Singularities of special Lagrangian submanifolds

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recommended reading:

math.DG/0111111
math.DG/0310460

These slides available at

`www.maths.ox.ac.uk/~joyce/talks.html`

Almost Calabi-Yau m-folds

An *almost Calabi-Yau* m*-fold* (M, J, g, Ω) is a compact complex m-fold (M, J) with a Kähler metric g with Kähler form ω, and a nonvanishing holomorphic $(m, 0)$-form Ω, the *holomorphic volume form*. It is a *Calabi-Yau* m*-fold* if $|\Omega|^2 \equiv 2^m$. Then $\nabla\Omega = 0$, the holonomy group $\mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$, and g is Ricci-flat.

Special Lagrangian m-folds

Let (M, J, g, Ω) be an almost Calabi-Yau m-fold. Let N be a real m-submanifold of M. We call N *special Lagrangian* (*SL*) if $\omega|_N \equiv \mathrm{Im}\,\Omega|_N \equiv 0$, and *SL with phase* $\mathrm{e}^{i\theta}$ if $\omega|_N \equiv (\cos\theta\,\mathrm{Im}\,\Omega - \sin\theta\,\mathrm{Re}\,\Omega)|_N \equiv 0$. If (M, J, g, Ω) is a Calabi-Yau m-fold then $\mathrm{Re}\,\Omega$ is a *calibration* on (M, g), and N is an SL m-fold iff it is calibrated with respect to $\mathrm{Re}\,\Omega$.

Let (M, J, g, Ω) be an almost Calabi–Yau m-fold and N a *compact* SL m-fold in M. Let $\mathcal{M}_N$ be the moduli space of *SL deformations* of N. We ask:

1. Is $\mathcal{M}_N$ a manifold, and of what dimension?

2. Does N persist under deformations of (J, g, Ω)?

3. Can we compactify $\mathcal{M}_N$ by adding a 'boundary' of *singular* SL m-folds? If so, what are the singularities like?

These questions concern the *deformations* of SL m-folds, *obstructions* to their existence, and their *singularities*.

Questions 1 and 2 are fairly well understood, and we shall discuss them in the first half of this lecture. Question 3 is an active area of research, and will be discussed in the second half, and next lecture.

The answer to Question 1, on *deformations* of SL m-folds, was given by McLean in 1990 (in the Calabi-Yau case).

Theorem. *Let* (M, J, g, Ω) *be an almost Calabi–Yau* m*-fold, and* N *a compact SL* m*-fold in* M*. Then the moduli space* $\mathcal{M}_N$ *of SL deformations of* N *is a smooth manifold of dimension* $b^1(N)$*, the first Betti number of* N*.*

Here is a sketch of the proof. Let $\nu \rightarrow N$ be the *normal bundle* of N in M. Then J identifies $\nu \cong TN$ and g identifies $TN \cong T^*N$. So $\nu \cong T^*N$. We can identify a small *tubular neighbourhood* T of N in M with a neighbourhood of the zero section in ν, identifying ω on M with the symplectic structure on T^*N.

Let $\pi : T \rightarrow N$ be the obvious projection.

Then graphs of small 1-forms α on N are identified with submanifolds N' in $T \subset M$ close to N. Which α correspond to *SL m-folds* N'?

Well, N' is special Lagrangian iff $\omega|_{N'} \equiv \operatorname{Im}\Omega|_{N'} \equiv 0$.

Now $\pi|_{N'} : N' \to N$ is a diffeomorphism, so this holds iff $\pi_*(\omega|_{N'}) = \pi_*(\operatorname{Im}\Omega|_{N'}) = 0$.

We regard $\pi_*(\omega|_{N'})$ and $\pi_*(\operatorname{Im}\Omega|_{N'})$ as functions of α.

Calculation shows that $\pi_*(\omega|_{N'}) = \mathrm{d}\alpha$ and $\pi_*(\mathrm{Im}\,\Omega|_{N'}) = F(\alpha, \nabla\alpha)$, where F is nonlinear. Thus, $\mathcal{M}_N$ is locally the set of small 1-forms α on N with $\mathrm{d}\alpha \equiv 0$ and $F(\alpha, \nabla\alpha) \equiv 0$. Now $F(\alpha, \nabla\alpha) \approx \mathrm{d}(*\alpha)$ for small α. So $\mathcal{M}_N$ is locally approximately the set of 1-forms α with $\mathrm{d}\alpha = \mathrm{d}(*\alpha) = 0$. But by Hodge theory this is the de Rham group $H^1(N, \mathbb{R})$, of dimension $b^1(N)$.

Question 2, on *obstructions* to the existence of SL m-folds, can locally be answered using the same methods.

Theorem. *Let $M_t : t \in (-\epsilon, \epsilon)$ be a family of almost Calabi–Yau m-folds, and N_0 a compact SL m-fold of M_0. If $[\omega_t|_{N_0}] = [\mathrm{Im}\,\Omega_t|_{N_0}] = 0$ in $H^*(N_0, \mathbb{R})$ for all t, then N_0 extends to a family $N_t : t \in (-\delta, \delta)$ of SL m-folds in M_t, for $0 < \delta \leqslant \epsilon$.*

Singular SL m-folds

General singularities of SL m-folds may be very bad, and difficult to study. Would like a class of singular SL m-folds with nice, well-behaved singularities to study in depth. Would like these to occur often in real life, i.e. of finite codimension in the space of all SL m-folds. SL m-folds with *isolated conical singularities* (*ICS*) are such a class.

Let N be an SL m-fold in M whose only singular points are $x_1, \ldots, x_n$. Near x_i we can identify M with $\mathbb{C}^m \cong T_{x_i}M$, and N near x_i approximates an SL m-fold in $\mathbb{C}^m$ with singularity at 0. We say N has *isolated conical singularities* if near x_i it converges with order $O(r^{\mu_i})$ for $\mu_i > 1$ to an SL cone C_i in $\mathbb{C}^m$ nonsingular except at 0.

SL m-folds with ICS have a rich theory.

• **Examples.** Many examples of SL cones C_i in $\mathbb{C}^m$ have been constructed. Rudiments of classification for $m = 3$.

• **Regularity near** $x_1, \ldots, x_n$**.** Let $\iota : N \to M$ be the inclusion. If $\nabla^k \iota$ converges to C_i near x_i with order $O(r^{\mu_i - k})$ for $k = 0, 1$ then it does so for all $k \geqslant 0$.

• **Deformation theory.** The moduli space $\mathcal{M}_N$ of deformations of N is locally homeomorphic to $\Phi^{-1}(0)$, for smooth $\Phi : \mathcal{I} \to \mathcal{O}$ and fin. dim. vector spaces $\mathcal{I}, \mathcal{O}$ with $\mathcal{I}$ the image of $H^1_{\rm cs}(N', \mathbb{R})$ in $H^1(N', \mathbb{R})$, $N' = N \setminus \{x_1, \ldots, x_n\}$, and $\dim \mathcal{O} = \sum_{i=1}^n \text{s-ind}(C_i)$. Here $\text{s-ind}(C_i) \in \mathbb{N}$ is the *stability index*, the obstructions from C_i. If $\text{s-ind}(C_i) = 0$ for all i then $\mathcal{M}_N$ is smooth.

• **Desingularization.** Let C be an SL cone in $\mathbb{C}^m$, nonsingular except at 0. A nonsingular SL m-fold L in $\mathbb{C}^m$ is *Asymptotically Conical* (*AC*) C if L converges to C at infinity with order $O(r^\nu)$ for $\nu < 1$. Then tL converges to C as $t \to 0_+$. Thus, AC SL m-folds model how families of nonsingular SL m-folds develop singularities modelled on C.

If N is an SL m-fold with ICS at $x_1, \ldots, x_n$ and cones C_i, and $L_1, \ldots, L_n$ are AC SL m-folds in $\mathbb{C}^m$ with cones C_i, then under cohomological conditions we can construct a family of compact nonsingular SL m-folds $\tilde{N}^t$ for small $t > 0$ converging to N as $t \to 0$, by gluing tL_i into N at x_i, all i.

Here is how this works. Let $B_\epsilon(0)$ be an open ball of small radius $\epsilon > 0$ in $\mathbb{C}^m$, and choose a local diffeomorphism $\Upsilon_i : B_\epsilon(0) \to M$ with $\Upsilon_i(0) = x_i$, that identifies C_i in $\mathbb{C}^m$ with the tangent cone to N at x_i, and $\Upsilon_i^*(\omega) = \omega_0$, for ω the Kähler form on M and ω_0 the Hermitian form on $\mathbb{C}^m$. Write $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$. Then $\iota_i : (\sigma, r) \mapsto r\sigma$ is a diffeomorphism $\iota_i : \Sigma_i \times (0, \infty) \to C_i \setminus \{0\}$.

For $0 < \epsilon' < \epsilon$ small there is a unique $\phi_i : \Sigma_i \times (0,\epsilon') \to \mathbb{C}^m$ such that $\mathrm{Im}(\Upsilon_i \circ \phi_i)$ coincides with $N \setminus \{x_i\}$ near x_i, and $(\phi_i - \iota_i)(\sigma, r)$ is perpendicular to $T_{r\sigma}C_i$ in $\mathbb{C}^m$ for all $(\sigma, r) \in \Sigma_i \times (0,\epsilon')$. These are distinguished coordinates on N near x_i. Regard $\phi_i - \iota_i$ as a small closed 1-form on C_i. Regularity theory gives $\nabla^k(\phi_i - \iota_i) = O(r^{\mu_i - k})$ as $r \to 0$ for some $\mu_i > 1$ and all $k \geqslant 0$.

Similarly, for $R \gg 0$ there is a unique $\psi_i : \Sigma_i \times (R, \infty) \to \mathbb{C}^m$ such that $\operatorname{Im} \psi_i$ coincides with L_i near ∞, and $(\phi_i - \iota_i)(\sigma, r)$ is perpendicular to $T_{r\sigma} C_i$ in $\mathbb{C}^m$ for all $(\sigma, r) \in \Sigma_i \times (R, \infty)$. These are distinguished coordinates on L_i near ∞. Regularity gives $\nabla^k(\psi_i - \iota_i) = O(r^{\nu_i - k})$ as $r \to \infty$ for some $\nu_i < 1$ and all $k \geqslant 0$. We assume $\nu_i < -1$ for no obstructions, or $\nu_i = -1$ and $m < 6$.

Fix $\tau \in (0,1)$. Let $t > 0$ with $2t^\tau < \epsilon'$ and $t^\tau > tR$. Define a compact, nonsingular Lagrangian N^t in M to be N outside $\Upsilon_i \circ \phi_i\bigl(\Sigma_i \times (0, 2t^\tau)\bigr)$ for all i, to be $\Upsilon_i(tL_i)$ outside $\psi_i\bigl(\Sigma_i \times (t^{\tau-1}, \infty)\bigr)$ in L_i, and to interpolate smoothly between these on $\Sigma_i \times [t^\tau, 2t^\tau]$. On $\Sigma_i \times [t^\tau, 2t^\tau]$ we have $\phi_i(\sigma, r) \equiv \iota_i(\sigma, \tau) + O(t^{\mu_i \tau})$ and $t\psi_i(\sigma, t^{-1}r) \equiv \iota_i(\sigma, r) + O(t^{\nu_i(\tau-1)+1})$, so $\bigl|\phi_i(\sigma, r) - t\psi_i(\sigma, t^{-1}r)\bigr|$ is small.

This N^t is *approximately* special Lagrangian, as $\omega|_{N^t} \equiv 0$ and $\operatorname{Im}\Omega|_{N^t}$ is small. Banach norms of $\operatorname{Im}\Omega|_{N^t}$ measure the 'error', e.g. $\|\operatorname{Im}\Omega|_{N^t}\|_{C^0} = O(t^{(\mu_i-1)\tau}) + O(t^{(\nu_i-1)(\tau-1)})$ for small t. But also, N^t is *nearly singular* for small t, with second fundamental form $\|B\|_{C^0} = O(t^{-1})$, Riemann curvature $\|R(g|_{N^t})\|_{C^0} = O(t^{-2})$ and injectivity radius $\delta(g|_{N^t}) = O(t)$.

We show using analysis that we can deform N^t to a nearby SL m-fold $\tilde{N}^t$. We must solve the nonlinear elliptic p.d.e. $Q(\tilde{N}^t) = \operatorname{Im}\Omega|_{\tilde{N}^t} \equiv 0$. We make the solution as the limit of a series of Lagrangians $(N^t_k)_{k=0}^\infty$ with $N^t_0 = N^t$, which roughly inductively satisfy $\mathrm{d}Q|_{N^t}(N^t_{k+1} - N^t_k) = -\operatorname{Im}\Omega|_{N^t_k}$. The series converges if the initial 'error' is small enough, in terms of $\|B\|_{C^0}, \|R(g|_{N^t})\|_{C^0}$, $\delta(g|_{N^t}), \ldots$.

Three things can go wrong in this proof:

(A) For the 'error' to be small and the series to converge, we need $\tau \approx 1$ and $\nu_i < -1$ for all i, or $\nu_i = -1$ and $m < 6$.

(B) To make the Lagrangian N^t we join $N \setminus \{x_1, \ldots, x_n\}$ and $\Upsilon(tL_1), \ldots, \Upsilon(tL_n)$. Effectively we must find a *closed* 1-form on $\Sigma_i \times [t^\tau, 2t^\tau]$ interpolating between small closed 1-forms $\phi_i(\sigma, r) - \iota_i(\sigma, \tau)$ and $t\psi_i(\sigma, t^{-1}r) - \iota_i(\sigma, r)$.

Now $\phi_i(\sigma, r) - \iota_i(\sigma, \tau)$ is exact, and $t\psi_i(\sigma, t^{-1}r) - \iota_i(\sigma, r)$ is exact if $\nu_i < -1$, but if $\nu_i \geqslant -1$ then we can have $[t\psi_i(\sigma, t^{-1}r) - \iota_i(\sigma, r)] \neq 0$ in $H^1(\Sigma_i, \mathbb{R})$. This is a *global topological obstruction* to making N^t Lagrangian. To overcome it, we modify $N' = N \setminus \{x_1, \ldots, x_n\}$ by a small closed 1-form α^t whose cohomology class $[\alpha^t] \in H^1(N', \mathbb{R})$ satisfies $[\alpha^t]|_{\Sigma_i} = [t\psi_i(\sigma, t^{-1}r) - \iota_i(\sigma, r)]$ in $H^1(\Sigma_i, \mathbb{R})$ for all i. Such α^t need not exist.

(C) Suppose N is connected, but $N' = N \setminus \{x_1, \ldots, x_n\}$ has $l > 1$ connected components, which meet at $x_1, \ldots, x_n$. Then the Laplacian Δ^t on functions on N^t has $l-1$ *small eigenvalues* of size $O(t^{m-2})$. The corresponding eigenfunctions are approximately constant on each component of N', and change on the 'necks' $\Upsilon(tL_i)$. The linearization $\mathrm{d}Q|_{N^t}$ of Q at N^t is basically Δ^t. So small eigenvalues of Δ^t can cause the series $(N_k^t)_{k=0}^{\infty}$ to diverge.

To overcome this, the components of $N_k^t - N^t$ in the directions of the $l-1$ eigenfunctions with small eigenvalues must remain small for all $k \geqslant 0$. There is a *global cohomological obstruction* to doing this, that there should be a small closed $(m-1)$-form β^t on N' whose cohomology class $[\beta^t] \in H^{m-1}(N', \mathbb{R})$ satisfies $[\beta^t]|_{\Sigma_i} = [*(t\psi_i(\sigma, t^{-1}r) - \iota_i(\sigma, r))]$ in $H^{m-1}(\Sigma_i, \mathbb{R})$ for all i. Such β^t need not exist.

We understand obstructions (B),(C) using *relative cohomology*. As $\omega|_{\tilde{N}^t} \equiv \operatorname{Im}\Omega|_{\tilde{N}^t} \equiv 0$, we have classes $[\omega], [\operatorname{Im}\Omega]$ in $H^k(M, N^t; \mathbb{R})$ for $k = 2, m$. Also we have $[\omega_0], [\operatorname{Im}\Omega_0]$ in $H^k(\mathbb{C}^m, L_i; \mathbb{R})$. An exact sequence gives $H^k(\mathbb{C}^m, L_i; \mathbb{R}) \cong H^{k-1}(L_i; \mathbb{R})$, and as Σ_i is the 'boundary' of L_i we restrict to $H^{k-1}(\Sigma_i; \mathbb{R})$. So $[\omega_0], [\operatorname{Im}\Omega_0]$ induce classes in $H^{k-1}(L_i; \mathbb{R})$ for all i, which must lie in the image of $H^{k-1}(N'; \mathbb{R})$.