold. It follows from Corollary 6.32 that the cone C must be Lagrangian also.

We have a bundle isomorphism $\flat_g J : N \to T^*X$ as in Section 2.3.4. Put $U := (\flat_g J)\tilde{U}$, which is an open subset of T^*X containing the zero section. We also define $U^\infty := (\flat_g J)\tilde{U}^\infty \subseteq C^\infty(T^*X)$ as in Section 2.3.4, and further put

$$\begin{aligned} U^k_\beta &:= \Big\{\xi \in C^k_\beta(T^*X) : \xi_x \in U \text{ for all } x \in X\Big\} \\ U^{k,a}_\beta &:= \Big\{\xi \in C^{k,a}_\beta(T^*X) : \xi_x \in U \text{ for all } x \in X\Big\} \\ U^\infty_\beta &:= \Big\{\xi \in C^\infty_\beta(T^*X) : \xi_x \in U \text{ for all } x \in X\Big\} \end{aligned}$$

which each contain 0. Note that the subsets $U^k_\beta \subseteq C^k_\beta(T^*X)$ and $U^{k,a}_\beta \subseteq C^{k,a}_\beta(T^*X)$ need not be open. We do, however, have the following useful result, which tells us that we may pick our tubular neighbourhood $\tilde{U}$ of $f : X \to \mathbb{C}^n$ in such a way that the submanifold $f : X \to \mathbb{C}^n$ has "room to move" within $\tilde{U}$ in a manner we should like.

Theorem 6.34 *We can always choose the above tubular neighbourhood $\tilde{U} \subseteq N$ so that there exists an $\varepsilon > 0$ with*

$$V^0_1 := \Big\{\eta \in C^0_1(T^*X) : \|\eta\|_{C^0_1(T^*X)} < \varepsilon\Big\} \subseteq U^0_1.$$

Proof: Recall the statement of the Inverse Function Theorem 2.7. It turns out that the open subset $\mathcal{V}$ of Theorem 2.7 can be made to contain balls of a certain size determined by the operator norms of $F'(0)$, $F'(0)^{-1}$ and $F''(x)$ for $\|x\|$ small. Estimates of this kind are, for example, to be found in the book [1, Proposition 2.5.6] of Abraham, Marsden and Ratiu.

Suppose that $k : Y \to Z$ is a submanifold of a (complete, say) Riemannian manifold (Z, e), with $k(Y) \subseteq Z$ a closed subset. Let N be the normal bundle of $k : Y \to Z$. The Tubular Neighbourhood Theorem 2.15 is proved by showing that the exponential map $\exp : N \to Z$ is a local diffeomorphism at each point $y \in Y$, so by the Inverse Function Theorem 2.7 one has a neighbourhood $V_y \subseteq N$ of each $y \in Y$ on which the exponential map is a diffeomorphism. If follows that

$$V := \bigcup_{y \in Y} V_y \xrightarrow{\exp} N \tag{6.52}$$

is an immersion, and as in Lang [42, Chapter IV, Theorem 9], there is a refinement $\{U_y : y \in Y\}$ of $\{V_y : y \in Y\}$ such that exp is a diffeomorphism on $\tilde{U} := \bigcup_{y \in Y} U_y$.

In our situation, with an asymptotically conical submanifold $f : X \to \mathbb{C}^n$, one can show that there exists a $C > 0$ such that the open subsets $\{V_x : x \in X\}$ can be chosen so to contain a ball of radius Ce^t about each $x = (t, \sigma) \in X_\infty$. The method here is to use an explicit coordinate description of the map $\exp : N \to \mathbb{C}^n$, and then the estimates of [1] mentioned above. This gives us an immersion as in (6.52) where V has the required growth properties, and one can show that the refinement $\{U_x : x \in X\}$ can be taken so as not to destroy this growth.

∎

For the rest of Section 6.2 we fix the $\varepsilon > 0$ and $\tilde{U} \subseteq N$ as given in Theorem 6.34. The point of the theorem is that if $(\flat_g J)\xi$ is any section of T^*X which has C^0_1 norm less than $\varepsilon > 0$ then the image of ξ lies inside the tubular neighbourhood $\tilde{U}$ and therefore $f_\xi : X \to \mathbb{C}^n$ defines a submanifold (provided ξ is smooth, say).

It follows immediately that we have

$$\begin{aligned} V^k_\beta &:= \Big\{\eta \in C^k_\beta(T^*X) : \|\eta\|_{C^0_1(T^*X)} < \varepsilon\Big\} &\subseteq\ & U^k_\beta \\ V^{k,a}_\beta &:= \Big\{\eta \in C^{k,a}_\beta(T^*X) : \|\eta\|_{C^0_1(T^*X)} < \varepsilon\Big\} &\subseteq\ & U^{k,a}_\beta \\ V^\infty_\beta &:= \Big\{\eta \in C^\infty_\beta(T^*X) : \|\eta\|_{C^0_1(T^*X)} < \varepsilon\Big\} &\subseteq\ & U^\infty_\beta \end{aligned}$$

for all $\beta \leqslant 1$. Obviously $V^k_\beta \subseteq C^k_\beta(T^*X)$ and $V^{k,a}_\beta \subseteq C^{k,a}_\beta(T^*X)$ are open subsets each containing 0.

We now relate the growth rate of a normal vector field ξ in the Euclidean norm with the growth rate of the 1-form $(\flat_g J)\xi$ on X.

Proposition 6.35 *Let ξ be a section of the normal bundle N, considered as a map $\xi : X \to \mathbb{C}^n$. Then*

1. $\xi \in C^k_\beta(X, \mathbb{C}^n)$ *precisely when* $(\flat_g J)\xi \in C^k_\beta(T^*X)$
2. $\xi \in \hat{C}^k_\beta(X, \mathbb{C}^n)$ *precisely when* $(\flat_g J)\xi \in \hat{C}^k_\beta(T^*X)$
3. $\xi \in C^{k,a}_\beta(X, \mathbb{C}^n)$ *precisely when* $(\flat_g J)\xi \in C^{k,a}_\beta(T^*X)$
4. $\xi \in \hat{C}^{k,a}_\beta(X, \mathbb{C}^n)$ *precisely when* $(\flat_g J)\xi \in \hat{C}^{k,a}_\beta(T^*X)$.

Proof: Write $\xi =: \sum_{j=1}^{2n} \xi_j e_j$ where $(e_1, \ldots, e_{2n})$ is the standard real frame for $\mathbb{C}^n$ and each $\xi_1, \ldots, \xi_{2n} : X \to \mathbb{R}$. Then $J\xi = \sum_{j=1}^{n}(\xi_j e_{j+n} - \xi_{j+n} e_j)$. Suppose that $x = (t, \sigma)$ are the usual coordinates on a patch $(T, \infty) \times U_\nu \subseteq X_\infty$. It is easy to show that

$$(\flat_g J)\xi = \sum_{j,k=1}^{n} \Big(-\frac{\partial f_j}{\partial x_k}\xi_{j+n} + \frac{\partial f_{j+n}}{\partial x_k}\xi_j \Big) \mathrm{d}x_k \tag{6.53}$$

in the coordinates $x = (t, \sigma)$. Working over $(T, \infty) \times U_\nu$, let $(\mathrm{d}f)^t$ be the $n \times 2n$ matrix whose (k, j)-entry is $\frac{\partial f_j}{\partial x_k}$ and let J be the $2n \times 2n$ matrix

$$J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

in $(n+n) \times (n+n)$-block form. Now define A to be the $2n \times 2n$ matrix

$$A = \begin{pmatrix} (\mathrm{d}f)^t J \\ (\mathrm{d}f)^t \end{pmatrix}$$

in $(n+n) \times (2n)$-block form. Since $f : X \to \mathbb{C}^n$ is a Lagrangian submanifold, the matrix A is invertible at each point of $(T, \infty) \times U_\nu$. Moreover, equation (6.53) can be neatly summarised as

$$\begin{pmatrix} (\flat_g J)\xi \\ 0 \end{pmatrix} = A\xi \tag{6.54}$$

and therefore the proof of this proposition comes down to controlling the decay in the derivatives of the entries of A, A^{-1}.

Since we have $f \in C_1^\infty(X)$ it is clear that the derivatives of the entries of the matrix A decay at rate $O(e^t)$ on the patches $(T,\infty) \times U_\nu$, and this is enough to prove the left implies right parts of the above assertions, because the $\mathrm{d}x_k$ parts of the 1-form (6.53) decay at rate $O(e^{-t})$ in the cone metric $\tilde{g}$.

To examine the properties of the matrix A^{-1}, consider the map $i : X \to \mathbb{C}^n$ and define

$$B = \begin{pmatrix} (\mathrm{d}i)^t J \\ (\mathrm{d}i)^t \end{pmatrix}$$

over the patch $(T,\infty) \times U_\nu$. Then B is invertible at each point of $(T,\infty) \times U_\nu$ since the cone C is Lagrangian. Further define

$$h := A - B \qquad D := e^{-t} B \qquad E := e^{-t} h$$

and then it is easy to check that $\sup_{\{t\}\times U_\nu} \left|\rho_\nu \partial^\lambda (D^{-1})\right| = O(1)$ and $\sup_{\{t\}\times U_\nu} \left|\rho_\nu \partial^\lambda E\right| = o(1)$ for each $1 \leqslant \nu \leqslant N$ and $|\lambda| \geqslant 0$. After a brief calculation we have

$$A^{-1} - B^{-1} = -e^{-t} D^{-1} E \sum_{k=0}^{\infty} \left(-D^{-1}E\right)^k D^{-1}$$

for each $1 \leqslant \nu \leqslant N$, where the power series converges on $(T+S,\infty) \times U_\nu$ for some large $S \geqslant 0$. It follows that

$$\sup_{\{t\}\times U_\nu} \left|\rho_\nu \partial^\lambda (A^{-1} - B^{-1})\right| = o(e^{-t})$$

for all $1 \leqslant \nu \leqslant N$, $|\lambda| \geqslant 0$ and hence

$$\sup_{\{t\}\times U_\nu} \left|\rho_\nu \partial^\lambda (A^{-1})\right| = O(e^{-t}) \tag{6.55}$$

for all $1 \leqslant \nu \leqslant N$, $|\lambda| \geqslant 0$. Now we are done: the right implies left parts of the above assertions follow from equation (6.54) and equation (6.55).

∎

The following corollary is now immediate because $f_\xi = f + \xi$ for $(\flat_g J)\xi \in U^\infty$, from equation (6.51).

Corollary 6.36 *Let $(\flat_g J)\xi \in U^\infty$. Then $f_\xi : X \to \mathbb{C}^n$ is (strongly) asymptotically conical with cone C (and rate $\tilde{\alpha}$) precisely when $(\flat_g J)\xi \in \hat{C}_1^\infty(T^*X)$ (respectively $(\flat_g J)\xi \in C_{\tilde{\alpha}}^\infty(T^*X)$).*

6.2.3 The moduli space of AC special Lagrangian submanifolds of $\mathbb{C}^n$

We now establish the deformation problem for asymptotically conical special Lagrangian submanifolds of $\mathbb{C}^n$: the material is conceptually very similar to that of Section 3.2.

As well as the assumptions we have already made in Section 6.2, we shall further assume that the submanifold $f : X \to \mathbb{C}^n$ is special Lagrangian and strongly asymptotically conical with cone C and rate $\tilde{\alpha} < 1$. It follows from Corollary 6.32 that C is special Lagrangian. We put $\alpha := \tilde{\alpha} - 1$. Also (J, e, Ω) denotes the standard Calabi-Yau structure on $\mathbb{C}^n$, with Kähler form denoted ω.

We are interested in the submanifolds of $\mathbb{C}^n$ which are near to $f : X \to \mathbb{C}^n$ in some sense. For us, "near" shall mean a submanifold of the form $f_\xi : X \to \mathbb{C}^n$ where $(\flat_g J)\xi$ lies in V_1^0. However, we shall only be interested in submanifolds $f_\xi : X \to \mathbb{C}^n$ which are strongly asymptotically conical with cone C and rate $\tilde{\alpha} = \alpha + 1$. Because of Corollary 6.36 we shall therefore restrict ourselves to contemplating submanifolds $f_\xi : X \to \mathbb{C}^n$ such that $(\flat_g J)\xi \in V_{\alpha+1}^\infty$. We now consider which of the $(\flat_g J)\xi \in V_{\alpha+1}^\infty$ give rise to submanifolds $f_\xi : X \to \mathbb{C}^n$ that are special Lagrangian.

To this end, fix some $k \geqslant 2$. Let $\beta + 1 \in \mathbb{R}^L$ with $\beta + 1 < 1$ and define a map $F_{\beta+1} : V_{\beta+1}^{k+1,a} \to C^0(\Lambda^* T^* X)$ by

$$F_{\beta+1}\big((\flat_g J)\xi\big) = *_g f_\xi^* \operatorname{Im}\Omega + f_\xi^* \omega \tag{6.56}$$

for all $(\flat_g J)\xi \in V_{\beta+1}^{k+1,a}$. Note that $F(0) = 0$ as $f : X \to \mathbb{C}^n$ is special Lagrangian, and furthermore

$$F_{\beta+1}^{-1}(0) = \Big\{(\flat_g J)\xi \in V_{\beta+1}^{k+1,a} : f_\xi : X \to \mathbb{C}^n \text{ is special Lagrangian}\Big\}.$$

It follows that we are interested in the structure of the subset $F_{\beta+1}^{-1}(0) \subseteq V_{\beta+1}^{k+1,a}$ and as in Section 3.2 the right tool to use is the Implicit Function Theorem 2.11. To invoke this theorem we need some further results.

Proposition 6.37 *The image of the map*

$$F_{\beta+1} : V_{\beta+1}^{k+1,a} \to C^0(\Lambda^* T^* X)$$

lies inside $C_\beta^{k,a}(X) \oplus C_\beta^{k,a}(\Lambda^2 T^* X)$.

Proof: If $(\flat_g J)\xi \in V_{\beta+1}^{k+1,a} \subseteq C_{\beta+1}^{k+1,a}(T^*X)$ then from Proposition 6.35 we have $\xi \in C_{\beta+1}^{k+1,a}(X, \mathbb{C}^n)$ if we consider ξ as a map $X \to \mathbb{C}^n$. But since $f_\xi - f = f + \xi - f = \xi$ and $f_\xi, f \in C_1^{k+1,a}(X, \mathbb{C}^n)$ we deduce from Proposition 6.31 that

$$f_\xi^* \omega = f_\xi^* \omega - f^* \omega \in C_\beta^{k,a}(\Lambda^2 T^* X).$$

The function $*_g f_\xi^* \operatorname{Im} \Omega$ is handled similarly.

∎

Theorem 6.38 *The map* $F_{\beta+1} : V_{\beta+1}^{k+1,a} \to C_\beta^{k,a}(X) \oplus C_\beta^{k,a}(\Lambda^2 T^* X)$ *is smooth.*

Proof: This result is very similar to Theorem 3.13, whose proof from Baier [5, Theorem 2.2.15] we explained there. We therefore omit the details.

∎

Proposition 6.39 *The smooth map* $F_{\beta+1} : V_{\beta+1}^{k+1,a} \to C_\beta^{k,a}(X) \oplus C_\beta^{k,a}(\Lambda^2 T^* X)$ *has derivative*

$$F'_{\beta+1}(0) : C_{\beta+1}^{k+1,a}(T^*X) \to C_\beta^{k,a}(X) \oplus C_\beta^{k,a}(\Lambda^2 T^* X)$$

at 0 *which acts as* $\mathrm{d}_g^* + \mathrm{d}$.

Proof: Given $x \in X$ let

$$\mathrm{ev}_x : C_\beta^{k,a}(\Lambda^* T^* X) \to \Lambda^* T_x^* X \tag{6.57}$$

denote the linear map which evaluates sections at $x \in X$. For $\eta \in C_\beta^{k,a}(\Lambda^* T^* X)$ we have

$$|\mathrm{ev}_x(\eta)| = |\eta_x| \leqslant C \cdot \|\eta\|_{C_\beta^0(\Lambda^* T^* X)} \leqslant C \cdot \|\eta\|_{C_\beta^{k,a}(\Lambda^* T^* X)}$$

where $C > 0$ is a constant independent of η (but not x). It follows that the map (6.57) is bounded. The rest of the proof is now identical to that of Proposition 3.14.

∎

Proposition 6.40 *The derivative* $F'_{\beta+1}(0) : C_{\beta+1}^{k+1,a}(T^*X) \to C_\beta^{k,a}(X) \oplus C_\beta^{k,a}(\Lambda^2 T^* X)$ *has a closed image in any of the following three situations:*

1. $\beta + 1 \in \mathbb{R}^L \setminus \mathcal{D}\big((\mathrm{d}_g^* + \mathrm{d})^{odd}\big)$
2. $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^2)$, $\beta + 2 > 2 - n$ *and no coordinate of* $\beta + 2$ *is* 0
3. $\beta + 1 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^1)$ *and* $\beta + 2 < 0$.

Proof: This follows immediately from Proposition 6.30. ∎

If any of the conditions 1, 2, 3 of Proposition 6.40 hold we shall say that $\beta+2 \in \mathbb{R}^L$ is *generic*. Clearly $\beta+2$ is generic for almost all $\beta+2 \in \mathbb{R}^L$.

Proposition 6.41 *If $\beta+2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$ with $\beta+2 > \max\{2-n+\alpha, 2-n-\lambda\}$ then the image $F_{\beta+1}\big(V_{\beta+1}^{k+1,a}\big)$ of the map $F_{\beta+1} : V_{\beta+1}^{k+1,a} \to C_\beta^{k,a}(X) \oplus C_\beta^{k,a}(\Lambda^2 T^*X)$ is contained inside*

$$F'_{\beta+1}(0)\big(C_{\beta+1}^{k+1,a}(T^*X)\big) = \mathrm{d}_g^*\big(C_{\beta+1}^{k+1,a}(T^*X)\big) \oplus \mathrm{d}\big(C_{\beta+1}^{k+1,a}(T^*X)\big). \tag{6.58}$$

Recall that the equality (6.58) is not automatic: we are using the fact that $\beta+2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$ with $\beta+2 > \max\{2-n+\alpha, 2-n-\lambda\}$, together with Lemma 6.27 and Corollary 6.29.

Proof: Write $\operatorname{Im}\Omega = \mathrm{d}\theta_{n-1}$, $\omega = \mathrm{d}\theta_1$, where $\theta_1 \in C^\infty(T^*\mathbb{C}^n)$ and $\theta_{n-1} \in C^\infty(\Lambda^{n-1}T^*\mathbb{C}^n)$ are forms whose components with respect to the usual coframe on $\mathbb{C}^n$ are linear functions. Let $(\flat_g J)\xi \in V_{\beta+1}^{k+1,a}$. Then it is easy to check that

$$\begin{aligned} \mathrm{d}_g^*\big((-1)^n *_g (f_\xi^*\theta_{n-1} - f^*\theta_{n-1})\big) &= *_g f_\xi^* \operatorname{Im}\Omega \\ \mathrm{d}(f_\xi^*\theta_1 - f^*\theta_1) &= f_\xi^*\omega. \end{aligned}$$

Now, by Proposition 6.31 we have

$$\begin{aligned} (-1)^n *_g (f_\xi^*\theta_{n-1} - f^*\theta_{n-1}) &\in C_{\beta+1}^{k,a}(T^*X) \\ f_\xi^*\theta_1 - f^*\theta_1 &\in C_{\beta+1}^{k,a}(T^*X) \end{aligned}$$

since $f_\xi - f = f + \xi - f = \xi \in C_{\beta+1}^{k+1,a}(X,\mathbb{C}^n)$, $f_\xi, f \in C_1^{k+1,a}(X,\mathbb{C}^n)$ and the components of θ_1, θ_{n-1} are linear. It follows that

$$*_g f_\xi^* \operatorname{Im}\Omega + f_\xi^*\omega \in \mathrm{d}_g^*\big(C_{\beta+1}^{k,a}(T^*X)\big) \oplus \mathrm{d}\big(C_{\beta+1}^{k,a}(T^*X)\big)$$

and so by Lemma 6.27 and Corollary 6.29 there exists $\eta \in C_{\beta+1}^{k,a}(T^*X)$ with $(\mathrm{d}_g^* + \mathrm{d})\eta = *_g f_\xi^* \operatorname{Im}\Omega + f_\xi^*\omega$. Unfortunately this is one less derivative than we need, but we can infer immediately that in fact we must have $\eta \in C_{\beta+1}^{k+1,a}(T^*X)$ from the elliptic regularity Theorem 4.21 and the fact that $\eta \in C_{\beta+1}^{1,a}(T^*X)$ and $*_g f_\xi^* \operatorname{Im}\Omega + f_\xi^*\omega \in C_\beta^{k,a}(\Lambda^* T^*X)$. Hence we are done. ∎

From now on we shall consider the bundles N and T^*X as being identified via the vector bundle isomorphism $\flat_g J$. Pick any closed subspace $\mathcal{A}_{\beta+1} \leqslant C_{\beta+1}^{k+1,a}(T^*X)$ such that

$$C_{\beta+1}^{k+1,a}(T^*X) = K_{\beta+1} \oplus \mathcal{A}_{\beta+1}$$

where $K_{\beta+1} = \operatorname{Ker} F'_{\beta+1}(0)$ is finite dimensional, with dimension as given in Table 5.1. Now suppose that $\beta+2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$ and that $\beta+2 > \max\{2-n+\alpha, 2-n-\lambda\}$. Further suppose that $\beta+2$ is generic. Considering $F_{\beta+1}$ as a smooth map

$$F_{\beta+1} : V_{\beta+1}^{k+1,a} \to \mathrm{d}_g^*\big(C_{\beta+1}^{k+1,a}(T^*X)\big) \oplus \mathrm{d}\big(C_{\beta+1}^{k+1,a}(T^*X)\big)$$

between open subsets of Banach spaces we see that $F'_{\beta+1}(0)$ is surjective. From the Implicit Function Theorem 2.11 there exist open subsets $W_1^{\beta+1} \subseteq K_{\beta+1}$, $\mathcal{W}_2^{\beta+1} \subseteq \mathcal{A}_{\beta+1}$ both containing 0, and a unique map $\chi_{\beta+1} : W_1^{\beta+1} \to \mathcal{W}_2^{\beta+1}$ such that

1. Each $\xi = (\xi_1, \xi_2) \in W_1^{\beta+1} \times \mathcal{W}_2^{\beta+1}$ gives a submanifold $f_\xi : X \to \mathbb{C}^n$ of class $C^{k+1,a}$.
2. $$F_{\beta+1}^{-1}(0) \cap \Big(W_1^{\beta+1} \times \mathcal{W}_2^{\beta+1}\Big) = \Big\{\big(\xi_1, \chi_{\beta+1}(\xi_1)\big) : \xi_1 \in W_1^{\beta+1}\Big\}$$

   in $W_1^{\beta+1} \times \mathcal{W}_2^{\beta+1}$. Furthermore, the map $\chi_{\beta+1}$ is smooth, and $\chi_{\beta+1}(0) = 0$.

It follows that there is a bijection

$$\begin{array}{rcl} W_1^{\beta+1} & \to & F_{\beta+1}^{-1}(0) \cap \big(W_1^{\beta+1} \times \mathcal{W}_2^{\beta+1}\big) \\ \xi_1 & \mapsto & \big(\xi_1, \chi_{\beta+1}(\xi_1)\big) \end{array} \tag{6.59}$$

and we can put the structure of a smooth manifold onto

$$\tilde{\mathcal{M}}_{\beta+1} := F_{\beta+1}^{-1}(0) \cap \big(W_1^{\beta+1} \times \mathcal{W}_2^{\beta+1}\big) \tag{6.60}$$

by declaring that the map (6.59) be a chart in the maximal smooth atlas for $\tilde{\mathcal{M}}_{\beta+1}$. The following lemma is now almost immediate.

Lemma 6.42 1. *The manifold $\tilde{\mathcal{M}}_{\beta+1}$ is diffeomorphic to an open subset of $K_{\beta+1}$ and consequently* $\dim \tilde{\mathcal{M}}_{\beta+1} = \dim K_{\beta+1}$ *where the dimensions* $\dim K_{\beta+1}$ *are as given in Table* 5.1.

2. *With the smooth structure on $\tilde{\mathcal{M}}_{\beta+1}$ defined above, the inclusion*

$$\tilde{\mathcal{M}}_{\beta+1} \to W_1^{\beta+1} \times \mathcal{W}_2^{\beta+1} \tag{6.61}$$

is a smooth injective map that is an immersion, and a homeomorphism onto its image. In other words, the inclusion (6.61) *is a smooth submanifold of $W_1^{\beta+1} \times \mathcal{W}_2^{\beta+1}$.*

Note that for all $\xi = (\xi_1, \xi_2) \in W_1^{\beta+1} \times \mathcal{W}_2^{\beta+1}$ we have

$$\Big[f_\xi : X \to \mathbb{C}^n \text{ is special Lagrangian}\Big] \iff \Big[\xi \in \tilde{\mathcal{M}}_{\beta+1}\Big] \iff \Big[\xi_2 = \chi_{\beta+1}(\xi_1)\Big].$$

The reason for the tilde on the smooth manifold $\tilde{\mathcal{M}}_{\beta+1}$ is that for $\xi \in \tilde{\mathcal{M}}_{\beta+1}$ the submanifold $f_\xi : X \to \mathbb{C}^n$ need not be strongly asymptotically conical with cone C and rate $\beta+1$: all we know is that the 1-form $(\flat_g J)\xi$ lies in $C^{k+1,a}_{\beta+1}(T^*X)$, so that $\xi \in C^{k+1,a}_{\beta+1}(X, \mathbb{C}^n)$, and furthermore $f - i \in C^\infty_{\alpha+1}(X, \mathbb{C}^n)$. Therefore

$$f_\xi - i = f + \xi - i \in C^{k+1,a}_{\gamma+1}(X, \mathbb{C}^n)$$

where $\gamma = \max\{\alpha, \beta\}$. Of course, we can say that f_ξ is necessarily smooth, from the regularity results of Section 2.3.2, but we do not know that f_ξ has the required decay in its high derivatives. We shall now address this point.

Theorem 6.43 *If $\xi \in V^{k+1,a}_{\beta+1}$ is such that*

$$F_{\beta+1}(\xi) := *_g f_\xi^* \operatorname{Im}\Omega + f_\xi^* \omega = 0 \tag{6.62}$$

then $\xi \in V^\infty_{\beta+1}$.

Note that the following argument does not rely on the fact that $\beta+2$ is generic, nor the fact that $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$ with $\beta + 2 > \max\{2 - n + \alpha, 2 - n - \lambda\}$.

Proof: For the purposes of this proof, we shall denote the map $F_{\beta+1}$ by F. Recall that we are identifying the bundles N and T^*X over X, and then F is a first order differential operator

$$F : V^{k+1,a}_{\beta+1} \to C^{k,a}_\beta(X) \oplus C^{k,a}_\beta(\Lambda^2 T^*X) \tag{6.63}$$

so that we may write $F(\xi) = \hat{F}(\xi, \nabla\xi)$ where the value of $\hat{F}(\xi_1, \xi_2)$ at $x \in X$ depends only on the values of ξ_1 and ξ_2 at x. An inspection of the operator F in local coordinates shows that (6.63) is a fully non-linear operator. For example, in our usual local coordinate system $x = (t, \sigma)$ on $(T, \infty) \times U_\nu \subseteq X_\infty$ we have

$$\begin{aligned} f_\xi^*\omega &= f_\xi^*\omega - f^*\omega \\ &= \sum_{k_1,k_2=1}^{n} \sum_{j_1,j_2=1}^{2n} \omega_{j_1 j_2}\big((\partial_{k_1}\xi_{j_1})(\partial_{k_2}\xi_{j_2}) + (\partial_{k_1}\xi_{j_1})(\partial_{k_2} f_{j_2}) + (\partial_{k_1} f_{j_1})(\partial_{k_2}\xi_{j_2})\big)\mathrm{d}x_{k_1} \wedge \mathrm{d}x_{k_2} \end{aligned}$$

which is obviously not linear in the $\partial_k \xi_j$, and similarly for the quantity $*_g f_\xi^* \operatorname{Im} \Omega$. *However*, note that applying the differentiation ∂_i to each component function

$$(\partial_{k_1}\xi_{j_1})(\partial_{k_2}\xi_{j_2}) + (\partial_{k_1}\xi_{j_1})(\partial_{k_2} f_{j_2}) + (\partial_{k_1} f_{j_1})(\partial_{k_2}\xi_{j_2})$$

of the above sum yields

$$(\partial^2_{ik_1}\xi_{j_1})(\partial_{k_2}\xi_{j_2} + \partial_{k_2} f_{j_2}) + (\partial^2_{ik_2}\xi_{j_2})(\partial_{k_1}\xi_{j_1} + \partial_{k_1} f_{j_1}) + (\partial^2_{ik_1} f_{j_1})(\partial_{k_2}\xi_{j_2}) + (\partial^2_{ik_2} f_{j_2})(\partial_{k_1}\xi_{j_1})$$

which *is* linear in the second order derivatives $\partial^2_{ik}\xi_j$ of ξ. It follows that the second order, non-linear differential operator

$$G : V^{k+1,a}_{\beta+1} \to C^{k-1,a}_{\beta-1}(T^*X)$$

defined by $G(\xi) := (\mathrm{d}_g^* + \mathrm{d})F(\xi) = \mathrm{d} *_g f_\xi^* \operatorname{Im}\Omega + \mathrm{d}_g^* f_\xi^* \omega$ is quasi-linear.

Now, by the regularity theory of Section 2.3.2, if ξ satisfies equation (6.62), then $f_\xi : X \to \mathbb{C}^n$ is special Lagrangian, so that ξ is smooth. In order to show that all derivatives of ξ decay at order $O(e^{(\beta+1)t})$, and hence prove our theorem, we can employ a standard technique from the theory of quasi-linear equations. First of all, write

$$G(\xi) = G_1(\xi, \nabla\xi)\nabla^2\xi + G_0(\xi, \nabla\xi)$$

where $G_0(\xi, \nabla\xi)$ consists of the parts of $G(\xi)$ which are of order less than or equal to 1, and $G_1(\xi, \nabla\xi)\nabla^2\xi$ contains the parts of order 2, so that the map

$$\eta \mapsto G_1(\xi, \nabla\xi)\nabla^2\eta \tag{6.64}$$

is linear. Now, one can easily check via a local coordinate calculation that $G_0(\xi, \nabla\xi) \in C^{k,a}_{\beta-1}(T^*X)$, and moreover that the operator (6.64) is order 2 and uniformly elliptic, asymptotically conical, with rate 2. Therefore

$$G_1(\xi, \nabla\xi)\nabla^2\xi = -G_0(\xi, \nabla\xi)$$

implies that $\xi \in C^{k+2,a}_{\beta+1}(T^*X)$, and a boot-strapping argument then shows $\xi \in C^\infty_{\beta+1}(T^*X)$, finishing the proof.

Actually, one must be a little careful, because although the coefficients of the operator (6.64) are smooth, only their first $k + a$ derivatives will decay at rate $O(e^{-2t})$, and in our definition of asymptotically conical operator we require all derivatives decay at the specified rate. However, this turns out not to be a problem because elliptic estimates, as in Theorem 4.21, for a uniformly elliptic, asymptotically conical operator

$$Q : C^{k+l,a}_{\beta+\gamma}(E) \to C^{k,a}_\beta(F)$$

of order $l \geqslant 1$ and rate $\gamma \in \mathbb{R}^L$ only require the coefficients of Q decay at rate $O(e^{-\gamma t})$ in their first $k + a$ derivatives. The previous statement is easily deduced from the corresponding local Schauder estimates for operators with Hölder continuous coefficients, as in [16, Theorem 1] for example.

∎

It follows from Theorem 6.43 that $\chi_{\beta+1}\big(W_1^{\beta+1}\big) \subseteq C^\infty_{\beta+1}(T^*X)$, and that

$$\tilde{\mathcal{M}}_{\beta+1} \subseteq C^\infty_{\beta+1}(T^*X).$$

Hence every element $\xi \in \tilde{\mathcal{M}}_{\beta+1}$ gives a submanifold $f_\xi : X \to \mathbb{C}^n$ which is special Lagrangian and strongly asymptotically conical with cone C and rate $\gamma + 1 < 1$, where $\gamma = \max\{\alpha, \beta\}$.

Let us now look at deformations ξ which have the growth rate we are interested in, namely $\tilde{\alpha} = \alpha + 1 < 1$. From now on we assume that $\alpha + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$ with $\alpha + 2 > 2 - n - \lambda$. Then we may choose generic $\beta_1 + 1, \beta_2 + 1 \in \mathbb{R}^L$ with $\beta_1 + 1 < \alpha + 1 < \beta_2 + 1 < 1$ and $\alpha - \beta_1 < n$, such that $\beta_1 + 2, \alpha + 2, \beta_2 + 2$ all lie in the same connected component of $\mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$. It follows that the previous discussion — from (6.56) onwards — applies to each of $\beta + 1 := \beta_1 + 1, \beta_2 + 1$ and moreover

$$K_{\beta_1+1} = K_{\alpha+1} = K_{\beta_2+1}.$$

The subsets of the above discussion can therefore be chosen so that

$$\begin{aligned}\mathcal{A}_{\beta_1+1} &= C^{k+1,a}_{\beta_1+1}(T^*X) \cap \mathcal{A}_{\beta_2+1}\\ W_1^{\beta_1+1} &= W_1^{\beta_2+1}\\ \mathcal{W}_2^{\beta_1+1} &= \mathcal{A}_{\beta_1+1} \cap \mathcal{W}_2^{\beta_2+1}\end{aligned}$$

and the mappings

$$\begin{aligned}\chi_{\beta_1+1} : W_1^{\beta_1+1} &\to \mathcal{W}_2^{\beta_1+1}\\ \chi_{\beta_2+1} : W_1^{\beta_2+1} &\to \mathcal{W}_2^{\beta_2+1}\end{aligned}$$

so that $\chi_{\beta_1+1}(\xi_1) = \chi_{\beta_2+1}(\xi_1)$ for all $\xi_1 \in W_1^{\beta_1+1} = W_1^{\beta_2+1}$. It follows that

$$\tilde{\mathcal{M}}_{\beta_1+1} = \tilde{\mathcal{M}}_{\beta_2+1}$$

and the smooth structures on $\tilde{\mathcal{M}}_{\beta_1+1}$ and $\tilde{\mathcal{M}}_{\beta_2+1}$ defined above coincide. *Now although $\alpha+2$ could be non-generic*, we can still define

$$\begin{aligned}\mathcal{A}_{\alpha+1} &:= C^{k+1,a}_{\alpha+1}(T^*X) \cap \mathcal{A}_{\beta_2+1}\\ W_1^{\alpha+1} &:= W_1^{\beta_2+1}\\ \mathcal{W}_2^{\alpha+1} &:= \mathcal{A}_{\alpha+1} \cap \mathcal{W}_2^{\beta_2+1}\end{aligned}$$

and then the map

$$\chi_{\alpha+1} : W_1^{\alpha+1} \to \mathcal{W}_2^{\alpha+1}$$

by $\chi_{\alpha+1}(\xi_1) := \chi_{\beta_1+1}(\xi_1) = \chi_{\beta_2+1}(\xi_1)$ for all $\xi_1 \in W_1^{\alpha+1} = W_1^{\beta_1+1} = W_1^{\beta_2+1}$. It follows easily that:

1. $\mathcal{A}_{\alpha+1} \leqslant C^{k+1,a}_{\alpha+1}(T^*X)$ is closed and $C^{k+1,a}_{\alpha+1}(T^*X) = K_{\alpha+1} \oplus \mathcal{A}_{\alpha+1}$.
2. $W_1^{\alpha+1} \subseteq K_{\alpha+1}$, $\mathcal{W}_2^{\alpha+1} \subseteq \mathcal{A}_{\alpha+1}$ are open subsets both containing 0 and the map $\chi_{\alpha+1} : W_1^{\alpha+1} \to \mathcal{W}_2^{\alpha+1}$ is smooth, with $\chi_{\alpha+1}(0) = 0$.
3. Each $\xi = (\xi_1, \xi_2) \in W_1^{\alpha+1} \times \mathcal{W}_2^{\alpha+1}$ gives a submanifold $f_\xi : X \to \mathbb{C}^n$ of class $C^{k+1,a}$.
4. $$F^{-1}_{\alpha+1}(0) \cap \left(W_1^{\alpha+1} \times \mathcal{W}_2^{\alpha+1}\right) = \left\{\left(\xi_1, \chi_{\alpha+1}(\xi_1)\right) : \xi_1 \in \mathcal{W}_1^{\alpha+1}\right\}$$
   in $W_1^{\alpha+1} \times \mathcal{W}_2^{\alpha+1}$.

We now see there is a bijection

$$\begin{aligned}W_1^{\alpha+1} &\to F^{-1}_{\alpha+1}(0) \cap \left(W_1^{\alpha+1} \times \mathcal{W}_2^{\alpha+1}\right)\\ \xi_1 &\mapsto \left(\xi_1, \chi_{\alpha+1}(\xi_1)\right)\end{aligned} \tag{6.65}$$

and we can put the structure of a smooth manifold onto

$$\mathcal{M}_{\alpha+1} := F^{-1}_{\alpha+1}(0) \cap \left(W_1^{\alpha+1} \times \mathcal{W}_2^{\alpha+1}\right) \tag{6.66}$$

by declaring that the map (6.65) be a chart in the maximal smooth atlas for $\mathcal{M}_{\alpha+1}$. The following lemma is now almost immediate.

Lemma 6.44 1. *The manifold $\mathcal{M}_{\alpha+1}$ is diffeomorphic to an open subset of $K_{\alpha+1}$ and consequently $\dim \mathcal{M}_{\alpha+1} = \dim K_{\alpha+1}$ where the dimensions $\dim K_{\alpha+1}$ are as given in Table* 5.1.

2. *With the smooth structure on $\mathcal{M}_{\alpha+1}$ defined above, the inclusion*

$$\mathcal{M}_{\alpha+1} \to W_1^{\alpha+1} \times \mathcal{W}_2^{\alpha+1} \tag{6.67}$$

is a smooth injective map that is an immersion, and a homeomorphism onto its image. In other words, the inclusion (6.67) *is a smooth submanifold of $W_1^{\alpha+1} \times \mathcal{W}_2^{\alpha+1}$.*

Note that

$$\mathcal{M}_{\alpha+1} = \tilde{\mathcal{M}}_{\beta_1+1} = \tilde{\mathcal{M}}_{\beta_2+1}$$

and the smooth structure defined on $\mathcal{M}_{\alpha+1}$ above coincides with the smooth structures already on $\tilde{\mathcal{M}}_{\beta_1+1}$ and $\tilde{\mathcal{M}}_{\beta_2+1}$. Note also that for all $\xi = (\xi_1, \xi_2) \in W_1^{\alpha+1} \times \mathcal{W}_2^{\alpha+1}$ we have

$$\Big[f_\xi : X \to \mathbb{C}^n \text{ is special Lagrangian}\Big] \Longleftrightarrow \Big[\xi \in \mathcal{M}_{\alpha+1}\Big] \Longleftrightarrow \Big[\xi_2 = \chi_{\alpha+1}(\xi_1)\Big]$$

and any $\xi \in W_1^{\alpha+1} \times \mathcal{W}_2^{\alpha+1}$ satisfying these conditions lies in $V_{\alpha+1}^{\infty}$, by Theorem 6.43, so that $f_\xi : X \to \mathbb{C}^n$ is strongly asymptotically conical with cone C and rate $\alpha + 1$.

We now summarise some of the results of the above discussion in the following theorem.

Theorem 6.45 *Let X be a manifold with ends, as described in Section 4.1. Let $f : X \to \mathbb{C}^n$ be a submanifold with normal bundle N, and suppose that $f : X \to \mathbb{C}^n$ is special Lagrangian and strongly asymptotically conical with cone $C \subseteq \mathbb{C}^n$ and rate $\alpha + 1 < 1$. Identify $N \cong T^*X$ via the vector bundle isomorphism $\flat_g J$. Define*

$$K_{\alpha+1} := \big\{\xi_1 \in C_{\alpha+1}^{\infty}(T^*X) : \mathrm{d}_g^*\xi_1 = 0 \textit{ and } \mathrm{d}\xi_1 = 0\big\}.$$

Then $K_{\alpha+1}$ has finite dimension as given in Table 5.1. Let $k \geqslant 2$ and suppose that $\alpha+2 > 2-n-\lambda$ with $\alpha+2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$. Then there exists a closed subspace $\mathcal{A} \leqslant C_{\alpha+1}^{k+1,a}(T^*X)$ such that $C_{\alpha+1}^{k+1,a}(T^*X) = K_{\alpha+1} \oplus \mathcal{A}$, and open subsets $W_1 \subseteq K_{\alpha+1}$, $\mathcal{W}_2 \subseteq \mathcal{A}$ both containing 0, and a smooth map $\chi : W_1 \to \mathcal{W}_2$ such that:

1. *Each $\xi = (\xi_1, \xi_2) \in W_1 \times \mathcal{W}_2$ gives a submanifold $f_\xi : X \to \mathbb{C}^n$ of class $C^{k+1,a}$.*

2. *For all $\xi = (\xi_1, \xi_2) \in W_1 \times \mathcal{W}_2$ we have*

$$\Big[f_\xi : X \to \mathbb{C}^n \textit{ is special Lagrangian}\Big] \Longleftrightarrow \Big[\chi(\xi_1) = \xi_2\Big]$$

   so that $\chi(W_1) \subseteq C_{\alpha+1}^{\infty}(T^*X)$ and $\chi(0) = 0$.

3. $$\mathcal{M}_{\alpha+1} := \Big\{\xi = (\xi_1, \xi_2) \in W_1 \times \mathcal{W}_2 : f_\xi : X \to \mathbb{C}^n \textit{ is special Lagrangian}\Big\}$$

   is a smooth manifold with dimension $\dim \mathcal{M}_{\alpha+1} = \dim K_{\alpha+1}$. Moreover,

$$\begin{array}{ccc} W_1 & \to & \mathcal{M}_{\alpha+1} \\ \xi_1 & \mapsto & \big(\xi_1, \chi(\xi_1)\big) \end{array}$$

   is a diffeomorphism and the inclusion $\mathcal{M}_{\alpha+1} \to W_1 \times \mathcal{W}_2$ is a smooth submanifold. Each element $\xi \in \mathcal{M}_{\alpha+1}$ gives rise to a special Lagrangian submanifold $f_\xi : X \to \mathbb{C}^n$ which is strongly asymptotically conical with cone C and rate $\alpha + 1$.

Before applying Theorem 6.45 to some examples in the next section, we note two things:

1. Up to now we have only considered submanifolds which are genuinely embedded in their ambient space, but the entire proof of Theorem 6.45 carries through to the case of immersed submanifolds $f : X \to \mathbb{C}^n$, so that the theorem holds in this situation also: the moduli space $\mathcal{M}_{\alpha+1}$ will then contain ξ such that the submanifolds $f_\xi : X \to \mathbb{C}^n$ are immersed.

2. In the proof of the Theorem 6.45 we used the fact that $\dim K_{\beta+1}$ is constant for $\beta + 2$ in a connected component of $\mathbb{R}^L \setminus \mathcal{D}(\Delta_g^0)$, and this allowed us to remove the genericity assumption on $\alpha + 2$ for the existence of a smooth moduli space $\mathcal{M}_{\alpha+1}$. In much the same way, one can easily show that a smooth moduli space $\mathcal{M}_{\alpha+1}$ exists for $\alpha + 2 \in \mathcal{D}(\Delta_g^0)$ with $\alpha + 2 > 2 - n - \lambda$, provided that $\dim K_{\beta+1}$ is constant for $\beta + 2$ in a small neighbourhood of $\alpha + 2$.

6.3 Applications of the deformation theory

6.3.1 Preliminary discussion

We begin with some general remarks: suppose that X is a manifold with ends as in Section 4.1 and that $f : X \to \mathbb{C}^n$ is a special Lagrangian submanifold which is strongly asymptotically conical with cone $C \subseteq \mathbb{C}^n$ and rate $\tilde{\alpha} = \alpha + 1 < 1$. Then, provided $\alpha + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$ and $\alpha + 2 > 2 - n - \lambda$ we have a smooth manifold $\mathcal{M}_{\alpha+1}$ which parameterizes the nearby special Lagrangian submanifolds $f_\xi : X \to \mathbb{C}^n$ which are strongly asymptotically conical with cone C and rate $\alpha + 1$. Moreover the manifold $\mathcal{M}_{\alpha+1}$ has dimension $\dim \mathcal{M}_{\alpha+1} = \dim K_{\alpha+1}$ got from Table 5.1. A study of Table 5.1 reveals that the deformations of $f : X \to \mathbb{C}^n$ have two sources:

1. The exact parts of $K_{\alpha+1}$. These are:

   (a) $\operatorname{Ker} \psi_{\alpha+1} = 0$ for $2 - n - \lambda < \alpha + 2 < 2 - n$.

   (b) $\operatorname{Ker} \psi_{\alpha+1} = \operatorname{Span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_L\}$ for $2 - n < \alpha + 2 < \lambda$. Essentially this space is the kernel of the natural projection $\phi_1 : H^1_c(X) \to H^1(X)$.

   (c) $\operatorname{Ker} \psi_{\alpha+1} = \mathrm{d}\operatorname{Ker}(\Delta^0_g)_{\alpha+2}$ for $\alpha + 2 > 0$. This space always contains $\mathrm{d}\operatorname{Ker}(\Delta^0_g)_0 = \operatorname{Span}\{\mathrm{d}h_1, \ldots, \mathrm{d}h_L\}$. However, for $\lambda < \alpha + 2 < 2$ there will also be contributions to $\mathrm{d}\operatorname{Ker}(\Delta^0_g)_{\alpha+2}$ coming from non-constant eigenfunctions of the link Laplacian $\Delta^0_{g_\Sigma}$. Recall that here $\lambda = (\lambda_1, \ldots, \lambda_L)$ where for each $1 \leqslant j \leqslant L$ we define $\lambda_j > 0$ to be such that $\lambda_j(\lambda_j + n - 2)$ is the smallest positive element of $\operatorname{Spec}(\Sigma_j, g_\Sigma, 0)$. It turns out that $\lambda \leqslant 1$ always: we prove this below by constructing eigenfunctions of the Laplacian on each Σ_j, which have eigenvalue $n - 1$. It is interesting to ask when $\lambda < 1$: in this case we can say that there exist unbounded harmonic functions on X which have sub-linear growth.

2. The "non-exact" parts $\frac{K_{\alpha+1}}{\operatorname{Ker} \psi_{\alpha+1}} \cong \operatorname{Im} \psi_{\alpha+1}$ of $K_{\alpha+1}$. These are:

   (a) $\operatorname{Im} \phi_1 \leqslant H^1(X)$ if $2 - n - \lambda < \alpha + 2 < 0$

   (b) $H^1(X)$ if $\alpha + 2 > 0$.

Note that there is an overlap in the cases 1(b) and 1(c) above, corresponding to $0 < \alpha + 2 < \lambda$. We now give an explanation of the claim in case 1(c) that $\lambda \leqslant 1$ always: we begin with some standard theory from symplectic geometry.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. If G acts on a symplectic manifold (M, ω) via symplectomorphisms then a *moment map* for the action is a map $m : M \to \mathfrak{g}^*$ such that

$$\mathrm{d}\langle m, \xi \rangle = \iota(v_\xi)\omega \tag{6.68}$$

for all $\xi \in \mathfrak{g}$. In equation (6.68) $\langle \ , \ \rangle$ is the natural pairing between the vector space $\mathfrak{g}$ and the dual space $\mathfrak{g}^*$, and v_ξ is the vector field on M induced by $\xi \in \mathfrak{g}$ and the action of G on M. Note that a moment map need not exist, nor be unique.

In the case that $M = \mathbb{C}^n$ with the usual symplectic form ω there are two actions as above which are particularly relevant for us:

1. If $G = \mathbb{C}^n$, acting on M via translations

$$\begin{array}{rcl} G \times M & \to & M \\ (w, z) & \mapsto & w + z, \end{array}$$

   then a moment map $m : \mathbb{C}^n \to \mathfrak{g}^*$ exists. Identifying $\mathfrak{g} \cong \mathbb{C}^n$ in the usual way, we may take m to be defined by

$$\langle m(z), \xi \rangle := \sum_{j=1}^{n} \operatorname{Im}(z_j \bar{\xi}_j)$$

   for all $z, \xi \in \mathbb{C}^n$, so that the components of m form a basis for the vector space of real linear maps $\mathbb{C}^n \to \mathbb{R}$.

2. If $G = \mathrm{SU}(n)$, acting on M via rotations

$$\begin{array}{rcl} G \times M & \to & M \\ (A, z) & \mapsto & Az, \end{array}$$

then a moment map $m : \mathbb{C}^n \to \mathfrak{g}^*$ exists. Identifying $\mathfrak{g}$ with the trace-free anti-hermitian $n \times n$ complex matrices we may take m to be defined by

$$\langle m(z), \xi^1_{jk} \rangle = \tfrac{1}{2}(|z_k|^2 - |z_j|^2) \tag{6.69}$$

$$\langle m(z), \xi^2_{jk} \rangle = -\operatorname{Re}(z_k \overline{z}_j) \tag{6.70}$$

$$\langle m(z), \xi^3_{jk} \rangle = -\operatorname{Im}(z_k \overline{z}_j) \tag{6.71}$$

for all $z \in \mathbb{C}^n$. In equations (6.69), (6.70), (6.71) we define, for each $1 \leqslant j < k \leqslant n$:

(a) $\xi^1_{jk} \in \mathfrak{g}$ to be the matrix whose only non-zero entries are i in the (j, j) place and $-i$ in the (k, k) places,

(b) $\xi^2_{jk} \in \mathfrak{g}$ to be the matrix whose only non-zero entries are i in the (j, k) and (k, j) places,

(c) $\xi^3_{jk} \in \mathfrak{g}$ to be the matrix whose only non-zero entries are 1 in the (j, k) place and -1 in the (k, j) place.

Note that $\{\xi^1_{1k} : 2 \leqslant k \leqslant n\} \cup \{\xi^2_{jk}, \xi^3_{jk} : 1 \leqslant j < k \leqslant n\}$ is a basis for $\mathfrak{g}$.

In both of the above actions, the group G preserves the standard Calabi-Yau structure on $M = \mathbb{C}^n$. The following result now allows us to obtain eigenfunctions for our Laplacian on the link Σ.

Proposition 6.46 *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let (M, J, g, Ω) be a Calabi-Yau manifold with Kähler form ω. Suppose that G acts on M preserving (J, g, Ω) and that $m : M \to \mathfrak{g}^*$ is a moment map for the action of G on the symplectic manifold (M, ω). Then for any special Lagrangian submanifold $f : X \to M$ and $\xi \in \mathfrak{g}$ the function*

$$f^* \langle m, \xi \rangle : X \to \mathbb{R}$$

is harmonic with respect to the induced metric on X.

Proof: It is easy to show that

$$\Delta^0_g\big(f^*\langle m, \xi\rangle\big) = \mathrm{d}^*_g\big(f^*(\iota(v_\xi)\omega)\big)$$

from the definition of a moment map. So to prove the proposition we must show that $f^*(\iota(v_\xi)\omega)$ is a coclosed 1-form on X. For this, let $\exp : \mathfrak{g} \to G$ denote the exponential map of the Lie group G. Viewing elements of G as diffeomorphisms $M \to M$ define $f_t := \exp(t\xi) \circ f$ for each $t \in \mathbb{R}$, so that

$$\begin{array}{rcl} F : \mathbb{R} \times X & \to & M \\ (t, x) & \mapsto & f_t(x) \end{array}$$

is a variation, in the sense of Section 2.2.2, with each $f_t : X \to M$ a special Lagrangian submanifold and $f_0 = f$. Note further that for each $t \in \mathbb{R}$ we have the infinitesimal variation $\xi^t = f_t^* v_\xi$ in $f_t^* TM$. Then we have

$$0 = \frac{\partial}{\partial t}\Big(f_t^* \operatorname{Im}\Omega\Big)\Big|_{t=0} = \mathrm{d}\big(f^*(\iota(\xi^0)\operatorname{Im}\Omega)\big) = -\mathrm{d}\big(*_g f^*(\iota(\xi^0)\omega)\big) \tag{6.72}$$

as required. In equation (6.72) we are using the variations material of Section 2.2.2 together with Corollary 2.28.

∎

The following result will also be useful for us:

Lemma 6.47 *Let G be a Lie group with Lie algebra $\mathfrak{g}$, and (M,ω) a symplectic manifold. Suppose that G acts on M via symplectomorphisms and that $m : M \to \mathfrak{g}^*$ is a moment map for the action. If $f : X \to M$ is a Lagrangian submanifold then $f^*m : X \to \mathfrak{g}^*$ is constant precisely when $(v_\xi)_x$ is tangent to X for each $x \in X$ and $\xi \in \mathfrak{g}$.*

Proof: Recall our assumption that all manifolds are connected unless explicitly stated otherwise. If $x \in X$ then we have that T_xX is equal to the subspace of $T_{f(x)}M$ which is ω_x-perpendicular to T_xX: this is because $f : X \to M$ is Lagrangian. Suppose further that $\xi \in \mathfrak{g}$ and $v \in T_xX$. Then the equation

$$v \cdot \big(f^*\langle m, \xi\rangle\big) = 2\omega(v_\xi, v)$$

proves the lemma.

∎

Let Σ_j be a component of the link Σ of C, and let $C_j \subseteq C \subseteq \mathbb{C}^n$ be the cone on Σ_j. Applying Proposition 6.46 with $M = \mathbb{C}^n$, acted on by translations, we see that the restriction of any real linear map $T : \mathbb{C}^n \to \mathbb{R}$ to the special Lagrangian cone $C_j \subseteq \mathbb{C}^n$ is harmonic on $C_j \setminus \{0\}$. Since any real linear map $T : \mathbb{C}^n \to \mathbb{R}$ is homogeneous of degree 1, have that $T|_{\Sigma_j}$ is an eigenfunction for $\Delta^0_{g_\Sigma}$ restricted to Σ_j, with eigenvalue $n-1$. In other words

$$\big\{T|_{\Sigma_j} : T : \mathbb{C}^n \to \mathbb{R} \text{ is linear}\big\} \leqslant \operatorname{Ker}\big(\Delta^0_{g_\Sigma} - (n-1)\big) \cap C^\infty(\Sigma_j),$$

which is a subspace of dimension $d_{\mathrm{tr}}(\Sigma_j)$, where

$$d_{\mathrm{tr}}(\Sigma_j) = \begin{cases} n & \text{if } \Sigma_j \text{ is a round unit } (n-1)\text{-sphere} \\ 2n & \text{otherwise.} \end{cases} \tag{6.73}$$

To see why equation (6.73) holds, suppose that Σ_j is not a round unit $(n-1)$-sphere, so that the cone C_j is not a linear subspace of $\mathbb{C}^n$ and has a singularity at $0 \in \mathbb{C}^n$. Let G be the group $\mathbb{C}^n$, acting on $\mathbb{C}^n$ by translations. Any non-trivial translation of $\mathbb{C}^n$ must move the submanifold $C_j \setminus \{0\} \to \mathbb{C}^n$, and it follows that for every non-zero $\xi \in \mathfrak{g}$ there exists an $x \in C_j \setminus \{0\}$ such that $(v_\xi)_x$ is not tangent to $C_j \setminus \{0\}$. So from Lemma 6.47 we deduce that $\langle m, \xi\rangle$ is not constant on $C_j \setminus \{0\}$, so that any non-zero linear function $\mathbb{C}^n \to \mathbb{R}$ is non-zero on Σ_j, and therefore $d_{\mathrm{tr}}(\Sigma_j) = 2n$.

Recall that the eigenvalue $n-1$ corresponds to the growth rate 0 on the manifold X, so that $K_{\beta+1}$ increases in dimension by at least

$$d_{\mathrm{tr}}(\Sigma) := \sum_{j=1}^{L} d_{\mathrm{tr}}(\Sigma_j) \tag{6.74}$$

as $\beta + 1$ crosses over 0. Because of the equations (6.73) note that $d_{\mathrm{tr}}(\Sigma)$ must be at least n, with equality precisely when C is a special Lagrangian plane in $\mathbb{C}^n$. Otherwise, C has a singularity at 0, and we can say that $d_{\mathrm{tr}}(\Sigma)$ is at least $2n$. The general idea is that increasing the growth rate $\beta+1$ above 0 picks up all the deformations of the submanifold $f : X \to \mathbb{C}^n$ which are got from translations: being special Lagrangian and strongly asymptotically conical with cone C and rate $\alpha + 1 > 0$ is preserved by translations. Note that in the cases $d_{\mathrm{tr}}(\Sigma) > 2n$ we have deformations of $f : X \to \mathbb{C}^n$ which are not just the usual translations on $\mathbb{C}^n$: one can think of our theory as saying that the ends of X may be "translated" independently of each other, to yield submanifolds which are still special Lagrangian and asymptotically conical with cone C and rate $\alpha + 1 > 0$. It is also interesting to consider if there are examples of special Lagrangian cones with link Σ such that $d_{\mathrm{tr}}(\Sigma) < \dim \operatorname{Ker}\big(\Delta^0_{g_\Sigma} - (n-1)\big)$.

We now look at the action of $\mathrm{SU}(n)$ on $\mathbb{C}^n$. Then Proposition 6.46 tells us that each of the functions on $\mathbb{C}^n$ given by equations (6.69), (6.70), (6.71) restrict to the components Σ_j of the link Σ to be an eigenfunction for $\Delta^0_{g_\Sigma}$, with eigenvalue $2n$. Let $V_j \leqslant \operatorname{Ker}(\Delta^0_{g_\Sigma} - 2n) \cap C^\infty(\Sigma_j)$ be the vector space of eigenfunctions got on the component Σ_j in this way, so that we have a surjective linear map

$$\begin{array}{rcl} \mathfrak{su}(n) & \to & V_j \\ \xi & \mapsto & \langle m, \xi\rangle|_{\Sigma_j}. \end{array} \tag{6.75}$$

The exact dimension $d_{\mathrm{rot}}(\Sigma_j)$ of the space V_j will depend upon the symmetries of the cone C_j under the group $\mathrm{SU}(n)$: for each $1 \leqslant j \leqslant L$ define

$$H_j := \{A \in \mathrm{SU}(n) : A(C_j) = C_j\}$$

the symmetry group of the cone $C_j \subseteq \mathbb{C}^n$. Then $H_j \leqslant \mathrm{SU}(n)$ is a closed, and therefore Lie, subgroup of $\mathrm{SU}(n)$. Let $\mathfrak{h}_j$ be the Lie algebra of H_j. Using Lemma 6.47, one can show that the kernel of the linear map (6.75) is $\mathfrak{h}_j \leqslant \mathfrak{su}(n)$, so that

$$d_{\mathrm{rot}}(\Sigma_j) = \dim V_j = \dim \mathrm{SU}(n) - \dim H_j \tag{6.76}$$

for each $1 \leqslant j \leqslant L$. We shall be less interested in the eigenfunctions with eigenvalue $2n$ because the corresponding growth rate is $\alpha + 1 = 1$, which is just outside the scope of Theorem 6.45. But, intuitively at least, the closed and coclosed 1-forms on X got from the $\mathrm{SU}(n)$ action on $\mathbb{C}^n$ and Proposition 6.46 correspond to rotating $f(X) \subseteq \mathbb{C}^n$ by elements of $\mathrm{SU}(n)$. As for translations, we have a contribution of the form (6.76) for each end of X, with a total dimension

$$d_{\mathrm{rot}}(\Sigma) := \sum_{j=1}^{L} d_{\mathrm{rot}}(\Sigma_j)$$

so that, in vague terms, the ends of X can be "rotated" independently by elements of $\mathrm{SU}(n)$. As for translations, it is interesting to consider if there are examples of special Lagrangian cones with link Σ such that $d_{\mathrm{rot}}(\Sigma) < \dim \mathrm{Ker}(\Delta^0_{g_\Sigma} - 2n)$. In actual fact, there are: see [31, Section 10.3], and also the $\mathrm{U}(1)^{n-1}$-invariant example of Harvey and Lawson [21] in dimension $n = 8$: the details are given below in Section 6.3.4.

We shall now look at some concrete examples. Note that in Section 6.3.2 onwards, we tend to use $\tilde{\alpha} = \alpha + 1$ to refer to the *minimal* decay rate at which a submanifold $X \to \mathbb{C}^n$ is strongly asymptotically conical, and then use $\tilde{\beta} = \beta + 1$ to refer to decay rates which are larger than $\alpha + 1$: certainly our submanifold $f : X \to \mathbb{C}^n$ will be strongly asymptotically conical with these $\beta + 1$ rates, and we can form the corresponding moduli spaces $\mathcal{M}_{\beta+1}$ accordingly.

6.3.2 Decay $\alpha + 1 = 1 - n - \lambda$

Unfortunately, we only have one example of a special Lagrangian submanifold $f : X \to \mathbb{C}^n$ which is strongly asymptotically conical with cone C and rate $\alpha + 1 = 1 - n - \lambda$: the special Lagrangian plane $\mathbb{R}^n \leqslant \mathbb{C}^n$, and this is a degenerate example because the submanifold $f : X \to \mathbb{C}^n$ obviously has arbitrarily negative decay. It would be interesting to find examples for which the relevant dimension $\dim K_{\alpha+1} = b^1_c(X) - L + 1$ is non-zero.

Example 1: special Lagrangian planes

Put $X = \mathbb{R}^n$ and define $f : X \to \mathbb{C}^n$ by

$$f(x_1, \ldots, x_n) := (x_1, \ldots, x_n).$$

Then $f : X \to \mathbb{C}^n$ is special Lagrangian and strongly asymptotically conical with cone $C = f(X)$ and arbitrarily negative growth rate $\alpha + 1 < 1$. We have

$$\begin{aligned} b^1_c(X) &= 0 \\ b^1(X) &= 0. \end{aligned}$$

Clearly C has $L = 1$ ends, and link the round unit sphere $S^{n-1} \subseteq \mathbb{R}^n$. It is a well-known fact that the spectrum of a round unit $(n-1)$-sphere consists of the points

$$\mathrm{Spec}(S^{n-1}, g_{\mathrm{rd}}, 0) = \Big\{\mu_l := l(l + n - 2) : l \geqslant 0\Big\} \tag{6.77}$$

and that the full $l(l+n-2)$-eigenspace is got from restricting homogeneous, harmonic polynomials of degree $l \geqslant 0$ on $\mathbb{R}^n$ to S^{n-1}: see [14] for example. Let P_n^l denote the homogeneous polynomials of degree $l \geqslant 0$ on $\mathbb{R}^n$, which is a vector space of dimension

$$\dim P_n^l = \binom{n+l-1}{l}.$$

Also let $H_n^l \leqslant P_n^l$ be the subspace consisting of those $p(x) \in P_n^l$ which are harmonic. Then, from [11, Segal, Section 10] we have $\dim H_n^l = \dim P_n^l - \dim P_n^{l-2}$ and a brief calculation yields

$$\dim \operatorname{Ker}(\Delta^0_{g_{\mathrm{rd}}} - \mu_l) = \frac{(n-2+2l)(n-3+l)!}{l!(n-2)!} \tag{6.78}$$

for $l \geqslant 0$. Note that the only element of $\operatorname{Spec}(S^{n-1}, g_{\mathrm{rd}}, 0)$ inside $(0, 2n)$ is $n-1$, and that

$$\dim \operatorname{Ker}\big(\Delta^0_{g_{\mathrm{rd}}} - (n-1)\big) = n.$$

The above collection of facts, together with Table 5.1, yield

$$\begin{array}{rcll} \dim \mathcal{M}_{\beta+1} & = & 0 & \text{for } \beta+1<0 \\ \dim \mathcal{M}_{\beta+1} & = & n & \text{for } 0<\beta+1<1. \end{array}$$

so that X is rigid up to growth rates $\beta+1=0$, and then for rates $0<\beta+1<1$ the only deformations are translations in the normal directions. Moreover, the full $(n-1)$-eigenspace is got from the $\mathbb{C}^n$ moment map construction given above.

Note that

$$\dim \operatorname{Ker}(\Delta^0_{g_{\mathrm{rd}}} - 2n) = \frac{(n+2)(n-1)}{2}$$

is the multiplicity of the eigenvalue corresponding to growth rate $\beta+1=1$. Now compare

$$d_{\mathrm{rot}}(S^{n-1}) = \dim \mathrm{SU}(n) - \dim \mathrm{SO}(n) = n^2 - 1 - \frac{n(n-1)}{2} = \frac{(n+2)(n-1)}{2}$$

where $SO(n) \leqslant \mathrm{SU}(n)$ is the symmetry group of the special Lagrangian plane $\mathbb{R}^n \leqslant \mathbb{C}^n$. Therefore the full $2n$-eigenspace is got from the $\mathrm{SU}(n)$ moment map construction given above.

6.3.3 Decay $\alpha+1=1-n$

Example 2: the cone construction of Castro and Urbano/Haskins/Joyce

We begin by quoting a result proved independently by Castro and Urbano [12, Remark 1, p. 81-82], Haskins [22, Theorem A] and Joyce [30, Theorem 6.4].

Theorem 6.48 *Let $C \subseteq \mathbb{C}^n$ be a special Lagrangian cone with link Σ. For each $a>0$ define*

$$X_a := \Big\{ zx : x \in \Sigma \text{ and } z \in \mathbb{C} \text{ with } \operatorname{Im}(z^n) = a \text{ and } 0 < \arg(z) < \tfrac{\pi}{n} \Big\}.$$

Then $X_a \to \mathbb{C}^n$ is an immersed special Lagrangian submanifold, diffeomorphic to $\mathbb{R}\times\Sigma$. Moreover, X_a is asymptotically conical with cone $\tilde{C} := C \cup e^{\frac{i\pi}{n}} C$ and rate $\alpha+1=1-n$.

Let $C \subseteq \mathbb{C}^n$ be a 1-ended special Lagrangian cone with link Σ. Applying Theorem 6.48 we have, for each $a>0$, an immersed, connected special Lagrangian submanifold X_a, with topology $\Sigma\times\mathbb{R}$. Then

$$b^1_c(X_a) = b^1_c(\mathbb{R}\times\Sigma) = b^0(\Sigma) = 1$$

by Bott and Tu [8, Proposition 4.7] and also

$$b^1(X_a) = b^1(\mathbb{R}\times\Sigma) = b^1(\Sigma)$$

by Bott and Tu [8, Proposition 4.1]. Now X_a is asymptotic to the cone $\tilde{C} = C \cup e^{\frac{i\pi}{n}} C$, which has $L = 2$ ends, and the link $\tilde{\Sigma}$ has components Σ and $e^{\frac{i\pi}{n}} \Sigma$, which are isometric. We therefore have

$$\begin{array}{rcll} \dim \mathcal{M}_{\beta+1} & = & 1 & \text{for } 1-n < \beta+1 < -1 \\ \dim \mathcal{M}_{\beta+1} & = & 1 + b^1(\Sigma) & \text{for } -1 < \beta+1 < \lambda - 1 \end{array}$$

with analytic terms $2\chi(\beta+2)$ from (Σ, g_Σ) contributing to $\dim \mathcal{M}_{\beta+1}$ if $\lambda - 1 < \beta + 1 < 1$. Note that if $b^1(\Sigma) > 0$ then we have new examples of special Lagrangian submanifolds in $\mathbb{C}^n$. More generally, if Σ is not a round unit $(n-1)$-sphere then we have $\dim \mathcal{M}_{\beta+1} \geqslant 1 + b^1(\Sigma) + 4n$ for $0 < \beta + 1 < 1$: this is an example of being able to "translate" the two ends of $f : X \to \mathbb{C}^n$ independently.

If we take $C := \mathbb{R}^n \leqslant \mathbb{C}^n$ the standard special Lagrangian plane then the above construction yields the X_a of Example 2.22 which are $\mathrm{SO}(n)$-invariant. Then $b^1(\Sigma) = 0$ and using equation (6.77) and equation (6.78) we deduce that

$$\begin{array}{rcll} \dim \mathcal{M}_{\beta+1} & = & 1 & \text{for } 1-n < \beta+1 < 0 \\ \dim \mathcal{M}_{\beta+1} & = & 1 + 2n & \text{for } 0 < \beta+1 < 1 \end{array}$$

because each component of the link $\tilde{\Sigma}$ is isometric to a round unit $(n-1)$-sphere. It follows that the examples $X_a \to \mathbb{C}^n$ are isolated, up to perturbations of $a > 0$ and translations of $\mathbb{C}^n$.

Example 3: Harvey/Joyce/Lawlor examples

The following examples are discussed in Harvey [20, p. 139-143], Joyce [26, Theorem 5.4] and Lawlor [44].

Let $a_1, \ldots, a_n > 0$ and define a polynomial $p(y) \in \mathbb{R}[y]$ by

$$p(y) := \frac{(1 + a_1 y^2) \ldots (1 + a_n y^2) - 1}{y^2}.$$

Now put

$$\begin{array}{rcl} r_k(y) & := & \sqrt{\dfrac{1}{a_k} + y^2} \\ \theta_k(y) & := & \displaystyle\int_0^y \frac{\mathrm{d}y}{(1 + a_k y^2)\sqrt{p(y)}} \\ z_k(y) & := & r_k(y) e^{i\theta_k(y)} \end{array}$$

for each $1 \leqslant k \leqslant n$ and $y \in \mathbb{R}$. Then from [20, Theorem 7.78] and [26, Section 5.4] we have

$$X := \Big\{ \big(x_1 z_1(y), \ldots, x_n z_n(y)\big) : y \in \mathbb{R}, x \in S^{n-1} \subseteq \mathbb{R}^n \Big\}$$

is a submanifold $X \to \mathbb{C}^n$ which is special Lagrangian (with respect to the calibration $\operatorname{Im} \Omega$), diffeomorphic to $\mathbb{R} \times S^{n-1}$, and asymptotic to a cone C at rate $\alpha + 1 = 1 - n$, where

$$C = \Pi_1 \cup \Pi_2$$

and $\Pi_1, \Pi_2 \leqslant \mathbb{C}^n$ are special Lagrangian planes with $\Pi_1 \cap \Pi_2 = \{0\}$. It is now easy complete the following:

$$\begin{array}{rcll} \dim \mathcal{M}_{\beta+1} & = & 1 & \text{for } 1-n < \beta+1 < 0 \\ \dim \mathcal{M}_{\beta+1} & = & 1 + 2n & \text{for } 0 < \beta+1 < 1. \end{array} \tag{6.79}$$

The reason why the parameters $a_1, \ldots, a_n > 0$ do not contribute n dimensions to the moduli spaces $\mathcal{M}_{\beta+1}$ above is that the cone C depends upon the $a_1, \ldots, a_n$. However, it turns out that C is unchanged under the dilation

$$(a_1, \ldots, a_n) \mapsto e^t (a_1, \ldots, a_n) \tag{6.80}$$

and that is where the 1 comes from in the equations (6.79). It follows that the Harvey/Joyce/Lawlor examples $X \to \mathbb{C}^n$ given above are isolated, modulo translations and the re-scaling (6.80).

6.3.4 Decay $\alpha + 1 = -1$

Example 4: Joyce examples by evolving quadrics

The following theorem is proved by Joyce [26, Theorem 5.9]:

Theorem 6.49 *Let $1 \leqslant k \leqslant n-1$. Then there is a countable set $\aleph$ such that for every point of $\aleph$ there are $a_1, \ldots, a_n \in \mathbb{R}$ and functions $u : \mathbb{R} \to \mathbb{R}, \theta_j : \mathbb{R} \to \mathbb{R}$ such that for every $c > 0$ the subset*

$$\left\{ \begin{pmatrix} x_1 e^{i\theta_1(t)} \sqrt{a_1 + u(t)} \\ \vdots \\ x_k e^{i\theta_k(t)} \sqrt{a_k + u(t)} \\ x_{k+1} e^{i\theta_{k+1}(t)} \sqrt{a_{k+1} + u(t)} \\ \vdots \\ x_n e^{i\theta_n(t)} \sqrt{a_n + u(t)} \end{pmatrix} : t \in \mathbb{R},\ x \in \mathbb{R}^n \text{ and } x_1^2 + \cdots + x_k^2 - x_{k+1}^2 - \cdots - x_n^2 = c \right\} \tag{6.81}$$

is an immersed special Lagrangian submanifold $X_c \to \mathbb{C}^n$, with X_c diffeomorphic to $S^{k-1} \times \mathbb{R}^{n-k} \times S^1$. Let C be the subset of $\mathbb{C}^n$ defined by (6.81) with $c = 0$. Then $C \subseteq \mathbb{C}^n$ is an immersed special Lagrangian cone with link Σ diffeomorphic to $S^{k-1} \times S^{n-k-1} \times S^1$. Each $X_c \to \mathbb{C}$ is strongly asymptotically conical with cone C and rate $\alpha + 1 = -1$.

The general idea for proving Theorem 6.49 is to contemplate the subset (6.81) for arbitrary u, θ_j, a_j. Then X_c being special Lagrangian comes down to the functions u, θ_j satisfying some first order system of ODEs. In order for the solutions to be such that $u(t)$ and the $e^{i\theta_j(t)}$ are periodic in t, and hence for X_c to sit nicely inside $\mathbb{C}^n$, we need to specify certain initial conditions for our ODEs: this restriction is where the countable set $\aleph$ comes from.

We work out some topological details when $n = 3$. The first case is $k = 1$. Then each $X_c \to \mathbb{C}^3$ is an immersed special Lagrangian submanifold which is diffeomorphic to two copies of $S^1 \times \mathbb{R}^2$, and strongly asymptotically conical with cone C and rate $\alpha + 1 = -1$, where the link Σ of C is diffeomorphic to two copies of $S^1 \times S^1$. Applying our deformation theory to a connected component of X_c' of X_c we have $b^1(X_c') = 1$ and $L = 1$ so that for $-1 < \beta + 1 < -1 + \lambda$ we have

$$\dim \mathcal{M}_{\beta+1} = b^1(X_c') + L - 1 = 1.$$

Of course the second connected component also has $\dim \mathcal{M}_{\beta+1} = 1$ for $-1 < \beta+1 < -1+\lambda$, and then both pieces can be deformed independently, to give a 2 dimensional family of deformations. However, this is rather explicit in the details of Theorem 6.49 above: the deformations come from the parameter $c > 0$ in the expression (6.81) for X_c given above.

The second case is $k = 2$. Then each $X_c \to \mathbb{C}^3$ is an immersed special Lagrangian submanifold which is diffeomorphic to $S^1 \times S^1 \times \mathbb{R}$, and strongly asymptotically conical with cone C and rate $\alpha + 1 = -1$ where the link Σ of C is diffeomorphic to two copies of $S^1 \times S^1$. Then $b^1(X_c) = 2$ and $L = 2$ so that for $-1 < \beta + 1 < -1 + \lambda$ we have

$$\dim \mathcal{M}_{\beta+1} = b^1(X_c) + L - 1 = 3$$

so that we have 2 dimensions worth of deformations which are *not* present in the explicit family given by Theorem 6.49 above. Note also that for any $0 < \beta + 1 < 1$ we have

$$\dim \mathcal{M}_{\beta+1} \geqslant 3 + 6 + 6 = 15$$

where the additional 12 dimensions come from "translating" the 2 ends of X_c independently.

Example 5: Harvey and Lawson $\mathrm{U}(1)^{n-1}$-invariant examples

The following family of special Lagrangian submanifolds are those of Example 2.22 which are $\mathrm{U}(1)^{n-1}$-invariant, first discovered by Harvey and Lawson in [21].

Given $a_1, \ldots, a_n, b \in \mathbb{R}$ define

$$X_{a_1,\ldots,a_n,b} := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \begin{array}{l} |z_1|^2 - a_1 = \cdots = |z_n|^2 - a_n \\ \operatorname{Im}(i^{n+1} z_1 \ldots z_n) = b \end{array} \right\}$$

which is invariant under the group of diagonal matrices $\mathrm{U}(1)^{n-1} \leqslant \mathrm{SU}(n)$. Further define $C := X_{0,\ldots,0,0}$ which is a special Lagrangian cone with link Σ diffeomorphic to 2 copies of T^{n-1}. We have the following result, from the author's dissertation [47, Proposition 3.5], and elsewhere.

Proposition 6.50 *Without loss of generality, suppose that* $\min\{a_1, \ldots, a_n\} = 0$.

1. *If* $b \neq 0$ *then* $X_{a_1,\ldots,a_n,b} \to \mathbb{C}^n$ *is a special Lagrangian submanifold diffeomorphic to* $\mathbb{R} \times T^{n-1}$.
2. *If exactly one* a_k *vanishes then* $X_{a_1,\ldots,a_n,0} \to \mathbb{C}^n$ *is a special Lagrangian submanifold diffeomorphic to* $\mathbb{R} \times T^{n-1}$.
3. *If exactly two* a_k *vanish then* $X_{a_1,\ldots,a_n,0}$ *is the union of two special Lagrangian submanifolds* $X^{\pm}_{a_1,\ldots,a_n,0} \to \mathbb{C}^n$ *each diffeomorphic to* $\mathbb{R}^2 \times T^{n-2}$, *with* $X^{+}_{a_1,\ldots,a_n,0} \cap X^{-}_{a_1,\ldots,a_n,0} \cong T^{n-2}$ *being the singular set of* $X_{a_1,\ldots,a_n,0}$.
4. C *is the union of two special Lagrangian cones* $C^{\pm}$, *each with link* $\Sigma^{\pm}$ *diffeomorphic to* T^{n-1}. *Also,* $C^+ \cap C^- = \{0\}$.

In case 1 and case 2 of Proposition 6.50 it is easy to check that $X_{a_1,\ldots,a_n,b} \to \mathbb{C}^n$ is strongly asymptotically conical with cone C and rate $\alpha + 1 = -1$. Also, in case 3 of Proposition 6.50 one can check that

$$X^{\pm}_{a_1,\ldots,a_n,0} := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \begin{array}{l} |z_1|^2 - a_1 = \cdots = |z_n|^2 - a_n \\ \operatorname{Im}(i^{n+1} z_1 \ldots z_n) = 0 \\ \pm \operatorname{Re}(i^{n+1} z_1 \ldots z_n) \geqslant 0 \end{array} \right\}$$

and that $C^{\pm} = X^{\pm}_{0,\ldots,0,0}$. Moreover, $X^{\pm}_{a_1,\ldots,a_n,0}$ is strongly asymptotically conical with cone $C^{\pm}$ and rate $\alpha + 1 = -1$.

We now apply our deformation theory to the above submanifolds. Clearly in cases 1 and 2 we have $b^1(X_{a_1,\ldots,a_n,b}) = n - 1$ and C having $L = 2$ ends. Therefore

$$\dim \mathcal{M}_{\beta+1} = n - 1 + 2 - 1 = n$$

for $-1 < \beta + 1 < \lambda - 1$ in the situation of cases 1 and 2. This is what we expect: there are explicitly n parameters in the family $X_{a_1,\ldots,a_n,b}$ defined above. Also, in case 3 we have $b^1(X^{\pm}_{a_1,\ldots,a_n,0}) = n - 2$ and $C^{\pm}$ has $L = 1$ ends. Therefore

$$\dim \mathcal{M}_{\beta+1} = n - 2 + 1 - 1 = n - 2$$

for $-1 < \beta + 1 < \lambda - 1$ in the situation of case 3: again what we expected.

In this example we can also compute the spectral data for the links $\Sigma^{\pm}$ of the cones $C^{\pm}$. Note that Σ^+ and Σ^- are isometric and $\Sigma = \Sigma^+ \cup \Sigma^-$ so we need only consider Σ^+. It is easy to see that

$$\Sigma^+ = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \begin{array}{l} |z_1| = \cdots = |z_n| = \frac{1}{\sqrt{n}} \\ \operatorname{Im}(i^{n+1} z_1 \ldots z_n) = 0 \\ \operatorname{Re}(i^{n+1} z_1 \ldots z_n) \geqslant 0 \end{array} \right\}.$$

Let g_{Σ^+} be the metric on Σ^+ induced by the Euclidean metric e on $\mathbb{C}^n$. Consider the map $\phi : \mathbb{R}^{n-1} \to \Sigma^+$ defined

$$\phi(t_1, \ldots, t_{n-1}) := \frac{1}{\sqrt{n}} \left(e^{it_1}, \ldots, e^{it_{n-1}}, (-i)^{n+1} e^{-i(t_1 + \cdots + t_{n-1})} \right).$$

Giving $\mathbb{R}^{n-1}$ the standard Euclidean metric, one can use ϕ to show that (Σ^+, g_{Σ^+}) is isometric to the quotient

$$\frac{\mathbb{R}^{n-1}}{L} \tag{6.82}$$

where $L \leqslant \mathbb{R}^{n-1}$ is the lattice generated over $\mathbb{Z}$ by the linearly independent vectors

$$2\pi\left(\sqrt{\tfrac{1}{n-1}}, 0, \ldots, 0, -\sqrt{\tfrac{j}{(j+1)n}}, \sqrt{\tfrac{1}{(j+1)(j+2)n}}, \sqrt{\tfrac{1}{(j+2)(j+3)n}}, \ldots, \sqrt{\tfrac{1}{(n-2)(n-1)n}}\right)^t \tag{6.83}$$

for $0 \leqslant j \leqslant n-2$: these are formed after a diagonalisation process on the usual basis for the lattice $2\pi\mathbb{Z}^n \leqslant \mathbb{R}^n$. When $j = 0$ there is no $-\sqrt{\frac{j}{(j+1)n}}$ term, and no zero terms in the vector (6.83). For $1 \leqslant j \leqslant n-2$ there are $j-1$ zero terms in the vector (6.83). Now define

$$L^* := \{y \in \mathbb{R}^{n-1} : x \cdot y \in \mathbb{Z} \text{ for all } x \in L\} \tag{6.84}$$

the *dual lattice* of L. In equation (6.84) we use the usual dot product $x \cdot y$ on $\mathbb{R}^{n-1}$. It is easy to show as in [14, Chapter II, Section 2] that if A is the $(n-1)\times(n-1)$ matrix whose columns are the vectors (6.83) then the dual lattice L^* is generated by the $(n-1)$ columns of the matrix $(A^T)^{-1}$. After some algebra, the relevant vectors turn out to be:

$$e_{j+1} := \frac{1}{2\pi}\left(\sqrt{\tfrac{1}{n-1}}, 0, \ldots, 0, -\sqrt{\tfrac{jn}{j+1}}, \sqrt{\tfrac{n}{(j+1)(j+2)}}, \sqrt{\tfrac{n}{(j+2)(j+3)}}, \ldots, \sqrt{\tfrac{n}{(n-2)(n-1)}}\right)^t \tag{6.85}$$

for $0 \leqslant j \leqslant n-2$, so that $\{e_1, \ldots, e_{n-1}\}$ is linearly independent and spans L^* over $\mathbb{Z}$. We denote the Laplacian on the torus $\frac{\mathbb{R}^{n-1}}{L}$ by Δ^0_L. Then, following Chavel [14] we have

$$\dim \operatorname{Ker}(\Delta^0_L - \mu) = \left|\{y \in L^* : \mu = 4\pi^2|y|^2\}\right|.$$

for any $\mu \geqslant 0$. This fact enables us to compute the spectrum of the Riemannian manifold (6.82), together with the eigenspace dimensions. We now illustrate the method with a simple example: in the case $n = 3$ we have

$$\begin{aligned} e_1 &= \tfrac{1}{2\pi}\left(\tfrac{1}{\sqrt{2}}, \tfrac{\sqrt{3}}{\sqrt{2}}\right) \\ e_2 &= \tfrac{1}{2\pi}\left(\tfrac{1}{\sqrt{2}}, -\tfrac{\sqrt{3}}{\sqrt{2}}\right) \end{aligned}$$

and then, given $m_1, m_2 \in \mathbb{Z}$ we have

$$4\pi^2\,|m_1e_1 + m_2e_2|^2 = \frac{(m_1+m_2)^2}{2} + \frac{3(m_1-m_2)^2}{2}$$

so that $0 < 4\pi^2|m_1e_1 + m_2e_2|^2 \leqslant 2n = 6$ precisely when

$$0 < (m_1+m_2)^2 + 3(m_1-m_2)^2 \leqslant 12. \tag{6.86}$$

One quickly shows that the integral solutions (m_1, m_2) of (6.86) are $\pm(0,1)$, $\pm(1,0)$, $\pm(1,1)$ each yielding eigenvalue $\mu = 2$, and $\pm(1,2)$, $\pm(2,1)$, $\pm(1,-1)$ each yielding eigenvalue $\mu = 6$. So when $n = 3$ the eigenvalues of $\Delta^0_{g_{\Sigma^+}}$ in the range $(0, 2n] = (0, 6]$ are $n - 1 = 2$, with multiplicity 6 and $2n = 6$, with multiplicity 6 also. Note that we haven't picked up any points of $\operatorname{Spec}(\Sigma^+, g_{\Sigma^+}, 0) \cap (0, 2n]$ which we didn't already know about: namely those got from the actions of the groups $G = \mathbb{C}^n$ and $G = \mathrm{SU}(n)$ on $\mathbb{C}^n$. In actual fact, this is not typical behaviour: using a computer, one is able to work out the points of $\operatorname{Spec}(\Sigma^+, g_{\Sigma^+}, 0)$ lying in $(0, 2n]$ for $n \geqslant 4$. We give in Table 6.1 below the results found for $3 \leqslant n \leqslant 13$. Notice there are no eigenfunctions corresponding to growth rates less than 0, so that $\lambda = 1$ in each case. Notice also that for $n \neq 8, 9$ the $\mathbb{C}^n$ and $\mathrm{SU}(n)$ moment map constructions discussed above give the full $(n-1)$- and $2n$-eigenspaces: in each case we have

$$\dim \operatorname{Ker}\big(\Delta^0_{\Sigma^+} - (n-1)\big) = 2n = d_{\mathrm{tr}}(\Sigma^+)$$

and also

$$\dim \operatorname{Ker}\big(\Delta^0_{\Sigma^+} - 2n\big) = n(n-1) = n^2 - 1 - (n-1) = \dim \mathrm{SU}(n) - \dim \mathrm{U}(1)^{n-1} = d_{\mathrm{rot}}(\Sigma^+)$$

where $\mathrm{U}(1)^{n-1} \leqslant \mathrm{SU}(n)$ is the symmetry group of the cone C^+. However, in dimensions $n = 8, 9$ we see that there are eigenfunctions with eigenvalue $2n$ on the link Σ^+ which are not got from the $\mathrm{SU}(n)$ moment map on $\mathbb{C}^n$, as $56 < 126$ in the case $n = 8$ and $72 < 240$ in the case $n = 9$.

A typical application of the results of Table 6.1 is as follows: when $n = 7$ and $\frac{1}{2}(\sqrt{73}-7) < \beta+1 < 1$ the submanifold $X^+_{a_1,\dots,a_7,0} \to \mathbb{C}^7$ moves in a family $\mathcal{M}_{\beta+1}$ of special Lagrangian submanifolds which are strongly asymptotically conical with cone C^+ and rate $\beta + 1$, where

$$\dim \mathcal{M}_{\beta+1} = b^1(X^+_{a_1,\dots,a_7,0}) + L - 1 + \chi(\beta + 2) = 7 - 2 + 14 + 42 + 70 = 131.$$

Also, when $b \neq 0$ the submanifold $X_{a_1,\dots,a_7,b} \to \mathbb{C}^7$ moves in a family $\mathcal{M}_{\beta+1}$ of special Lagrangian submanifolds which are strongly asymptotically conical with cone C and rate $\beta + 1$, where

$$\dim \mathcal{M}_{\beta+1} = b^1(X_{a_1,\dots,a_7,b}) + L - 1 + \chi(\beta + 2) = 7 + 2 \cdot (14 + 42 + 70) = 259. \tag{6.87}$$

The $2 \cdot 14$ in equation (6.87) corresponds to being able to "translate" the two ends of $X_{a_1,\dots,a_n,b}$ independently.

6.3.5 Higher decay rates

Example 7: Joyce $O(r^{\frac{1}{2}})$ decay

The main purpose of this final example is to demonstrate that there are special Lagrangian submanifolds $f : X \to \mathbb{C}^n$ which are strongly asymptotically conical whose minimal rates $\alpha + 1$ are not equal to $1 - n$ or -1. The following theorem is proved by Joyce [27, Theorem 11.6]:

Theorem 6.51 *For each $s \in (0, \frac{1}{2}) \cap \mathbb{Q}$ write $s = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ are coprime with $0 < 2p < q$. Define*

$$a_1 := p^2 - q^2 \qquad a_2 := q^2 - 2pq \qquad a_3 := 2pq - p^2$$

and

$$C := \Big\{(ie^{ia_1t}x_1, e^{ia_2t}x_2, e^{ia_3t}x_3) : x \in \mathbb{R}^3,\ t \in \mathbb{R} \text{ with } x_1 \geqslant 0 \text{ and } a_1x_1^2 + a_2x_2^2 + a_3x_3^2 = 0\Big\}.$$

Then $C \subseteq \mathbb{C}^3$ is a special Lagrangian cone with link Σ diffeomorphic to T^2. Moreover, there are explicit formulae for a 13-dimensional family of immersed special Lagrangian submanifolds $X_a \to \mathbb{C}^3$ which are strongly asymptotically conical with cone $\tilde{C}$ a double cover of C, and rate $\alpha + 1 = \frac{1}{2}$. The manifolds X_a are diffeomorphic to $S^1 \times \mathbb{R}^2$.

Referring to Theorem 6.51, we have $X_a \to \mathbb{C}^3$ a special Lagrangian submanifold which is strongly asymptotically conical with cone $\tilde{C}$ and rate $\frac{1}{2}$. Inspecting the proof of Theorem 6.51 we see that the cone $\tilde{C}$ has $L = 1$ ends, and is not a special Lagrangian plane in $\mathbb{C}^n$. Recalling definition (6.74) we deduce that $d_{\mathrm{tr}}(\Sigma) = 2n = 6$. Also

$$b^1(X_a) = b^1(S^1 \times \mathbb{R}^2) = b^1(S^1) = 1$$

by Bott and Tu [8, Proposition 4.1]. Now, from our deformation result Theorem 6.45 we have

$$b^1(X_a) + L - 1 + \chi(\beta + 2) \geqslant 13$$

for $\beta + 2 \in \mathbb{R}^L \setminus \mathcal{D}(\Delta^0_g)$ with $\beta + 1 \geqslant \frac{1}{2}$, and taking $\beta + 1$ only slightly larger than $\frac{1}{2}$ gives

$$\sum_{0<\mu\leqslant\hat{\mu}} \dim \mathrm{Ker}(\Delta^0_{g_\Sigma} - \mu) \geqslant 12$$

where $\hat{\mu} = (\frac{1}{2}+1)(\frac{1}{2}+2) = \frac{15}{4}$ corresponds to the growth rate $\alpha+1 = \frac{1}{2}$. Taking away the contribution $d_{\mathrm{tr}}(\Sigma)$ of the eigenfunctions got from the $\mathbb{C}^3$ moment map shows that there are 6 eigenvalues (counted with multiplicities) of the link Laplacian $\Delta^0_{g_\Sigma}$ lying in $(0, \frac{15}{4}]$ which we didn't already know about. In the previous examples we have used the existence of eigenfunctions on the link Σ to infer the existence of special Lagrangian deformations. In this example, we turn the argument on its head and use the existence of special Lagrangian deformations to infer the existence of eigenfunctions of the link Laplacian.

Dimension n	Eigenvalue μ	Eigenspace dimension	Growth rate $\beta+1$
3	2	6	0
3	6	6	1
4	3	8	0
4	4	6	$\sqrt{5}-2$
4	8	12	1
5	4	10	0
5	6	20	$\frac{1}{2}(\sqrt{33}-5)$
5	10	20	1
6	5	12	0
6	8	30	$\sqrt{12}-3$
6	9	20	$\sqrt{13}-3$
6	12	30	1
7	6	14	0
7	10	42	$\frac{1}{2}(\sqrt{65}-7)$
7	12	70	$\frac{1}{2}(\sqrt{73}-7)$
7	14	42	1
8	7	16	0
8	12	56	$\sqrt{21}-4$
8	15	112	$\sqrt{24}-4$
8	16	126	1
9	8	18	0
9	14	72	$\frac{1}{2}(\sqrt{105}-9)$
9	18	240	1
10	9	20	0
10	16	90	$\sqrt{32}-5$
10	20	90	1
11	10	22	0
11	18	110	$\frac{1}{2}(\sqrt{153}-11)$
11	22	110	1
12	11	24	0
12	20	132	$\sqrt{45}-6$
12	24	132	1
13	12	26	0
13	22	156	$\frac{1}{2}(\sqrt{209}-13)$
13	26	156	1

Table 6.1: The points $0 < \mu \leqslant 2n$ of $\operatorname{Spec}(\Sigma^+, g_{\Sigma^+}, 0)$ for each $3 \leqslant n \leqslant 13$ together with the dimensions of the relevant eigenspaces and the corresponding growth rates $\beta+1$

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