# U(1)-**invariant special Lagrangian 3-folds in $\mathbb{C}^3$ and special Lagrangian fibrations**

Dominic Joyce
Oxford University

based on
math.DG/0011179
math.DG/0111324
math.DG/0111326
math.DG/0204343
math.DG/0206016

## Calibrated geometry

Let $(M, g)$ be a Riemannian manifold. A *tangent $k$-plane* $V$ on $M$ is a vector subspace $V$ of some tangent space $T_xM$ to $M$ with $\dim V = k$.

A *calibration* on $M$ is a closed $k$-form $\varphi$ with $|\varphi|_V| \leqslant 1$ for every $k$-plane $V$ on $M$.

Let $N$ be a submanifold of $M$ with $\dim N = k$. We call $N$ *calibrated* if $\left|\varphi|_{T_xN}\right| = 1$ for all $x \in N$. Then $N$ is automatically a *minimal submanifold* of $M$. If $N$ is compact, then $\mathrm{vol}(N) = [\varphi]\cdot[N]$, where $[\varphi] \in H^k(M,\mathbb{R})$ and $[N] \in H_k(M,\mathbb{Z})$.

# SL $m$-folds in $\mathbb{C}^m$

Let $\mathbb{C}^m$ have coordinates $(z_1, \ldots, z_m)$, Kähler metric $g = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2$, Kähler form $\omega$, and $\Omega = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m$.

Then $\mathrm{Re}\,\Omega$ is a *calibration*.

A real $m$-submanifold $N$ in $\mathbb{C}^m$ is called *special Lagrangian* if it is calibrated w.r.t. $\mathrm{Re}\,\Omega$.

Equivalently, $N$ is an SL $m$-fold iff $\omega|_N \equiv \mathrm{Im}\,\Omega|_N \equiv 0$.

## Almost Calabi-Yau $m$-folds

An *almost Calabi-Yau $m$-fold* $(M, J, g, \Omega)$ is a compact complex $m$-fold $(M, J)$ with a Kähler metric $g$ with Kähler form $\omega$, and a nonvanishing holomorphic$(m, 0)$-form $\Omega$, the *holomorphic volume form*.

It is a *Calabi-Yau $m$-fold* if $|\Omega|^2 \equiv 2^m$. Then $\nabla\Omega = 0$ and $g$ is Ricci-flat.

## SL $m$-folds in ACY $m$-folds

Let $(M,J,g,\Omega)$ be an almost Calabi-Yau $m$-fold. Let $N$ be a real $m$-submanifold of $M$. We call $N$ *special Lagrangian* if $\omega|_N \equiv \mathrm{Im}\,\Omega|_N \equiv 0$.

If $(M,J,g,\Omega)$ is a Calabi-Yau $m$-fold then $\mathrm{Re}\,\Omega$ is a *calibration* on $(M,g)$, and $N$ is an SL $m$-fold iff it is calibrated with respect to $\mathrm{Re}\,\Omega$.

## Mirror Symmetry

String theorists believe that each Calabi–Yau 3-fold $X$ has a quantization, a *SCFT*. Calabi–Yau 3-folds $X, \hat{X}$ are a *mirror pair* if their SCFT's are related by a certain involution of SCFT structure. Then invariants of $X, \hat{X}$ are related in surprising ways. For instance,
$H^{1,1}(X) \cong H^{2,1}(\hat{X})$ and
$H^{2,1}(X) \cong H^{1,1}(\hat{X})$.

Using physics, Strominger, Yau and Zaslow proposed:

**The SYZ Conjecture.** *Let $X, \hat{X}$ be mirror Calabi–Yau 3-folds. There is a compact 3-manifold $B$ and continuous, surjective $f : X \to B$ and $\hat{f} : \hat{X} \to B$, such that*

(i) *For $b$ in a dense $B_0 \subset B$, the fibres $f^{-1}(b), \hat{f}^{-1}(b)$ are dual SL 3-tori $T^3$ in $X, \hat{X}$.*

(ii) *For $b \notin B_0$, $f^{-1}(b)$ and $\hat{f}^{-1}(b)$ are singular SL 3-folds in $X, \hat{X}$.*

We call $f, \hat{f}$ *special Lagrangian fibrations*, and $\Delta = B \backslash B_0$ the *discriminant*.

In (i), the nonsingular fibres $T, \hat{T}$ of $f, \hat{f}$ are supposed to be *dual tori*. Topologically, this means an isomorphism $H^1(T, \mathbb{Z}) \cong H_1(\hat{T}, \mathbb{Z})$. But the metrics on $T, \hat{T}$ should really be dual as well. This only makes sense in the 'large complex structure limit', when the fibres are small and nearly flat.

## U(1)-**invariant SL 3-folds**

Let U(1) act on $\mathbb{C}^3$ by $(z_1,z_2,z_3)\mapsto(\mathrm{e}^{i\theta}z_1,\mathrm{e}^{-i\theta}z_2,z_3)$. Let $N$ be a U(1)-invariant SL 3-fold. Then locally we can write $N$ in the form

$$\big\{(z_1,z_2,z_3):|z_1|^2-|z_2|^2=2a,\quad z_1z_2=v(x,y)+iy,\quad z_3=x+iu(x,y),\ x,y\in\mathbb{R}\big\},$$

where $u,v:\mathbb{R}^2\to\mathbb{R}$ satisfy

$$u_x=v_y \quad\text{and}\quad v_x=-2(v^2+y^2+a^2)^{1/2}u_y. \qquad (*)$$

Since $u_x = v_y$, there exists a potential function $f$ with $u = f_y$ and $v = f_x$. The 2nd equation of $(*)$ becomes

$$f_{xx} + 2(f_x^2 + y^2 + a^2)^{1/2} f_{yy} = 0. \qquad (+)$$

This is a second-order quasi-linear equation. When $a \neq 0$ it is locally uniformly elliptic. When $a = 0$ it is non-uniformly elliptic, except at *singular points* $f_x = y = 0$.

**Theorem A.** Let $S$ be a compact domain in $\mathbb{R}^2$ satisfying some convexity conditions.
Let $\phi \in C^{3,\alpha}(\partial S)$.
If $a \neq 0$ there exists a unique $f \in C^{3,\alpha}(S)$ satisfying $(+)$ with $f|_{\partial S} = \phi$. If $a = 0$ there exists a unique $f \in C^1(S)$ satisfying $(+)$ with weak second derivatives, with $f|_{\partial S} = \phi$.
Also $f$ depends continuously in $C^1(S)$ on $a, \phi$.

Theorem A shows that the Dirichlet problem for (+) is uniquely solvable in certain convex domains. The induced solutions $u, v \in C^0(S)$ of $(*)$ yield U(1)-invariant SL 3-folds in $\mathbb{C}^3$ satisfying certain boundary conditions over $\partial S$. When $a \neq 0$ these SL 3-folds are nonsingular, when $a = 0$ they are singular when $v = y = 0$.

**Theorem B.**
Let $\phi, \phi' \in C^{3,\alpha}(\partial S)$, let $a \in \mathbb{R}$ and let $f, f' \in C^{3,\alpha}(S)$ or $C^1(S)$ be the solutions of (+) from Theorem A with $f|_{\partial S} = \phi$, $f'|_{\partial S} = \phi'$. Let $u = f_y$, $v = f_x$, $u' = f'_y$, $v' = f'_x$. Suppose $\phi - \phi'$ has $k+1$ local maxima and $k+1$ local minima on $\partial S$. Then $(u, v) - (u', v')$ has no more than $k$ zeroes in $S^\circ$, counted with multiplicity.

**Theorem C.**
Let $u, v \in C^0(S)$ be a singular solution of $(*)$ with $a = 0$, e.g. from Theorem A. Then **either** $u(x, y) \equiv u(x, -y)$ and $v(x, y) \equiv -v(x, -y)$, so that $u, v$ is singular on the $x$-axis, **or** the singularities $(x, 0)$ of $u, v$ in $S^\circ$ are *isolated*, with a *multiplicity* $n > 0$. Multiplicity $n$ singularities occur in codimension $n$ of boundary data. All multiplicities occur.

**Theorem D.**
Let $U \subset \mathbb{R}^3$ be open, $S$ as above, and $\Phi : U \to C^{3,\alpha}(\partial S)$ continuous such that if $(a, b, c) \neq (a, b', c') \in U$ then $\Phi(a, b, c) - \Phi(a, b', c')$ has 1 local maximum and 1 local minimum.
For $\alpha = (a, b, c) \in U$, let $f_\alpha \in C^1(S)$ be the solution of (+) from Theorem A with $f_\alpha|_{\partial S} = \Phi(\alpha)$.

Set $u_\alpha = (f_\alpha)_y$ and $v_\alpha = (f_\alpha)_x$. Let $N_\alpha$ be the SL 3-fold

$$\bigl\{(z_1, z_2, z_3) : |z_1|^2 - |z_2|^2 = 2a,\ z_1 z_2 = v_\alpha(x,y) + iy,\ z_3 = x + iu_\alpha(x,y),\ (x,y) \in S^\circ\bigr\}.$$

Then there exists an open $V \subset \mathbb{C}^3$ and a continuous map $F : V \to U$ with $F^{-1}(\alpha) = N_\alpha$. This is a U(1)-*invariant special Lagrangian fibration*. It can include *singular fibres*, of every multiplicity $n > 0$.

**Example.** Define $f : \mathbb{C}^3 \to \mathbb{R} \times \mathbb{C}$ by $f(z_1, z_2, z_3) = (a, b)$, where $2a = |z_1|^2 - |z_2|^2$ and

$$b = \begin{cases} z_3, & z_1 = z_2 = 0, \\ z_3 + \bar{z}_1 \bar{z}_2 / |z_1|, & a \geqslant 0, \ z_1 \neq 0, \\ z_3 + \bar{z}_1 \bar{z}_2 / |z_2|, & a < 0. \end{cases}$$

Then $f$ is a piecewise-smooth SL fibration of $\mathbb{C}^3$. It is not smooth on $|z_1| = |z_2|$.
The fibres $f^{-1}(a, b)$ are $T^2$-cones when $a = 0$, and non-singular $\mathcal{S}^1 \times \mathbb{R}^2$ when $a \neq 0$.

## Conclusions

Using these SL fibrations as local models, if $X$ is a *generic* ACY 3-fold and $f : X \to B$ an SL fibration, I predict:

- $f$ is only piecewise smooth.
- All fibres have finitely many singular points.
- $\Delta$ is codim 1 in $B$. Generic singularities are modelled on the example above.
- Some codim 2 singularities are also locally U(1)-invariant.

• Codim 3 singularities are not locally U(1)-invariant.

• If $f : X \to B$, $\hat{f} : \hat{X} \to B$ are dual SL fibrations of mirror C-Y 3-folds, the discriminants $\Delta, \hat{\Delta}$ have different topology near codim 3 singular fibres, so $\Delta \neq \hat{\Delta}$.

This contradicts some statements of the SYZ Conjecture. I regard SYZ as primarily a limiting statement about the 'large complex structure limit'.