

RESEARCH ARTICLE

The sharp doubling threshold for approximate convexity

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Abstract

We show for $A, B \subset \mathbb{R}^d$ of equal volume and $t \in (0, 1/2]$ that if $|tA + (1-t)B| < (1+t^d)|A|$, then (up to translation) $|\text{co}(A \cup B)|/|A|$ is bounded. This establishes the sharp threshold for the quantitative stability of the Brunn–Minkowski inequality recently established by Figalli, van Hintum, and Tiba, the proof of which uses our current result. We additionally establish a similar sharp threshold for iterated sumsets.

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1 | INTRODUCTION

The Brunn–Minkowski inequality asserts that for sets $A, B \subset \mathbb{R}^d$ with equal volume and $t \in (0, 1/2]$, we have

$$|tA + (1-t)B| \geq |A|,$$

with equality exactly if $A = B$ is a convex set. The stability of this inequality has sparked a rich body of research (e.g., [2, 4, 6–12, 14–16]). These results variously control (up to translation[†]) $|A \triangle B|$, $|\text{co}(A) \setminus A|$, and $|\text{co}(A \cup B) \setminus A|$ (where $\text{co}(A)$ is the convex hull of A , i.e., the smallest convex set containing A) in terms of the parameter:

$$\delta_t(A, B) := \frac{|tA + (1-t)B|}{|A|} - 1 \geq 0.$$

[†] That is, there exists $x \in \mathbb{R}^n$ so that $|A \triangle (B+x)|$ or $|\text{co}(A \cup (B+x)) \setminus A|$ is small.

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In [7], Figalli and Jerison showed that there exist $a_{d,t}, c_{d,t}, \Delta_{d,t} > 0$, so that if $\delta = \delta_t(A, B) \leq \Delta_{d,t}$, then (up to translation)

$$|\text{co}(A \cup B) \setminus A| \leq c_{d,t} \delta^{a_{d,t}} |A|.$$

The question of determining the optimal values of $a_{d,t}$ and $c_{d,t}$ has received a lot of attention. The optimal values for general $A, B \subset \mathbb{R}^d$ were recently determined to be $a_{d,t} = 1/2$ and $c_{d,t} = O_d(t^{-1/2})$ by Figalli, van Hintum, and Tiba [12] concluding a long line of partial results for specific classes of sets [1, 3, 9, 10]. In their proof, they use our Theorem 1.1 as a crucial tool (cited as [12, Proposition 8.3]). For the particular case $A = B$, the stronger result with $a_{d,t} = 1$ and $c_{d,t} = t^{-1} \exp(O(d \log(d)))$ was established in [14] proving a conjecture from [8].

In this paper, we determine the optimal value of $\Delta_{d,t}$ for these results. We establish this bound both for general $A, B \subset \mathbb{R}^d$ and for iterated sumsets. Both can be extended to quantitative stability results for all doublings below this threshold.

Theorem 1.1. *For all $d \in \mathbb{N}, t \in (0, 1/2]$, there are $C_{d,t} > 0$ so that if $A, B \subset \mathbb{R}^d$ of the same volume have $|tA + (1 - t)B| < (1 + t^d)|A|$, then (after possibly translating) $|\text{co}(A \cup B)| \leq C_{d,t}|A|$.*

In fact, we can choose $C_{d,t} = t^{-O(d^2)}$. The second theorem determines this threshold for iterated sumsets. For $X \subset \mathbb{R}^d$ and $k \in \mathbb{N}$, we write $k \cdot X := \underbrace{X + \dots + X}_{k \text{ terms}}$.

Theorem 1.2. *For all $d, k \in \mathbb{N}$, there are $C_{d,k} > 0$ so that if $A \subset \mathbb{R}^d$ satisfies $|k \cdot A| < (1^d + \dots + k^d)|A|$, then $|\text{co}(A)| \leq C_{d,k}|A|$.*

Remark 1.3. Cole Hugelmeyer, Hunter Spink, and Jonathan Tidor established Theorem 1.2 independently [17]. Theorem 1.1 for $t = 1/2$ coincides with Theorem 1.2 for $k = 2$. This result as well as Corollary 1.4 for $t = 1/2$ are established through independent methods in [13, Corollary 1.9].

Combining Theorem 1.1 with the main result from [12] (included here as Theorem 5.1), we find the following quantitative stability of the Brunn–Minkowski inequality.

Corollary 1.4. *For all $d \in \mathbb{N}, t \in (0, 1)$, there exist $a_{d,t}, C_{d,t} > 0$ so that if $A, B \subset \mathbb{R}^d$ of the same volume satisfy $\delta := \delta_t(A, B) < t^d$, then (up to translation) we have*

$$|\text{co}(A \cup B) \setminus A| \leq C_{d,t} \delta^{1/2} |A|, \text{ and}$$

$$|\text{co}(A) \setminus A| + |\text{co}(B) \setminus B| \leq C_{d,t} \delta |A|.$$

All these results are sharp as shown by $A = [0, 1]^d$ and $B = A \cup \{v\}$, where $v \in \mathbb{R}^d$ is some arbitrarily large vector. For these A, B , we have

$$tA + (1 - t)B = A \cup ([0, t]^d + (1 - t)v),$$

so that $\delta_t(A, B) = t^d$, while $\frac{|\text{co}(B) \setminus B|}{|B|} \rightarrow \infty$ as $\|v\|_2 \rightarrow \infty$.

In Section 2, we establish strong versions of the results for $d = 1$, which are instrumental in the proof of the general results. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 1.1. Note that the proof of Theorem 1.2 in Section 3 is a more accessible version of the proof of Theorem 1.1 in Section 4. Finally, in Section 5, we include a proof of the corollary.

The idea in both proofs is to find two points in A (or B) for each coordinate direction that are very far apart, which can be done by increasing $|\text{co}(A)|$ in combination with Lemma 4.2. We then distinguish two cases; either A contains long fibres in all coordinate directions or not. In the former case, we find a lower bound on the doubling using Plünnecke’s inequality[†] as the sum of those long fibres is large (see Claim 4.3). In the latter case, we fix a direction in which the fibres of A are short and show that using an optimal transport map, we can pair up the fibres from A and B whose (weighted) sum form a reference set of size $|A|$ (see Lemma 4.1). Finally we show that summing fibres of B with the two far removed points from A gives a set disjoint from the reference set of the required size (see Lemma 2.1).

2 | STRONG VERSIONS OF FREIMAN’S $3k - 4$ THEOREM

We use two versions of the following lemma. This lemma implies continuous versions of Freiman’s $3k - 4$ theorem for distinct sets.

Lemma 2.1. *Given subsets $X, Y, Z \subset [0, 1]$, we have*

$$|(X + Y) \cup (\{0, 1\} + Z)| \geq \min\{1, |X| + |Y|\} + |Z|.$$

Proof. Let $S := (X + Y) \cup (\{0, 1\} + Z)$. Let $f : \mathbb{R} \rightarrow [0, 1]$; $x \mapsto x - [x]$ be the canonical quotient map. Note that for $z \in Z$, we have $|f^{-1}(z) \cap S| \geq 2$. Moreover, note that $|f(X + Y)| = |f(X) + f(Y)| \geq \min\{1, |f(X)| + |f(Y)|\}$ by Cauchy–Davenport in the torus, so we find:

$$\begin{aligned} |S| &\geq |S \cap f^{-1}(Z)| + |S \setminus f^{-1}(Z)| \\ &\geq |(\{0, 1\} + Z) \cap f^{-1}(Z)| + |(X + Y) \setminus f^{-1}(Z)| \\ &\geq 2|Z| + |f(X + Y) \setminus Z| \\ &\geq 2|Z| + (\min\{1, |X| + |Y|\} - |Z|) \\ &= \min\{1, |X| + |Y|\} + |Z|. \end{aligned}$$

The lemma follows. □

We will only apply Lemma 2.1 to sets with $|X|, |Y|, |Z| \leq \frac{1}{2}$, so that the bound gives $|X| + |Y| + |Z|$.

Note that for sets $A, B \subset \mathbb{R}$ with $|\text{co}(A)| \geq |\text{co}(B)|$, up to rescaling and translating, we can assume $\text{co}(A) = [0, 1] \supset B$. In particular, we may assume $\{0, 1\} \subset A$, so that $|A + B| = |(A + B) \cup (\{0, 1\} + B)|$. Setting $X = A$ and $Y = Z = B$, we find that Lemma 2.1 implies

$$|A + B| \geq |B| + \min\{|\text{co}(A)|, |A| + |B|\} = |A| + |B| + \min\{|\text{co}(A) \setminus A|, |B|\},$$

which can be seen as a stronger version of the one-dimensional instance of Theorem 1.1.

[†] Plünnecke’s inequality [20] states that $|X + Y| \leq \lambda|Y|$ implies $|d \cdot X| \leq \lambda^d|Y|$.

For iterated sumsets, we have the following version of this lemma.

Lemma 2.2. *Suppose $Y_i \subset [0, 1]$ with $|Y_i| \leq 1/k$ for $1 \leq i \leq k$. Let $S := \bigcup_{i=1}^k \{0, 1, \dots, (k - i)\} + i \cdot Y_i$. Then, we have $|S| \geq \sum_i i|Y_i|$.*

Proof. Let $f : \mathbb{R} \rightarrow \mathbb{T}; x \mapsto x - [x]$ be the canonical quotient map from the line to the torus. Note that for $y \in f(i \cdot Y_i)$, we have $|f^{-1}(y) \cap S| \geq k - i + 1$. Moreover, note that $|f(i \cdot Y_i)| \geq i|Y_i|$ by Cauchy–Davenport. Let $Z_i := f(i \cdot Y_i) \setminus \bigcup_{j < i} f(j \cdot Y_j)$. With a little thought (e.g., by induction on k), we find $\sum_i (k - i + 1)|Z_i| \geq \sum_i |f(i \cdot Y_i)| \geq \sum_i i|Y_i|$. Combining these, we find

$$\begin{aligned} |S| &\geq \sum_i |S \cap f^{-1}(Z_i)| \\ &\geq \sum_i \left| (\{0, 1, \dots, (k - i)\} + i \cdot Y_i) \cap f^{-1}(Z_i) \right| \\ &\geq \sum_i (k - i + 1)|Z_i| \\ &\geq \sum_i i|Y_i|. \end{aligned}$$

The lemma follows. □

Though we shall only apply the lemma in its current form, the proof actually gives the following (stronger) result when all the Y_i 's are the same, but not necessarily small in their convex hull. For $A \subset \mathbb{R}$, rescaling and translating, we may assume $\{0, 1\} \subset A \subset [0, 1]$, so that

$$\bigcup_{i=1}^k \{0, 1, \dots, (k - i)\} + i \cdot A = \bigcup_{i=1}^k (k - i) \cdot \{0, 1\} + i \cdot A = k \cdot A.$$

Using the more general Cauchy–Davenport bound $|f(i \cdot A)| \geq \min\{| \text{co}(A) |, i|A|\}$, we find

$$|k \cdot A| \geq \sum_{i=1}^k \min\{i|A|, | \text{co}(A) |\} = \binom{\ell + 1}{2} |A| + (k - \ell) | \text{co}(A) |,$$

where $\ell := \min \left\{ \left\lfloor \frac{| \text{co}(A) |}{|A|} \right\rfloor, k \right\}$. This can be seen as the continuous version of Corollary 1 from [18].

In the most dense situation (i.e., $\ell = 1$), this gives that if $|k \cdot A| \leq (k + 1)|A|$, then $| \text{co}(A) \setminus A | \leq \frac{1}{k-1} (|k \cdot A| - k|A|)$. In the least dense situation (i.e., $\ell = k$), this is a sharp version of Theorem 1.2 in one dimension. For $k = 2$, this reduces to the continuous version of Freiman’s $3k - 4$ theorem. These results are sharp as shown by a union of an interval with a point.

3 | ITERATED SUMSETS: THEOREM 1.2

Proof of Theorem 1.2. For $i = 1, \dots, d$, let $\pi_i : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ be the projection onto the coordinate hyperplane spanned by the basis vectors $e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_d$. Let $L = L_{d,k}$ be sufficiently large in terms of k and d and $C_{d,k}$ sufficiently large in terms of L .

Aiming for a contradiction, we assume $|\text{co}(A)| > C_{d,k}|A|$.

Note that the statement of the theorem is affine invariant, so that we may normalize so that $|A| = 1$. We will now apply further transformations to put A in a standard form; these are not hard to justify directly and also follow from Lemma 4.2 below. First, we can apply a volume preserving affine transformation to $X = Y = \text{co}(A)$ to find that a new set (which for notational convenience we will still call A) which contains fibres in all coordinate directions, such that the product of the lengths of these fibres is the volume of a parallelotope containing $\text{co}(A)$. Second, after rescaling each direction while preserving the volume, all of these fibres can be assumed to have the same length $L' > L$. Hence, we find for $i = 1, \dots, d$ points $x^i \in \mathbb{R}^{d-1}$ so that $|\pi_i^{-1}(x^i) \cap \text{co}(A)| = \max_{x \in \mathbb{R}^{d-1}} |\pi_i^{-1}(x) \cap \text{co}(A)| = L' > L$.

Claim 3.1. If for all i , there exists a $y_i \in \mathbb{R}^{d-1}$ with $|\pi_i^{-1}(y_i) \cap A| \geq L/k$, then $|k \cdot A| \geq (1^d + 2^d + \dots + k^d)|A|$

Proof. For a contradiction assume $|k \cdot A| < (1^d + 2^d + \dots + k^d)|A|$. Then by Plünnecke’s inequality, we have $|d \cdot A| < (1^d + 2^d + \dots + k^d)^d |A|$. However,

$$|d \cdot A| \geq \left| \sum_{i=1}^d (\pi_i^{-1}(y_i) \cap A) \right| \geq (L/k)^d > (1^d + 2^d + \dots + k^d)^d.$$

This contradiction proves the claim. □

Hence, we may assume there is a coordinate direction i with $|\pi_i^{-1}(y_i) \cap A| \leq L/k$ for all $y_i \in \mathbb{R}^{d-1}$. Rotating if necessary, we may assume $i = 1$. For notational convenience, write $A_x := \pi_1^{-1}(x) \cap A$ and $S_x := \pi_1^{-1}(x) \cap (k \cdot A)$.

Translate A so that $x^1 = 0$, and $A_0 \supset \{0, L'\} \times (0, \dots, 0)$. Now we find that

$$S_x \supset \bigcup_{i=1}^k i \cdot A_{x/i} + (k - i) \cdot A_0$$

which by Lemma 2.2 implies

$$|S_x| \geq \sum_{i=1}^k i |A_{x/i}|.$$

We conclude:

$$\begin{aligned} |k \cdot A| &= \int_{x \in \mathbb{R}^{d-1}} |S_x| dx \\ &\geq \int_{x \in \mathbb{R}^{d-1}} \sum_{i=1}^k i |A_{x/i}| dx \\ &= \sum_{i=1}^k i^d \int_{x \in \mathbb{R}^{d-1}} |A_x| dx \\ &= (1^d + \dots + k^d) |A|. \end{aligned}$$

This concludes the proof of the theorem. □

4 | DISTINCT SETS: THEOREM 1.1

We use the following standard lemma that establishes the existence of a large subset of $tA + (1 - t)B$. For an exposition, see, for example, Section 3, Step 1 in [5], for a proof of this specific lemma, see [19, Appendix D].

Lemma 4.1 [19]. *Let $\mu_A, \mu_B : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be two probability measures and $T : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}$ the optimal transport map so that for all measurable $X \subset \mathbb{R}^{d-1}$, we have $\mu_A(X) = \mu_B(T(X))$. For $t \in (0, 1)$, let $\rho_t : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be defined by $\rho_t(x) := t\mu_A(y) + (1 - t)\mu_B(T(y))$, where $y \in \mathbb{R}^{d-1}$ is the unique element so that $x = ty + (1 - t)T(y)$. Then $\int \rho_t \geq 1$.*

We also need the following lemma that translates any two sets into a common bounding polytope in which each perpendicular height corresponds to a fibre of one of the sets. We note that the case when $X = Y$ is straightforward and was used in the previous section.

Lemma 4.2. *Given convex sets $X, Y \subset \mathbb{R}^d$ and an orthonormal basis e_1, \dots, e_d , there exists a translation $v \in \mathbb{R}^d$ and a volume-preserving affine transformation $T : \mathbb{R}^d \rightarrow \mathbb{R}^d$, so that if we let $U := T(X)$ and $V := v + T(Y)$, then there are points $p^i \in U \cup V$, $\lambda_i \in \mathbb{R}$ and hyperplanes $H_i \subset \mathbb{R}^d$ (for $1 \leq i \leq d$) so that:*

1. $p^i, p^i + \lambda_i e_i \in U$, or $p^i, p^i + \lambda_i e_i \in V$,
2. $U \cup V \subset H_i + [0, \lambda_i]e_i$, and
3. $|\bigcap_i (H_i + [0, \lambda_i]e_i)| = \prod_i \lambda_i$.

Proof. We proceed by induction on k ; assume after an affine transformation and translation, there are (for $1 \leq i \leq k$) $p^i \in X \cup Y$, $\lambda_i \in \mathbb{R}$, and hyperplanes $H_i \subset \mathbb{R}^d$ so that:

1. if $p^i \in X$, then $p^i + \lambda_i e_i \in X$ and if $p^i \in Y$, then $p^i + \lambda_i e_i \in Y$,
2. $X \cup Y \subset H_i + [0, \lambda_i]e_i$,
3. $|\mathbb{R}^k \times \{0\}^{d-k} \cap \bigcap_{1 \leq i \leq k} (H_i + [0, \lambda_i]e_i)| = \prod_{1 \leq i \leq k} \lambda_i$.

Translating one of the sets along a multiple of e_{k+1} , we may assume that the points $q, r \in X \cup Y$ minimizing $\langle e_{k+1}, q \rangle$ and maximizing $\langle e_{k+1}, r \rangle$ belong to the same set X or Y . Note that translating along e_{k+1} does not affect any of the properties of p^i, λ_i , and H_i with $i \leq k$ (up to the appropriate translations). For notational convenience, translate both sets by $-q$ (i.e., assume $q = 0$). Let $H_{k+1} := \mathbb{R}^k \times \{0\} \times \mathbb{R}^{d-k-1}$ and $\lambda_{k+1} := \langle e_{k+1}, r \rangle$. Clearly, we have $X \cup Y \subset H_{k+1} + [0, \lambda_{k+1}]e_{k+1}$ and

$$\begin{aligned} \left| \mathbb{R}^{k+1} \times \{0\}^{d-k-1} \cap \bigcap_{1 \leq i \leq k+1} (H_i + [0, \lambda_i]e_i) \right| &= \left| \mathbb{R}^k \times \{0\}^{d-k} \cap \bigcap_{1 \leq i \leq k} (H_i + [0, \lambda_i]e_i) \right| \cdot |\lambda_{k+1}e_{k+1}| \\ &= \prod_{1 \leq i \leq k+1} \lambda_i. \end{aligned}$$

The only issue is that though $r \in H_{k+1} + \lambda_{k+1}e_{k+1}$, it might not coincide with $\lambda_{k+1}e_{k+1}$. Hence, we apply the affine transformation $T : \mathbb{R}^d \rightarrow \mathbb{R}^d, x \mapsto x - \langle x, e_{k+1} \rangle (\lambda_{k+1}^{-1} r - e_{k+1})$ to X, Y, p^i , and H_i (for $i \leq k$). T preserves all planes parallel to $\mathbb{R}^k \times \{0\} \times \mathbb{R}^{d-k-1} = H_{k+1}$, and takes r to $\lambda_{k+1}e_{k+1}$. As the basis vectors e_i ($i \neq k+1$) are preserved by T , the inductive hypotheses are not affected. Choosing $p^{k+1} = (0, \dots, 0)$ concludes the induction. \square

Proof of Theorem 1.1. Let $C_{d,t} := (L/t)^d$ where $L = L_{d,t} := \left(\frac{4}{(1-t)t}\right)^{2d}$. We will prove the contrapositive, so let $|\text{co}(A \cup (B + x))| > C_{d,t}|A|$ for all translates $x \in \mathbb{R}^n$. Normalize so that $|A| = |B| = 1$.

For $i = 1, \dots, d$, let $\pi^i : \mathbb{R}^d \rightarrow \mathbb{R}$ be the coordinate projections and $\pi_i : \mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$ the complementary projections.

Apply Lemma 4.2 to $\text{co}(tA)$ and $\text{co}((1-t)B)$ to find a volume preserving affine transformation T , translation v , points p^i , scalars λ_i , and hyperplanes H_i . Since the theorem is affine invariant, we may assume T is the identity and $v = \vec{0}$. We find that

$$C_{d,t}t^d \leq |\text{co}((tB) \cup (tA))| \leq |\text{co}(((1-t)B) \cup (tA))| \leq \left| \bigcap_i (H_i + [0, \lambda_i]e_i) \right| = \prod_i \lambda_i.$$

After another volume preserving affine transformation, we may assume all λ_i to be equal to some $L' \geq L$.

Hence, for $i = 1, \dots, d$, find the points $x^i \in \mathbb{R}^{d-1}$ so that

$$\begin{aligned} & \max \{ |\pi_i^{-1}(x^i) \cap \text{co}(tA)|, |\pi_i^{-1}(x^i) \cap \text{co}((1-t)B)| \} \\ &= \max_{x \in \mathbb{R}^{d-1}} \max \{ |\pi_i^{-1}(x) \cap \text{co}(tA)|, |\pi_i^{-1}(x) \cap \text{co}((1-t)B)| \}. \end{aligned}$$

Note that for these x^i , we in particular have

$$\begin{aligned} & \max \{ |\text{co}(\pi_i^{-1}(x^i) \cap tA)|, |\text{co}(\pi_i^{-1}(x^i) \cap (1-t)B)| \} \\ &= \max \{ |\pi_i^{-1}(x^i) \cap \text{co}(tA)|, |\pi_i^{-1}(x^i) \cap \text{co}((1-t)B)| \} \\ &= \max_{x \in \mathbb{R}^{d-1}} \{ |\pi_i^{-1}(x) \cap \text{co}(tA)|, |\pi_i^{-1}(x) \cap \text{co}((1-t)B)| \} \\ &= L' > L. \end{aligned}$$

Let $a^i, b^i \in \mathbb{R}^{d-1}$ be such that

$$|\pi_i^{-1}(a^i) \cap tA| = \max_{a \in \mathbb{R}^{d-1}} |\pi_i^{-1}(a) \cap tA| \text{ and } |\pi_i^{-1}(b^i) \cap (1-t)B| = \max_{b \in \mathbb{R}^{d-1}} |\pi_i^{-1}(b) \cap (1-t)B|.$$

Claim 4.3. If for all $i = 1, \dots, d$, we have $\max \{ |\pi_i^{-1}(a^i) \cap tA|, |\pi_i^{-1}(b^i) \cap (1-t)B| \} \geq \sqrt{L}$, then $|tA + (1-t)B| \geq 2|A|$.

Proof. For a contradiction, assume $|tA + (1-t)B| < 2|A|$. Distinguish two cases, either for all i ,

$$\min \{ |\pi_i^{-1}(a^i) \cap tA|, |\pi_i^{-1}(b^i) \cap (1-t)B| \} \geq 1$$

or not.

In the latter case, consider the i_0 so that $\min \{ |\pi_{i_0}^{-1}(a^{i_0}) \cap tA|, |\pi_{i_0}^{-1}(b^{i_0}) \cap (1-t)B| \} \leq 1$. Assume $|\pi_{i_0}^{-1}(a^{i_0}) \cap tA| \leq 1$ (the other case follows analogously). As $|tA| = t^d$, this implies $|\pi_{i_0}(tA)| \geq t^d$, so that

$$|tA + (1-t)B| \geq |\pi_{i_0}(tA)| \cdot \left| \left(\pi_{i_0}^{-1}(b^{i_0}) \cap (1-t)B \right) \right| \geq t^d \sqrt{L} > 2,$$

a contradiction.

Hence, we may assume $\min \{ |\pi_i^{-1}(a^i) \cap tA|, |\pi_i^{-1}(b^i) \cap (1-t)B| \} \geq 1$ for all i . Note that by Plünnecke’s inequality $|tA + (1-t)B| < 2|A| = 2t^{-d}|tA|$ implies

$$|d \cdot (1-t)B| \leq (2t^{-d})^d |tA| = 2^d t^{-d^2} t^d < \left(\frac{2}{t}\right)^{d^2}.$$

Analogously we find the same bound on $|d \cdot tA| \leq \left(\frac{2}{1-t}\right)^{d^2}$. On the other hand, we find

$$|d \cdot (1-t)B| \geq \left| \sum_{i=1}^d (\pi_i^{-1}(b^i) \cap (1-t)B) \right| = \prod_{i=1}^d |\pi_i^{-1}(b^i) \cap (1-t)B|,$$

and the analogous result for tA . Combining these bounds on the iterated sumsets of tA and $(1-t)B$, we find

$$\left(1 \cdot \sqrt{L}\right)^d \leq \prod_{i=1}^d |\pi_i^{-1}(a^i) \cap tA| \cdot |\pi_i^{-1}(b^i) \cap (1-t)B| \leq |d \cdot tA| \cdot |d \cdot (1-t)B| < \left(\frac{4}{(1-t)t}\right)^{d^2},$$

a contradiction. □

Hence, we may assume there is a coordinate direction i with $|\pi_i^{-1}(y) \cap tA|, |\pi_i^{-1}(y) \cap (1-t)B| \leq \sqrt{L}$ for all $y \in \mathbb{R}^{d-1}$. Rotating if necessary, we may assume $i = 1$. For notational convenience, for any set $X \subset \mathbb{R}^d$, write $X_x := \pi_1^{-1}(x) \cap X$.

By the application of Lemma 4.2, we have that at least one of $|\text{co}(tA_{x^1})|$ and $|\text{co}((1-t)B_{x^1})|$ is larger than $L > 2 \max_{x \in \mathbb{R}^{d-1}} \{t|A_x|, (1-t)|B_x|\}$. Henceforth assume the latter.[†]

Translate B so that $x^1 = 0$ and $(1-t)\text{co}(B_0) = [0, L'] \times (0, \dots, 0)$.

Write $\mu_A : \mathbb{R}^{d-1} \rightarrow \mathbb{R}, x \mapsto |A_x|$ and μ_B analogously. Let $T : \mathbb{R}^{d-1} \rightarrow \mathbb{R}^{d-1}$ be the optimal transport map that takes μ_A to μ_B . For $x, y \in \mathbb{R}^{d-1}$ define

$$F(x, y) := (t \min(I) + (1-t)J) \cup (tI + (1-t) \max(J)) \times \{tx + (1-t)y\},$$

where $I, J \subset \mathbb{R}$ are so that $A_x = I \times \{x\}$ and $B_y = J \times \{y\}$. Using this function $F(x, y)$, define

$$S^1 := \bigcup_{x \in \pi_1(A)} F(x, T(x)).$$

Note that $|S_x^1| = t\mu_A(y) + (1-t)\mu_B(y)$ where $y \in \mathbb{R}^{d-1}$ is the unique element with $x = ty + (1-t)T(y)$. Hence, by Lemma 4.1, we find $|S^1| = \int |S_x^1| dx \geq |A|$.

Define $S^2 := tA + \{(0, \dots, 0), (L', 0, \dots, 0)\}$ and note that $S^1, S^2 \subset S := tA + (1-t)B$. We will use these two particular subsets of S to show that $|S|$ is large.

For any $x \in \mathbb{R}^{d-1}$, we note that

$$S_{tx} \supset (tA_y + (1-t)B_{T(y)}) \cup (tA_x + \{(0, \dots, 0), (L', 0, \dots, 0)\}),$$

[†] Though the other case follows analogously, there is an asymmetry between t and $1-t$ that gives a stronger result in the other case.

where y is such that $ty + (1 - t)T(y) = tx$ (if such a y exists). Since $|tA_y|, |(1 - t)B_{T(y)}|, |tA_x| < L'/2$, we can apply Lemma 2.1 to find that

$$|S_{tx}| \geq |tA_y| + |(1 - t)B_{T(y)}| + |tA_x| = |S_{tx}^1| + |tA_x|,$$

so that

$$|S_{tx} \setminus S^1| \geq |tA_x|.$$

Integrating over all x , we find

$$\begin{aligned} |tA + (1 - t)B \setminus S^1| &= \int_{x \in \mathbb{R}^{d-1}} |S_x \setminus S^1| \\ &= t^{d-1} \int_{x \in \mathbb{R}^{d-1}} |S_{tx} \setminus S^1| \\ &\geq t^{d-1} \int_{x \in \mathbb{R}^{d-1}} |tA_x| \\ &= t^d \int_{x \in \mathbb{R}^{d-1}} |A_x| \\ &= t^d |A|. \end{aligned}$$

Recalling that $|S_1| \geq |A|$, this concludes the proof of the theorem. □

5 | PROOF OF THE COROLLARY

First recall the main theorems (Theorem 1.3 and Theorem 1.4) from [12].

Theorem 5.1 [12]. *For all $d \in \mathbb{N}$, there exist C_d so that for all $t \in (0, 1)$, there exist $C_{d,t}, \Delta_{d,t} > 0$ so that if $A, B \subset \mathbb{R}^d$ of the same volume satisfy $\delta := \delta_t(A, B) < \Delta_{d,t}$, then (up to translation) we have*

$$|\text{co}(A \cup B) \setminus A| \leq C_d \sqrt{\frac{\delta}{t}} |A| \text{ and}$$

$$|\text{co}(A) \setminus A| + |\text{co}(B) \setminus B| \leq C_{d,t} \delta |A|.$$

The corollary follows quickly.

Proof of Corollary 1.4. Let $C_{d,t} := \max\{c_{d,t}(\Delta_{d,t})^{-1/2}, c'_d t^{-1/2}\}$, where c'_d and $\Delta_{d,t}$ are the constants from Theorem 5.1 and $c_{d,t}$ is the constant from Theorem 1.1.

Distinguish two cases; either $\delta < \Delta_{d,t}$ or $\Delta_{d,t} \leq \delta < t^d$. In the former case, Theorem 5.1 gives (after translation)

$$|\text{co}(A \cup B) \setminus A| \leq c'_d t^{-1/2} \delta^{1/2} |A| \leq C_{d,t} \delta^{1/2} |A|.$$

In the latter case, we find by Theorem 1.1 that

$$|\operatorname{co}(A \cup B) \setminus A| \leq |\operatorname{co}(A \cup B)| \leq c_{d,t} |A| \leq C_{d,t} (\Delta_{d,t})^{1/2} |A| \leq C_{d,t} \delta^{1/2} |A|.$$

Combining the cases gives the first result in the corollary. The second result follows similarly. \square

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