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ISOPERIMETRIC STABILITY IN LATTICES

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Abstract. We obtain isoperimetric stability theorems for general Cayley digraphs on $\mathbb{Z}^d$. For any fixed B that generates $\mathbb{Z}^d$ over $\mathbb{Z}$, we characterise the approximate structure of large sets A that are approximately isoperimetric in the Cayley digraph of B: we show that A must be close to a set of the form $kZ \cap \mathbb{Z}^d$, where for the vertex boundary Z is the conical hull of B, and for the edge boundary Z is the zonotope generated by B.

1. Introduction

An important theme at the interface of Geometry, Analysis and Combinatorics is understanding the structure of approximate minimisers to isoperimetric problems. These problems take the form of minimising surface area of sets with a fixed volume, for various meanings of 'area' and 'volume'. The usual meanings give the Euclidean Isoperimetric Problem considered since the ancient Greek mathematicians, where balls are the measurable subsets of $\mathbb{R}^d$ with a given volume which minimize the surface area. There is a large literature on its stability, i.e. understanding the structure of approximate minimisers, culminating in the sharp quantitative isoperimetric inequality of Fusco, Maggi and Pratelli [9].

In the discrete setting, isoperimetric problems form a broad area that is widely studied within Combinatorics (see the surveys [2,15]) and as part of the Concentration of Measure phenomenon (see [16,26]). Certain particular settings have been intensively studied due to their applications; for example, there has been considerable recent progress (see [12–14,23]) on isoperimetric stability in the discrete cube $\{0,1\}^n$, which is intimately connected to the Analysis of Boolean Functions (see [21]) and the Kahn–Kalai Conjecture (see [11]) on thresholds for monotone properties (see [8] for the recent solution of Talagrand's fractional version). This paper concerns the setting of integer lattices, which is widely studied in Additive Combinatorics, where the Polynomial Freiman–Ruzsa Conjecture (see [10]) predicts the structure of sets with small doubling.

For an isoperimetric problem on a digraph (directed graph) G, we measure the 'volume' of $A \subseteq V(G)$ by its size $|A|$, and its 'surface area' either by the *edge boundary* $\partial_{e,G}(A)$, which is the number of edges $\overrightarrow{xy} \in E(G)$ with $x \in A$ and $y \in V(G) \setminus A$, or by the *vertex boundary* $\partial_{v,G}(A)$, which is the number of vertices $y \in V(G) \setminus A$ such that $\overrightarrow{xy} \in E(G)$ for some $x \in A$. Here we consider Cayley digraphs: given a generating set B of $\mathbb{Z}^d$, we write G_B for the digraph on $\mathbb{Z}^d$ with edges $E(G_B) = \{\overrightarrow{uv} : v - u \in B\}$.

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It is an open problem to determine the minimum possible value of $\partial_{v,G_B}(A)$ or $\partial_{e,G_B}(A)$ for $A \subseteq \mathbb{Z}^d$ of given size, let alone any structural properties of (approximate) minimisers; exact results are only known for a few instances of B (see [3, 4, 24, 27]). It is therefore natural to seek asymptotics. For ease of reference we collect here our notation for the various sets involved in stating the following results.

$C(B) \subseteq \mathbb{R}^d$ The *conical hull* $C(B)$ of B is the convex hull of $B \cup \{0\}$.

$B_n \subseteq \mathbb{Z}^d$ The sets $kC(B) \cap \mathbb{Z}^d$ are increasing as a function of $k > 0$. Write B_n for the smallest of these sets with at least n elements.

$[B] \subseteq \mathbb{Z}^d$ Write $[B] = \left\{ \sum_{b \in B'} b : B' \subseteq B \right\}$ for the set of all sums of subsets of B. Thus $|[B]| \leq 2^{|B|}$, where the bound is strict if multiple subsets of B have equal sums.

$Z(B) \subseteq \mathbb{R}^d$ The *zonotope* generated by B is $\left\{ \sum_{b \in B} x_b b : x \in [0,1]^B \right\}$. Equivalently, $Z(B)$ is the convex (or conical, as $[B]$ contains 0) hull of $[B]$.

For $A \subseteq \mathbb{Z}^d$ of size $n \to \infty$, Ruzsa [25] showed that the minimum value of the vertex boundary $\partial_{v,G_B}(A)$ is asymptotic to that achieved by a set of the form $kC(B) \cap \mathbb{Z}^d$. A corresponding result for the edge boundary was obtained in [1]: the minimum value of $\partial_{e,G_B}(A)$ is asymptotic to that achieved by a set of the form $kZ(B) \cap \mathbb{Z}^d$.

We will prove stability versions of both these results, describing the approximate structure of asymptotic minimisers for both the vertex and edge isoperimetric problems in G_B. We use μ to denote Lebesgue measure.

Theorem 1.1. *Let $d \geq 2$. For every generating set B of $\mathbb{Z}^d$, there is a $K \in \mathbb{N}$ such that whenever*

- *$A \subseteq \mathbb{Z}^d$ with $|A| = n \geq K$,*
- *$Kn^{-1/2d} < \varepsilon < K^{-1}$ and*
- *$\partial_{v,G_B}(A) \leq d\mu(C(B))^{1/d} n^{1-1/d}(1+\varepsilon)$,*

there is a $v \in \mathbb{Z}^d$ with $|A \triangle (v + B_n)| < Kn\sqrt{\varepsilon}$.

Theorem 1.2. *Let $d \geq 2$. For every generating set B of $\mathbb{Z}^d$ and $\delta > 0$, there are $K \in \mathbb{N}$ and $\epsilon > 0$ such that whenever*

- *$A \subseteq \mathbb{Z}^d$ with $|A| = n \geq K$ and*
- *$\partial_{e,G_B}(A) \leq d\mu(Z(B))^{1/d} n^{1-1/d}(1+\varepsilon)$,*

there is a $v \in \mathbb{Z}^d$ with $|A \triangle (v + [B]_n)| < \delta n$.

The square root dependence in Theorem 1.1 is tight, as may be seen from an example where B consists of the corners of a cube and A is an appropriate cuboid.

Our statement of Theorem 1.2 is qualitiative, as our proof does not give good quantitative bounds. For certain B, namely those for which G_B is equivalent to the ℓ^1-grid (see Theorem 3.2), we do obtain good bounds, giving a new proof of a result of Ellis, Friedgut, Kindler and Yehudayoff [6].

Besides drawing on the methods of [25] (particularly Plünnecke's inequality for sumsets) and [1] (a probabilistic reduction to [25]), the most significant new contribution of our paper is a technique for transforming discrete problems to a continuous setting where one can apply results from Geometric Measure Theory. We will employ the sharp estimate on asymmetric index in terms of anisotropic perimeter with respect to any convex set K due to Figalli, Maggi and Pratelli [7] (building on the case when K is a ball, established in [9]). We consider vertex isoperimetry in the

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next section and then edge isoperimetry in the following section. We conclude the paper by discussing some potential directions for further research.

2. Vertex isoperimetry

This section contains the proof of our sharp tight quantitative stability result for the vertex isoperimetric inequality in general Cayley digraphs. We start in the first subsection with a summary of Ruzsa's approach in [25], during which we record some key lemmas on sumsets and fundamental domains of lattices that we will also use in our proof. In the second subsection we state the Geometric Measure Theory result of [7] (in a simplified setting that suffices for our purposes). The third subsection contains a technical lemma in elementary Real Analysis. We conclude in the final subsection by proving Theorem 1.1.

2.1. **Ruzsa's approach.** The sumset of $A, B \in \mathbb{Z}^d$ is defined by $A+B := \{a+b : a \in A, b \in B\}$. The vertex isoperimetric problem in the Cayley digraph G_B is equivalent to finding the minimum of $|A+B|$ over all sets A of given size. The following result of Ruzsa [25, Theorem 2] implies an asymptotic for this minimum.

Theorem 2.1. *Let B be a generating set of $\mathbb{Z}^d$ with $d \geq 2$. Then for any $A \subseteq \mathbb{Z}^d$ with $|A| = n$ large we have $|A+B| \geq d\mu(C(B))^{1/d} n^{1-1/d}\big(1 - O(n^{-1/2d})\big)$.*

Ruzsa aims to deduce this inequality from the Brunn–Minkowski inequality (in the form due to Lusternik [17]) $\mu(U+V)^{1/d} \geq \mu(U)^{1/d} + \mu(V)^{1/d}$, which is tight when U and V are closed, convex and homothetic (that is, agree up to scaling and translation).

Passing from a discrete inequality to a continuous one can be achieved by adding a *fundamental set* Q to each side; that is, a measurable Q such that any $x \in \mathbb{R}^d$ has a unique representation as $x = z + q$ with $z \in \mathbb{Z}^d$ and $q \in Q$. This ensures that $\mu(X+Q) = |X|$ for any $X \subseteq \mathbb{Z}^d$. One example of a fundamental set is the half-open unit cube $[0,1)^d$, but we will prefer a fundamental set tailored to B rather than to the standard coordinate axes.

Typically $B + Q$ will be far from convex, so a naive application of Brunn–Minkowski gives poor results. Ruzsa smooths out B by using a version of Plünnecke's inequality [22] to replace B by its sumset. We write $\Sigma_k(A)$ for the k-fold sumset of A rather than the commonly used kA, which in this paper denotes the dilate of A by factor k.

Theorem 2.2 (See [25, Statement 6.2])**.** *Let $k \in \mathbb{N}$ and $A, B \subseteq \mathbb{Z}^d$ with $|A| = n$ and $|A+B| = \alpha n$. Then there is a non-empty subset $A' \subseteq A$ with $|A' + \Sigma_k(B)| \leq \alpha^k |A'|$.*

To return to a bound on to discrete sets Ruzsa uses Lemma 2.3. By *nice* we mean that a set is a finite union of bounded convex polytopes.

Lemma 2.3 ([25, Lemma 11.2])**.** *Let B be a generating set of $\mathbb{Z}^d$ with $d \geq 2$ and $0 \in B$. Then there are $p \in \mathbb{N}$, $z \in \mathbb{Z}^d$ and a nice fundamental set $Q \subseteq Z(B)$ such that $kC(B) + Q + z \subseteq \Sigma_{k+p}(B) + Q$ for any $k \in \mathbb{N}$.*

The fact that Q may be chosen to be nice and such that $Q \subseteq Z(B)$ is not stated in [25], but it can be read out of the proof. With a little care Q can be taken to be a parallelepiped, but we make no use of this observation.

Chaining together the inequalities in this section and optimising over k proves Theorem 2.1. A similar process, taking notice of the stability of our application of the Brunn–Minkowski inequality, will prove Theorem 1.1.

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2.2. **Some geometric measure theory.** Here we give a brief account of the quantitative isoperimetric stability result of Figalli, Maggi and Pratelli [7]. We adopt simplified definitions that suffice for sets that are nice, as defined in the previous subsection; see [18, 19] for the general setting of sets of finite perimeter.

For a closed convex polytope $K \subseteq \mathbb{R}^d$ and a union E of disjoint (possibly non-convex) closed polytopes, the perimeter of E with respect to K is given by

$$\mathrm{Per}_K(E) = \lim_{\varepsilon \to 0^+} \frac{\mu(E + \varepsilon K) - \mu(E)}{\varepsilon}. \tag{2.1}$$

In our setting, given a nice set A, for all $r \geq 0$ the measure of $A + rK$ and its closure $\overline{A + rK}$ are the same; that is $\mu(A + rK) = \mu(\overline{A + rK})$. Thus for all $r \geq 0$, (2.1) gives

$$\mathrm{Per}_K(\overline{A + rK}) = \lim_{\varepsilon \to 0^+} \frac{\mu(A + (r + \varepsilon)K) - \mu(A + rK)}{\varepsilon}. \tag{2.2}$$

The anisotropic isoperimetric problem was posed in 1901 by Wulff [28], who conjectured that minimisers of Per_K up to null sets are homothetic copies of K, giving $\mathrm{Per}_K(E) \geq d\mu(K)^{1/d}\mu(E)^{1-1/d}$. This was established for sets E with continuous boundary by Dinghas [5] and for general sets E of finite perimeter by Gromov [20]. It is equivalent to non-negativity of the *isoperimetric deficit* $\delta_K(E)$ of E with respect to K, defined by

$$\delta_K(E) := \frac{\mathrm{Per}_K(E)}{d\mu(K)^{1/d}\mu(E)^{1-1/d}} - 1.$$

We quantify the structural similarity between K and E via the *asymmetric index* (also known as Fraenkel asymmetry) of E with respect to K, which is given by

$$\mathcal{A}_K(E) = \inf\left\{ \frac{\mu(E \triangle (x_0 + rK))}{\mu(E)} : x_0 \in \mathbb{R}^d \text{ and } r^d\mu(K) = \mu(E) \right\}.$$

Theorem 2.4 ([7, Theorem 1.1])**.** *For any $d \in \mathbb{N}$ there exists $D = D(d)$ such that for any bounded convex open set $K \subseteq \mathbb{R}^d$ and $E \subseteq \mathbb{R}^d$ of finite perimeter we have*

$$\mathcal{A}_K(E) \leq D\sqrt{\delta_K(E)}.$$

2.3. **Some real analysis.** In this subsection we establish the following technical lemma in elementary Real Analysis, which will allow us to pass to the setting of perimeters in Ruzsa's approach as described in Section 2.1, so that we can apply the result of Section 2.2. We presume the result is well-known, but we include a proof for completeness.

Lemma 2.5. *Let $f : [a, b] \to \mathbb{R}$ be continuous and right differentiable. Then for any $\varepsilon > 0$ there is $x \in [a, b)$ with $f'_+(x) \leq \frac{f(b)-f(a)}{b-a} + \varepsilon$.*

Proof. Without loss of generality we may assume $a = 0$, $b = 1$ and $f(0) = f(1) = 0$. Suppose for contradiction that we have $f'_+(x) \geq \varepsilon > 0$ for all $x \in [0, 1]$. Let $B = \{x : f(x) \geq \varepsilon x/2\}$. As $f(1) = 0$ we have $1 \notin B$. As f is continuous, B is closed. The required contradiction will thus follow if we show that B is open to the right, i.e. for any $x \in B$ there is $\delta > 0$ such that $(x, x+\delta) \subseteq B$. To see this, note that for small enough δ, by definition of $f'_+(x)$ we have $f(y) \geq f(x) + (y - x)\varepsilon/2 \geq \varepsilon y/2$ for any $y \in (x, x + \delta)$, so $y \in B$. $\square$

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2.4. **Stability.** In this final subsection we prove our theorem on stability for vertex isoperimetry in G_B. For convenience, we work with sumsets, which is an equivalent setting via the identity $\partial_{v,G_B}(n) = |A + (B \cup \{0\})| - |A|$ for $A, B \subseteq \mathbb{Z}^d$.

Proof of Theorem 1.1. Let B be a generating set of $\mathbb{Z}^d$ with $d \geq 2$, and assume without loss of generality that $0 \in B$. Suppose that $K = K(B, d)$ is sufficiently large and $A \subseteq \mathbb{Z}^d$ is such that $|A| = n \geq K$ and $|A + B| \leq \alpha|A|$, where

$$\alpha = 1 + (1 + \varepsilon)\beta n^{-1/d}, \text{ with } \beta = d\mu(C(B))^{1/d} \text{ and } Kn^{-1/2d} < \varepsilon < K^{-1}.$$

We need to find $v \in \mathbb{Z}^d$ with $|A \triangle (v + B_n)| < Kn\sqrt{\varepsilon}$.

By Lemma 2.3, there are $p \in \mathbb{N}$, $z \in \mathbb{Z}^d$ and a nice fundamental set $Q \subseteq Z(B)$ such that, for every k, $kC(B) + Q + z \subseteq \Sigma_{k+p}(B) + Q$. With foresight, we choose $k = \lceil n^{1/2d} \rceil$. By Lemma 2.2, there is a non-empty subset $A' \subseteq A$ with $|A' + \Sigma_{k+p}(B)| \leq \alpha^{k+p}|A'|$. It now suffices to prove Claim 2.6.

Claim 2.6. $|A'| \geq (1 + 2\varepsilon)^{-d}n$ and $|A' \triangle (v + B_n)| \leq \frac{1}{2}Kn\sqrt{\varepsilon}$ for some $v \in \mathbb{Z}^d$.

To see the bound on $|A'|$, we use the choice of Q and Brunn–Minkowski to get

$$\begin{aligned}\alpha^{k+p}|A'| &\geq |A' + \Sigma_{k+p}(B)| = \mu(A' + \Sigma_{k+p}(B) + Q) \geq \mu(A' + kC(B) + Q) \\ &\geq \big(\mu(A' + Q)^{1/d} + \mu(kC(B))^{1/d}\big)^d = (|A'|^{1/d} + k\mu(C(B))^{1/d})^d.\end{aligned}$$

Expanding the last expression and dividing throughout by $|A'|$ then gives

$$1 + k\beta|A'|^{-1/d} \leq \alpha^{k+p} = 1 + (1 + \varepsilon)k\beta n^{-1/d} + O(n^{-1/d}) < 1 + (1 + 2\varepsilon)k\beta n^{-1/d},$$

from which the first part of the claim follows.

For the second part of the claim, let $A_r := \overline{A' + Q + rC(B)}$ and $f(r) = \mu(A_r)$. Since Q, and so A_r, is nice, by (2.2) we have $\mathrm{Per}_{C(B)}(A_r) = f'_+(r)$ for all $r \geq 0$.

Now $f(k) - f(0) < (\alpha^{k+p} - 1)|A'| < (1 + 2\varepsilon)k\beta n^{-1/d}|A'|$, so by Lemma 2.5 with $\varepsilon = 1$ there is an $r \in [0, k)$ such that

$$\mathrm{Per}_{C(B)}(A_r) \leq \frac{f(k) - f(0)}{k} + 1 < (1 + 3\varepsilon)\beta n^{-1/d}|A'|.$$

Then by Theorem 2.4, the asymmetric index $\mathcal{A}_{C(B)}(A_r)$ of A_r with respect to $C(B)$ is at most $D\sqrt{3\varepsilon}$. Thus, there is $t \in \mathbb{R}^d$ such that

$$\mu\big(A_r \triangle (t + r'C(B))\big) \leq D\sqrt{3\varepsilon}\mu(A_r),$$

where $r' = (\mu(A_r)/\mu(C(B)))^{1/d}$. As $\mu(A_r) \leq \mu(A_k) \leq \alpha^{k+p}|A'|$, we have

$$r' < q := (\alpha^{k+p}|A'|/\mu(C(B)))^{1/d}.$$

Since $Q \subseteq Z(B)$, by increasing D if necessary we may assume that $Q \subseteq DC(B)$. Let v be the unique lattice point in $t + Q$. We will show that v satisfies the claim.

To see this, we start by applying the triangle inequality to get

$$|A' \triangle (v + B_n)| \leq \mu((A' + Q) \triangle (t + qC(B))) + \mu((t + qC(B)) \triangle (v + B_n + Q)). \tag{2.3}$$

We will use the inequality $\mu(X \triangle Y) \leq 2\mu(X \setminus Y) + |\mu(X) - \mu(Y)|$ holding for any measurable X and Y to estimate both terms on the right of (2.3). Using

$$\mu\big((A' + Q) \setminus (t + qC(B))\big) \leq \mu\big(A_r \setminus (t + r'C(B))\big) \leq D\sqrt{3\varepsilon}\alpha^{k+p}|A'|,$$

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we bound the first term as

$$\begin{aligned}\mu\big((A'+Q)\triangle(t+qC(B))\big) &\le 2\mu\big((A'+Q)\setminus(t+qC(B))\big)+|\mu\big(qC(B))\big)-|A'||\\ &\le \big(2D\sqrt{3\varepsilon}\alpha^{k+p}+\alpha^{k+p}-1\big)|A'|\le 4D\sqrt{\epsilon}n,\end{aligned}$$

since $\alpha^{k+p}=1+O(n^{-1/2d})$. For the second, we observe that

$$\begin{aligned}\mu\big((v+B_n+Q)\setminus(t+qC(B))\big) &\le \mu\big((t+B_n+2Q)\setminus(t+qC(B))\big)\\ &\le \mu\big((\kappa_B(n)+2D)C(B))\setminus qC(B)\big),\end{aligned}$$

so the second term on the right of (2.3) is $O(n^{1-1/2d})$, as $|B_n|$, $\mu((\kappa_B(n)+2D)C(B))$ and $\mu(qC(B))$ are all $n+O(n^{1-1/2d})$. This proves the claim, and so the theorem. □

3. Edge isoperimetry

In this short section we deduce our stability result for edge isoperimetry from our stability result for vertex isoperimetry proved in the previous section. We use the reduction in [1] linking small edge boundaries to small vertex boundaries in a related graph.

Lemma 3.1. *Let B be a generating set of $\mathbb{Z}^d$ and let $\gamma>0$. Then there exists an $s\in\mathbb{N}$ sufficiently large depending on γ, an $\epsilon>0$ sufficiently small depending on s, and a $K\in\mathbb{N}$ sufficiently large depending on ϵ, such that the following holds: Whenever*

- *$A\subseteq\mathbb{Z}^d$ with $|A|=n\ge K$ and*
- *$\partial_{e,G_B}(A)\le d\mu(Z(B))^{1/d}n^{1-1/d}(1+\varepsilon)$,*

we have that $\partial_{v,G_{\Sigma_s[B]}}(A)\le s\partial_{e,G_B}(A)(1+\gamma)$.

Lemma 3.1 is combined with Ruzsa's lower bound on $\partial_{v,G_{\Sigma_s[B]}}(A)$ to obtain a lower bound on $\partial_{e,G_B}(A)$ and prove [1, Theorem 1]. The statement above is obtained by stripping out the application of Ruzsa's theorem from that proof.

Proof of Theorem 1.2. Let B be a generating set of $\mathbb{Z}^d$ with $d\ge 2$ and let $\delta>0$. Let $K'\in\mathbb{N}$ be the K given by applying Theorem 1.1 to $[B]$, and fix $\gamma=\frac{\delta^2}{8K'^2}$. Given generating set B and γ, we adopt the parameter hierarchy given in Lemma 3.1, namely $K^{-1}\ll\varepsilon\ll s^{-1}\ll\gamma$, i.e. let s be large given γ, let ε be small given s, and let K be large given ε (where we further impose $K\ge K'$).

Now suppose $A\subseteq\mathbb{Z}^d$ is such that $|A|=n\ge K$ and $\partial_{e,G_B}(A)\le\beta n^{1-1/d}(1+\varepsilon)$, where $\beta=d\mu(Z)^{1/d}$ with $Z=Z(B)=C([B])$. We need to find $v\in\mathbb{Z}^d$ with $|A\triangle(v+[B]_n)|<\delta n$.

From Lemma 3.1 we have $\partial_{v,G_{\Sigma_s[B]}}(A)\le s\beta n^{1-1/d}(1+\varepsilon)(1+\gamma)$. Rewriting $\partial_{v,G_{\Sigma_s[B]}}(A)$ as $|A+\Sigma_s([B])|-|A|$, we see from the telescoping sum

$$|A+\Sigma_s([B])|-|A|=\sum_{j=0}^{s-1}|A+\Sigma_{j+1}([B])|-|A+\Sigma_j([B])|,$$

that we can fix $A_+=A+\Sigma_j([B])$ for some $j<s$ such that

$$|A_++[B]|-|A_+|\le\beta n^{1-1/d}(1+\varepsilon)(1+\gamma)\le\beta n_+^{1-1/d}(1+2\gamma),$$

where $n_+=|A_+|\le n+O(n^{1-1/d})$. By Theorem 1.1 we have $|A_+\triangle(v+B_{n_+})|\le K'n_+\sqrt{2\gamma}=\frac12\delta n_+$ for some $v\in\mathbb{Z}^d$. Then $|A\triangle(v+[B]_n)|\le|A_+\triangle(v+[B]_{n_+})|+|A_+\setminus A|+|[B]_{n_+}\setminus[B]_n|\le\delta n$. □

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Our proof of Theorem 1.2 does not give a tight quantitative result, but we will now demonstrate a simple trick that provides such a result when the generating set B takes the form $\{\pm v : v \in \mathcal{B}\}$ for some integral basis $\mathcal{B}$ of $\mathbb{Z}^d$ (which may as well be the standard basis $\{e_1, \dots, e_d\}$).

Theorem 3.2. *Let $B = \{\pm e_i : i \in [d]\} \subseteq \mathbb{Z}^d$ with $d \geq 2$. Then there is $K \in \mathbb{N}$ so that for any $A \subseteq \mathbb{Z}^d$ such that $|A| = n \geq K$ and $\partial_{e,G_B}(A) \leq 2dn^{1-1/d}(1+\varepsilon)$, where $Kn^{-1/2d} < \varepsilon < K^{-1}$, there exists $v \in \mathbb{Z}^d$ with $|A \triangle (v + [B]_n)| < Kn\sqrt{\varepsilon}$.*

Proof. Let $A' = A + [-1/2, 1/2]^d$. Then the edges of G_B counted by $\partial_{e,G_B}(A)$ are in bijection with those $(d-1)$-cubes that occur exactly once as $x + C$ with $x \in A$ and C a facet of $[-1/2, 1/2]^d$. Thus $\partial_{e,G_B}(A) = \mathrm{Per}_Z(A')$, where $Z = Z(B) = [-1,1]^d$. By Theorem 2.4 we have $\mathcal{A}_Z(A') \leq D\sqrt{\varepsilon}$, i.e. there is $x \in \mathbb{R}^d$ with $\mu(A' \triangle (x + rZ)) \leq nD\sqrt{\varepsilon}$, where $\mu(rZ) = (2r)^d = n$. We fix $v \in \mathbb{Z}^d$ with $x + (r+1)Z \subseteq v + (r+2)Z$. As

$$(A \setminus (x + (r+1)Z)) + [-1/2, 1/2]^d \subseteq A' \setminus (x + rZ)$$

we have $|A \setminus (v + (r+2)Z)| \leq nD\sqrt{\varepsilon}$. The theorem now follows from

$$A \triangle (v + [B]_n)| = 2|A \setminus (v + (r+2)Z)| + O(n^{1-1/d}) < Kn\sqrt{\varepsilon}. \qquad \square$$

We remark that by considering $A' = A + [-1/2, 1/2]^d$ as in the previous proof one can obtain a bound for the edge isoperimetric problem in G_B that is tight in some cases. Indeed, by the anisotropic isoperimetric inequality, $\partial_{e,G_B}(A) = \mathrm{Per}_Z(A') \geq d\mu(Z)^{1/d}\mu(A')^{1-1/d} = 2dn^{1-1/d}$, which is tight whenever $n = k^d$ for some $k \in \mathbb{N}$. Bollobás and Leader [4] gave a tight result for general n, although they used compression techniques that alter structure, so it is interesting that exact results can also be obtained by geometric methods, and moreover in two ways: by the argument above, or by the Loomis-Whitney inequality as in [6].

4. Concluding remarks

As mentioned in the introduction, there are several challenging and important open problems in isoperimetric stability, such as the Kahn–Kalai Conjecture and the Polynomial Freiman–Ruzsa Conjecture. We therefore find it rather striking that in this short paper we have been able to characterise isoperimetric stability for general Cayley graphs in lattices. Of course, the brevity of our paper masks the fact that we have greatly relied on previous work, particularly an analogous stability result of [7] in Geometric Measure Theory. This naturally suggests that further investigation of transformations between the discrete and continuous settings may be fruitful in future research.

For the isoperimetric problems considered in this paper, it is natural to ask if one can obtain tighter estimates than those in the asymptotic results of [1, 25], particularly for the edge boundary, where the probabilistic reduction in [1] introduces error terms that are presumably far from optimal. Does our improved estimate for the edge isoperimetric inequality in the remark following Theorem 3.2 hold for general B, i.e. do we always have $\partial_{e,G_B}(A) \geq d\mu(Z(B))^{1/d}|A|^{1-1/d}$? Do we always have $\partial_{v,G_B}(A) \geq d\mu(C(B))^{1/d}|A|^{1-1/d}$?

It would also be interesting to quantify the dependence on the dimension d of the constants K in our theorems. Our use of [7] gives $K = O(d^7)$, whereas in the case of the ℓ^1-grid the authors of [6] show $K = O(d^{5/2})$ and conjecture $K = O(\sqrt{d})$.

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Finally, a challenging direction for further research is to understand the structure of large sets A for which $|A+B|-|A|$ is within a multiplicative $O(1)$ factor of its minimum value. We conjecture that any such A can be covered by $O(1)$ homothetic copies of $C(B)$ with total volume $O(|A|)$.

More generally, in the spirit of the Kahn–Kalai and Polynomial Freiman–Ruzsa Conjectures, we pose the following (somewhat vague) question for any (natural) isoperimetric problem: can any set with boundary $O(\cdot)$ of the minimum possible be 'almost' covered by $O(1)$ 'canonical examples' of size $O(|A|)$, perhaps even with polynomially-related constants?

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